A New Malmquist Index Based on a Standard Technology for Measuring Total Factor Productivity Changes

Aparicio, Juan and Santín, Daniel

Center of Operations Research (CIO), Miguel Hernandez University, Complutense Institute of Economic Analysis (ICAE), Complutense University of Madrid

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Juan Aparicio[1] and Daniel Santín[2*]

[1] Center of Operations Research (CIO), Miguel Hernandez University, 03202, Elche, Spain

[2] Complutense Institute of Economic Analysis (ICAE), Complutense University of Madrid, 28223, Pozuelo de Alarcón, Spain

Abstract

The Malmquist productivity index is one of the best known and most widely used measures in the economic literature to quantify and decompose changes in productivity of multi-input multi-output production processes over time. Two main approaches are used to calculate this index: the adjacent Malmquist index and the base period Malmquist index. No base period is required to calculate the adjacent Malmquist index, but it fails to comply with the circularity property. The base period Malmquist index uses the technology of a base period and is circular, but the base period choice is arbitrary. There is, therefore, a trade-off between the choice of one or other version of the Malmquist index. The aim of this paper is to propose a new total factor productivity index that is simultaneously circular and does not need to resort to a base period or ad hoc reference. To this end, as in other sciences, we propose a new multi-input multi-output reference production technology for use as a standard for measuring and decomposing total factor productivity changes. As discussed, the standard production technology is conceptually attractive. Also, its parameterization is versatile and adaptable to the evolution of a set of firms performing any multi-input multi-output production process. Additionally, the new approach can bring about a true total factor productivity index, which can be decomposed into an output change and an input change. Finally, the new index can be used to decompose the traditional technical change component into a global technical change applicable across the industry under study and a locally specific technical change dependent on the assessed firm.

Keywords: Productivity change, Malmquist index, technical change, efficiency change, standard.
JEL-Code: C43; D24; O47

[*] Corresponding author: Email: dsantin@ccee.ucm.es; Tel.: +34913942433

1. Introduction

In the early 1980s, Caves et al. (1982) introduced the Malmquist productivity index (MPI), also called the adjacent Malmquist index (Adj-MI), as an adaptation to production theory of the index originally defined by Malmquist (1953) for consumer theory. As opposed to other classical alternatives in economics like the Törnqvist, Laspeyres, Paasche and Fisher indices, which all rely on price information, the MPI is a non-price dependent approach for determining productivity change over time. Later, Färe et al. (1992, 1994a) showed how to implement the Adj-MI under data envelopment analysis (DEA). DEA is a non-parametric technique strongly based on mathematical linear programming for estimating production frontiers and technical efficiency. Moreover, Färe et al. (1992) suggested an initial decomposition of the Adj-MI into a catching-up effect (efficiency change) and a frontier-shift effect (technical change) in order to derive direct drivers of productivity change. Later, other alternative decompositions were developed (see, for example, Färe et al, 1994b, Ray and Desli, 1997, Lovell, 2003 and Zofio, 2007).

Under the axiomatic test approach to index number theory, the definition of an appropriate index for capturing productivity change requires the establishment of a set of properties (also called tests) that the index formula should satisfy. In the 1920s, Fisher (1922) proposed a sizeable number of such, economically and mathematically intuitive, tests, which any approach had to pass to be considered a suitable index number. Later, Eichhorn and Voeller (1976) provided a summary of the main tests published in the literature. More up-to-date summaries are Balk (1995), Althin (2001) and Diewert and Fox (2017). Within this axiomatic test approach, the circular test (circularity) is a desirable property for any productivity change index. Simply speaking, a productivity change index (*I*) computed at three points in time, *t, t+1* and *t+2,* passes the circular test when ${}^{t}I^{t+1} \cdot {}^{t+1}I^{t+2} = {}^{t}I^{t+2}$.

As Frisch (1936) and other authors (see, for example, Balk and Althin, 1996, Pastor and Lovell, 2007) pointed out, an attractive property of a productivity change index covering a long period of time is that it is possible to chain it (i.e., circularity). Chaining is not possible if the reference technology changes over time. Recently, Färe and Zelenyuk (2021) clarified the similarities and differences between circularity and transitivity, since there do not appear to be commonly accepted notions of these two terms in the index number literature. Circularity and transitivity are not exactly the same. However, the two notions are equivalent once the so-called identity test is satisfied (Balk and Althin, 1996; Althin, 2001). This happens to be the case with the Adj-MI, which trivially satisfied the identity test. So, circularity and transitivity can be used interchangeably in the case of the Adj-MI.

The theoretical formulation of the Adj-MI by Färe et al. (1992) only considers two periods of time. In real applications where the available panel data covers more than two periods, the Adj-MI is repeatedly determined for each pair of consecutive periods of time, as in Färe et al. (1992). In this case, the Adj-

MI is calculated as the geometric average of two terms (each using a different period as a reference technology) and does not satisfy the circularity test.

Therefore, Berg et al. (1992) were the first to introduce in their celebrated paper a version of the MPI that satisfied circularity: the base period Malmquist productivity index (BP-MI). In their definition, the index is not an average of two terms. Instead, it is the ratio of a pair of distance functions calculated from a common reference technology across all the considered years. In the case of the BP-MI, the reference technology is empirically determined. Indeed, it represents the production possibility set estimated from the data observed in a certain period of time. In this regard, Berg et al. (1992) recommended the use of the first period associated with the corresponding panel data because productivity change estimation aims to measure improvements or deteriorations with respect to the initial period.

Since the contribution by Berg et al. (1992), many researchers have taken this research avenue in order to determine productivity change in applications or propose new theoretical solutions in the context of productivity measurement under the satisfaction of the circularity property. Regarding new approaches based on Berg et al. (1992), worthy of note are, for example, Pastor and Lovell (2005), who introduced a circular global Malmquist index based on an empirically determined reference technology for all the considered periods. In this case, the empirically determined reference technology does not correspond to the first period of the series, but, instead, matches the convex hull of all the estimated technologies in the periods under evaluation. Additionally, Pastor et al. (2011) defined the biennial Malmquist productivity index, as a refinement of the previous global Malmquist index. Other recent contributions are Aparicio and Santín (2018), who adapted the so-called Camanho-Dyson index (Camanho and Dyson, 2006) in order to compare the performance of groups of decision-making units (DMUs) over time subject to the circularity property, and Camanho et al. (2021) and Walheer (2022), who defined the global counterpart of the MPI for group contexts. All in all, the Adj-MI and the BP-MI have attracted the interest of many scholars in economics for empirical applications over the last decades[1].

Returning to the axiomatic test approach, another interesting test guarantees the independency of the reference period considered for index determination. Indices that do not satisfy this condition can draw different conclusions about productivity change depending on the arbitrarily selected base period (Althin, 2001). In this regard, Berg et al.'s approach (1992) is base period dependent, whereas the Adj-MP satisfies independency. Althin (2001) proved that the BP-MI is independent of the reference period if and only if the marginal rate of substitution of inputs is independent of time. Unfortunately, this is a

[1] Note that, based on a Google Scholar web search conducted at the end of August 2022, we found that, measured by the number of citations, the Adj-MI is more successful than the BP-MI. Caves et al. (1982) accumulates 5,674 citations for the Adj-MI, whereas its famous decomposition proposed by Färe et al. (1994b) has 6,666 citations. On the other hand, Berg et al. (1992) receives a "mere" 841 citations.

very hard condition to enforce with regard to a production technology. Some more recent contributions partially dealing with this issue are Asmild and Tam (2007) and Otsuki (2013).

Overall, practitioners measuring productivity changes nowadays face a trade-off between circularity and base period independency. The use of the BP-MI rather than the Adj-MI has the plus of satisfying the circularity test, but the downside of reference period dependency. On the other hand, the use of the Adj-MI for our empirical application instead of the BP-MI will satisfy base period independency, but the circularity property will not hold.

Furthermore, since Caves et al. (1982) proposed the Adj-MI, the Malmquist index has been commonly considered as a productivity index. However, the standard formulation of the Adj-MI does not comply with what is traditionally understood in the literature as a total factor productivity index (TFPI), that is, the ratio of an output quantity index to an input quantity index. Recently, O'Donnell (2012) formally verified that the Adj-MI cannot be expressed as a TFPI (see also Pastor et al., 2020). On this ground, Ang and Kerstens (2017) recently claimed that the MPI is not multiplicatively complete. A possible solution for this weakness of the Adj-MI is the so-called Hicks–Moorsteen productivity index introduced by Bjurek (1996). Nevertheless, under Bjurek's approach, the original formulation of the Adj-MI is abandoned in favor of another expression based upon the ratio of an aggregate output quantity index to an aggregate input quantity index. In particular, the Hicks–Moorsteen productivity index measures the change in input quantities in the input direction using input distance functions and the change in output quantities in the output direction using output distance functions rather than just implementing an input orientation based upon input distance functions or an output orientation grounded on output distance functions, as the traditional Adj-MI does.

Finally, and regarding the decomposition of the Malmquist index into its main drivers, that is, efficiency change and technical change, some authors claim that the frontier shift over time should be considered as a global phenomenon affecting the entire technologies in the two periods for the corresponding sector (see Balk and Althin, 1996, Asmild and Tam, 2007, and Otsuki, 2013). In contrast, under the traditional decomposition of the Malmquist index, the component associated with technical change is calculated locally with respect to the units under evaluation, showing frontier shift as a firm-specific phenomenon. For this reason, Balk and Althin (1996), Asmild and Tam (2007) and Otsuki (2013) introduced technical change components that should reflect a global frontier shift over time. In our opinion, however, the technical change experienced by a firm depends on two drivers, which should be individually identified: a global sector change, which is common for all production units, and how each company locally experiences technological change over the two considered periods.

In this paper, we suggest a new approach to overcome all three weaknesses of the Malmquist productivity index pointed out above: i) the trade-off between circularity or independency of the considered reference time period; ii) the issue of whether or not the Malmquist index is a total factor productivity index, and iii) the dilemma between measuring technical change as a local or as a global phenomenon with a view to the decomposition of the Malmquist index. To simultaneously solve all the above problems, we venture to propose, as is common practice in other sciences, the use of a reference production technology or simply a standard to be systematically used in production economics for enhancing the measurement of productivity changes.

International standards are systematically utilized as fundamental measurement references in different disciplines. For example, the International System of Units (SI) defines seven well-known base units for measuring mass (kilogram), length (meter), time (second), electric current (ampere), thermodynamic temperature (kelvin), luminous intensity (candela) and amount of substance (mole). There have also been some attempts at using standards in economics, for example the quality adjusted life years (QALYs), some American oil companies report their production in terms of oil barrels and the World Bank currently defines the international poverty line in low-income countries, used to calculate the number of people leaving in extreme poverty, as \$1.90 a day. In our production context, the aim of this paper is to define this standard together with an appropriate decomposition able to overcome all the aforementioned drawbacks of the two most important versions of the Malmquist index: the Adj-MI and the BP-MI.

The paper is organized as follows. Section 2 briefly introduces the background. In Section 3, we define the new standard total factor productivity index (STFPI), together with its decomposition, and the steps for applying this methodology in practice. Section 3 also provides a numerical example to illustrate the theoretical ideas. In Section 4, we exemplify the new decomposition by applying the approach to the group of 42 Swedish pharmacies operating from 1980 to 1989 previously used in Färe et al. (1992) and then in Althin (2001) for the purposes of comparison with the Adj-MI and the BP-MI. Finally, Section 5 outlines the conclusions and points out some future research lines.

2. Background

In this section, we briefly introduce the notation for the Adj-MI and the BP-MI. Additionally, we introduce the formal definition of a total factor productivity index (TFPI) and briefly discuss the locality of the technical change component of the Adj-MI. Although other techniques could be used to estimate the corresponding technologies and technical efficiency, our calculations are based on data envelopment analysis (DEA) (see Charnes et al., 1978 and Banker et al., 1984) because DEA has the flexibility to handle multi-input multi-output production processes.

Let us consider a panel of $j = 1,...,J$ decision-making units (DMUs) and, at least, two different time periods, t and $t+1$. In this context, x_{ji}^{τ} denotes the quantity of the i-th input, $i = 1,...,m$, consumed by the j-th DMU, $j = 1,...,J$, in the period τ, $\tau = t, t+1$. And y_{jr}^{τ} denotes the quantity of the r-th output,

$r = 1,...,n$, produced by the j-th DMU, $j = 1,...,J$, in the period τ, $\tau = t, t+1$. In vectorial notation, we use (x_j^τ, y_j^τ) to denote the input-output bundle corresponding to the j-th DMU, $j = 1,...,J$. Additionally, we need to define two technologies to establish the relationship between inputs $x = (x_1,...,x_m) \in R_+^m$ and outputs $y = (y_1,...,y_n) \in R_+^n$. Within this framework, the *contemporaneous* benchmark technology is defined as $T^\tau = \{(x^\tau, y^\tau) : x^\tau \text{ can produce } y^\tau\}$, $\tau = t, t+1$. Moreover, we use a subscript linked to the technology to denote which type of returns to scale is being assumed: T_c^τ for constant returns to scale (CRS), the so-called benchmark technology, and T_v^τ for variable returns to scale (VRS), the so-called best practice technology.

As previously mentioned, there are very different approaches to estimate a technology from a sample of observed bundles of inputs and outputs. One, classified as a non-parametric methodology, is DEA. Within this approach, T_c^τ and T_v^τ are estimated as follows (see Charnes et al., 1978 and Banker et al., 1984):

$$T_c^\tau = \left\{ \begin{array}{l} (x^\tau, y^\tau) \in R_+^{m+n} : y_r^\tau \leq \sum_{j=1}^{J} \lambda_j y_{jr}^\tau, \forall r = 1,...,n, \\ x_i^\tau \geq \sum_{j=1}^{J} \lambda_j x_{ji}^\tau, \forall i = 1,...,m, \lambda_j \geq 0, \forall j = 1,...,J \end{array} \right\} \quad (1)$$

$$T_v^\tau = \left\{ \begin{array}{l} (x^\tau, y^\tau) \in R_+^{m+n} : y_r^\tau \leq \sum_{j=1}^{J} \lambda_j y_{jr}^\tau, \forall r = 1,...,n, \\ x_i^\tau \geq \sum_{j=1}^{J} \lambda_j x_{ji}^\tau, \forall i = 1,...,m, \sum_{j=1}^{J} \lambda_j = 1, \lambda_j \geq 0, \forall j = 1,...,J \end{array} \right\}. \quad (2)$$

Without identifying the assumed type of returns to scale, given the input-output bundle (x_o^l, y_o^l) observed in period l, l taking the values t or $t+1$, the Shephard output distance function (Shephard, 1970) with respect to the technology T^τ, $\tau = t, t+1$, is generically defined as:

$$D^\tau (x_o^l, y_o^l) = \inf \{ \gamma > 0 : (x_o^l, y_o^l / \gamma) \in T^\tau \}. \quad (3)$$

Additionally, we use a subscript to denote which type of returns to scale is being assumed on the technology —$D_c^\tau (x_o^l, y_o^l)$ for CRS and $D_v^\tau (x_o^l, y_o^l)$ for VRS—, that is, whether T_c^τ or T_v^τ is used to determine the distance function, respectively.

An equivalent representation of the technology, which is useful when an output-oriented distance function is utilized, is associated with the partially oriented notion of output production possibility set $P^{\tau}\left(x^{\tau}\right)=\left\{y^{\tau}\in R_{+}^{n}:\left(x^{\tau},y^{\tau}\right)\in T^{\tau}\right\}$. In this way, the Shephard output distance function in (3) can be equivalently rewritten as:

$$D^{\tau}\left(x_{o}^{l},y_{o}^{l}\right)=\inf\left\{\gamma>0:\left(y_{o}^{l}/\gamma\right)\in P^{\tau}\left(x_{o}^{l}\right)\right\}. \tag{4}$$

Under DEA, the value of the Shephard output distance function $D_{c}^{\tau}\left(x_{o}^{l},y_{o}^{l}\right)$ is calculated as follows:

$$\begin{aligned}\left[D_{c}^{\tau}\left(x_{o}^{l},y_{o}^{l}\right)\right]^{-1} = \quad & \max \quad \phi \\ & s.t. \\ & \qquad \sum_{j=1}^{J}\lambda_{j}x_{ji}^{\tau}\leq x_{oi}^{l}, \qquad i=1,\ldots,m \\ & \qquad \sum_{j=1}^{J}\lambda_{j}y_{jr}^{\tau}\geq\phi y_{or}^{l}, \quad r=1,\ldots,n \\ & \qquad \lambda_{j}\geq 0, \qquad\qquad j=1,\ldots,J\end{aligned} \quad . \tag{5}$$

The distance $D_{v}^{\tau}\left(x_{o}^{l},y_{o}^{l}\right)$ is computed in the same way as $D_{c}^{\tau}\left(x_{o}^{l},y_{o}^{l}\right)$, albeit adding the constraint $\sum_{j=1}^{J}\lambda_{j}=1$ to (5).

The (simple) Adj-MI grounded on the technology of period τ is defined as follows.

$$M^{\tau}\left(x_{o}^{t},y_{o}^{t},x_{o}^{t+1},y_{o}^{t+1}\right)=\frac{D_{c}^{\tau}\left(x_{o}^{t+1},y_{o}^{t+1}\right)}{D_{c}^{\tau}\left(x_{o}^{t},y_{o}^{t}\right)},\ \tau=t,t+1. \tag{6}$$

Except for trivial cases, the assumption of CRS or VRS when calculating Shephard's distance functions leads to very different results. In recent years, however, some voices have called attention to the unsuitability of using the expression of the traditional Malmquist index according to the type of returns to scale that best fits the technology estimated from the data. For example, Grifell-Tatjé and Lovell (1995) used a two-dimensional example to show that, in the presence of variable returns to scale, the Malmquist productivity index does not adequately measure productivity change. In the same vein, Ray and Desli (1997) stated that *the Malmquist productivity index is correctly measured by the ratio of CRS distance functions even when the technology exhibits variable return to scale*. Lovell (2003) is another author who supports this same thinking. This is the reason why we assume CRS in expression (6).

Given that the choice of either of the above two indices, expression (6) for $\tau = t, t+1$, is arbitrary with respect to the measurement of productivity change over time, and possibly leading to different results Caves et al. (1982) suggested taking the geometric mean of both expressions, that is,

$$M\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = \sqrt{M^t\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) \cdot M^{t+1}\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right)}. \tag{7}$$

Equation (7) is also known as the Färe, Grosskopf, Lindgren and Roos (1989) adjacent Malmquist output-based productivity index. This index was originally decomposed into efficiency change and technical change as follows (Färe et al., 1992):

$$M\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = EC_o \cdot TC_o, \tag{8}$$

where $EC_o = \dfrac{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^t\left(x_o^t, y_o^t\right)}$ and $TC_o = \left[\dfrac{D_c^t\left(x_o^t, y_o^t\right)}{D_c^{t+1}\left(x_o^t, y_o^t\right)} \cdot \dfrac{D_c^t\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}\right]^{1/2}$.

However, the above index does not satisfy circularity (see, for example, Pastor and Lovell, 2007). A desirable property of any productivity change index covering a long period of time is that it is possible to chain it. This is known as circularity in index number theory. Chaining is out of the question if the reference technology changes over time, which is the case with the Adj-MI. Fortunately, Berg et al. (1992) proposed an index that compares adjacent period data using a technology from a unique base period. This BP-MI satisfies circularity, generates a single measure of productivity change, and can also be decomposed into efficiency change and technical change, as demonstrated by Berg et al. In this regard, the 'chain' version of the Malmquist productivity index, known as the BP-MI, is defined as follows (Berg et al., 1992):

$$M^b\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = \frac{D_c^b\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^b\left(x_o^t, y_o^t\right)}, \tag{9}$$

where b denotes the base time period (usually the first period in a series). Additionally, (9) can be decomposed into a term interpreted as efficiency change and a component interpreted as technical change:

$$M^b\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = EC_o \cdot TC_o^b, \tag{10}$$

where $EC_o = \dfrac{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^t\left(x_o^t, y_o^t\right)}$ and $TC_o^b = \dfrac{D_c^t\left(x_o^t, y_o^t\right) / D_c^b\left(x_o^t, y_o^t\right)}{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right) / D_c^b\left(x_o^{t+1}, y_o^{t+1}\right)}$.

In the decomposition, the expression of the efficiency change component is the same as the expression of the same term in (8), whereas the frontier shift over time is measured in (10) by the distance between technology t and t + 1, albeit as a relative distance to the base reference technology T_c^b. However, and in contrast to the Adj-MI, the values of Berg et al.'s index (1992) are not independent of the selected base period.

Along these lines, Asmild and Tam (2007) recently revisited this property of independency, introducing a global base period Malmquist grounded on the geometric mean of all the shifts experienced by each of the units in the sample and utilizing each of the time periods in the available dataset as the base period. In this way, the index is independent of the selection of base period (in the considered time windows) because all the time periods used as references are averaged, and the choice of a specific base period is no longer arbitrary.

Regarding the two decompositions of the Malmquist indices in both (8) and (10), there is a coincident component that measures efficiency change and another term that captures the frontier shift. As these expressions are defined in (8) and (10), technical change is determined locally, exclusively assessing the input-output bundles of the evaluated company. Therefore, under the Adj-MI, the geometric average of individual technical changes is calculated as a way of capturing the overall pattern for the whole industry in a single value. In contrast, as Asmild and Tam (2007) and Otsuki (2013) claim, technical change is expected to be a global phenomenon, not simply the average of local measures or technical change. Accordingly, Asmild and Tam (2007) and Otsuki (2013) directly introduce aggregated measures of technical change to estimate the global trend. In the case of Asmild and Tam (2007), one of their proposals is the geometric mean of the ratios $D_c^t\left(x_j^h, y_j^h\right) / D_c^{t+1}\left(x_j^h, y_j^h\right)$ for all the observed firms $j = 1, ..., J$ in all the periods considered in the panel ($h = 1, 2, ...$). In place of a full-blown total factor productivity index with its decomposition, Otsuki (2013) proposes a similar idea focused exclusively on measuring a global technical change. Instead of using all the observations as in Asmild and Tam (2007), his approach adopts a grid of synthetic points using directional vectors. Both approaches are inspired by the seminal paper by Balk and Althin (1996).

Although we agree that technical change is a global phenomenon, we also believe that, due to its intrinsic characteristics and its relative location in the input-output space with respect to the production frontiers in each time period, each firm experiences technological change locally. Therefore, our proposal lies somewhere in between the above two strategies: the frontier shift associated with a firm is decomposed into an average global effect and a local change component.

3. Standard total factor productivity index

Over the last few decades, the MPI has been under the spotlight as a measure of productivity change over time for several reasons. One is related to the fact that it does not need information on market prices, which makes this index suitable for benchmarking public services produced by public production units like education or health (for a review, see De Witte and López-Torres, 2017 and Hollingsworh, 2008, respectively). Although much has already been written about this index, no approach has, to the best of our knowledge, so far succeeded in simultaneously satisfying the following properties: circularity, independency of a considered reference period for index calculation, the determination of a TFPI and, finally, the possibility of considering a technical change term as the mixture of two components (a global and a local frontier shift). In this section, we introduce a solution aimed at filling this gap of the literature.

3.1. Standard technology

Our approach is fundamentally based upon the definition of a standard technology, that is, a common reference or base technology for use in any empirical problem to measure productivity change over time. We are aware that many standards could be defined as possible alternatives. However, once a standard has been accepted by most practitioners in a field, its associated advantages far outweigh any possible disadvantages. In our productivity context, the main benefit of using a standard reference will be an increase in the comparability of the results achieved by different researchers and ease of use. Additionally, as far as the direct benefits of the new Malmquist productivity index defined based on our proposal of a standard reference are concerned, the properties of circularity, independency of the considered reference period for index calculation and the determinateness test will be naturally satisfied. Moreover, due to the features of the standard that we propose, the new approach is rewritten as a ratio of an output quantity change index to an input quantity change index. In this regard, the new index could be reinterpreted as a total factor productivity change (TFPC) index. This is something that the Adj-MI or the BP-MI cannot guarantee (O'Donnell, 2012). As a direct consequence of its application, the new approach is also able to decompose traditional technical change into two components: an average global technical change and a local technical change.

Our approach is particularly grounded on the expression of the BP-MI, see (9), where the base technology b is a synthetic reference set rather than a particular set corresponding to the first, the last or any period or combination of time periods in the empirical series. In this respect, given the difficulty for dealing parametrically with multi-input multi-output production processes, we select the following simple formulation of a multi-output constant elasticity of substitution function, introduced by Färe and Primont (1995, p. 155), which is based on the notion of distance function:

$$D_{FP}\left(x_o, y_o\right)=\beta_0\left(\sum_{r=1}^{n}\beta_r y_{or}^{\delta}\right)^{\frac{1}{\delta}} \Big/ \alpha_0\left(\sum_{i=1}^{m}\alpha_i x_{oi}^{\rho}\right)^{\frac{1}{\rho}}. \tag{11}$$

From (11), we can naturally define the synthetic technology $S:=\left\{\left(x,y\right)\in R_{+}^{m+n}:D_{FP}\left(x,y\right)\le 1\right\}$. Next, we prove that if this technology is defined through the output production possibility sets $P_S\left(x\right)=\left\{y\in R_{+}^{n}:\left(x,y\right)\in S\right\}=\left\{y\in R_{+}^{n}:D_{FP}\left(x,y\right)\le 1\right\}$, $\forall x\in R_{+}^{m}$, then $D_{FP}\left(x_o,y_o\right)$ matches the definition of the Shephard output distance function in (4).

Proposition 1. Let $P_S\left(x\right)=\left\{y\in R_{+}^{n}:D_{FP}\left(x,y\right)\le 1\right\}$, $\forall x\in R_{+}^{m}$. If $\beta_0\left(\sum_{r=1}^{n}\beta_r y_{or}^{\delta}\right)^{\frac{1}{\delta}} \Big/ \alpha_0\left(\sum_{i=1}^{m}\alpha_i x_{oi}^{\rho}\right)^{\frac{1}{\rho}}>0$, then $D_{FP}\left(x_o,y_o\right)=\inf\left\{\gamma>0:\left(y_o/\gamma\right)\in P_S\left(x\right)\right\}$.

Proof. Based on the expression of the Shephard output distance function (4), we can take the following steps:

$$\inf\left\{\gamma>0:\left(y_o/\gamma\right)\in P_S\left(x_o\right)\right\}=\inf\left\{\gamma>0:D_{FP}\left(x_o,y_o/\gamma\right)\le 1\right\}=$$

$$\inf\left\{\gamma>0:\beta_0\left(\sum_{r=1}^{n}\beta_r\left(\frac{y_{or}}{\gamma}\right)^{\delta}\right)^{\frac{1}{\delta}} \Big/ \alpha_0\left(\sum_{i=1}^{m}\alpha_i x_{oi}^{\rho}\right)^{\frac{1}{\rho}}\le 1\right\}=$$

$$\inf\left\{\gamma>0:\gamma\ge\beta_0\left(\sum_{r=1}^{n}\beta_r y_{or}^{\delta}\right)^{\frac{1}{\delta}} \Big/ \alpha_0\left(\sum_{i=1}^{m}\alpha_i x_{oi}^{\rho}\right)^{\frac{1}{\rho}}\right\}=$$

$$\beta_0\left(\sum_{r=1}^{n}\beta_r y_{or}^{\delta}\right)^{\frac{1}{\delta}} \Big/ \alpha_0\left(\sum_{i=1}^{m}\alpha_i x_{oi}^{\rho}\right)^{\frac{1}{\rho}}=D_{FP}\left(x_o,y_o\right).$$

■

Next, we prove that the output production possibility sets $P_S\left(x\right)$ satisfy the set of usual axioms in microeconomics under a specification of the parameters in (11) (for more details, see Färe and Primont, 1995, p. 27).

Proposition 2. Let $x\in R_{+}^{m}$ and $\rho>0$. Let $\beta_0>0$, $\beta_r\ge 0$, $r=1,...,n$, $\alpha_0>0$, $\alpha_i\ge 0$, $i=1,...,m$, and $\delta\ge 1$. Then, $P_S\left(x\right)$ fulfills the following axioms:

(A1) $0_s\in P_S\left(x\right)$.

(A2) If $y\in P_S\left(x\right)$ and $0<\theta\le 1$, then $\theta y\in P_S\left(x\right)$ (weak disposability of outputs).

(A3) If $y\in P_S\left(x\right)$ and $y'\le y$, then $y'\in P_S\left(x\right)$ (strong disposability of outputs).

(A4) $P_S(x)$ is a bounded set (scarcity).

(A5) $P_S(x)$ is a closed set.

(A6) $P_S(x)$ is a convex set.

Proof. We will use that $P_S(x) = \{y \in R_+^n : D_{FP}(x,y) \le 1\} = \left\{ y \in R_+^n : \beta_0 \left(\sum_{r=1}^{n} \beta_r y_r^{\delta} \right)^{\frac{1}{\delta}} \le \alpha_0 \left(\sum_{i=1}^{m} \alpha_i x_i^{\rho} \right)^{\frac{1}{\rho}} \right\}$ $= \left\{ y \in R_+^n : h(y) \le \alpha_0 \left(\sum_{i=1}^{m} \alpha_i x_{oi}^{\rho} \right)^{\frac{1}{\rho}} \right\}$, where $h(y) := \beta_0 \left(\sum_{r=1}^{n} \beta_r y_r^{\delta} \right)^{\frac{1}{\delta}}$. Then, (A1) is true because $h(0_n) = 0$ and $\alpha_0 \left(\sum_{i=1}^{m} \alpha_i x_{oi}^{\rho} \right)^{\frac{1}{\rho}} \ge 0$. (A2) $h(\theta y) = \beta_0 \left(\sum_{r=1}^{n} \beta_r (\theta y_r)^{\delta} \right)^{\frac{1}{\delta}} = \theta\beta_0 \left(\sum_{r=1}^{n} \beta_r y_r^{\delta} \right)^{\frac{1}{\delta}} =$ $\theta h(y) \le h(y) \le \alpha_0 \left(\sum_{i=1}^{m} \alpha_i x_{oi}^{\rho} \right)^{\frac{1}{\rho}}$, which implies that $\theta y \in P_S(x)$. (A3) is true because $h(y') \le h(y)$. (A4) holds because $0 \le y_r \le \dfrac{\alpha_0 \left(\sum_{i=1}^{m} \alpha_i x_{oi}^{\rho} \right)^{\frac{1}{\rho}}}{\beta_0 \beta_r^{\frac{1}{\delta}}}$, for all $r = 1,...,s$. (A5) is satisfied because $h(y)$ is a continuous function. Finally, to prove (A6), notice that $h(y)$ is a quasi-convex function. This last claim is true because $\beta_0 > 0$ and $\left(\sum_{r=1}^{n} \beta_r y_r^{\delta} \right)^{\frac{1}{\delta}} = f[g(y)]$, with $g(y) = \sum_{r=1}^{n} \beta_r y_r^{\delta}$ and $f(z) = z^{\frac{1}{\delta}}$. $f(z)$ is a monotone increasing function and $g(y)$ is a convex function because it is the sum of convex functions, since $\beta_r \ge 0$, $r = 1,...,s$, and $\delta \ge 1$. Then, we can apply Theorem 13.8(b) in Madden (1986) (because a convex function is also a quasi-convex function) and we get that $f[g(y)]$ is a quasi-convex function. Therefore, $h(y)$ is also a quasi-convex function. As a consequence, any lower contour set is convex. In particular, $P_S(x)$ is a lower contour set. ■

Furthermore, S is a conical technology and, therefore, it exhibits (global) constant returns to scale. To demonstrate this point, notice that, by (11), $D_{FP}(\sigma x_o, \sigma y_o) = D_{FP}(x_o, y_o)$ for any $\sigma > 0$ and, consequently, if $(x_o, y_o) \in S$, then $\sigma(x_o, y_o) \in S$ due to the definition of S. In this way, we have that $S = \sigma S$ for all $\sigma > 0$, which is the definition of global CRS (see Färe and Primont, 1995, p. 23).

Additionally, (11) can be plugged into the Malmquist expression in (6) to determine productivity change. Before doing so, a previous indispensable step is to set values for the parameters that appear in (11) in order to define a specific standard technology. This technology will be used to compute all distance functions and productivity measures. For the sake of simplicity, our proposal sets the parameters that define the output side, the numerator in (11), as the standard or unit n-sphere or, in other words, the surface of the unit ball in the positive orthant. To define the input side, the denominator in (11), we suggest the use of a standard or unit m-simplex, that is, the convex hull of its $m+1$ vertices in the positive orthant. All in all, this implies defining the following parameter values[2]:

$$\begin{gathered}\beta_r = 1, \forall r = 1,...,n, \\ \alpha_i = 1, \forall i = 1,...,m, \\ \delta = 2, \rho = 1, \\ \alpha_0 = \beta_0\end{gathered} \tag{12}$$

Now, by plugging (11) with parameters defined in (12) into (6), we get the definition of the so-called standard total factor productivity index (STFPI):

$$M^S\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = \frac{D_{FP}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_{FP}\left(x_o^t, y_o^t\right)}, \tag{13}$$

which, by applying (11) and (12), can be explicitly expressed as follows:

$$M^S\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = \frac{\left(\sum_{r=1}^{n}\left(y_{or}^{t+1}\right)^2\right)^{\frac{1}{2}} \Big/ \left(\sum_{i=1}^{m} x_{oi}^{t+1}\right)}{\left(\sum_{r=1}^{n}\left(y_{or}^{t}\right)^2\right)^{\frac{1}{2}} \Big/ \left(\sum_{i=1}^{m} x_{oi}^{t}\right)} = \frac{\left[\sum_{r=1}^{n}\left(y_{or}^{t+1}\right)^2 \Big/ \sum_{r=1}^{n}\left(y_{or}^{t}\right)^2\right]^{\frac{1}{2}}}{\sum_{i=1}^{m} x_{oi}^{t+1} \Big/ \sum_{i=1}^{m} x_{oi}^{t}}. \tag{14}$$

A TFPC index is formally defined in the literature as the ratio of an output quantity change index to an input quantity change index. According to this definition, the Adj-MI and the BP-MI are not TFPC indices (an exception is the Hicks–Moorsteen productivity index, see Bjurek, 1996), implying that they cannot always be interpreted as measures of actual productivity change over time (O'Donnell, 2012). However, the new STFPI can be easily expressed as a ratio of an output quantity change index to an input quantity change index, which means that it can be really understood as an actual TFPC index.

Although, at this point, the new index could be naturally decomposed through a direct application of (10), where the reference technology associated with the base period is substituted by the standard technology, we, like other authors (see Balk and Althin, 1996, Asmild and Tam, 2007, and Otsuki, 2013), believe that the component of technical change should represent the frontier shift over time as a

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[2] After considering many possibilities, we decided to base the standard technology on geometric grounds. Accordingly, a unit n-sphere evokes the production possibilities frontier illustrated in practically all efficiency measurement handbooks, whereas the m-simplex evokes an isoquant where inputs are perfect substitutes.

global phenomenon affecting the studied industry rather than an average of how the observed DMUs locally experience technical change. For this reason, inspired mainly by Asmild and Tam (2007) and Otsuki (2013), we prefer to generate a set of *K* artificial DMUs uniformly distributed on the surface of the standard technology and determine, through the projection of these synthetic units on to the two empirical technologies at *t* and *t+1*, a global technical change as the aggregation of all the distances calculated based on this procedure. The larger the number of generated DMUs, the better our estimate of technical change for the sector between periods should be. Nevertheless, as we will use a geometric average to aggregate all these calculations, there should be a point at which it will no longer be necessary to increase the size of this artificial data set. Following Asmild and Tam (2007) and Otsuki (2013), we implement this strategy to improve the estimation accuracy of the overall frontier shift over time. Furthermore, our decomposition will allow us to identify an additional term devoted to capturing how a company locally experiences technological change over time.

Let us assume that $\left\{\left(x_k^S, y_k^S\right)\right\}_{k=1}^{K}$ is the set of *K* artificial production units uniformly distributed on the surface of the standard technology. Later, we will show how to generate this grid of points. Our proposal for decomposing the STFPI is as follows:

$$M^S\left(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1}\right) = EC_o \cdot TC_o^s = EC_o \cdot GTC_o^s \cdot LTC_o^s, \qquad (15)$$

where

$$EC_o = \frac{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^t\left(x_o^t, y_o^t\right)}, \qquad GTC_o^S = \left(\prod_{k=1}^{K} \frac{D_c^t\left(x_k^S, y_k^S\right)}{D_c^{t+1}\left(x_k^S, y_k^S\right)}\right)^{\frac{1}{K}} \quad \text{and}$$

$$LTC_o^S = \left[\left(\frac{D_{FP}\left(x_o^{t+1}, y_o^{t+1}\right)}{D_c^{t+1}\left(x_o^{t+1}, y_o^{t+1}\right)}\right) \cdot \left(\frac{D_c^t\left(x_o^t, y_o^t\right)}{D_{FP}\left(x_o^t, y_o^t\right)}\right) \cdot \left(\prod_{k=1}^{K} \frac{D_c^{t+1}\left(x_k^S, y_k^S\right)}{D_c^t\left(x_k^S, y_k^S\right)}\right)^{\frac{1}{K}}\right].$$

In (15), we identify the term EC_o as the typical component for measuring efficiency change between periods t and $t+1$ that also appears in (8) and (10). The traditional technical change TC_o^S in (10) is now decomposed in two new components, a global technical change GTC_o^S and a local technical change LTC_o^S. The component GTC_o^S should capture the frontier shift of the corresponding sector over time. To this end, GTC_o^S averages the global frontier shift between *t* and *t+1* evaluated for every synthetic point $\left(x_k^S, y_k^S\right)$, $k = 1,...,K$. Asmild and Tam (2007) propose a similar idea, using, in place of a grid of artificial points, all the observations over all the periods of the panel data in order to determine the global technical change. Asmild and Tam (2007, p. 143) also pointed out the possibility of using a set of artificial DMUs as a future extension of their approach. Otsuki (2013) is another author that defines a measure of overall technical change, in this case based upon a grid of synthetic points. However, none of these authors made use of a standard technology, nor did they suggest generating the artificial points at the boundary of a reference technology, as we do. Moreover, no previous contribution has, to the best of our knowledge, proposed simultaneously decomposing technical change into global (GTC) and local (LTC) components. In (15), the local technical change (LTC) is the measurement of the relative position of each DMU at t+1 with respect to the synthetic average global technology change. Note that the traditional Adj-MI and BP-MI propose local technical change components, whereas Asmild and Tam (2007) and Otsuki (2013) exclusively identify global technical change. We stand astride these two currents in the literature and propose both.

Regarding the axiomatic properties of the STFPI, this new approach is assessed resorting to the tests shown in Frisch (1936) and more recently in Althin (2001). Most of the properties met by the STFPI, for example, circularity, are directly inherited from the BP-MI. Other properties, like independency of base period, are satisfied due to the use of a standard as a reference technology for all the calculations and any panel data. Next, we summarize all the tests that the new approach satisfies by stating a proposition.

Proposition 3. The STFPI passes the tests of identity, time reversal, circularity, commensurability, determinateness, inverse proportionality of inputs, proportionality of outputs and independency of base period.

Proof. The STFPI satisfies the tests of identity, time reversal, circularity, and commensurability because it uses a reference technology as the base period. Regarding independency of base period, it is trivially satisfied due to the use of a standard technology. By expression (14), the STFPI satisfies both inverse proportionality of inputs and proportionality of outputs. Regarding determinateness, following Frisch (1936), if only one individual quantity becomes zero, then the STFPI is well-defined and, consequently, it does not become zero, infinite, or indeterminate. ■

Table 1 compares the traditional adjacent and base period versions of the Malmquist index with respect to the STFPI approach and the satisfaction of the usual tests[3]. In this regard, with respect to this list of properties, the results reported in the columns labelled Adj_MI and BP-MI in Table 1 are familiar. Regarding the results in Table 1, we find that the new index outperforms the most common MPI indices in the literature. The STFPI meets all the tests except proportionality of inputs (as applies to the classical approaches), although all three indices satisfy the test of inverse proportionality of inputs under CRS (because the output distance function is homogeneous of degree -1 in inputs if and only if the technology

[3] Under output-oriented distance functions, the indices pass the inverse proportionality test of inputs given CRS. In contrast, if we use input-orientation (as assumed in Althin, 2001), then the indices pass the proportionality test of outputs given CRS.

exhibits CRS – see Färe, 1988, p. 52). This makes economic sense since productivity is expected to be half if inputs are doubled. The proportionality of outputs is another property that all three indices satisfy without special conditions, thanks to the fact that the output distance function is homogeneous of degree +1 for outputs (see, for example, Färe and Primont, 1995, p. 17). Finally, on top of the satisfaction of all these desirable properties, we should add that the standard-based approach is also a TFPC index, to which neither the Adj-MI or BP-MI can lay claim.

Table 1. Index tests for different productivity indices

Test	**Adj-MI**	**BP-MI**	**STFPI**
T1: Identity	Yes	Yes	Yes
T2: Time reversal	Yes	Yes	Yes
T3: Circularity	No	Yes	Yes
T4: Commensurability	Yes	Yes	Yes
T5: Determinateness	No	No	Yes[4]
T6: Inverse Proportionality of inputs given CRS	Yes	Yes	Yes
T7: Proportionality of outputs	Yes	Yes	Yes
T8: Independent of base period	Yes	No	Yes

3.2. Steps for using the standard technology in practice

In empirical problems, inputs and outputs are quantitative variables measured in different units and often expressed in very different orders of magnitude. This is not a problem in traditional applications of the BP-MI where the base technology is defined ad hoc using exactly the same variables and units of measurement as the inputs and outputs managed by the production units to be evaluated. However, under the new framework, the standard technology is parametrically defined in pure numbers with no units of measurement. As it makes no sense to add apples and oranges, our first task in order to solve the empirical problem on our hands is to normalize the initial empirical database with the twofold aim of erasing the units of measurement and preventing variables measured with higher values weight more in the productivity change calculus in expression (14).

To do this, we propose dividing all the output and input variables by the highest value observed for each variable in the panel data. By doing this for every variable, all values will be naturally bounded between zero and one, where the maximum observed values will be equal to one for all outputs and inputs in the

[4] Following the definition of determinateness introduced by Frisch (1936), the new index strictly takes positive values, where some of the input or the output values equal zero as the test requires. Irrespective of the definition of the determinateness test, mathematically speaking, problems do arise when all the components of any of the input or output vectors observed in periods t or $t+1$ for DMU_0 are zero. However, this case does not make economic sense in a real production process.

transformed database. For instance, if the maximum value over all the periods in the database for the r-th output is 20 tons of apples, then we will divide all observed values of this r output by 20 tons of apples. In this way, the result would be a (pure) number between zero and one free of units of measurement. As model (5) is units invariant (Lovell and Pastor, 1995), the division of all values of a specific input or output variable by a number does not change the calculus of the distances or technical efficiency values. Therefore, the two traditional Malmquist indices will remain the same after this transformation.

Once the original database is transformed, we can directly plug the resulting data into (14) to output the STFPI[5]. The calculus of EC_o in (15) is straightforward through (5). Therefore, the next step is to calculate the GTC_o^S using the standard technology. To do this, the set of the m inputs $x^S = \left(x_1^S, x_2^S, \ldots x_m^S\right)$ and n outputs $y^S = \left(y_1^S, y_2^S, \ldots y_n^S\right)$ has to be generated according to the n outputs and m inputs in the empirical problem for K synthetic DMUs that should be plugged into (15) in order to calculate all necessary distances. Empirically, for the output side, that is, the numerator in (11), we define a unit n-sphere centered at the origin in the positive orthant where the distance of its points to the origin, that is, the radius, equals one. For empirical purposes, K synthetic production units have to be uniformly distributed at random on the boundary of an $n+1$-dimesional ball. To do this, we resort to Marsaglia's algorithm (1972), which is composed of the following steps.

First, and obviously according to the empirical problem at hand, we generate n independent outputs $\left(y_1, y_2, \ldots y_n\right)$ using the absolute values from a big number K of draws of a normal distribution $y_r \sim \left|N\left(0,1\right)\right|$, $r = 1,...,n$, where K is the number of synthetic production units. Second, we calculate the radius of this point as $P = \sqrt{\left(y_1^2 + y_2^2 + \ldots + y_n^2\right)}$. The final values that will be used as output for the reference are $y^S = \left(y_1^S, y_2^S, \ldots y_n^S\right) = \left(\frac{y_1}{P}, \frac{y_2}{P}, \ldots, \frac{y_n}{P}\right)$.

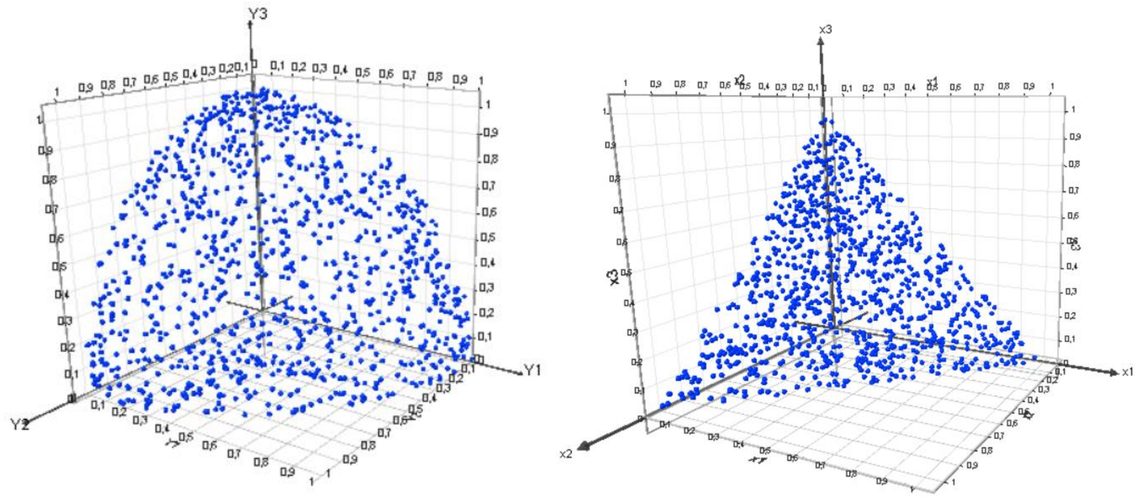
To generate input data for the same K number of synthetic DMUs on the surface of the unit m-simplex, we follow Onn and Weissman (2011) who propose a simple procedure that can be adapted for this purpose. First, generate m independent inputs $\left(x_1, x_2, \ldots x_m\right)$ from an exponential distribution. This can easily be done by drawing independent variables from a big number K of draws of a uniform distribution in the [0,1] interval and then computing the negative natural logarithm of all generated variables.

[5] Equation (13) suggests that we might project the normalized empirical data in t and $t+1$ against the reference technology defined by K synthetic DMUs generated. This would be equivalent to the traditional procedure that we follow for the BP-MI in (9). The difference is that there are no parameters for the base technology in (9) because the parametric technology in this case is unknown and must be estimated. Within the new framework, however, we do know the function and the parameters of the standard technology so we can simply plug the normalized data into (14).

Second, sum all variables to get $Q = (x_1 + x_2 + \ldots + x_m)$. The final input vector that will be used for the standard is $x^S = (x_1^S, x_2^S, \ldots x_m^S) = \left(\frac{x_1}{Q}, \frac{x_2}{Q}, \ldots, \frac{x_m}{Q}\right)$.

Panel A in Figure 1 illustrates a mesh of K=1,000 synthetic DMUs that define the standard technology for the output side in the case of three outputs and one equal input, while panel B represents the surface of a technology of three inputs producing one equal output.

Figure 1: The surface of the output and the input sides using a standard unit 3-sphere (Panel A) and a standard unit 3-simplex (Panel B) generated using a mesh of 1,000 uniformly random draws.



Panel A: The unit 3-sphere. Draws using Marsaglia's algorithm.

Panel B: The unit 3-simplex. Draws using Onn and Weissman's algorithm.

Once the set of inputs and outputs are generated from the standard technology, it is straightforward to combine the vectors of inputs and outputs in (x^S, y^S) [6] and to project the K synthetic DMUs against any technology. For instance, we calculate the distances $D_c^t(x_k^S, y_k^S)$ and $D_c^{t+1}(x_k^S, y_k^S)$ for the synthetic DMU k and, thus, the geometric averages that define GTC_o^S in (15). Once $M^S(x_o^t, y_o^t, x_o^{t+1}, y_o^{t+1})$, EC_o and GTC_o^S have been calculated, the LTC_o^S term can be computed as a residual.

[6] Note that $D_{FP}(x^S, y^S) = 1$ from both data generation processes so the K synthetic production units are on the surface of the frontier of the reference technology.

3.3. A numerical example

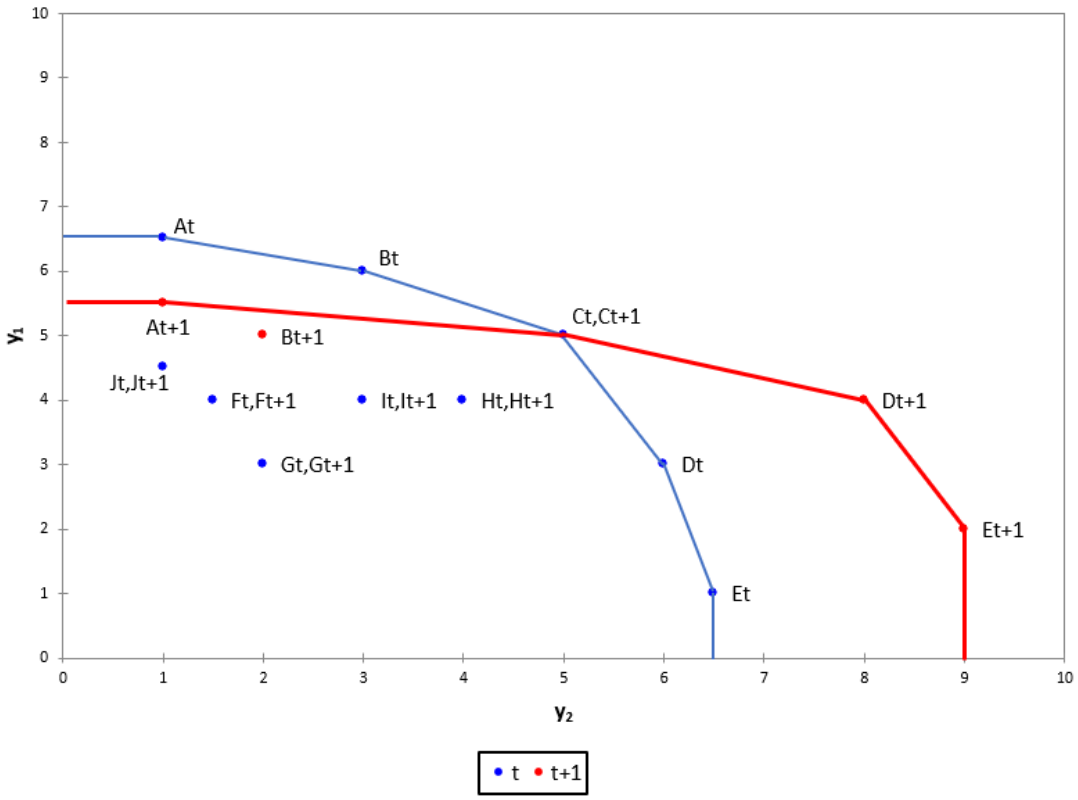
To illustrate how the index works in practice, we set out the following simple numerical example. Let us assume an industry composed of ten DMUs, *A, B, C, D, E, F, G, H, I* and *J* that produce two outputs (y_2 and y_1) using one equal input (x) in two time periods t and $t+1$. The data, together with the outputs normalized by the maximum observed output values in the two periods (the input side is irrelevant in this example) are listed in Table 2.

Table 2. Production data for ten DMUs in two periods.

	t			$t+1$			t			$t+1$		
DMU	y_2	y_1	x	y_2	y_1	x	y_2 /*MAX(y2)*	y_1/*MAX (y1)*	x	y_2 /*MAX(y2)*	y_1/*MAX (y1)*	x
A	1	6.5	1	1	5.5	1	0.1111	1.0000	1	0.1111	0.84615	1
B	3	6	1	2	5	1	0.3333	0.9231	1	0.2222	0.76923	1
C	5	5	1	5	5	1	0.5556	0.7692	1	0.5556	0.76923	1
D	6	3	1	8	4	1	0.6667	0.4615	1	0.8889	0.61538	1
E	6.5	1	1	9	2	1	0.7222	0.1538	1	1.0000	0.30769	1
F	1.5	4	1	1.5	4	1	0.1667	0.6154	1	0.1667	0.61538	1
G	2	3	1	2	3	1	0.2222	0.4615	1	0.2222	0.46154	1
H	4	4	1	4	4	1	0.4444	0.6154	1	0.4444	0.61538	1
I	3	4	1	3	4	1	0.3333	0.6154	1	0.3333	0.61538	1
J	1	4.5	1	1	4.5	1	0.1111	0.6923	1	0.1111	0.69231	1
MAX	6.5	**6.5**	1	**9**	5.5	1						

For illustrative purposes, Figure 2 represents the production frontiers for the two periods using the raw data without normalization. The piece-wise linear form of the non-parametric frontier in a DEA model under constant returns to scale reveals that *A, B, C, D* and *E* are efficient in period t, while only *A, B, C* and *E* remain fully efficient in period t+1. Regarding Table 2 and Figure 2, we also observe that DMUs *C, F, G, H, I* and *J* have the same input-output information in both periods. It is clear from Figure 2 that the productivity of firm *A* is lower at $t+1$ than in period t, although it still belongs to the production frontier. The opposite applies for firms *D* and *E*, which are efficient in both periods but have higher output values in period $t+1$ with respect to t.

Figure 2: Production frontiers for 10 DMUs producing in a two outputs one equal input setting and two time periods.



The next step is to use Equations (8), (10) and (15) to compute and analyze the differences between the three productivity indices: the two traditional Malmquist indices plus the new standard total factor productivity index, alongside the components of these indices described above. Table 3 reports these results.

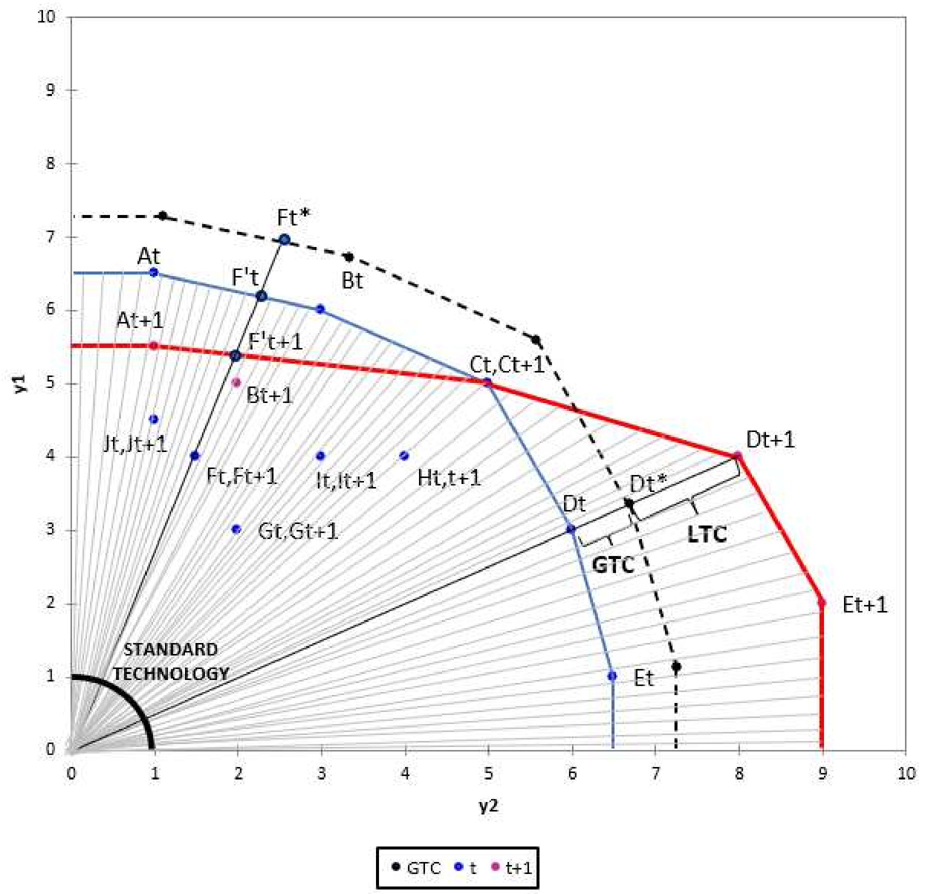
Table 3: Total factor productivity indices and their components

	Adjacent MI			**Base Period MI**			**Standard TFPI**				
DMU	Adj-MI	EC	TC	BP-MI	EC	TC	STFPI	EC	TC	GTC	LTC
A	0.8490	1	0.8490	0.8519	1	0.8519	0.8482	1	0.8488	1.1177	0.7595
B	0.8192	0.9333	0.8777	0.8148	0.9333	0.8730	0.8158	0.9333	0.8748	1.1177	0.7827
C	1	1	1	1	1	1	1	1	1	1.1177	0.8954
D	1.3333	1	1.3333	1.3333	1	1.3333	1.3333	1	1.3343	1.1177	1.1938
E	1.3960	1	1.3960	1.4074	1	1.4074	1.4169	1	1.4179	1.1177	1.2686
F	1	1.1647	0.8586	1	1.1647	0.8586	1	1.1647	0.8592	1.1177	0.7688
G	1	1.0833	0.9231	1	1.0833	0.9231	1	1.0833	0.9237	1.1177	0.8265
H	1	1	1	1	1	1	1	1	1	1.1177	0.8954
I	1	1.0606	0.9429	1	1.0606	0.9429	1	1.0606	0.9435	1.1177	0.8442
J	1	1.1684	0.8559	1	1.1684	0.8559	1	1.1684	0.8565	1.1177	0.7663
Mean	1.0261	1.0385	0.9881	1.0268	1.0385	0.9887	1.0272	1.0385	0.9897	1.1177	0.8856

Looking at Table 3, note firstly that the efficiency change (EC) is calculated in the same way and is the same for all three indices. Secondly, on average, all three indices and their technical changes are very similar. Paired correlation coefficients among the three indices and the technical changes exceed 0.9997 in both cases. Mean technical changes are below one in all three cases: 0.9881, 0.9887 and 0.9897 for the Adj-MI, the BP-MI and the STFPI, respectively, leading to the conclusion that, on average, the firms of this sector suffered a technical regress between the two periods. It is worth mentioning here that, in the case of the STFPI, the abovementioned average technical change $TC_o^S = 0.9897$ is the component in (15) that evolves from the technical change of BP-MI in (10), where the standard technology substitutes the base period. However, the new STFPI allows a further valuable decomposition of the technical change TC_o^S between *t* and *t+1* into two components: an average global technical change GTC_o^S and a local technical change LTC_o^S.

Figure 3 reproduces Figure 2 but also illustrates the new technical change decomposition. Firstly, the GTC_o^S of this industry is represented in Figure 3 as the dashed production frontier. This global technical change is obtained after averaging the distances calculated after projecting 1,000,000 synthetic DMUs generated from the surface of the reference technology using the steps developed in Section 3.2 against the production frontiers in *t* and *t*+1. In Figure 3, some of these distances $D_c^t\left(x_k^S, y_k^S\right)$ and $D_c^{t+1}\left(x_k^S, y_k^S\right)$ are represented as the ray vectors from the surface of the standard to the technologies in *t* and *t+1*.

Figure 3: The global and the local technical changes in the standard total factor productivity index



In this example, the GTC_o^S shows that the production frontier at t+1 has experienced, on average, a shift upwards with respect to the technology in t. In Figure 3, this result can be intuitively understood using firm C as a reference. Regarding y_2, on the right (left) of unit C, the technology in t+1 (t) dominates the technology at t (t+1), where the area comprised between the two technologies is bigger on the right than on the left of C. The GTC_o^S is the average shift upwards of the technology at t and, being an average, is equal for all firms. For example, GTC for DMUs D, a fully efficient firm in both periods, and F, inefficient at t and t+1, are graphically measured in Figure 3 by the radial distances $OF_t^* / OF_t^{'} = OD_t^* / OD_t$. On average, there was a non-negligible GTC equal to 1.1177 (Table 2), considering the whole industry between the two periods. The GTC is global in the sense that it accounts for the technical change that has occurred on average across the two production frontiers between the two periods and not only the geometric mean of those distances between t and t+1 calculated locally for the observed firms.

Secondly, the local technical change (LTC) is the measurement of the relative position of each firm at t+1 with respect to the synthetic average technology change. In our case, LTC for DMUs D and F is graphically measured in Figure 3 by the radial distances $OF_{t+1}^{'} / OF_t^*$ and OD_{t+1} / OD_t^*, respectively. In sum, the technical change of firm D between t and t+1 is the distance $OD_{t+1}/OD_t = 1.3333$. This change is decomposed into two parts: i) *GTC*, $OD_t^*/OD_t = 1.1177$, that is, an equal average global technical change for all DMUs belonging to this industry, and ii) *LTC* ranging from the average synthetic production frontier up to the production frontier at t+1, $OD_{t+1}/OD_t^* = 1.1938$. In this case, the *LTC* shows that DMU *D* managed to perform above average compared to the *GTC* of its industry due to its particular output combination. Regarding firm *F*, its technical change is $OF_{t+1}/OF_t = 0.8592$ suggesting a technical regress. However, the decomposition of the technical change for unit *F* highlights an average positive global technical change $OF_t^*/OF_t^{'} = 1.1177$ for the whole industry together with a local technical regress $OF_{t+1}^{'}/OF_t^* = 0.7688$. This means that the radial projection to the frontier of the output-input combination chosen by *F* is located in a region of the production frontier that is currently suffering a technical regress. This finding is highly useful for avoiding misleading results, for example, both traditional indices might conclude by averaging the observed technical changes that the industry had a technical regress. The new decomposition may suggest that this industry made technical progress globally, although most firms were producing closer to the region where the technology was experiencing a technical regress.

Thirdly, as discussed earlier, the STFPI can be used to calculate an output change over an input change as in Bjurek's index, which is out of the question using either the Adj-MI or the BP-MI. In this example, as there is no input side, the output change is obtained by plugging the normalized output values of Table 2 into (14), and the results match the STFPI in Table 3.

4. Empirical application

In this empirical application, we use the same panel data of 42 Swedish pharmacies between 1980 and 1989 as was previously analyzed in the seminal papers by Färe et al. (1992) and Althin (2001). This set of pharmacies produce four outputs employing four inputs. The four inputs are 'Labor input for pharmacist' (X1); 'Labor input for technical staff' (X2); 'Equipment services' (X3) and 'Building services' (X4). Both labor inputs (X1 and X2) are measured in number of hours per year, X3 is measured using the annual depreciation of pharmacy equipment measured in 1980 prices and X4 is assumed to be proportional to the available floor space, measured in square meters. The four outputs are 'Drug deliveries to hospitals' (O1); 'Prescription drugs for outpatient care' (O2); 'Medical appliances for the handicapped' (O3) and 'Over the counter goods' (O4). The first three outputs (O1, O2 and O3) are measured in number of times, whereas the fourth (O4) is measured in 1980 prices[7].

[7] Färe et al. (1992) provide more details about the database, the variables, descriptive statistics and disaggregated Malmquist index and its decomposition over time for the set of 42 pharmacies.

Table 4 shows the productivity changes and their decomposition for the Adj-MI, the BP-MI, using the first year (1980) as the fixed reference technology, and the new STFPI. The modeled technologies assume constant returns to scale and strong disposability of inputs, and distances are calculated using an output orientation. The global technical change is calculated using 1,000,000 synthetic DMUs. The results provided are the geometric means of the 42 analyzed pharmacies[8].

Table 4. Results for the Adj-MI, BP-MI and STFPI and their decompositions.

	Adjacent-MI			Base Period-MI			Standard TFPI				
Year	Adj-MI	EC	TC	BP MI	EC	TC	STFPI	EC	TC	GTC	LTC
8081	0.9911	1.0220	0.9698	1.0561	1.0220	1.0334	1.0049	1.0220	0.9833	1.0123	0.9713
8182	1.0740	0.9353	1.1483	1.1072	0.9353	1.1838	1.0758	0.9353	1.1503	1.2063	0.9536
8283	1.0246	1.0548	0.9713	1.0586	1.0548	1.0036	1.0380	1.0548	0.9841	0.9467	1.0394
8384	0.9424	0.9875	0.9544	0.9923	0.9875	1.0049	0.9559	0.9875	0.9680	0.9073	1.0669
8485	1.0435	1.0039	1.0395	1.0495	1.0039	1.0454	1.0376	1.0039	1.0336	1.0215	1.0118
8586	1.0189	0.9868	1.0325	1.0470	0.9868	1.0610	1.0067	0.9868	1.0201	1.0096	1.0104
8687	1.0665	1.0015	1.0649	1.1035	1.0015	1.1018	1.0814	1.0015	1.0797	1.0211	1.0574
8788	1.0435	1.0133	1.0298	1.0365	1.0133	1.0229	1.0274	1.0133	1.0140	1.0004	1.0136
8889	1.0513	1.0056	1.0455	1.0572	1.0056	1.0513	1.0330	1.0056	1.0272	1.0216	1.0055
Mean	1.0277	1.0007	1.0269	1.0559	1.0007	1.0552	1.0283	1.0007	1.0241	1.0136	1.0138
Accum.	1.2786	1.0065	1.2703	1.6320	1.0065	1.6214	1.2859	1.0065	1.2775	1.1290	1.1316
Direct	1.1922	1.0065	1.1844	1.6320	1.0065	1.6214	1.2859	1.0065	1.2775	1.1290	1.1316

To interpret the results, note that a value of one means no change, a number greater than one means progress and less than one is equivalent to regress. By construction, efficiency changes coincide for all three indices. Therefore, the differences among indices stem from the calculus of technical change. Another straight result is that, as expected, the BP-MI and the STFPI satisfy the circularity test, and this implies that the accumulated productivity and technical changes are equal to the direct productivity and technical changes of the first period (1980) relative to the last period (1989). Table 5 shows that Pearson's correlation coefficients for the nine average standard total factor productivity and traditional technical change values for this industry are positive, greater than 0.92 and statistically significant. In fact, correlations between the two traditional indices are slightly lower than when they are correlated with the STFPI.

Table 5: Pearson correlation coefficients of average productivity change and technical change values.

	Productivity indices				Technical changes		
	Adj-MI	BP-MI	STFPI		TC-Adj	TC-BP	TC-STFPI
Adj-MI	1	0.8292	0.9405	TC-Adj	1	0.9246	0.9730
BP MI	0.8292	1	0.9335	TC-BP	0.9246	1	0.9704
SMI	0.9405	0.9335	1	TC-S	0.9730	0.9704	1

[8] The Adj-MI and the BP-MI results coincide exactly with Althin (2001, Table 4).

The new index provides more, rich information than the traditional approaches. Firstly, the accumulated average global technical change for this industry over the 1980-1989 period (1.1290) is much smaller than the resulting technical change calculated as the average of observed DMUs by the Adj-MI (1.2703) and the BP-MI (1.6214). Around half of the observed technical change is driven by the local technical change (1.1316) of a bunch of DMUs, which on average push the technology up above the average technical change. This new decomposition is useful for disentangling how the technical change evolved in every period. For example, according to the Adj-MI and BP-MI, technical change was measured in the period 80-81 as 0.9698 and 1.0334, respectively, leading to inconclusive results. However, the STFPI concludes that, on average, the industry made slight technical progress equal to 1.0123, although some pharmacies were located in the region where the production frontier suffered a technical regress, and they pull back the technology shift enough (0.9713) to offset the average positive global effect.

Other valuable information that was previously discussed in Section 3.1 is that normalized output and input values can be plugged into (14) using the STFPI in order to calculate how it decomposes into an aggregate output and an aggregate input change. Table 6 provides this decomposition.

Table 6. Productivity results for the STFPI including its decomposition into output and input changes.

Year	STFPI	OUTPUT CHANGE	INPUT CHANGE
8081	1.0049	0.9880	0.9832
8182	1.0758	1.0589	0.9843
8283	1.0380	1.0782	1.0387
8384	0.9559	0.9071	0.9490
8485	1.0376	1.0323	0.9949
8586	1.0067	1.0372	1.0303
8687	1.0814	1.0683	0.9879
8788	1.0274	1.0178	0.9906
8889	1.0330	0.9982	0.9663
Average	1.0283	1.0194	0.9913
Accum.	1.2859	1.1890	0.9247
Direct	1.2859	1.1890	0.9247

The measures of output change and input change can be useful from a managerial point of view, especially in sectors like the public services, where there are no prices for many variables and individual information about each DMU is important for decision-making by policymakers. Output changes that are greater than one indicate an increase in outputs from period t to period $t + 1$, while values of less than one denote a decline. The values related to input changes can be interpreted in the same manner with respect to the inputs. From Table 6, we observe that the clear decline in productivity in the 83-84 period is explained by the fact that, on average, the decreases in outputs (0.9071) were greater than the reduction in inputs (0.9490). Years 81-82, 84-85, 86-87 and 87-88 were especially interesting because, on average, the industry managed to increase productivity by producing more (output change greater than one) with fewer resources (input change less than one). Interestingly, the opposite result, producing less with more, never happened in the above period, although the productivity gains in periods 82-83 and 85-86 were due to positive output changes that were greater than the positive input increases.

Before concluding this empirical section, let us look at how many synthetic DMUs should be used to calculate global technical change. Although this paper does not set out to analyze and find a large enough number of DMUs as of which the results are almost identical, Table 7 shows that the results remain quite stable when the number of DMUs increases over 1,000,000 synthetic units.

Table 7. Global and local technical changes estimated with a different number *K* of synthetic DMUs on the standard technology surface.

K	10,000		100,000		1,000,000		10,000,000	
Year	GTC	LTC	GTC	LTC	GTC	LTC	GTC	LTC
8081	1.0109	0.9727	1.0120	0.9716	1.0123	0.9713	1.0123	0.9714
8182	1.2107	0.9501	1.2067	0.9532	1.2063	0.9536	1.2061	0.9537
8283	0.9455	1.0408	0.9467	1.0394	0.9467	1.0394	0.9468	1.0394
8384	0.9056	1.0688	0.9068	1.0674	0.9073	1.0669	0.9072	1.0670
8485	1.0222	1.0111	1.0215	1.0118	1.0215	1.0118	1.0215	1.0118
8586	1.0103	1.0097	1.0097	1.0103	1.0096	1.0104	1.0095	1.0105
8687	1.0213	1.0572	1.0216	1.0569	1.0211	1.0574	1.0212	1.0573
8788	1.0004	1.0135	1.0007	1.0133	1.0004	1.0136	1.0004	1.0135
8889	1.0217	1.0054	1.0216	1.0055	1.0216	1.0055	1.0216	1.0055
Average	1.0137	1.0137	1.0136	1.0138	1.0136	1.0138	1.0135	1.0139
Accum.	1.1298	1.1307	1.1293	1.1313	1.1290	1.1316	1.1287	1.1318
Direct	1.1298	1.1307	1.1293	1.1313	1.1290	1.1316	1.1287	1.1318

Table 7 shows that the results are highly correlated but slightly different depending on the number of synthetic DMUs projected against the technologies at t and $t+1$. Regarding average results, there are reductions (increases) of ten-thousandths in the average global (local) technical change as the number of synthetic DMUs increases. The more units we generate the better the results will be, although, at the end of the day, the choice of a final number of K artificial units will depend on the power of our computer and how much time is available.

5. Conclusions

In this paper, we introduce a new standard total factor productivity index based on defining a standard reference technology for all required calculations. As is the case in other fields, the acceptance of a common standard improves the comparability of calculations, ease of use and satisfaction of interesting properties. In our case, we defined the standard reference technology as a unit m-simplex input side able to produce a unit n-sphere output side. This standard does not need to be estimated from data and is grounded on a particular parametric specification of a general multi-output multi-input production technology approach previously defined by Färe and Primont (1995). Färe and Primont (1995) introduced an extension of the CES production function for multi-output multi-input production contexts, on which we base our definition of the standard technology.

The STFPI satisfies more typical tests or axioms in production theory than the Adj-MI and the BP-MI. To date, there has been a trade-off in the literature between circularity and the base period independency test. In particular, if we opted for the BP-MI instead of the Adj-MI, the gain in terms of circularity was offset by reference period dependency. On the other hand, if we utilized the Adj-MI instead of the BP-MI, our approach passed the base period independency test, but the circularity property did not hold. From now on, though, we can use the STFPI, which passes both these two tests. Additionally, the new approach also satisfies the determinateness test thanks to the formulation of the standard technology. This is something to which neither of the above Malmquist indices can lay claim. Another interesting property satisfied by the STFPI is that it can be identified as a true total factor productivity index, that is, it can be expressed as the ratio of an aggregated change in outputs and an aggregated change in inputs. In contrast, the most common versions of the Malmquist productivity index do not fulfill this property.

Regarding the decomposition of the new approach into typical drivers, efficiency change and technical change, we follow the ideas established by Balk and Althin (1996), Asmild and Tam (2007) and Otsuki (2013), where the technical change component is regarded as a global phenomenon affecting the frontier shift of the entire sector. Nevertheless, and unlike the above authors, our proposal is based upon two subcomponents: a global technical change across the industry and how a company locally experiences technical change over time depending on its particular position with respect to technologies at t and $t+1$. We identified both drivers in our approach by exploiting a grid of synthetic DMUs (in line with Otsuki, 2013) located on the frontier surface of the standard technology. The methodology is easy to apply following the instructions provided in Marsaglia (1972) and Ohn and Weissman (2011), also briefly described in this paper. Once synthetic data are generated, the remaining distances can be calculated using different DEA models and resorting to the parametric standard technology. To facilitate the application and interpretation of this new approach, we also described a numerical example with ten DMUs and two periods outlining all necessary steps for calculating all the measurements under DEA.

In our empirical application, we use the same panel data of 42 Swedish pharmacies between 1980 and 1989 that was previously analyzed in the seminal papers by Färe et al. (1992) and Althin (2001). Results yielded by the STFPI provide new information about this industry, adding a local and global technical change together with aggregate output and input changes over time. We also found that global and local technical changes were stable from 1,000,000 synthetic DMUs onwards, although further research is necessary to define a suitable number for dealing with the trade-off between computation time and precision.

This paper should open up a good number of research avenues for the future. The standard can be extended to other Malmquist-type indices, like the Camanho and Dyson (2006) index for dealing with productivity gaps between groups of DMUs and its time-based version proposed by Aparicio and Santín (2018). Another challenge is to extend the decomposition of the global and local technical changes for production technologies operating under variable returns to scale over time. Finally, the use of standards is rare in economics compared with other sciences, and this research could open a door to the proposal of more standards not only in production economics but also in other fields.

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