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Are Progressive Income Taxes Stabilizing?–A Reply*

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Abstract

Dromel and Pintus [Are Progressive Income Taxes Stabilizing?, Journal of Public Economic Theory 10, (2008) 329-349] have shown that labor-income tax progressivity reduces the likelihood of local indeterminacy, sunspots and cycles in a one sector monetary economy with constant returns to scale. In this note, we extend Dromel and Pintus (2008) into a two sector monetary economy with constant returns to scale studied by Bosi et al. (2007) and reassess the stabilizing effect of progressive income taxes. We show that the result in Dromel and Pintus (2008) is robust to this extension, which means that changes of the production structure won't affect the stabilizing effect of progressive income taxes, i.e., tax progressivity (regressivity) reduces (increases) the likelihood of local indeterminacy, sunspots and cycles.

1. Introduction

In a recent article, Dromel and Pintus (2008) [Are Progressive Income Taxes Stabilizing?, Journal of Public Economic Theory 10, (2008) 329-349] have shown that labor-income tax progressivity reduces

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the likelihood of local indeterminacy, sunspots and cycles in a one sector monetary economy with constant returns to scale. They state in the conclusion that "...we show that similar results hold with capital income taxes, or in an OLG economy with consumption in old age...Moreover, extending the analysis to introducing increasing returns to scale does not change the main message of this paper...". In their paper, they mainly compare the results of their model with those of the models of Christiano and Harrison (1999) and Guo and Lansing (1998) in the one sector framework.

In this note, we complete Dromel and Pintus's analysis by asking whether their main result extends to the two sector framework studied by Bosi et. al. (2007). This extension is necessary since changes in the production framework may influence the stabilizing effects of progressive income taxes. For example, Guo and Harrison (2001) extend the tax policy analysis into a two sector real business cycle model with strong investment externalities and find that a regressive tax policy can stabilize the economy against the sunspot shocks. On the contrary, Guo and Lansing (1998) show that regressive taxes can destabilize the economy in the Benhabib-Farmer (1994) one sector model with productive externalities. But in a two sector monetary economy with constant returns to scale, we find that the results in Dromel and Pintus (2008) still hold, i.e., tax progressivity (regressivity) reduces (increases) the likelihood of local indeterminacy, sunspots and cycles. More precisely, as in Bosi et. al., when the consumption good is sufficiently capital intensive, local indeterminacy arises while the elasticities of capital-labor substitution in both sectors are slightly greater than unity and the elasticity of the offer curve is low enough. The tax progressivity reduces the likelihood of local indeterminacy, sunspots and cycles.

It should be pointed out that the results in Dromel and Pintus (2008) are valid in the two sector framework, as we consider here a setting which, unlike Guo and Harrison (2001)'s, doesn't allow for sufficiently strong externalities in the investment goods sector. As Guo and Harrison pointed out, a regressive tax policy can destabilize the economy with an aggregate constant returns-to scale technology or a low investment externality. In the two sector version of the Dromel and Pintus (2008) model, without investment externalities, the results of Dromel and Pintus still hold, i.e., tax progressivity reduces, in parameter space, the likelihood of local indeterminacy.

2. Progressive Income Taxes in a Two Sector Monetary Economy with Constant Returns to Scale

In this section, in the spirit of Bosi et. al (2007), we describe our model to which we add progressive income taxes as in Dromel and Pintus (2008). The economy consists of two types of infinite-lived agents, workers and capitalists, each of them identical within their own type. The agents called workers consume, supply labor and are subject to a financial constraint: their expenditures must be financed out of their initial money balances or out of the returns earned on productive capital. Capitalists do not work and are subject to their budget constraint. In the production side, contrary to the aggregate formulation of Dromel and Pintus, we use the framework of Bosi et al. (2007) who assume two different technologies producing a consumption good and an investment good, respectively.

2.1. Capitalists

The problem of the capitalists is to maximize their logarithmic intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t \ln c_t^c, \tag{1}$$

where c_t^c denotes their consumption and $\beta \in (0,1)$ the discount factor. Because capitalists do not work, their budget constraint can be stated as follows

$$c_t^c + p_t \left[k_{t+1}^c - (1-\delta)\, k_t^c\right] + q_t M_{t+1}^c \leq r_t k_t^c + q_t M_t^c, \tag{2}$$

where we require that $c_t^c \geq 0$, $k_{t+1}^c \geq 0$, and $M_{t+1}^c \geq 0$ (given that $k_t^c > 0$). Here p denotes the price of investment, k the capital, $\delta \in (0,1)$ the depreciation rate of capital, M the money balances, q the price of money and r the interest rate in terms of the numeraire consumption good.

As in Bosi et. al (2007), we focus on the case where $c_t^c > 0$ holds for all t, and then impose the restriction that $\frac{p_{t+1}(1-\delta)+r_{t+1}}{p_t} > \frac{q_{t+1}}{q_t}$ holds for all t. That means, the gross rate of return on capital is higher than the returns of money holding and capitalists choose to hold capital and no money ($M_{t+1}^c = 0$). The optimal policy function can be stated as follows

$$k_{t+1}^c = \beta \left[r_t/p_t + (1-\delta) \right] k_t^c, \tag{3}$$

and the corresponding consumption path is given by $c_t^c = (1-\beta)[r_t + (1-\delta)\, p_t] k_t^c$.

2.2. Workers

The workers' problem is to maximize their intertemporal utility function

$$\max \sum_{t=0}^{\infty} \gamma^t \left[u\left(c_t^w\right) - \gamma v\left(l_t\right) \right] \tag{4}$$

where c_t^w, l_t denote the consumption and labor supply, u, v the per-period utility function from consumption and per-period disutility of labor supply and $\gamma \in (0,1)$ the discount factor. On the functions u and v, we assume the following.

Assumption 1 $u(c)$ and $v(l)$ are C^r, with r large enough, for, respectively, $c > 0$ and $0 \leq l < l^*$, where $l^* > 0$ is the (maybe infinite) workers' endowment of labor. They satisfy $u'(c) > 0$, $u''(c) < 0$, $v'(l) > 0$, $v''(l) > 0$ with $\lim_{c \to 0} u'(c) = +\infty$, $\lim_{c \to +\infty} u'(c) = 0$, $\lim_{l \to 0} v'(l) = 0$ and $\lim_{l \to l^*} v'(l) = \infty$. Consumption and leisure are assumed to be gross substitutes, i.e., $u'(c) + cu''(c) > 0$. In other words, gross substitutability means that the elasticity of intertemporal substitution in consumption,

$\epsilon_c = -u'(c)/u"(c)c$, is larger than unity. This means that the labor supply is an increasing function of the real wage. (The same assumption appears on page 315 in Bosi et. al (2007).)

We impose a second assumption that capitalists are more patient than workers.

Assumption 2 $\beta > \gamma$

Workers are subject to the resource constraint

$$c_t^w + p_t \left[k_{t+1}^w - (1-\delta) k_t^w\right] + q_t M_{t+1}^w \leq r_t k_t^w + \phi\left(w_t l_t\right) + q_t M_t^w, \tag{5}$$

and the borrowing constraint

$$c_t^w + p_t \left[k_{t+1}^w - (1-\delta) k_t^w\right] \leq r_t k_t^w + q_t M_t^w, \tag{6}$$

where we require that $c_t^w \geq 0$, $l_t \geq 0$ and $M_{t+1}^w \geq 0$ (given that $k_t^w \geq 0$). Here all the variables represent those used in the capitalists' budget constraint and the wage variable w_t is in terms of the numeraire good. The borrowing constraint shows that the workers cannot borrow against future labor income. As in Dromel and Pintus (2008), we introduce the fiscal policy by mapping labor income x_t into disposable income $\phi\left(x_t\right)$ and requiring that $x_t \geq \phi\left(x_t\right)$. In addition, $\phi\left(x\right)$ satisfies the following assumption.

Assumption 3 Disposable income $\phi\left(x\right)$ is a continuous, positive function of market income $x \geq 0$, with $x \geq \phi(x)$, $\phi'(x) > 0$ and $0 \geq \phi''(x)$, for $x > 0$. The income tax-and-transfer scheme exhibits weak progressivity, that is, $\phi(x)/x$ is nonincreasing for $x > 0$ or, equivalently, $1 \geq \psi(x) \equiv x\phi'(x)/\phi(x)$. Then $\pi(x) \equiv 1 - \psi(x)$ is a measure of income tax progressivity. In particular, the fiscal schedule is linear when $\pi(x) = 0$, or $\psi(x) = 1$, for $x > 0$, and the higher $\pi(x)$, the more progressive the fiscal schedule (see the same assumption on page 333 in Dromel and Pintus (2008)).

It is easy to verify that workers' capital holdings are zero at all dates ($k_t^w = 0$) if and only if

$$u'\left(c_t^w\right) > \frac{\gamma u'\left(c_{t+1}^w\right)\left[r_{t+1} + (1-\delta)\, p_{t+1}\right]}{p_t}. \tag{7}$$

Here we focus on the case where $c_t^w > 0$ and $l_t > 0$ hold for all t. Workers (at the equilibrium) choose to hold money instead of capital ($k_t^w = 0$). In this setting, there is a constant money supply, i.e. $M_t = \overline{M} > 0$ for any t. And the borrowing constraint is binding at the optimum, i.e.,

$$c_t^w = q_t \overline{M}. \tag{8}$$

Once the borrowing constraint binds, the resource constraint can be expressed as follows

$$q_t M_{t+1}^w = q_t \overline{M} = \phi\left(w_t l_t\right) = w_t l_t - g$$

where $g = x - \phi(x)$ denotes the amount of the public goods and $x \equiv wl$. As in Dromel and Pintus (2008), we assume that the proceeds of taxes, net of transfers, are used to produce a flow of public goods g. Therefore, the government budget is balanced.

From appendix 1, we can easily have the following equation

$$v'(l_t) = w_t \phi'\left(w_t l_t\right) u'\left(c_{t+1}^w\right) \frac{q_{t+1}}{q_t}. \tag{9}$$

We then manipulate equation (9) in the following way: $c_{t+1}^w u'\left(c_{t+1}^w\right) \phi'\left(w_t l_t\right) \frac{q_{t+1} w_t l_t}{q_t c_{t+1}^w} = l_t v'\left(l_t\right)$, or else, $U\left(c_{t+1}^w\right) \psi\left(x_t\right) = V\left(l_t\right)$, where $U(c) = cu'(c)$, $V(l) = lv'(l)$ and $\psi(x) = x\phi'(x)/\phi(x)$. Moreover, $c_t^w = q_t \overline{M} = \phi\left(w_t l_t\right)$. Equation $U\left(c_{t+1}^w\right) \psi\left(x_t\right) = V\left(l_t\right)$ can then be stated as follows $U\left(\phi(x_{t+1})\right) \psi\left(x_t\right) = V\left(l_t\right)$, where $x \equiv wl$.

2.3. Production Side

Following Bosi et. al (2007), there are two sectors in the production side of the economy: one for the consumption good Y^0, the other for the investment good Y^1:

$$Y^i = F^i\left(K^i, L^i\right)$$

F^i $(i = 0, 1)$ represent two different constant returns to scale technologies using capital and labor as inputs. At equilibrium, $K^0 + K^1 = K = N^c k^c$ and $L^0 + L^1 = L = N^w l$ hold, where (K^0, K^1) and (L^0, L^1) denote the amounts of capital and labor inputs in consumption and investment sectors. K and L are total capital and labor inputs, N^c and N^w denote the number of capitalists and workers. k^c and l denote the capital stock of each capitalist and the labor supply of each worker respectively.

All the variables need to be normalized as in Bosi et. al by dividing Y^i, K^i and L^i by the size N^w of the labor force:

$$y^i \equiv Y^i/N^w,\ k^i \equiv K^i/N^w,\ l^i \equiv L^i/N^w,$$

$$k \equiv K/N^w = N^c k^c/N^w,\ l \equiv L/N^w,\ i = 0, 1,$$

at the equilibrium, $k^0 + k^1 = k$ and $l^0 + l^1 = l$ hold. For simplicity, we assume a constant ratio $n = 1$ between capitalists and workers: $k = N^c k^c/N^w = nk^c = k^c$.

According to the homogeneity of production functions, the per-worker production functions in sector $(i = 0, 1)$ are given by,

$$y^i = f^i(k^i, l^i),$$

where $f^i \equiv F^i/N^w$.

We need the following assumption for the per-worker production functions as in Bosi et. al.

Assumption 3 The production function $f^i : R_+^2 \rightarrow R_+$, $i = 0, 1$, is C^r, with r large enough, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $lim_{y\rightarrow 0} f^i_1(y,x) = lim_{y\rightarrow 0} f^i_2(x,y) = +\infty$, $lim_{y\rightarrow +\infty} f^i_1(y,x) = lim_{y\rightarrow +\infty} f^i_2(x,y) = 0$.

We derive the social production function by solving the problem of optimal resources allocation problem between the two sectors:

$$\max_{\{k^0,k^1,l^0,l^1\}} f^0(k^0,l^0)$$

$$\begin{aligned} \text{such that } y^1 &\leq f^1(k^1,l^1), \\ k^0+k^1 &\leq k, \\ l^0+l^1 &\leq l, \\ k^0,\, k^1,\, l^0,\, l^1 &\geq 0. \end{aligned}$$

Define the Lagrangian as follows

$$\begin{aligned} L^f &= f^0(k^0,l^0) + p\left[f^1(k^1,l^1) - y^1\right] + r\left(k - k^0 - k^1\right) \\ &\quad +w\left(l - l^0 - l^1\right) \end{aligned}$$

The value function or the social production function is

$$T(k,y^1,l) = f^0\left(k^0(k,y^1,l), l^0\left(k,y^1,l\right)\right),$$

which is derived by using the optimal demand functions for capital and labor $k^0(k,y^1,l)$, $l^0\left(k,y^1,l\right)$, $k^1(k,y^1,l)$ and $l^1\left(k,y^1,l\right)$. We can easily find that T is homogeneous of degree one and non-strictly concave.

The first order conditions imply that the rental rate of capital, the price of the investment good and the wage rate are

$$T_1(k, y^1, l) = r,\ T_2(k, y^1, l) = -p,\ T_3(k, y^1, l) = w$$

The concavity of T implies that

$$T_{11}(k, y^1, l) \leq 0,\ T_{22}(k, y^1, l) \leq 0, \text{ and } T_{33}(k, y^1, l) \leq 0.$$

Define the relative capital intensity difference across sectors as follows

$$b \equiv a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) \tag{10}$$

with

$$a_{00} \equiv \frac{l^0}{y^0},\ a_{10} \equiv \frac{k^0}{y^0},\ a_{01} \equiv \frac{l^1}{y^1},\ a_{11} \equiv \frac{k^1}{y^1}$$

the input coefficients in each of the two sectors as in Bosi et.al.

From Bosi et al. (2007), we have

$$T_{12} = -T_{11}b,\ T_{31} = -T_{11}a \geq 0,\ T_{32} = T_{11}ab$$

with $a \equiv k^0/l^0 > 0$ the capital-labor ratio in the consumption good sector.

And we can also obtain the following equations,

$$T_{22} = T_{11}b^2,\ T_{33} = T_{11}a^2.$$

2.4. Intertemporal Equilibrium and Steady State

We focus on the symmetric equilibrium in which all agents are identical within their own type. In order to simplify notation, we let $c = c^w$ and $k = k^c$.

We introduce the intertemporal equilibrium with perfect foresight in terms of k and l, that is a sequence $\{k_{t+1}, l_t\}_{t=0}^{\infty} > 0$ satisfying (k_t is a predetermined variable and $k_0 > 0$ is given.)

$$\{ \begin{array}{c} k_{t+1} - \beta \left[1 - \delta - \frac{T_1(k_t, k_{t+1}-(1-\delta)k_t, l_t)}{T_2(k_t, k_{t+1}-(1-\delta)k_t, l_t)}\right] k_t = 0 \\ U\left(\phi(T_3\left(k_t, k_{t+1} - (1-\delta)\, k_t, l_t\right) l_t)\right) \psi \left(T_3\left(k_{t-1}, y_{t-1}^1, l_{t-1}\right) l_{t-1}\right) - V\left(l_{t-1}\right) = 0 \end{array} \tag{11}$$

together with the transversality condition

$$\lim_{t \to +\infty} \beta^t \left(p_t/c_t\right) k_{t+1} = 0. \tag{12}$$

We follow Dromel and Pintus (2008) by introducing tax progressivity through a constant parameter $\pi = 1 - \psi$ with $0 \leq \pi < 1$. It is easy for us to have $\phi(x) = mx^{1-\pi}$ ($m > 0$ is a scaling parameter that plays a minor role in the analysis) since $\psi(x) = x\phi'(x)/\phi(x) = 1 - \pi$. The above equilibrium conditions are therefore

$$\begin{aligned} k_{t+1} - \beta \left[1 - \delta - \frac{T_1\left(k_t, k_{t+1} - (1-\delta)\, k_t, l_t\right)}{T_2\left(k_t, k_{t+1} - (1-\delta)\, k_t, l_t\right)}\right] k_t &= 0, \\ U\left(\phi(T_3\left(k_t, k_{t+1} - (1-\delta)\, k_t, l_t\right) l_t)\right)(1-\pi) - V\left(l_{t-1}\right) &= 0,\ \phi(T_3) = mT_3^{1-\pi}. \end{aligned}$$

The first step is to prove the existence of the steady state.

Definition 1 The steady state of the system (11) is a stationary sequence $\{k_{t+1}, l_t\}_{t=0}^{\infty} = \{k^*, l^*\}_{t=0}^{\infty} > 0$ that satisfies the following equalities

$$\left\{\begin{array}{c} 1 = \beta\left[1 - \delta - \frac{T_1(k,\delta k,l)}{T_2(k,\delta k,l)}\right] \\ U\left(m(T_3\left(k,\delta k,l\right)l)^{1-\pi}\right) = V\left(l\right)/(1-\pi),\ \phi(T_3) = mT_3^{1-\pi} \end{array}\right. \tag{13}$$

Before analyzing the stability properties of the dynamical system, we need prove the uniqueness and existence of the steady state.

Proposition 1 Under the above three assumptions, there exists a unique steady state $(k^*, l^*) > 0$ in the above system (13).[1]

2.5. Characteristic polynomial and geometric method

In order to analyze the dynamical system (11) around the unique steady state, we shall introduce three elasticity parameters all evaluated at this steady state: the elasticity of the interest rate

$$\varepsilon_r \equiv -\frac{T_{11}k}{T_1} \in (0, +\infty),$$

the elasticity of the real wage

$$\varepsilon_w \equiv -\frac{T_{33}l}{T_3} \in (0, +\infty),$$

and the elasticity of the offer curve $\lambda(l) \equiv U^{-1}\left(V\left(l\right)/\left(1-\pi\right)\right)$

$$\varepsilon \equiv \frac{1}{1-\pi}\frac{V'l}{U'c} \in (1, +\infty).$$

Like Pintus and Dromel (2008), we can first fix the technology at the steady state and then consider the parameterized curve $(T(\varepsilon),\ D(\varepsilon))$ when ε varies in the open interval $(1, +\infty)$.

Before we solve the model, we show that the elasticity of the labor supply with respect to the

[1] The proof is given in appendix 2.

real wage can be stated as follows[2]

$$\varepsilon_{lw} = \frac{1-\pi}{\varepsilon - (1-\pi)}.$$

We then show that the elasticity of the offer curve ε can be expressed in terms of the elasticity of intertemporal substitution in consumption ($\varepsilon_c = -\frac{u'(c)}{cu''(c)}$) and the inverse of the elasticity of the marginal disutility of labor ($\varepsilon_l = \frac{v'(l)}{lv''(l)}$), that is, $\varepsilon = \frac{1+1/\varepsilon_l}{1-1/\varepsilon_c}$.[3]

Moreover, the following lemma is useful as we characterize the equilibrium conditions of the model.

Lemma 1. By linearizing (11) around the unique steady state, the Jacobian matrix can be expressed as follows: $J^* = \begin{bmatrix} 1-b\vartheta\varepsilon_r & -a\vartheta\varepsilon_r \\ -b\varepsilon_w(1-\pi) & a(1-\varepsilon_w)(1-\pi) \end{bmatrix}^{-1} \begin{bmatrix} 1-[1+(1-\delta)b]\vartheta\varepsilon_r & 0 \\ -(1-\pi)[1+(1-\delta)b]\varepsilon_w & a\varepsilon \end{bmatrix}$ with $\theta = \beta^{-1} - (1-\delta)$ and $\vartheta = \beta\theta(1-b\theta)$.[4]

Using the above lemma, we have the following proposition.

Proposition 2 The characteristic polynomial of the Jacobian matrix J is $P(\lambda) = \lambda^2 - T\lambda + D$ with

$$T = 1 + D + \frac{\delta b\vartheta\varepsilon_r - \vartheta\varepsilon_r}{1-\varepsilon_w - b\vartheta\varepsilon_r}\left[1 - \frac{\varepsilon}{(1-\pi)}\right], \tag{14}$$

and

$$D = \varepsilon\frac{1-\vartheta\varepsilon_r - b\vartheta\varepsilon_r + b\delta\vartheta\varepsilon_r}{(1-\pi)(1-\varepsilon_w-b\vartheta\varepsilon_r)}. \tag{15}$$

Moreover, as ε is equal to 1, one has $T_1 = 1 + D_1 + \Lambda$, where $\Lambda = -\frac{\delta b\vartheta\varepsilon_r - \vartheta\varepsilon_r}{1-\varepsilon_w-b\vartheta\varepsilon_r}\frac{\pi}{1-\pi}$.

The two elasticities ε_w and ε_r are linked through the following relationship:

$$\varepsilon_w = \varepsilon_r(1-\delta b)^2\frac{s}{1-s}, \tag{16}$$

[2] The proof is in appendix 3.
[3] The proof is in appendix 3.
[4] The proof is in appendix 4.

where s is the share of capital in total income (i.e., $s = rk/(rk + wl)$).[5]

Using the equation $\varepsilon_w = \varepsilon_r (1-\delta b)^2 \frac{s}{1-s}$, T and D can be stated as follows:

$$T = 1 + D - \left[1 - \frac{\varepsilon}{1-\pi}\right] \frac{\beta\theta\varepsilon_r (1-\theta b)(1-\delta b)}{1 - \varepsilon_r \left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta (1-\theta b) b\right]}, \tag{17}$$

$$D = \frac{\varepsilon}{1-\pi} \frac{1 - \varepsilon_r \beta\theta (1-\theta b)[1 + b(1-\delta)]}{1 - \varepsilon_r \left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta (1-\theta b) b\right]}. \tag{18}$$

As $\varepsilon = 1$, $T_1 = 1 + D_1 + \Lambda$ holds where

$$D^1 = \frac{1}{1-\pi} \frac{1 - \varepsilon_r \beta\theta (1-\theta b)[1 + b(1-\delta)]}{1 - \varepsilon_r \left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta (1-\theta b) b\right]} \tag{19}$$

$$\Lambda = \frac{\pi}{1-\pi} \frac{\beta\theta\varepsilon_r (1-\theta b)(1-\delta b)}{1 - \varepsilon_r \left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta (1-\theta b) b\right]}. \tag{20}$$

Lemma 2 Notice that both T and D are linear with respect to ε. When ε varies in the open interval $(1, +\infty)$, the graph of $[(T(\varepsilon), D(\varepsilon))]$ is a half-line $\Delta(T)$ with slope[6]

$$\psi = 1 - \frac{\beta\theta\varepsilon_r (1-\theta b)(1-\delta b)}{1 - \beta\theta\varepsilon_r (1-\theta b) b}. \tag{21}$$

Now we study the variations of T and D in the (T, D) plane as we allow the elasticity of the offer curve to vary continuously within $(1, +\infty)$. In other words, we fix the technology parameters (i.e., ε_r, δ, β and s) at the steady state and consider the parameterized curve $(T(\varepsilon), D(\varepsilon))$ when the domain of ε is $(1, +\infty)$. It is easy to verify that this locus is a half-line Δ that starts close to (T_1, D_1) when ε is close to 1, and whose slope is ψ. The value of $\Lambda = T_1 - 1 + D_1$, on the other hand, measures the deviation of the point (T_1, D_1) from the line (AC) of equation $D = T - 1$, in the

[5] The proof is in appendix 5.
[6] The proof is in appendix 6.

(T, D) plane.

Before analyzing the half-line Δ, we need to characterize its origin $\left(T^1, D^1\right)$, its slope ψ and its endpoint $(T(\infty), D(\infty))$. Following Bosi et al. (2007), for a fixed value of s, we vary independently both ε_r and b. Based on the fact that T and D in equations (17) and (18) are first order polynomials in ε_r, we can proceed by first fixing the value of b and then considering variations of ε_r. By repeating this procedure with different values of b, we can know about the evolution of the local dynamics and bifurcations. Since we need the prices to be positive, b must fall within the range $(-\infty, 1/\theta)$ (see Bosi et al. (2007)). We prove that two types of geometrical configurations, associated with different properties of the slope ψ, can appear.

Lemma 3 Under assumptions 1, 2 and 3, the following properties hold: 1. The slope satisfies $\psi(\varepsilon_r) \in (1-\delta+1/b, 1)$ for any $\varepsilon_r > 0$. 2. $lim_{\varepsilon \to +\infty} D(\varepsilon) = +(-)\infty$ if and only if $D^1 > (<) 0$.[7]

If $b < -1/(1-\delta)$, the slope $\psi(\varepsilon_r)$ is in the interval $(0,1)$ for any $\varepsilon_r \geq 0$. If $b \in (-1/(1-\delta), 1/\theta)$, the sign of the slope depends on the value of ε_r. In our model, we consider the indeterminate case: $\Lambda < 0$, that is to say,

$$1 - \varepsilon_r \left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta(1-\theta b) b\right] < 0. \tag{22}$$

Following Bosi et al., we set $b < -1/(1-\delta)$. Under the restriction (22), we have $D^1 < 0$.

Assumption 4 $b < -1/(1-\delta)$

Under assumption 4 and (22), Lemma 3 implies that the slope ψ is positive and less than one, and D^1 is less than zero. To get indeterminacy, we need find conditions for $D^1 \in (-1, 0)$. To this end, examining Lemma 3 and (22) allows us to show that there exist some critical values for, respectively, the share of capital in total income s^* and the elasticity of interest rate ε_r^*, such that if $s \leq s^*$ or $\varepsilon_r \in (0, \varepsilon_r^*)$, the slope of $\Delta(T)$ is positive and lower than one and $D^1 < -1$ $(\lim_{\varepsilon \to +\infty} D(\varepsilon) = -\infty)$. As a result, $\Delta(T)$ remains in the saddle path region and the steady state is locally determinate.

[7] The proof is in appendix 6.

Conversely, when $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, we have $D^1 \in (-1, 0)$ and $\lim_{\varepsilon \to +\infty} D(\varepsilon) = -\infty$. It follows that for low elasticities of the offer curve ε, the half-line $\Delta(T)$ crosses the interior of the triangle ABC and therefore the steady state is locally indeterminate. Then $\Delta(T)$ intersects the line $D = -T - 1$ at $\varepsilon = \varepsilon_F$ and a flip bifurcation generically occurs. Lastly, for $\varepsilon > \varepsilon_F$ the steady state becomes a saddle, thus locally determinate. Notice that the triangle ABC can be found on page 321 in Bosi et. al. (2007).

We summarize these results in the following proposition.

Proposition 3 Let Assumption 1, 2, 3 ,4 and (22) hold. Then there exist $s^* \in (0, 1)$ and $\varepsilon_r^* > 0$, such that:

(1) If $s \leq s^*$ or $\varepsilon_r \in (0, \varepsilon_r^*)$, then the steady state is a saddle (locally determinate) for all $\varepsilon > 0$.

(2) If $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, then these exists $\varepsilon_F > 1$ such that the steady state is a sink (locally indeterminate) when $\varepsilon \in (0, \varepsilon_r^*)$ and a saddle when $\varepsilon > \varepsilon_F$. A flip bifurcation generically occurs at $\varepsilon = \varepsilon_F$.[8]

As in Bosi et. al. (2007), it is easy to verify that when the consumption good is sufficiently capital intensive and π is small, local indeterminacy arises while the elasticities of capital-labor substitution in both sectors are slightly greater than unity and the elasticity of the offer curve is low enough.

The next proposition is the key to the present paper. It shows how the critical values in the above proposition move with π.

Proposition 4 (Income Tax Progressivity and Local Indeterminacy) Let Assumption 1, 2, 3 ,4 and (22) hold. When $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, the critical value for the flip bifurcation is,

$$\varepsilon_F = \frac{2(1-\pi)\left\{\varepsilon_r\left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta(1-\theta b)b\right] - 1\right\}}{2 - \varepsilon_r \beta\theta(1-\theta b)[1 + (2-\delta)b]}, \tag{23}$$

[8] The proof is in the appendix.

which is a decreasing function of $\pi \in (0,1)$. Therefore, income tax progressivity reduces the set of parameter values that are associated with local indeterminacy.

If we consider negative values of π, this means that income taxes are regressive. Then decreasing π from zero would enlarge the set of parameter values that are associated with local indeterminacy, as in Dromel and Pintus (2008).

We are now in a position to intuitively explain why Dromel and Pintus's results still hold in the two sector framework. As we know, Guo and Harrison (2001) conclude that a regressive tax policy can **stabilize** the economy in a two sector model while Guo and Lansing (1998) show that such a policy can **destabilize** the economy in a one sector model. The reversal of the stabilizing effects depends on the assumption of strong externalities in the investment goods sector. In other words, if the aggregate economy exhibits constant returns to scale or there is a low investment externality in the two sector model, the regressive tax policy is still destabilizing. In the two sector version of the Dromel and Pintus (2008) model, their results hold since there are no externalities in the investment goods sector.

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3. Appendix

3.1. Part 1–Capitalists' choices and Workers' choices

For the capitalists, we define the Lagrangian as follows (ε_t, ε_t^1, ε_t^2 and ε_t^3 are complementary slackness variables.)

$$\begin{aligned} L^c &= \sum_{t=0}^{\infty} \beta^t \{\ln c_t^c + \varepsilon_t \left[r_t k_t^c + q_t M_t^c - c_t^c + p_t \left[k_{t+1}^c - (1-\delta) k_t^c\right] + q_t M_{t+1}^c\right] \\ &\quad + \varepsilon_t^1 c_t^c + \varepsilon_t^2 k_{t+1}^c + \varepsilon_t^3 M_{t+1}^c\}. \end{aligned}$$

The first order conditions are given by

$$0 \geq \frac{1}{c_t^c} - \varepsilon_t, \ = 0 \text{ if } c_t^c > 0;$$

$$0 \geq -p_t \varepsilon_t + \beta \varepsilon_{t+1} \left[r_{t+1} + (1-\delta) p_{t+1}\right], \ = 0 \text{ if } k_{t+1}^c > 0;$$

$$0 \geq -q_t \varepsilon_t + \beta q_{t+1} \varepsilon_{t+1}, \ = 0 \text{ if } M_{t+1}^c > 0.$$

For the workers, we define the Lagrangian as follows (λ_t, μ_t, λ_t^1, λ_t^2, λ_t^3 and λ_t^4 are complementary slackness variables.)

$$\begin{aligned}L^W &= \sum_{t=0}^{\infty}\gamma^t\{[u(c_t^w)-\gamma v(l_t)]+\lambda_t[r_tk_t^w+\phi(w_tl_t)+q_tM_t^w-c_t^w-q_tM_{t+1}^w\\&\quad -p_t\left(k_{t+1}^w-(1-\delta)k_t^w\right)]+\mu_t[r_tk_t^w+q_tM_t^w-c_t^w-p_t\left(k_{t+1}^w-(1-\delta)k_t^w\right)]\\&\quad +\lambda_t^1c_t^w+\lambda_t^2l_t+\lambda_t^3k_{t+1}^w+\lambda_t^4M_{t+1}^w\}.\end{aligned}$$

The first order conditions are given by

$$0\geq u'(c_t^w)-\lambda_t-\mu_t,\ =0 \text{ if } c_t^w>0\ (\lambda_t^1=0);$$

$$0\geq -\gamma v'(l_t)+\lambda_tw_t\phi'(w_tl_t),\ =0 \text{ if } l_t>0\ (\lambda_t^2=0);$$

$$0\geq -p_t(\lambda_t+\mu_t)+\gamma\left(\lambda_{t+1}+\mu_{t+1}\right)[r_{t+1}+(1-\delta)p_{t+1}],\ =0 \text{ if } k_{t+1}^w>0;$$

$$0\geq -\lambda_tq_t+\gamma q_{t+1}\left(\lambda_{t+1}+\mu_{t+1}\right),\ =0 \text{ if } M_{t+1}^w>0;$$

and

$$\mu_t\geq 0,\ =0 \text{ if the borrowing constraint is not binding.}$$

At the optimum, $u'(c_t^w)=\lambda_t+\mu_t$, $\mu_t>0$ and $\lambda_t=\frac{\gamma v'(l_t)}{w_t\phi'(w_tl_t)}$ hold. From the first order condition $\mu_t=u'(c_t^w)-\lambda_t=u'(c_t^w)-\frac{\gamma v'(l_t)}{w_t\phi'(w_tl_t)}>0$, we have the following inequality $u'(c_t^w)w_t\phi'(w_tl_t)>\gamma v'(l_t)$. $M_{t+1}^w>0$ implies that the fourth inequality of the first order conditions is binding,

$$\lambda_t=\frac{\gamma q_{t+1}u'\left(c_{t+1}^w\right)}{q_t}.$$

Using $\lambda_t = \frac{\gamma v'(l_t)}{w_t \phi'(w_t l_t)}$and $\lambda_t = \frac{\gamma q_{t+1} u'\left(c_{t+1}^w\right)}{q_t}$, we can have

$$v'(l_t) = w_t \phi'\left(w_t l_t\right) u'\left(c_{t+1}^w\right) \frac{q_{t+1}}{q_t}.$$

3.2. Part 2–Proof of proposition 1

Proof. Following the method in Bosi. et .al. In this case, $T\left(k, \delta k, l\right)$ is homogenous of degree one, we have

$$-\frac{T_1\left(\kappa, \delta\kappa, 1\right)}{T_2\left(\kappa, \delta\kappa, 1\right)} = \frac{1}{\beta} - \left(1 - \delta\right),$$

with $\kappa = k/l$. Notice that the steady state value of κ only depends on the technology. Using the technique of theorem 3.1 in Becker and Tsyganov (2002), it implies that there exists a unique solution κ^*. Considering the definition of U and V with the fact that $c^* = \phi\left(wl^*\right)$ and $\phi(x) = mx^{1-\pi}$, we rewrite the second equation in (13) as

$$mT_3\left(\kappa^*, \delta\kappa^*, 1\right)^{1-\pi} u'\left(m(T_3\left(k, \delta k, l\right) l)^{1-\pi}\right) = \frac{l^\pi v'\left(l\right)}{1-\pi}.$$

As $0 \leq \pi < 1$, $\lim_{l\to 0} \frac{l^\pi v'(l)}{1-\pi} = 0$, $\lim_{l\to l^*} \frac{l^\pi v'(l)}{1-\pi} = +\infty$ and $mT_3\left(\kappa^*, \delta\kappa^*, 1\right)^{1-\pi}$ is a constant. Under the first assumption, we know that such an equation has a unique solution l^*.

3.3. Part 3– Proofs of two relationships

Proof. By definition $c = \phi\left(wl\right) = U^{-1}\left(V(l) / \left(1 - \pi\right)\right)$. Taking total differentiation in both sides of this equation, we have

$$\phi' w dl + \phi' l dw = \frac{dU^{-1}\left(V\left(l\right) / \left(1 - \pi\right)\right)}{dl} dl = \frac{V'}{\left(1 - \pi\right) U'} dl.$$

Manipulating the above equation leads to $\frac{dl}{dw} = \frac{\phi' l}{\frac{V'}{(1-\pi)U'} - \phi' w}$. Then the elasticity of the labor supply with respect to the real wage is

$$\begin{aligned} \frac{wdl}{ldw} &= \frac{\phi' wl}{\frac{V'l}{(1-\pi)U'} - \phi' wl} = \frac{\phi' x}{\frac{V'l}{(1-\pi)U'} - \phi' x} \\ &= \frac{\phi' x/\phi}{\frac{1}{1-\pi}\frac{V'l}{U'\phi} - \frac{\phi' x}{\phi}} = \frac{1-\pi}{\varepsilon - (1-\pi)}, \end{aligned}$$

with $x = wl$.

Proof. $V' = (lv'(l))' = v'[1 + lv''/v'] = v'[1 + 1/\varepsilon_l]$, $U' = (cu'(c))' = u'[1 + lu''/u'] = u'[1 - 1/\varepsilon_c]$. Then $\varepsilon = \frac{1}{1-\pi}\frac{V'l}{U'c} = \frac{1}{1-\pi}\frac{V}{U}\frac{1+1/\varepsilon_l}{1-1/\varepsilon_c} = \frac{1+1/\varepsilon_l}{1-1/\varepsilon_c}$ since $U(1-\pi) = V$ holds at the steady state.

3.4. Part 4–the Jacobian matrix

Proof. Taking total differentiation in the first equation of (11), we have

$$dk_{t+1} = \beta\left[1 - \delta - \frac{T_1}{T_2}\right] dk_t - \beta k_t d\Gamma_t,$$

with $\Gamma_t \equiv \frac{T_1(k_t, k_{t+1}-(1-\delta)k_t, l_t)}{T_2(k_t, k_{t+1}-(1-\delta)k_t, l_t)}$. The total differentiation of $\Gamma_t = \frac{T_1(k_t, k_{t+1}-(1-\delta)k_t, l_t)}{T_2(k_t, k_{t+1}-(1-\delta)k_t, l_t)}$ is

$$d\Gamma_t = A_1 dk_t + A_2 dl_t + A_3 dk_{t+1},$$

with $A_1 = \frac{1}{T_2}\left[(T_{11} - (1-\delta)T_{12}) - (T_{21} - (1-\delta)T_{22})\frac{T_1}{T_2}\right]$, $A_2 = \frac{1}{T_2}\left(T_{13} - T_{23}\frac{T_1}{T_2}\right)$ and $A_3 = \frac{1}{T_2}\left[T_{12} - T_{22}\frac{T_1}{T_2}\right]$. At the steady state, $\beta\left[1 - \delta - \frac{T_1}{T_2}\right] = 1$ and $U(c^*) = V(l^*)/(1-\pi)$ hold. Let's define $\theta \equiv -\frac{T_1}{T_2} = \beta^{-1} - (1-\delta)$, $\vartheta = \beta\theta(1 - b\theta)$, the share of capital in total income $s = \frac{rk}{T + py^1} \in (0,1)$ and the relative capital intensity across sectors b, all evaluated at the steady state. We then have $E_{11}dk_{t+1} + E_{12}dl_t = F_{11}dk_t + F_{12}dl_{t-1}$ with $F_{11} = 1 - \beta k^* A_1 = 1 - \frac{\beta k^* T_{11}}{T_1}\frac{T_1}{T_2}(1 + b\frac{T_1}{T_2})[1 + (1-\delta)b] =$

$1-[1+(1-\delta)b]\vartheta\varepsilon_r$, $F_{12}=0$, $E_{11}=1+\beta k^* A_3=1+\beta k^*\frac{1}{T_2}(T_{12}-T_{22}\frac{T_1}{T_2})=1-b\vartheta\varepsilon_r$ and $E_{12}=\beta k^* A_2=\beta k^*\frac{1}{T_2}\left(T_{13}-T_{23}\frac{T_1}{T_2}\right)=-a\vartheta\varepsilon_r$. Totally differentiating the second equation in the dynamical system implies that $B_1 dk_{t+1}+B_2 dl_t+B_3 dk_t=\frac{V'}{1-\pi}dl_{t-1}(*)$, with $B_1=U'\phi' l_t T_{32}$, $B_2=U'\phi'(T_3+l_t T_{33})$ and $B_3=U'\phi'[T_{31}-(1-\delta)T_{32}]l_t$. Manipulating the equation (*) by dividing U' in both sides and evaluating B_1, B_2 and B_3 at the steady state leads to the following equation $\phi' l T_{32} dk_{t+1}+\phi'(T_3+l_t T_{33})dl_t=-\phi' l[T_{31}-(1-\delta)T_{32}]dk_t+\frac{V'}{(1-\pi)U'}dl_{t-1}(**)$. Using $c^*=\phi(T_3 l^*)$ and multiplying $a\frac{l}{c}$ in both sides of (**), we have $E_{21}dk_{t+1}+E_{22}dl_t=F_{21}dk_t+F_{22}dl_{t-1}$ with $F_{21}=\phi'\frac{lT_{33}}{a}[1+(1-\delta)b]\frac{al}{\phi}=-\varepsilon_w(1-\pi)[1+(1-\delta)b]$, $F_{22}=\frac{aV'l}{(1-\pi)U'c}=a\varepsilon$, $E_{21}=\phi' l T_{33}\frac{b}{a}\frac{al}{\phi}=-b\varepsilon_w(1-\pi)$ and $E_{22}=\phi' T_3(1-\varepsilon_w)\frac{al}{\phi}=a(1-\varepsilon_w)(1-\pi)$.

3.5. Part 5–Proof of the relationship $\varepsilon_w=\varepsilon_r(1-\delta b)^2\frac{s}{1-s}$

Proof. At the steady state, we have

$$\begin{aligned}
\varepsilon_w &= -\frac{T_{33}l}{T_3}=-\frac{T_{11}a^2 l}{T_3},\ (T_{33}=T_{11}a^2)\\
\varepsilon_r &\equiv -\frac{T_{11}k}{T_1},\\
\frac{\varepsilon_w}{\varepsilon_r} &= \frac{a^2 l T_1}{kT_3}=\left(a\frac{l}{k}\right)^2\frac{T_1 k}{T_3 l}=\left(a\frac{l}{k}\right)^2\frac{s}{1-s}.
\end{aligned}$$

The parameter b satisfies,

$$\begin{aligned}
b &\equiv a_{01}\left(\frac{a_{11}}{a_{01}}-\frac{a_{10}}{a_{00}}\right)\\
&= \frac{l^1}{y^1}\left(\frac{k^1}{l^1}-\frac{k^0}{l^0}\right)\\
&= \frac{l^1}{\delta k}\left(\frac{k^1}{l^1}-\frac{k^0}{l^0}\right), \text{ since } y^1=\delta k \text{ holds at the steady state.}
\end{aligned}$$

So $1-\delta b = 1-\frac{k^1}{k}+\frac{l^1k^0}{l^0k}$. In addition, we have the following equation

$$
\begin{aligned}
a\frac{l}{k} &= \frac{k^0}{l^0}\frac{l}{k}=\frac{k^0}{l^0}\frac{l^0+l^1}{k} \\
&= \frac{k^0}{k}+\frac{l^1k^0}{l^0k}=1--\frac{k^1}{k}+\frac{l^1k^0}{l^0k} \\
&= 1-\delta b.
\end{aligned}
$$

Therefore,

$$
\varepsilon_w = \varepsilon_r\left(1-\delta b\right)^2\frac{s}{1-s}.
$$

3.6. Part 6-Proofs of Lemma 2 and Lemma 3

Proof of lemma 2. We let $D = c_1\frac{\varepsilon}{1-\pi}$ and $T = 1+D-c_2\left[1-\frac{\varepsilon}{1-\pi}\right]$, where c_1 and c_2 do not rely on ε (see equation 17 and 18). Then, the slope of $\Delta\left(T\right)$ can be solved by,

$$
\begin{aligned}
\psi &= \frac{D-D^1}{T-T^1} \\
&= \frac{\frac{c_1}{1-\pi}\left(\varepsilon-1\right)}{\frac{c_1}{1-\pi}\left(\varepsilon-1\right)+\frac{c_2}{1-\pi}\left(\varepsilon-1\right)} \\
&= 1-\frac{c_2}{c_1+c_2},\ (\text{as } \varepsilon>1) \\
&= 1-\frac{\beta\theta\varepsilon_r\left(1-\theta b\right)\left(1-\delta b\right)}{1-\beta\theta\varepsilon_r\left(1-\theta b\right)b}.
\end{aligned}
$$

Proof of lemma 3. Since $\theta=\beta^{-1}+\delta-1>0$ and $\beta^{-1}>1$, we know that $\theta>\delta$. Multiplying b in both sides of $\theta>\delta$, we have $\delta b<\theta b<1$. The latter inequality holds since $b<1/\theta$. Therefore, $1-\delta b>0$. $\psi\left(\varepsilon_r\right)$ is a decreasing function with respect to ε_r. When ε_r approaches to 0 and ∞, we have $lim_{\varepsilon_r\to 0}\psi\left(\varepsilon_r\right)=1$ and $lim_{\varepsilon_r\to+\infty}\psi\left(\varepsilon_r\right)=1-\delta+1/b$. Thus claim (1) is proved. Claim (2) follows directly by checking (18).

3.7. Part 7–Proofs of propositions 3 and 4

Proof of proposition 3 As $b < -1/(1-\delta)$, when ε_r moves from zero to $+\infty$, ψ decreases continuously from one to $1-\delta+1/b \in (0, 1-\delta)$. This implies that $\psi \in (0,1)$ holds for all $\varepsilon_r > 0$. As $\varepsilon = 1$, we have $T_1 = 1 + D_1 + \Lambda$. Therefore indeterminacy emerges only for the case: $\Lambda < 0$.[9] Now, let us consider the following

$$z \equiv \frac{1-\delta b}{1-\theta b}\frac{1}{\beta\theta}\frac{s}{1-s},$$

$$z_1 \equiv \frac{1-\varepsilon_r \beta\theta(1-\theta b) b}{\varepsilon_r \beta\theta(1-\theta b)(1-\delta b)} > 1,$$

$$z_2 \equiv \frac{2-\varepsilon_r \beta\theta(1-\theta b)[1+(2-\delta) b]}{\varepsilon_r \beta\theta(1-\theta b)(1-\delta b)} + \frac{1}{2}\frac{\pi}{1-\pi} > z_1.$$

- $\Lambda < 0$ implies that $z > z_1$. Then, $D_1 < 0$ and $\lim_{\varepsilon\to+\infty} D(\varepsilon) = -\infty$.
- When $z_1 < z < z_2$, then $D_1 < -1$ and $\lim_{\varepsilon\to+\infty} D(\varepsilon) = -\infty$.
- When $z > z_2$, then $D^1 \in (-1, 0)$ and $\lim_{\varepsilon\to+\infty} D(\varepsilon) = -\infty$.

Therefore, we face two possible subcases:

(1) If $z < z_2$, then $\Delta-$line does not cross the triangle ABC and the steady state is a saddle.

(2) If $z > z_2$, then $\Delta-$line does not cross the triangle ABC and there exists $\varepsilon_F > 1$ such that the steady state is a sink when $\varepsilon \in (0, \varepsilon_r^*)$ and a saddle when $\varepsilon > \varepsilon_F$.

It is easy to prove that $z \leq z_2$ if $s \leq s^*$ with

$$\begin{aligned} & s^* \\ \equiv \; & \frac{\beta\theta(1-\theta b)[1+(2-\delta) b] - \frac{1}{2}\frac{\pi}{1-\pi}\beta\theta(1-\theta b)(1-\delta b)}{\beta\theta(1-\theta b)[1+(2-\delta) b] - \frac{1}{2}\frac{\pi}{1-\pi}\beta\theta(1-\theta b)(1-\delta b) - (1-\delta b)^2} \\ \in \; & (0,1) \end{aligned}$$

[9] The uniqueness of the steady state rules out the occurence of transcritical bifurcations, and we only consider the case: $\Lambda < 0$.

and that when $s > s^*$, then $z < z_2$ if and only if $\varepsilon_r \in (0, \varepsilon_r^*)$, with

$$\begin{aligned} & \varepsilon_r^* \\ \equiv \; & \frac{2(1-s)}{s(1-\delta b)^2 (1-s) \beta\theta (1-\theta b) \left[1 + (2-\delta) b - \frac{1}{2}\frac{\pi}{1-\pi}(1-\delta b)\right]} \\ > \; & 0. \end{aligned}$$

Proof of proposition 4. When $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, the flip bifurcation occurs if $\Delta(\varepsilon)$ line crosses the segment AB, that is

$$\left\{ \begin{array}{c} T(\varepsilon_F) = -D(\varepsilon_F) - 1 \\ T(\varepsilon_F) = 1 + D(\varepsilon_F) + \Lambda(\varepsilon_F) \end{array} \right. .$$

This gives rise to $\varepsilon_F = \frac{2(1-\pi)\left\{\varepsilon_r\left[(1-\delta b)^2 \frac{s}{1-s} + \beta\theta(1-\theta b) b\right] - 1\right\}}{2 - \varepsilon_r \beta\theta(1-\theta b)[1+(2-\delta)b]}$.