Holoreturns And Holothetic Invariance in Economics

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Abstract: *Hicks neutral technical progress implies that in $Y = AF(K, L)$ any technical progress only changes the technological knowledge, A, but does not affect inputs and this is the Hicks-neutral holothetic result given by Sato in which the technical progress transforms into a scale effect without changing the map of isoquants to non-holothetic trajectory. This holothetic nature of production functions is possible under Lie-group like transformations with invariance properties for production functions under a given technical progress. This paper proposes a Farkas Lemma-like character of this holothetic transformation. Kostant's Convexity Theorem is a well-generalized result of convexity on Lie-group transformations concerning holotheticity. There's an 'equilibrium pricing' way found here for a convexity assumption implying 'holothetic invariance'.*

Preliminaries:

Separating Hyperplane Theorem:
Minkowski: Let K be a convex subset of $\mathbb{R}$^n and z a point of $\mathbb{R}$^n. There is a hyperplane H through z and bounding for K if and only if z is not interior to K. [1]

Farkas' Lemma: If C is a closed convex cone, then for any $b \in \mathbb{R}$^n \ C there is $n \in C*$ such that $n, b < 0$. Where $C*$ is the polar dual of C.

Corollary: If C is a convex cone, Rc is the closure of C.
Proof Of course $C \subset Rc$, and Rc is closed. For any b outside the closure of C Farkas' lemma gives an n such that $n, b < 0$ and $n, x \geq 0$ for all $x \in C$, so $b \notin Rc$. [2]

Proposition1: *Given a current technical optimal T, a truly above-optimal technical change should be such that $T+ > T$ so that equilibrium prices PT must not imply $PT+$; therefore, a priced $T+$ on new equilibrium prices should be Hicks neutral and thus holothetic [3]. And if $T+$ is not priced in the new equilibrium prices then it must violate the underlying homotheticity.*

Remark1: *In the production and asset pricing cases homotheticity of production functions and no-arbitrage of asset pricing serve as the equivalent bounding conditions respectively for production and asset pricing. Hicks neutrality here makes homotheticity equivalent to holotheticity.*

Via Brouwer's and No Retraction Theorems and Farkas Lemma** [4]**:

Theorem1: *Let X be a closed convex cone* $\in \mathbb{R}^n$ *while* $0 \in X$. *Let every* $x \in X$ *be an optimal trading strategy with its respective payoff while 0 is also a no-trade but optimal strategy. The No-Arbitrage condition makes X convex and closed under an equilibrium security prices' set Y as a polar projection of X separated by a separating hyperplane H with prices and strategy payoffs orthogonal to H.*

Assume H to be a closure of X as a boundary of a unit disk. There cannot be a strategy $b \in H$ *with a retraction* $h : X \rightarrow H$ *such that:*

$h(b) = b$, *for any* $b \in H$ *(No Retraction for a unit disk)*

So $b \notin X$.

And there cannot be a point $b \in H$ *such that* $(b, x) > 0$ *(Farkas Lemma)*

Proof1:

Economic Proof (Proposing a 'No-trade, Holoreturns Lemma'):

Given the No-Arbitrage condition and from [5] implying optimality of no-trade, in terms of pre-trade being equal to post-trade for equilibrium prices, implies Farkas Lemma and Brouwer's Fixed-point Theorem. The h in H, given the concave utility of consumption and risky returns, implies as per the "law of iterated expectations" that $\Delta h = 0$. *And if equilibrium consumption of the consumer i is Ci then for an above-optimal h-strategy payoff the* $C_{i,h} = (C_i + h)$ *which should be higher than Ci; but* $C_{i,h}$ *is bounded by the concave utility of risky return with the corresponding disutility of that risk such that there is no* $C_{i,h} > C_i$.

Holothetic production functions given by Sato and Economic Invariance:

Let X be a set of optimal technologies' closed convex cone with H as its boundary and with vertex zero implying "possibility of inaction, as 0 also being a maximizer" [1].

Via No Retraction and Farkas Lemma:

Theorem2: *There cannot be an above-optimal technology x+ in a unit disk X such that it is not priced (which is so by definition of being above-optimal). Therefore x+ must be lying on the boundary and therefore must not be positively priced. That an x+ must transform into scale effect by preserving holothetic isoquant maps given the equilibrium prices.*

Here pricing of technology implies it being adopted for having developed a market.

Proof2:
Here Farkas Lemma guarantees the Lie group-like invariance under transformation.
[3]: I quote,
"Lemma (Fundamental Lemma of Holotheticity). A family of production functions is holothetic under a given type of technical progress if and only if it is invariant under a group." [3] [6]

From Theorem1, there cannot be a retraction to the boundary, a condition conserving the convexity of X. (Brouwer's Fixed-point Theorem)

Proposition2: *The pricing implications for convexity and homotheticity of the technology set of production functions, given a Farkas Lemma condition of nonnegative equilibrium price set, homotheticity is a holotheticity.*

Remark2: *On Proposition2, at least in terms of an always possible "product-augmenting Hicks neutrality", the isoquant map must be holothetic.*

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