Profit Maximization Strategy in an Industry: A Sustainable Procedure

Mohajan, Devagit and Mohajan, Haradhan

Department of Civil Engineering, Chittagong University of Engineering and Technology, Chittagong, Bangladesh, Assistant Professor, Department of Mathematics, Premier University, Chittagong, Bangladesh

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Devajit Mohajan

Department of Civil Engineering, Chittagong University of Engineering and Technology, Chittagong, Bangladesh

Email: devajit1402@gmail.com

Mobile: +8801866207021

Haradhan Kumar Mohajan

Assistant Professor, Department of Mathematics, Premier University, Chittagong, Bangladesh

Email: haradhan1971@gmail.com

Mobile: +8801716397232

Abstract

This article tries to discuss profit maximization policies by using four variable inputs, such as capital, labor, principal raw materials, and other inputs in an industry, where mathematical economic models are applied by considering budget constraint. For the sustainable production, every industry should apply scientific method, such as mathematical techniques to obtain more accurate results. In the study Cobb-Douglas production function is analyzed with detail mathematical analysis. The sensitivity analysis is included in the operation to show profit maximization that is a pivotal goal of the industry. The economic predictions of future production are provided through the comparative statics to become sure of profit maximization before starting of production in the industry. In the study Lagrange multiplier technique is applied to achieve optimal result in every step of industrial operation.

Keywords: Lagrange multiplier, Cobb-Douglas production function, profit maximization

1. Introduction

Mathematical modeling in economics is considered as the application of mathematics to represent theories and analyze problems in economics, which began in the 19th century with the use of calculus to represent and explain economic behavior of optimization (Samuelson, 1947). In the 21st century, many analytic problems of economic theory are presented in terms of mathematical economic models (Carter, 2001). The firms and industries of every nation are continuously trying to maximize their profits; subject to their production functions, input costs, and market demand (Dixit, 1990).

In economics and business activities, profit is depended on easily available necessary inputs, their marginal productivities, factor shares in total output, and degree of returns to scale (Khatun & Afroze, 2016). Profit maximization is the capability of an industry to earn the maximum profit with minimum cost. It is a financial act to achieve the highest revenue by the efficient use of raw materials. It directly effects on the industry and indirectly plays a role in economy and social wellbeing (Eaton & Lipsey, 1975).

Cobb-Douglas production function is one of the most widely used production function in mathematical economics. It helps the industry to take rational decision on the quantity of each factor inputs to employ so as to minimize the production cost for its profit maximization (Cobb & Douglas, 1928; Mohajan, 2021a). The method of Lagrange multiplier is a very useful and powerful technique in multivariable calculus, which transfers a constrained problem to a higher dimensional unconstrained problem (Islam et al. 2010a, b; Mohajan, 2021a).

In this study we have tried to display mathematical calculations elaborately. We have introduced some theorems with proof where necessary to make the article interesting to the common readers. To obtain maximum profit, an industry must be very sincere in every step of its operation, such as in production, financial balance, demand and supply strategy, transportation, total management system, etc. In this study we have stressed only on production sector, so that the profit of the industry is maximized.

2. Literature Review

In economics, the literature review section is an introductory region of research, which shows the works of previous researchers in the same field within the existing knowledge (Polit & Hungler, 2013). Abhishek Tripathi has defined the economic profits as the difference between total revenue and economic costs. He observes that profit maximization is generally used in explaining managerial decision making (Tripathi, 2019). Two American scholars; mathematician Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976), have worked on production functions in a seminal paper in 1928 that delivers maximum profit (Cobb & Douglas, 1928). Later, another two American professors; mathematician John V. Baxley and economist John C. Moorhouse, have given the production functions in an outstanding and extraordinary paper with sufficient mathematical techniques (Baxley & Moorhouse, 1984). Steven E. Landsburg has tried to highlight on the profit maximization techniques (Landsg, 2002). Famous mathematician and physicist Jamal Nazrul Islam (1939-2013) and his coauthors have discussed both profit maximization and utility maximization in two different articles (Islam et al., 2010a, b). Pahlaj Moolio and his coworkers have considered the Cobb-Douglas production functions to determine maximum profit (Moolio et al., 2009).

Waheed Hussain believes that corporations and their managers should maximize profits because they will lead to an “economically efficient” or “welfare maximizing” outcome. He argues that the fundamental problem with this argument is really not, because the markets in the real world are less than the perfect (Hussain, 2006). Steven D. Levitt shows that profit maximizing behavior by firms is one of the most fundamental and widely applied policies in all of economics (Levitt, 2006). Lia Roy and her coauthors have worked on cost minimization by the use of Cobb-Douglas production function (Roy et al., 2021). On the other hand, Haradhan Kumar Mohajan has considered the Cobb-Douglas production function to predict the cost minimization policies (Mohajan, 2021a). In the earlier, he and his coworkers have considered the optimization techniques to discuss social welfare economics (Mohajan et al., 2013). Shaiara Husain and Md. Shahidul Islam have used the Cobb-Douglas production functions for impressive average annual growth of manufacturing sector of Bangladesh (Husain & Islam, 2016).

3. Research Methodology of the Study

Research is an essential and influential device to the academicians for the leading in academic kingdom (Pandey & Pandey, 2015). Methodology is a guideline for performing a good research. It helps the researchers to increase the trust of a reader in the research findings that create credibility of the research (Kothari, 2008). It tries to describe the types of research and the types of data, which are gathered to do the research efficiently and successfully. It generally encompasses the theoretical concepts that further provide information about the methods selection and application. It examines the purposes, problems, and questions of a research (Somekh & Lewin, 2005). It highlights the criteria of the experiment to measure the validity and reliability of the prevailing research. It also tries to create new knowledge with the existing knowledge (Goddard & Melville, 2001).

In this article, the profit maximization mathematical model is developed using a quantity of capital, b quantity of labor, c quantity of principal raw materials, and d quantity of other inputs. The Cobb-Douglas production function plays an important role in the optimization techniques of mathematical economics, and we have discussed it with detail mathematical analysis. In this section, we have tried to obtain optimum values of a, b, c, d, λ, and P. We have used 5×5 Hessian matrix to discuss the profit maximization policies. Then we have tried to predict on future production for the maximization of profit, where we have used the determinant of 5×5 Hessian and Jacobian matrices. We have tested three comparative statics, such as $\frac{\partial b}{\partial k}$, $\frac{\partial a}{\partial B}$, $\frac{\partial a}{\partial k}$ in three theorems to predict on the efficient production of the industry.

In this study we have depended on the optimization related mathematical secondary data sources. The data are collected from the internal and external secondary data sources, such as published research papers, books of well-known authors, handbooks of expert researchers, research reports, internet, websites, etc. (Mohajan, 2017a, 2018b). In the study we have tried to maintain the rules of reliability and validity, and have tried to cite the references properly throughout our research area (Mohajan, 2017b, 2018a, 2020).

4. Objective of the Study

The chief objective of this study is to discuss profit maximization strategy of an industry. Other subordinate objectives are as follows:

- to display the calculation more clearly and accurately,
- to highlight on Cobb-Douglas production function analysis, and
- to predict the nature of future effects in sustainable production.

5. Combination of Profit Function and Lagrange Multiplier

Let us consider the authority of an industry wants to produce and distribute its products within a year; say for 2023, using *a* quantity of capital, *b* quantity of labor, *c* quantity of principal raw materials, and *d* quantity of other inputs. Here *a*, *b*, *c*, and *d* are exogenous variables. The objective of the factory is to maximize the profit function (Islam et al., 2010a, b, Moolio et al., 2009),

$$P = f\left(a, b, c, d\right), \tag{1}$$

subject to the budget constraint,

$$B(a,b,c,d) = ka + lb + mc + nd\ , \qquad (2)$$

where k is rate of interest or services of capital per unit of capital a; l is the wage rate per unit of labor b; m is the cost per unit of principal raw material; and n is the cost per unit of other inputs d; while f is a suitable profit function. We assume that second order partial derivatives of the function f with respect to the independent variables (factors) a, b, c, and d exist.

Now we introduce a single Lagrange multiplier λ, as a device, by using equations (1) and (2); and define the Lagrangian function U, in a five-dimensional unconstrained problem as follows (Mohajan et al., 2013; Mohajan, 2015):

$$U\left(a,b,c,d,\lambda\right) = P\left(a,b,c,d\right) + \lambda\left(B - ka - lb - mx - nd\right). \qquad (3)$$

We consider that the industry maximizes its profits, the optimal quantities a^*, b^*, c^*, d^*, and λ^* of a, b, c, d, and λ that necessarily satisfy the first order conditions; which we obtain by partial differentiation of the Lagrangian function (3) with respect to five variables a, b, c, d, and λ; and setting them equal to zero by the condition of optimization, we obtain;

$$U_\lambda = B - ka - lb - mc - nd = 0\ , \qquad (4a)$$

$$U_a = P_a - \lambda k = 0\ , \qquad (4b)$$

$$U_b = P_b - \lambda l = 0\ , \qquad (4c)$$

$$U_c = P_c - \lambda m = 0\ , \qquad (4d)$$

$$U_d = P_d - \lambda n = 0\ , \qquad (4e)$$

where $U_\lambda = \dfrac{\partial U}{\partial \lambda}$, $U_a = \dfrac{\partial U}{\partial a}$, etc., specify partial derivatives. Now we have a desire to establish a relationship of Lagrange multiplier λ with the profit function P. The relation will benefit the industry to take quick decision about the various inputs and expected profit. Now we will try to provide a theorem with proof which will hunt for the effects of future production for maximum profit with the change of budget (Mohajan et al., 2012, 2013).

Theorem 1: Lagrange multiplier λ can be interpreted as the marginal profit as, $\frac{dP}{dB} = \lambda$.

Proof: Let the industry wants to make a maximum profit \$*P*. It uses *a* quantity of capital, *b* quantity of labor, *c* quantity of principal raw material, and *d* quantity of other inputs. It has total budget \$*B* that supports for sustainable production with maximum profit *P*. In this theorem we use the optimization techniques of (4) (Chiang, 1984; Mohajan et al., 2012, 2013).

From (4b) we get, $$\lambda = \frac{P_a}{k}. \tag{5a}$$

From (4c) we get, $$\lambda = \frac{P_b}{l}. \tag{5b}$$

From (4d) we get, $$\lambda = \frac{P_c}{m}. \tag{5c}$$

From (4e) we get, $$\lambda = \frac{P_d}{n}. \tag{5d}$$

Combining (5a-d) we can express the Lagrange multiplier λ as;

$$\frac{P_a}{k} = \frac{P_b}{l} = \frac{P_c}{m} = \frac{P_d}{n} = \lambda. \tag{6}$$

Now we consider that the industry makes infinitesimal changes *da*, *db, dc and, dd* in *a*, *b*, *c*, and *d* respectively. Also it makes infinitesimal changes *dP* and *dB* in *P* and *B*, respectively; we yield the relations,

$$dP = P_a da + P_b db + P_c dc + P_d dd \ , \tag{7}$$

$$dB = B_a da + B_b db + B_c dc + B_d dd$$

$$dB = kda + ldb + mdc + ndd \ . \tag{8}$$

Dividing (7) by (8) we get,

$$\frac{dP}{dB} = \frac{P_a da + P_b db + P_c dc + P_d dd}{kda + ldb + mdc + ndd} \ . \tag{9}$$

Also from (5a-d) we get,

$$\lambda = \frac{P_a da}{kda} = \frac{P_b db}{ldb} = \frac{P_c dc}{mdc} = \frac{P_d dd}{ndd} = \frac{P_a da + P_b db + P_c dc + P_d dd}{kda + ldb + mdc + ndd} \ . \tag{10}$$

Combining (9) and (10) we get,

$$\frac{dP}{dB} = \lambda \ . \qquad \text{Q. E. D.} \tag{11}$$

From (11) we have observed that the Lagrange multiplier λ can be interpreted as the marginal production. From (11) we see that profit, *P* will be increased λ units from the increase of a unit of budget, *B*. The industry will deprive from λ units of profit if it decreases one units of its budget; and will gain additional λ units of profit from more one unit increase in budget. Therefore, it will proceed to produce its products efficiently for the sustainable strategy (Mohajan, 2021b, c; Roy et al., 2021).

6. Cobb-Douglas Production Function

Let us consider the Cobb-Douglas production function *f* as (Cobb & Douglas, 1928; Mohajan, 2021a; Roy et al., 2021),

$$P = f(a,b,c,d) = Aa^x b^y c^z d^w, \quad (12)$$

where A is the efficiency parameter that reflects the level of technology, i.e., technical process, economic system, etc., which represents total factor productivity. Moreover, A also reflects the skill and efficient level of the workforce. Here x, y, z, and w are parameters; x indicates the output of elasticity of capital measures the percentage change in P for 1% change in a, while b, c, and d are held constants; y indicates the output of elasticity of labor, z indicates the output of elasticity of principal raw materials, and w indicates the output of elasticity of other inputs in the production process, are exactly parallel to x. The values of x, y, z, and w are determined by available technologies. Now these four parameters x, y, z, and w must satisfy the following four inequalities (Mohajan, 2021a; Mohajan et al., 2013):

$$0 < x < 1, 0 < y < 1, \; 0 < z < 1, \text{ and } 0 < w < 1. \quad (13)$$

A strict Cobb-Douglas production function, in which $\Omega = x + y + z + w < 1$ indicates decreasing returns to scale, $\Omega = 1$ indicates constant returns to scale, and $\Omega > 1$ indicates increasing returns to scale, can be obtained by using (4), (6), and (12) in (8) as (Moolio et al., 2009; Islam et al., 2010a,b; Roy et al., 2021);

$$U(a,b,c,d,\lambda) = Aa^x b^y c^z d^w + \lambda(B - ka - lb - mc - nd). \quad (14)$$

For maximization, first order differentiation equals to zero; then from (14) we can write,

$$U_\lambda = B - ka - lb - mc - nd = 0, \quad (15a)$$

$$U_a = xAa^{x-1} b^y c^z d^w - \lambda k = 0, \quad (15b)$$

$$U_b = yAa^x b^{y-1} c^z d^w - \lambda l = 0, \quad (15c)$$

$$U_c = zAa^x b^y c^{z-1} d^w - \lambda m = 0, \tag{15d}$$

$$U_d = wAa^x b^y c^z d^{w-1} - \lambda n = 0. \tag{15e}$$

From (15a) we get,

$$B = ka + lb + mc + nd. \tag{16}$$

In the next step we want to provide a theorem with the help of Cobb-Douglas production function, $P = f(a, b, c, d)$, to achieve maximum profit. In this study, we want that the industry will use its raw materials efficiently in all times. To establish Theorem 2, we take help from equations (13) to (16) where necessary (Humphery, 1997; Moolio et al., 2009).

Theorem 2: A factory can make profit $\$P$ in 2023 using a quantity of capital, b quantity of labor, c quantity of principal raw materials, and d quantity of other inputs. Using Cobb-Douglas production function; Lagrange multiplier can be expressed as, $\lambda = \dfrac{Ax^x y^y z^z w^w B^{\Omega-1}}{k^x l^y m^z n^w \Omega^{\Omega-1}}$, and the maximum profit function of the industry is, $P = \dfrac{Ax^x y^y z^z B^{\Omega}}{k^x l^y m^z n^w \Omega^{\Omega}}$, where $\Omega = x + y + z + w$ and B is the budget of the industry.

Proof: The industry uses a quantity of capital, b quantity of labor, c quantity of principal raw materials, and d quantity of other inputs within its budget B (Mohajan, 2021b; Moolio et al., 2009). To prove the theorem we use the mathematical tools of equations (14), (15) and (16).

From (15b) we get,

$$\lambda = \frac{xAa^x b^y c^z d^w}{ka} \tag{17a}$$

$$\Rightarrow ka = \frac{xAa^x b^y c^z d^w}{\lambda}. \tag{17b}$$

From (15c) we get,

$$\lambda = \frac{yAa^x b^y c^z d^w}{lb} \tag{18a}$$

$$\Rightarrow lb = \frac{yAa^x b^y c^z d^w}{\lambda}. \tag{18b}$$

From (15d) we get,

$$\lambda = \frac{zAa^x b^y c^z d^w}{mc} \tag{19a}$$

$$\Rightarrow mc = \frac{zAa^x b^y c^z d^w}{\lambda}. \tag{19b}$$

From (15e) we get,

$$\lambda = \frac{wAa^x b^y c^z d^w}{nd} \tag{20a}$$

$$\Rightarrow nd = \frac{wAa^x b^y c^z d^w}{\lambda}. \tag{20b}$$

Now using the necessary values from (17b), (18b), (19b), and (20b) in (16) we get,

$$B = \frac{xAa^x b^y c^z d^w}{\lambda} + \frac{yAa^x b^y c^z d^w}{\lambda} + \frac{zAa^x b^y c^z d^w}{\lambda} + \frac{wAa^x b^y c^z d^w}{\lambda}$$

$$B = \frac{Aa^x b^y c^z d^w}{\lambda}(x + y + z + w)$$

$$B = \frac{Aa^x b^y c^z d^w \Omega}{\lambda} \tag{21}$$

where $\Omega = x + y + z + w$. Now we use the value of λ from (17a) in (21) to obtain;

$$B = \frac{Aa^x b^y c^z d^w \Omega}{\frac{xAa^x b^y c^z d^w}{ka}}$$

$$B = \frac{ka\Omega}{x}. \tag{22a}$$

$$\Rightarrow a = a^* = \frac{xB}{k\Omega} \tag{22b}$$

where $\Omega = x + y + z + w$. In the next step we use the value of λ from (18a) in (21) to yield;

$$B = \frac{Aa^x b^y c^z d^w \Omega}{\frac{yAa^x b^y c^z d^w}{lb}}$$

$$B = \frac{lb\Omega}{y}. \tag{23a}$$

$$\Rightarrow b = b^* = \frac{yB}{l\Omega}. \tag{23b}$$

Now we use the value of λ from (19a) in (21) and hence we get;

$$B = \frac{Aa^x b^y c^z d^w \Omega}{\frac{zAa^x b^y c^z d^w}{mc}}$$

$$B = \frac{mc\Omega}{z}. \tag{24a}$$

$$\Rightarrow c = c^* = \frac{zB}{m\Omega}. \tag{24b}$$

Now we use the value of λ from (20a) in (21) to get;

$$B = \frac{Aa^x b^y c^z d^w \Omega}{\frac{wAa^x b^y c^z d^w}{nd}}$$

$$B = \frac{nd\Omega}{w}. \qquad \text{(25a)}$$

$$\Rightarrow d = d^* = \frac{wB}{n\Omega}. \qquad (25b)$$

The stationary point for the production can be written as;

$$\left(a^*, b^*, c^*, d^*\right) = \left(\frac{xB}{k\Omega}, \frac{yB}{l\Omega}, \frac{zB}{m\Omega}, \frac{wB}{n\Omega}\right). \qquad (26)$$

Putting the values of a, b, c, and d from (22b), (23b), and (24b) in (17a) we get,

$$\lambda = \frac{xA\left(\frac{xB}{k\Omega}\right)^x \left(\frac{yB}{l\Omega}\right)^y \left(\frac{zB}{m\Omega}\right)^z \left(\frac{wB}{n\Omega}\right)^w}{\frac{xB}{\Omega}}$$

$$= A\frac{x^x B^x}{k^x \Omega^x} \frac{y^y B^y}{l^y \Omega^y} \frac{z^z B^z}{m^z \Omega^z} \frac{w^w B^w}{n^w \Omega^w} \frac{\Omega}{B}$$

$$\lambda = \lambda^* = \frac{Ax^x y^y z^z B^{\Omega-1}}{k^x l^y m^z n^w \Omega^{\Omega-1}}. \qquad \text{Q. E. D.} \qquad (27)$$

Now substituting the values of a, b, c, and d from (22b), (23b), (24b) and (25b) into (12), we get the optimal value of the profit function,

$$P = A\left(\frac{xB}{k\Omega}\right)^x \left(\frac{yB}{l\Omega}\right)^y \left(\frac{zB}{m\Omega}\right)^z \left(\frac{wB}{n\Omega}\right)^w$$

$$\Rightarrow P = \frac{Ax^x y^y z^z w^w B^\Omega}{k^x l^y m^z n^w \Omega^\Omega}. \qquad \text{Q. E. D.} \qquad (28)$$

Equation (28) is the production function in terms of $k, l, m, n, A, B > 0$, $x, y, z, w > 0$, and $\Omega = x + y + z + w > 0$. All the parameters in right hand side of (28) are known to the industry and can easily calculate its maximum profit.

7. Verification of the Maximum Profit

In this section we will try with the second-order partial differentiation with sufficient conditions of optimization. Of course, we will test whether the maximum profit of the industry is possible or not, by using mathematical economics model (Baxley & Moorhouse, 1984). Let us consider the determinant of the 5×5 Hessian matrix,

$$|H| = \begin{vmatrix} 0 & -B_a & -B_b & -B_c & -B_d \\ -B_a & U_{aa} & U_{ab} & U_{ac} & U_{ad} \\ -B_b & U_{ba} & U_{bb} & U_{bc} & U_{bd} \\ -B_c & U_{ca} & U_{cb} & U_{cc} & U_{cd} \\ -B_d & U_{da} & U_{db} & U_{dc} & U_{dd} \end{vmatrix}. \qquad (29)$$

For maximum profit, determinant of 5×5 Hessian matrix, $|H| > 0$. Now we want to confirm that the profit function that is obtained in equation (28) is really maximum (Moolio et al., 2009, Mohajan et al., 2013). The following Theorem 3 clarifies the concept of maximum profit.

Theorem 3: The profit by the use of Cobb-Douglas production function is indeed a maximum, i.e., for maximum profit, determinant of 5×5 Hessian matrix, $|H| > 0$.

Proof: In equation (28), we have obtained the maximum profit function. Now we will try to verify that it is really a maximum. Here, we use the results obtain in equations (2) and (15). Taking first-order partial differentiations of (2) we get,

$$B_a = k\,,\ B_b = l\,,\ B_c = m\,,\ \text{and} \qquad B_d = n\,. \tag{30}$$

Taking second-order and cross-order partial derivatives of (15a-d) we get,

$$\begin{aligned}
U_{aa} &= x(x-1)Aa^{x-2}b^{y}c^{z}d^{w},\\
U_{bb} &= y(y-1)Aa^{x}b^{y-2}c^{z}d^{w},\\
U_{cc} &= z(z-1)Aa^{x}b^{y}c^{z-2}d^{w},\\
U_{dd} &= w(w-1)Aa^{x}b^{y}c^{z}d^{w-2},\\
U_{ab} &= U_{ba} = xyAa^{x-1}b^{y-1}c^{z}d^{w},\\
U_{ac} &= U_{ca} = xzAa^{x-1}b^{y}c^{z-1}d^{w},\\
U_{ad} &= U_{da} = xwAa^{x-1}b^{y}c^{z}d^{w-1},\\
U_{bc} &= U_{cb} = yzAa^{x}b^{y-1}c^{z-1}d^{w},\\
U_{bd} &= U_{db} = ywAa^{x}b^{y-1}c^{z}d^{w-1},\\
U_{cd} &= U_{dc} = zwAa^{x}b^{y}c^{z-1}d^{w-1}.
\end{aligned} \tag{31}$$

Now we expand the Hessian (29) as,

$$|H| = B_a \begin{vmatrix} -B_a & U_{ab} & U_{ac} & U_{ad} \\ -B_b & U_{bb} & U_{bc} & U_{bd} \\ -B_c & U_{cb} & U_{cc} & U_{cd} \\ -B_d & U_{db} & U_{dc} & U_{dd} \end{vmatrix} - B_b \begin{vmatrix} -B_a & U_{aa} & U_{ac} & U_{ad} \\ -B_b & U_{ba} & U_{bc} & U_{bd} \\ -B_c & U_{ca} & U_{cc} & U_{cd} \\ -B_d & U_{da} & U_{dc} & U_{dd} \end{vmatrix} + B_c \begin{vmatrix} -B_a & U_{aa} & U_{ab} & U_{ad} \\ -B_b & U_{ba} & U_{bb} & U_{bd} \\ -B_c & U_{ca} & U_{cb} & U_{cd} \\ -B_d & U_{da} & U_{db} & U_{dd} \end{vmatrix}$$

$$- B_d \begin{vmatrix} -B_a & U_{aa} & U_{ab} & U_{ac} \\ -B_b & U_{ba} & U_{bb} & U_{bc} \\ -B_c & U_{ca} & U_{cb} & U_{cc} \\ -B_d & U_{da} & U_{db} & U_{dc} \end{vmatrix}$$

$$= B_a \left\{ -B_a \begin{vmatrix} U_{bb} & U_{bc} & U_{bd} \\ U_{cb} & U_{cc} & U_{cd} \\ U_{db} & U_{dc} & U_{dd} \end{vmatrix} - U_{ab} \begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} + U_{ac} \begin{vmatrix} -B_b & U_{bb} & U_{bd} \\ -B_c & U_{cb} & U_{cd} \\ -B_d & U_{db} & U_{dd} \end{vmatrix} - U_{ad} \begin{vmatrix} -B_b & U_{bb} & U_{bc} \\ -B_c & U_{cb} & U_{cc} \\ -B_d & U_{db} & U_{dc} \end{vmatrix} \right\}$$

$$- B_b \left\{ -B_a \begin{vmatrix} U_{ba} & U_{bc} & U_{bd} \\ U_{ca} & U_{cc} & U_{cd} \\ U_{da} & U_{dc} & U_{dd} \end{vmatrix} - U_{aa} \begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} + U_{ac} \begin{vmatrix} -B_b & U_{ba} & U_{bd} \\ -B_c & U_{ca} & U_{cd} \\ -B_d & U_{da} & U_{dd} \end{vmatrix} - U_{ad} \begin{vmatrix} -B_b & U_{ba} & U_{bc} \\ -B_c & U_{ca} & U_{cc} \\ -B_d & U_{da} & U_{dc} \end{vmatrix} \right\}$$

$$+ B_c \left\{ -B_a \begin{vmatrix} U_{ba} & U_{bb} & U_{bd} \\ U_{ca} & U_{cb} & U_{cd} \\ U_{da} & U_{db} & U_{dd} \end{vmatrix} - U_{aa} \begin{vmatrix} -B_b & U_{bb} & U_{bd} \\ -B_c & U_{cb} & U_{cd} \\ -B_d & U_{db} & U_{dd} \end{vmatrix} + U_{ab} \begin{vmatrix} -B_b & U_{ba} & U_{bd} \\ -B_c & U_{ca} & U_{cd} \\ -B_d & U_{da} & U_{dd} \end{vmatrix} - U_{ad} \begin{vmatrix} -B_b & U_{ba} & U_{bb} \\ -B_c & U_{ca} & U_{cb} \\ -B_d & U_{da} & U_{db} \end{vmatrix} \right\}$$

$$- B_d \left\{ -B_a \begin{vmatrix} U_{ba} & U_{bb} & U_{bc} \\ U_{ca} & U_{cb} & U_{cc} \\ U_{da} & U_{db} & U_{dc} \end{vmatrix} - U_{aa} \begin{vmatrix} -B_b & U_{bb} & U_{bc} \\ -B_c & U_{cb} & U_{cc} \\ -B_d & U_{db} & U_{dc} \end{vmatrix} + U_{ab} \begin{vmatrix} -B_b & U_{ba} & U_{bc} \\ -B_c & U_{ca} & U_{cc} \\ -B_d & U_{da} & U_{dc} \end{vmatrix} - U_{ac} \begin{vmatrix} -B_b & U_{ba} & U_{bb} \\ -B_c & U_{ca} & U_{cb} \\ -B_d & U_{da} & U_{db} \end{vmatrix} \right\}$$

$$\begin{aligned}
&= -B_a^2 \{ U_{bb} (U_{cc} U_{dd} - U_{dc} U_{cd}) + U_{bc} (U_{db} U_{cd} - U_{cb} U_{dd}) + U_{bd} (U_{cb} U_{dc} - U_{db} U_{cc}) \} \\
&- B_a U_{ab} \{ -B_b (U_{cc} U_{dd} - U_{dc} U_{cd}) + U_{bc} (-B_d U_{cd} + B_c U_{dd}) + U_{bd} (-B_c U_{dc} + B_d U_{cc}) \} \\
&+ B_a U_{ac} \{ -B_b (U_{cb} U_{dd} - U_{db} U_{cd}) + U_{bb} (-B_d U_{cd} + B_c U_{dd}) + U_{bd} (-B_c U_{db} + B_d U_{cb}) \} \\
&- B_a U_{ad} \{ -B_b (U_{cb} U_{dc} - U_{db} U_{cc}) + U_{bb} (-B_d U_{cc} + B_c U_{dc}) + U_{bc} (-B_c U_{db} + B_d U_{cb}) \} \\
&+ B_a B_b \{ U_{ba} (U_{cc} U_{dd} - U_{dc} U_{cd}) + U_{bc} (U_{da} U_{cd} - U_{ca} U_{dd}) + U_{bd} (U_{ca} U_{dc} - U_{da} U_{cc}) \} \\
&+ B_b U_{aa} \{ -B_b (U_{cc} U_{dd} - U_{dc} U_{cd}) + U_{bc} (-B_d U_{cd} + B_c U_{dd}) + U_{bd} (-B_c U_{dc} + B_d U_{cc}) \} \\
&- B_b U_{ac} \{ -B_b (U_{ca} U_{dd} - U_{da} U_{cd}) + U_{ba} (-B_d U_{cd} + B_c U_{dd}) + U_{bd} (-B_c U_{da} + B_d U_{ca}) \} \\
&+ B_b U_{ad} \{ -B_b (U_{ca} U_{dc} - U_{da} U_{cc}) + U_{ba} (-B_d U_{cc} + B_c U_{dc}) + U_{bc} (-B_c U_{da} + B_d U_{ca}) \}
\end{aligned}$$

$$
\begin{aligned}
&- B_a B_c \{U_{ba}(U_{cb}U_{dd} - U_{db}U_{cd}) + U_{bb}(U_{da}U_{cd} - U_{ca}U_{dd}) + U_{bd}(U_{ca}U_{db} - U_{da}U_{cb})\} \\
&- B_c U_{aa}\{- B_b(U_{cb}U_{dd} - U_{db}U_{cd}) + U_{bb}(- B_d U_{cd} + B_c U_{dd}) + U_{bd}(- B_c U_{db} + B_d U_{cb})\} \\
&+ B_c U_{ab}\{- B_b(U_{ca}U_{dd} - U_{da}U_{cd}) + U_{ba}(- B_d U_{cd} + B_c U_{dd}) + U_{bd}(- B_c U_{da} + B_d U_{ca})\} \\
&- B_c U_{ad}\{- B_b(U_{ca}U_{db} - U_{da}U_{cb}) + U_{ba}(- B_d U_{cb} + B_c U_{db}) + U_{bb}(- B_c U_{da} + B_d U_{ca})\} \\
&+ B_a B_d \{U_{ba}(U_{cb}U_{dc} - U_{db}U_{cc}) + U_{bb}(U_{da}U_{cc} - U_{ca}U_{dc}) + U_{bc}(U_{ca}U_{db} - U_{da}U_{cb})\} \\
&+ B_d U_{aa}\{- B_b(U_{cb}U_{dc} - U_{db}U_{cc}) + U_{bb}(- B_d U_{cc} + B_c U_{dc}) + U_{bc}(- B_c U_{db} + B_d U_{cb})\} \\
&- B_d U_{ab}\{- B_b(U_{ca}U_{dc} - U_{da}U_{cc}) + U_{ba}(- B_d U_{cc} + B_c U_{dc}) + U_{bc}(- B_c U_{da} + B_d U_{ca})\} \\
&+ B_d U_{ac}\{- B_b(U_{ca}U_{db} - U_{da}U_{cb}) + U_{ba}(- B_d U_{cb} + B_c U_{db}) + U_{bb}(- B_c U_{da} + B_d U_{ca})\}
\end{aligned}
$$

$$
\begin{aligned}
&= - B_a^2 U_{bb}U_{cc}U_{dd} + B_a^2 U_{bb}U_{dc}U_{cd} - B_a^2 U_{bc}U_{db}U_{cd} + B_a^2 U_{bc}U_{cb}U_{dd} - B_a^2 U_{bd}U_{cb}U_{dc} \\
&+ B_a^2 U_{bd}U_{db}U_{cc} + B_a B_b U_{ab}U_{cc}U_{dd} - B_a B_b U_{ab}U_{dc}U_{cd} + B_a B_d U_{ab}U_{bc}U_{cd} - B_a B_c U_{ab}U_{bc}U_{dd} \\
&+ B_a B_c U_{ab}U_{bd}U_{dc} - B_a B_d U_{ab}U_{bd}U_{cc} - B_a B_b U_{ac}U_{cb}U_{dd} + B_a B_b U_{ac}U_{db}U_{cd} - B_a B_d U_{ac}U_{bb}U_{cd} \\
&+ B_a B_c U_{ac}U_{bb}U_{dd} - B_a B_c U_{ac}U_{bd}U_{db} + B_a B_d U_{ac}U_{bd}U_{cb} + B_a B_b U_{ad}U_{cb}U_{dc} - B_a B_b U_{ad}U_{db}U_{cc} \\
&+ B_a B_d U_{ad}U_{bb}U_{cc} - B_a B_c U_{ad}U_{bb}U_{dc} + B_a B_c U_{ad}U_{bc}U_{db} - B_a B_d U_{ad}U_{bc}U_{cb} + B_a B_b U_{ba}U_{cc}U_{dd} \\
&- B_a B_b U_{ba}U_{dc}U_{cd} + B_a B_b U_{bc}U_{da}U_{cd} - B_a B_b U_{bc}U_{ca}U_{dd} + B_a B_b U_{bd}U_{ca}U_{dc} - B_a B_b U_{bd}U_{da}U_{cc} \\
&- B_b^2 U_{aa}U_{cc}U_{dd} + B_b^2 U_{aa}U_{dc}U_{cd} - B_b B_d U_{aa}U_{bc}U_{cd} + B_b B_c U_{aa}U_{bc}U_{dd} - B_b B_c U_{aa}U_{bd}U_{dc} \\
&+ B_b B_d U_{aa}U_{bd}U_{cc} + B_b^2 U_{ac}U_{ca}U_{dd} - B_b^2 U_{ac}U_{da}U_{cd} + B_b B_d U_{ac}U_{ba}U_{cd} - B_b B_c U_{ac}U_{ba}U_{dd} \\
&+ B_b B_c U_{ac}U_{bd}U_{da} - B_b B_d U_{ac}U_{bd}U_{ca} - B_b^2 U_{ad}U_{ca}U_{dc} + B_b^2 U_{ad}U_{da}U_{cc} - B_b B_d U_{ad}U_{ba}U_{cc} \\
&+ B_b B_c U_{ad}U_{ba}U_{dc} - B_b B_c U_{ad}U_{bc}U_{da} + B_b B_d U_{ad}U_{bc}U_{ca} - B_a B_c U_{ba}U_{cb}U_{dd} + B_a B_c U_{ba}U_{db}U_{cd} \\
&- B_a B_c U_{bb}U_{da}U_{cd} + B_a B_c U_{bb}U_{ca}U_{dd} - B_a B_c U_{bd}U_{ca}U_{db} + B_a B_c U_{bd}U_{da}U_{cb} + B_b B_c U_{aa}U_{cb}U_{dd} \\
&- B_b B_c U_{aa}U_{db}U_{cd} + B_c B_d U_{aa}U_{bb}U_{cd} - B_c^2 U_{aa}U_{bb}U_{dd} + B_c^2 U_{aa}U_{bd}U_{db} - B_c B_d U_{aa}U_{bd}U_{cb} \\
&- B_b B_c U_{ab}U_{ca}U_{dd} + B_b B_c U_{ab}U_{da}U_{cd} - B_c B_d U_{ab}U_{ba}U_{cd} + B_c^2 U_{ab}U_{ba}U_{dd} - B_c^2 U_{ab}U_{bd}U_{da} \\
&+ B_c B_d U_{ab}U_{bd}U_{ca} + B_b B_c U_{ad}U_{ca}U_{db} - B_b B_c U_{ad}U_{da}U_{cb} + B_c B_d U_{ad}U_{ba}U_{cb} - B_c^2 U_{ad}U_{ba}U_{db} \\
&+ B_c^2 U_{ad}U_{bb}U_{da} - B_c B_d U_{ad}U_{bb}U_{ca} + B_a B_d U_{ba}U_{cb}U_{dc} - B_a B_d U_{ba}U_{db}U_{cc} + B_a B_d U_{bb}U_{da}U_{cc} \\
&- B_a B_d U_{bb}U_{ca}U_{dc} + B_a B_d U_{bc}U_{ca}U_{db} - B_a B_d U_{bc}U_{da}U_{cb} - B_b B_d U_{aa}U_{cb}U_{dc} + B_b B_d U_{aa}U_{db}U_{cc} \\
&- B_d^2 U_{aa}U_{bb}U_{cc} + B_c B_d U_{aa}U_{bb}U_{dc} - B_c B_d U_{aa}U_{bc}U_{db} + B_d^2 U_{aa}U_{bc}U_{cb} + B_b B_d U_{ab}U_{ca}U_{dc} \\
&- B_b B_d U_{ab}U_{da}U_{cc} + B_d^2 U_{ab}U_{ba}U_{cc} - B_c B_d U_{ab}U_{ba}U_{dc} + B_c B_d U_{ab}U_{bc}U_{da} - B_d^2 U_{ab}U_{bc}U_{ca} \\
&- B_b B_d U_{ac}U_{ca}U_{db} + B_b B_d U_{ac}U_{da}U_{cb} - B_d^2 U_{ac}U_{ba}U_{cb} + B_c B_d U_{ac}U_{ba}U_{db} - B_c B_d U_{ac}U_{bb}U_{da} \\
&+ B_d^2 U_{ac}U_{bb}U_{ca}
\end{aligned}
$$

$$
\begin{aligned}
&= - B_a^2 U_{bb}U_{cc}U_{dd} + B_a^2 U_{bb}U_{cd}^2 - B_a^2 U_{bc}U_{db}U_{cd} + B_a^2 U_{bc}^2 U_{dd} - B_a^2 U_{bd}U_{cb}U_{dc} + B_a^2 U_{bd}^2 U_{cc} \\
&+ B_a B_b U_{ab}U_{cc}U_{dd} - B_a B_b U_{ab}U_{cd}^2 + B_a B_d U_{ab}U_{bc}U_{cd} - B_a B_c U_{ab}U_{bc}U_{dd} + B_a B_c U_{ab}U_{bd}U_{dc}
\end{aligned}
$$

$$
\begin{aligned}
&- B_aB_dU_{ab}U_{bd}U_{cc} - B_aB_bU_{ac}U_{cb}U_{dd} + B_aB_bU_{ac}U_{db}U_{cd} - B_aB_dU_{ac}U_{bb}U_{cd} + B_aB_cU_{ac}U_{bb}U_{dd} \\
&- B_aB_cU_{ac}U_{bd}^2 + B_aB_dU_{ac}U_{bd}U_{cb} + B_aB_bU_{ad}U_{cb}U_{dc} - B_aB_bU_{ad}U_{db}U_{cc} + B_aB_dU_{ad}U_{bb}U_{cc} \\
&- B_aB_cU_{ad}U_{bb}U_{dc} + B_aB_cU_{ad}U_{bc}U_{dc} - B_aB_dU_{ad}U_{bc}^2 + B_aB_bU_{ba}U_{cc}U_{dd} - B_aB_bU_{ba}U_{cd}^2 \\
&+ B_aB_bU_{bc}U_{da}U_{cd} - B_aB_bU_{bc}U_{ca}U_{dd} + B_aB_bU_{bd}U_{ca}U_{dc} - B_aB_bU_{bd}U_{da}U_{cc} - B_b^2U_{aa}U_{cc}U_{dd} + B_b^2U_{aa}U_{cd}^2 \\
&- B_bB_dU_{aa}U_{bc}U_{cd} + B_bB_cU_{aa}U_{bc}U_{dd} - B_bB_cU_{aa}U_{bd}U_{dc} + B_bB_dU_{aa}U_{bd}U_{cc} + B_b^2U_{ac}^2U_{dd} - B_b^2U_{ac}U_{da}U_{cd} \\
&+ B_bB_dU_{ac}U_{ba}U_{cd} - B_bB_cU_{ac}U_{ba}U_{dd} + B_bB_cU_{ac}U_{bd}U_{da} - B_bB_dU_{ac}^2U_{bd} - B_b^2U_{ad}U_{ca}U_{dc} + B_b^2U_{ad}^2U_{cc} \\
&- B_bB_dU_{ad}U_{ba}U_{cc} + B_bB_cU_{ad}U_{ba}U_{dc} - B_bB_cU_{ad}^2U_{bc} + B_bB_dU_{ad}U_{bc}U_{ca} - B_aB_cU_{ba}U_{cb}U_{dd} \\
&+ B_aB_cU_{ba}U_{db}U_{cd} - B_aB_cU_{bb}U_{da}U_{cd} + B_aB_cU_{bb}U_{ca}U_{dd} - B_aB_cU_{bd}^2U_{ca} + B_aB_cU_{bd}U_{da}U_{cb} \\
&+ B_bB_cU_{aa}U_{cb}U_{dd} - B_bB_cU_{aa}U_{db}U_{cd} + B_cB_dU_{aa}U_{bb}U_{cd} - B_c^2U_{aa}U_{bb}U_{dd} + B_c^2U_{aa}U_{bd}^2 - B_cB_dU_{aa}U_{bd}U_{cb} \\
&- B_bB_cU_{ab}U_{ca}U_{dd} + B_bB_cU_{ab}U_{da}U_{cd} - B_cB_dU_{ab}^2U_{cd} + B_c^2U_{ab}^2U_{dd} - B_c^2U_{ab}U_{bd}U_{da} + B_cB_dU_{ab}U_{bd}U_{ca} \\
&+ B_bB_cU_{ad}U_{ca}U_{db} - B_bB_cU_{ad}^2U_{cb} + B_cB_dU_{ad}U_{ba}U_{cb} - B_c^2U_{ad}U_{ba}U_{db} + B_c^2U_{ad}^2U_{bb} - B_cB_dU_{ad}U_{bb}U_{ca} \\
&+ B_aB_dU_{ba}U_{cb}U_{dc} - B_aB_dU_{ba}U_{db}U_{cc} + B_aB_dU_{bb}U_{da}U_{cc} - B_aB_dU_{bb}U_{ca}U_{dc} + B_aB_dU_{bc}U_{ca}U_{db} \\
&- B_aB_dU_{bc}^2U_{da} - B_bB_dU_{aa}U_{cb}U_{dc} + B_bB_dU_{aa}U_{db}U_{cc} - B_d^2U_{aa}U_{bb}U_{cc} + B_cB_dU_{aa}U_{bb}U_{dc} \\
&- B_cB_dU_{aa}U_{bc}U_{db} + B_d^2U_{aa}U_{bc}^2 + B_bB_dU_{ab}U_{ca}U_{dc} - B_bB_dU_{ab}U_{da}U_{cc} + B_d^2U_{ab}^2U_{cc} - B_cB_dU_{ab}^2U_{dc} \\
&+ B_cB_dU_{ab}U_{bc}U_{da} - B_d^2U_{ab}U_{bc}U_{ca} - B_bB_dU_{ac}^2U_{db} + B_bB_dU_{ac}U_{da}U_{cb} - B_d^2U_{ac}U_{ba}U_{cb} + B_cB_dU_{ac}U_{ba}U_{db} \\
&- B_cB_dU_{ac}U_{bb}U_{da} + B_d^2U_{ac}^2U_{bb}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^3a^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\Big\{ -k^2a^2y(y-1)z(z-1)w(w-1) + k^2a^2y(y-1)z^2w^2 - k^2a^2y^2z^2w^2 \\
&+ k^2a^2y^2z^2w(w-1) - k^2a^2y^2z^2w^2 + k^2a^2y^2z(z-1)w^2 + klabxyz(z-1)w(w-1) - klabxyz^2w^2 \\
&+ knadxy^2z^2w - kmacxy^2zw(w-1) + kmacxy^2zw^2 - knadxy^2z(z-1)w - klabxyz^2w(w-1) \\
&+ klabxyz^2w^2 - knadxy(y-1)z^2w + kmacxy(y-1)zw(w-1) - kmacxy^2zw^2 + knadxy^2z^2w \\
&+ klabxyz^2w^2 - klabxyz(z-1)w^2 + knadxy(y-1)z(z-1)w - kmacxy(y-1)zw^2 + kmacxy^2zw^2 \\
&- knadxy^2z^2w + klabxyz(z-1)w(w-1) - klabxyz^2w^2 + klabxyz^2w^2 - klabxyz^2w(w-1) \\
&+ klabxyz^2w^2 - klabxyz(z-1)w^2 - l^2b^2x(x-1)z(z-1)w(w-1) + l^2b^2x(x-1)z^2w^2 \\
&- nlbdx(x-1)yz^2w + lmbcx(x-1)yzw(w-1) - lmbcx(x-1)yzw^2 + nlbdx(x-1)yz(z-1)w \\
&+ l^2b^2dx^2z^2w(w-1) - l^2b^2x^2z^2w^2 + nlbdx^2yz^2w - lmbcx^2yzw(w-1) + lmbcx^2yzw^2 - nlbdx^2yz^2w \\
&- l^2b^2x^2z^2w^2 + l^2b^2x^2z(z-1)w^2 - nlbdx^2yz(z-1)w + lmbcx^2yzw^2 - lmbcx^2yzw^2 + nlbdx^2yz^2w \\
&- kmacxy^2zw(w-1) + kmacxy^2zw^2 - kmacxy(y-1)zw^2 + kmacxy(y-1)zw(w-1) - kmacxy^2zw^2 \\
&+ kmacxy^2zw^2 + lmbcx(x-1)yzw(w-1) - lmbcx(x-1)yzw^2 + mncdx(x-1)y(y-1)zw \\
&- m^2c^2x(x-1)y(y-1)w(w-1) + m^2c^2x(x-1)y^2w^2 - mncdx(x-1)y^2zw - lmbcx^2yzw(w-1) \\
&+ lmbcx^2yzw^2 - mncdx^2y^2zw + m^2c^2x^2y^2w(w-1) - m^2c^2x^2y^2w^2 + mncdx^2y^2zw + lmbcx^2yzw^2 \\
&- lmbcx^2yzw^2 + mncdx^2y^2zw - m^2c^2x^2y^2w^2 + m^2c^2x^2y(y-1)w^2 - mncdx^2y(y-1)zw \\
&+ knadxy^2z^2w - knadxy^2z(z-1)w + knadxy(y-1)z(z-1)w - knadxy(y-1)z^2w + knadxy^2z^2w
\end{aligned}
$$

$$-knadxy^2z^2w-nlbdx(x-1)yz^2w+nlbdx(x-1)yz(z-1)w-n^2d^2x(x-1)y(y-1)z(z-1)$$
$$+mncdx(x-1)y(y-1)zw-mncdx(x-1)y^2zw+n^2d^2x(x-1)y^2z^2+nlbdx^2yz^2w-nlbdx^2yz(z-1)w$$
$$+n^2d^2x^2y^2z(z-1)-mncdx^2y^2zw+mncdx^2y^2zw-n^2d^2x^2y^2z^2-nlbdx^2yz^2w+nlbdx^2yz^2w$$
$$-n^2d^2x^2y^2z^2+mncdx^2y^2zw-mncdx^2y(y-1)zw+n^2d^2x^2y(y-1)z^2\Big\}$$

$$=\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\Big\{-\frac{1}{x}k^2a^2(y-1)(z-1)(w-1)+\frac{1}{x}k^2a^2(y-1)zw-\frac{1}{x}k^2a^2yzw$$
$$+\frac{1}{x}k^2a^2yz(w-1)-\frac{1}{x}k^2a^2yzw+\frac{1}{x}k^2a^2y(z-1)w+klab(z-1)(w-1)-klabzw+knadyz$$
$$-kmacy(w-1)+kmacyw-knady(z-1)-klabz(w-1)+klabzw-knad(y-1)z$$
$$+kmac(y-1)(w-1)-kmacyw+knadyz+klabzw-klab(z-1)w+knad(y-1)(z-1)$$
$$-kmac(y-1)w+kmacyw-knadyz+klab(z-1)(w-1)-klabzw+klabzw-klabz(w-1)$$
$$+klabzw-klab(z-1)w-\frac{1}{y}l^2b^2(x-1)(z-1)(w-1)+\frac{1}{y}l^2b^2(x-1)zw-nlbd(x-1)z$$
$$+lmbc(x-1)(w-1)-lmbc(x-1)w+nlbd(x-1)(z-1)+\frac{1}{y}l^2b^2dxz(w-1)-\frac{1}{y}l^2b^2xzw+nlbdxz$$
$$-lmbcx(w-1)+lmbcxw-nlbdxz-\frac{1}{y}l^2b^2xzw+\frac{1}{y}l^2b^2x(z-1)w-nlbdx(z-1)+lmbcxw$$
$$-lmbcxw+nlbdxz-kmacy(w-1)+kmacyw-kmac(y-1)w+kmac(y-1)(w-1)-kmacyw$$
$$+kmacyw+lmbc(x-1)(w-1)-lmbc(x-1)w+mncd(x-1)(y-1)-\frac{1}{z}m^2c^2(x-1)(y-1)(w-1)$$
$$+\frac{1}{z}m^2c^2(x-1)yw-mncd(x-1)y-lmbcx(w-1)+lmbcxw-mncdxy+\frac{1}{z}m^2c^2xy(w-1)$$
$$-\frac{1}{z}m^2c^2xyw+mncdxy+lmbcxw-lmbcxw+mncdxy-\frac{1}{z}m^2c^2xyw+\frac{1}{z}m^2c^2x(y-1)w$$
$$-mncdx(y-1)+knadyz-knady(z-1)+knad(y-1)(z-1)-knad(y-1)z+knadyz-knadyz$$
$$-nlbd(x-1)z+nlbd(x-1)(z-1)-\frac{1}{w}n^2d^2(x-1)(y-1)(z-1)+mncd(x-1)(y-1)-mncd(x-1)y$$
$$+\frac{1}{w}n^2d^2(x-1)yz+nlbdxz-nlbdx(z-1)+\frac{1}{w}n^2d^2xy(z-1)-mncdxy+mncdxy-\frac{1}{w}n^2d^2xyz$$
$$-nlbdxz+nlbdxz-\frac{1}{w}n^2d^2xyz+mncdxy-mncdx(y-1)+\frac{1}{w}n^2d^2x(y-1)z\Big\}$$

$$=\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\Big\{-x^{-1}k^2a^2(y-1)(z-1)(w-1)+x^{-1}k^2a^2(y-1)zw-x^{-1}k^2a^2yzw$$
$$+x^{-1}k^2a^2yz(w-1)-x^{-1}k^2a^2yzw+x^{-1}k^2a^2y(z-1)w-y^{-1}l^2b^2(x-1)(z-1)(w-1)+y^{-1}l^2b^2(x-1)zw$$

$$
\begin{aligned}
&+ y^{-1}l^2b^2dxz(w-1) - y^{-1}l^2b^2xzw - y^{-1}l^2b^2xzw + y^{-1}l^2b^2x(z-1)w - z^{-1}m^2c^2(x-1)(y-1)(w-1) \\
&+ z^{-1}m^2c^2(x-1)yw + z^{-1}m^2c^2xy(w-1) - z^{-1}m^2c^2xyw - z^{-1}m^2c^2xyw + z^{-1}m^2c^2x(y-1)w \\
&- w^{-1}n^2d^2(x-1)(y-1)(z-1) + w^{-1}n^2d^2(x-1)yz + w^{-1}n^2d^2xy(z-1) - w^{-1}n^2d^2xyz - w^{-1}n^2d^2xyz \\
&+ w^{-1}n^2d^2x(y-1)z + klab(z-1)(w-1) - klabzw - klabz(w-1) + klabzw + klabzw - klab(z-1)w \\
&+ klab(z-1)(w-1) - klabzw + klabzw - klabz(w-1) + klabzw - klab(z-1)w - kmacy(w-1) \\
&+ kmacyw + kmac(y-1)(w-1) - kmacyw - kmac(y-1)w + kmacyw - kmacy(w-1) + kmacyw \\
&- kmac(y-1)w + kmac(y-1)(w-1) - kmacyw + kmacyw + knadyz - knady(z-1) - knad(y-1)z \\
&+ knadyz + knad(y-1)(z-1) - knadyz + knadyz - knady(z-1) + knad(y-1)(z-1) - knad(y-1)z \\
&+ knadyz - knadyz + lmbc(x-1)(w-1) - lmbc(x-1)w - lmbcx(w-1) + lmbcxw + lmbcxw \\
&- lmbcxw + lmbc(x-1)(w-1) - lmbc(x-1)w - lmbcx(w-1) + lmbcxw + lmbcxw - lmbcxw \\
&- nlbd(x-1)z + nlbd(x-1)(z-1) + nlbdxz - nlbdxz - nlbdx(z-1) + nlbdxz - nlbd(x-1)z \\
&+ nlbd(x-1)(z-1) - nlbdxz + nlbdxz + nlbdxz - nlbdx(z-1) + mncd(x-1)(y-1) - mncd(x-1)y \\
&+ mncdxy + mncdxy - mncdx(y-1) + mncd(x-1)(y-1) - mncd(x-1)y - mncdxy + mncdxy \\
&+ mncdxy - mncdx(y-1) - mncdxy\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\{y^{-1}k^2a^2[-(y-1)(z-1)(w-1) + (y-1)zw - 2yzw + yz(w-1) + y(z-1)w] \\
&+ y^{-1}l^2b^2[-(x-1)(z-1)(w-1) + (x-1)zw - 2xzw + x(z-1)w + xz(w-1)] \\
&+ z^{-1}m^2c^2[-(x-1)(y-1)(w-1) + (x-1)yw - 2xyw + xy(w-1) + x(y-1)w] \\
&+ w^{-1}n^2d^2[-(x-1)(y-1)(z-1) + (x-1)yz + xy(z-1) - 2xyz + x(y-1)z] \\
&+ 2klab[(z-1)(w-1) - z(w-1) - (z-1)w + zw] + 2knad[yz - y(z-1) - (y-1)z + (y-1)(z-1)] \\
&+ 2kmac[-y(w-1) + yw + (y-1)(w-1) - (y-1)w] + 2lmbc[(x-1)(w-1) - (x-1)w - x(w-1) \\
&+ xw] + 2nlbd[-(x-1)z + (x-1)(z-1) - x(z-1) + xz] + 2mncd[(x-1)(y-1) - (x-1)y + xy \\
&-(x-1)y]\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\{-x^{-1}k^2a^2(y+z+w-1) - y^{-1}l^2b^2(x+z+w-1) - z^{-1}m^2c^2(x+y+w-1) \\
&- w^{-1}n^2d^2(x+y+z-1) + 2lmbc + 2nlbd + 2mncd + 2klab + 2knad + 2kmac\}
\end{aligned}
$$

$$
= \frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\left\{-\frac{2B^2}{\Omega^2}\left(xy+xz+yz+xw+yw+zw-\frac{1}{2}\Omega\right) + \frac{2B^2}{\Omega^2}(xy+xz+yz+xw+yw+zw)\right\}
$$

$$= \frac{A^3 xyzw a^{3x} b^{3y} c^{3z} d^{3w}}{a^2 b^2 c^2 d^2} \frac{B^2}{\Omega} > 0 \tag{32}$$

Since $A > 0$, x, y, z, and w are the rates of inputs of *a*, *b*, *c*, and *d*, respectively and hence all are positive; while B is budget, which will never be negative, therefore, $|H| > 0$, as required for maximum profit. Thus, the value of the profit function obtained in (28) is indeed a relative maximum value. Thus, the theorem is proved (Islam et al., 2010a,b; Moolio et al., 2009).

8. Prediction of Production by Economic Analysis

We have observed that the second order condition is satisfied, so that the determinant of (28) survive at the optimum, i.e., $|J| = |H|$; Hence, we can apply the implicit function theorem. Let *G* be the vector-valued function of ten variables $\lambda^*, a^*, b^*, c^*, d^*, k, l, m, n$, and *B*, and we define the function *G* for the point $(\lambda^*, a^*, b^*, c^*, d^*, k, l, m, n, B) \in R^{10}$, and take the values in R^5. By the Implicit Function Theorem of multivariable calculus, the equation,

$$F(\lambda^*, a^*, b^*, c^*, d^*, k, l, m, n, B) = 0, \tag{33}$$

may be solved in the form of

$$\begin{bmatrix} \lambda^* \\ a^* \\ b^* \\ c^* \\ d^* \end{bmatrix} = G(k, l, m, n, B). \tag{34}$$

Also the Jacobian matrix for *G* is given by,

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial k} & \frac{\partial \lambda^*}{\partial l} & \frac{\partial \lambda^*}{\partial m} & \frac{\partial \lambda^*}{\partial n} & \frac{\partial \lambda^*}{\partial B} \\ \frac{\partial a^*}{\partial k} & \frac{\partial a^*}{\partial l} & \frac{\partial a^*}{\partial m} & \frac{\partial a^*}{\partial n} & \frac{\partial a^*}{\partial B} \\ \frac{\partial b^*}{\partial k} & \frac{\partial b^*}{\partial l} & \frac{\partial b^*}{\partial m} & \frac{\partial b^*}{\partial n} & \frac{\partial b^*}{\partial B} \\ \frac{\partial c^*}{\partial k} & \frac{\partial c^*}{\partial l} & \frac{\partial c^*}{\partial m} & \frac{\partial c^*}{\partial n} & \frac{\partial c^*}{\partial B} \\ \frac{\partial d^*}{\partial k} & \frac{\partial d^*}{\partial l} & \frac{\partial d^*}{\partial m} & \frac{\partial d^*}{\partial n} & \frac{\partial d^*}{\partial B} \end{bmatrix} = -J^{-1} \begin{bmatrix} -a^* & -b^* & -c^* & -d^* & 1 \\ -\lambda^* & 0 & 0 & 0 & 0 \\ 0 & -\lambda^* & 0 & 0 & 0 \\ 0 & 0 & -\lambda^* & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^* \end{bmatrix}. \qquad (35)$$

In (35), $J^{-1} = \frac{1}{|J|} C^T$, where $C = \left(C_{ij}\right)$. Now (35) can be expressed as,

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial k} & \frac{\partial \lambda^*}{\partial l} & \frac{\partial \lambda^*}{\partial m} & \frac{\partial \lambda^*}{\partial n} & \frac{\partial \lambda^*}{\partial B} \\ \frac{\partial a^*}{\partial k} & \frac{\partial a^*}{\partial l} & \frac{\partial a^*}{\partial m} & \frac{\partial a^*}{\partial n} & \frac{\partial a^*}{\partial B} \\ \frac{\partial b^*}{\partial k} & \frac{\partial b^*}{\partial l} & \frac{\partial b^*}{\partial m} & \frac{\partial b^*}{\partial n} & \frac{\partial b^*}{\partial B} \\ \frac{\partial c^*}{\partial k} & \frac{\partial c^*}{\partial l} & \frac{\partial c^*}{\partial m} & \frac{\partial c^*}{\partial n} & \frac{\partial c^*}{\partial B} \\ \frac{\partial d^*}{\partial k} & \frac{\partial d^*}{\partial l} & \frac{\partial d^*}{\partial m} & \frac{\partial d^*}{\partial n} & \frac{\partial d^*}{\partial B} \end{bmatrix} = -\frac{1}{|J|} C^T \begin{bmatrix} -a^* & -b^* & -c^* & -d^* & 1 \\ -\lambda^* & 0 & 0 & 0 & 0 \\ 0 & -\lambda^* & 0 & 0 & 0 \\ 0 & 0 & -\lambda^* & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^* \end{bmatrix}$$

$$= -\frac{1}{|J|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{bmatrix} -a^* & -b^* & -c^* & -d^* & 1 \\ -\lambda^* & 0 & 0 & 0 & 0 \\ 0 & -\lambda^* & 0 & 0 & 0 \\ 0 & 0 & -\lambda^* & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^* \end{bmatrix}$$

$$= -\frac{1}{|J|} \begin{bmatrix} -a^*C_{11} - \lambda^*C_{21} & -b^*C_{11} - \lambda^*C_{31} & -c^*C_{11} - \lambda^*C_{41} & -d^*C_{11} - \lambda^*C_{51} & C_{11} \\ -a^*C_{12} - \lambda^*C_{22} & -b^*C_{12} - \lambda^*C_{32} & -c^*C_{12} - \lambda^*C_{42} & -d^*C_{12} - \lambda^*C_{52} & C_{12} \\ -a^*C_{13} - \lambda^*C_{23} & -b^*C_{13} - \lambda^*C_{33} & -c^*C_{13} - \lambda^*C_{43} & -d^*C_{13} - \lambda^*C_{53} & C_{13} \\ -a^*C_{14} - \lambda^*C_{24} & -b^*C_{14} - \lambda^*C_{34} & -c^*C_{14} - \lambda^*C_{44} & -d^*C_{14} - \lambda^*C_{54} & C_{14} \\ -a^*C_{15} - \lambda^*C_{25} & -b^*C_{15} - \lambda^*C_{35} & -c^*C_{15} - \lambda^*C_{45} & -d^*C_{15} - \lambda^*C_{55} & C_{15} \end{bmatrix}. \qquad (36)$$

In (36) total 25 comparative statics are available and we will try three of them in Theorems 4 to 6 to predict the economic analysis for the maximum profit (Baxley & Moorhouse, 1984; Islam et al., 2010a,b, Mohajan, 2020a).

Theorem 4: There is no effect on the level of labor b, if the interest rate of capital a increases, i.e., $\frac{\partial b^*}{\partial k} = 0$.

Proof: Now we examine the effects on labor b when the interest rate of capital a increases. From equation (36) we can write,

$$\frac{\partial b^*}{\partial k} = -\frac{1}{|J|}\left[-a^* C_{13} - \lambda^* C_{23}\right]$$

$$= \frac{a^*}{|J|}[C_{13}] + \frac{\lambda^*}{|J|}[C_{23}]$$

$$= \frac{a^*}{|J|}\text{Cofactor of } C_{13} + \frac{\lambda^*}{|J|}\text{Cofactor of } C_{23}$$

$$= \frac{a^*}{|J|}\begin{vmatrix} -B_a & U_{aa} & U_{ac} & U_{ad} \\ -B_b & U_{ba} & U_{bc} & U_{bd} \\ -B_c & U_{ca} & U_{cc} & U_{cd} \\ -B_d & U_{da} & U_{dc} & U_{dd} \end{vmatrix} - \frac{\lambda^*}{|J|}\begin{vmatrix} 0 & -B_a & -B_c & -B_d \\ -B_b & U_{ba} & U_{bc} & U_{bd} \\ -B_c & U_{ca} & U_{cc} & U_{cd} \\ -B_d & U_{da} & U_{dc} & U_{dd} \end{vmatrix}$$

$$= \frac{a^*}{|J|}\left\{ -B_a\begin{vmatrix} U_{ba} & U_{bc} & U_{bd} \\ U_{ca} & U_{cc} & U_{cd} \\ U_{da} & U_{dc} & U_{dd} \end{vmatrix} - U_{aa}\begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} + U_{ac}\begin{vmatrix} -B_b & U_{ba} & U_{bd} \\ -B_c & U_{ca} & U_{cd} \\ -B_d & U_{da} & U_{dd} \end{vmatrix} - U_{ad}\begin{vmatrix} -B_b & U_{ba} & U_{bc} \\ -B_c & U_{ca} & U_{cc} \\ -B_d & U_{da} & U_{dc} \end{vmatrix} \right\}$$
$$- \frac{\lambda^*}{|J|}\left\{ B_a\begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} - B_c\begin{vmatrix} -B_b & U_{ba} & U_{bd} \\ -B_c & U_{ca} & U_{cd} \\ -B_d & U_{da} & U_{dd} \end{vmatrix} + B_d\begin{vmatrix} -B_b & U_{ba} & U_{bc} \\ -B_c & U_{ca} & U_{cc} \\ -B_d & U_{da} & U_{dc} \end{vmatrix} \right\}$$

$$
\begin{aligned}
&= \frac{a^*}{|J|}\Big[-B_a\{U_{ba}(U_{cc}U_{dd}-U_{dc}U_{cd})+U_{bc}(U_{da}U_{cd}-U_{ca}U_{dd})+U_{bd}(U_{ca}U_{dc}-U_{da}U_{cc})\}\\
&-U_{aa}\{-B_b(U_{cc}U_{dd}-U_{dc}U_{cd})+U_{bc}(-B_dU_{cd}+B_cU_{dd})+U_{bd}(-B_cU_{dc}+B_dU_{cc})\}\\
&+U_{ac}\{-B_b(U_{ca}U_{dd}-U_{da}U_{cd})+U_{ba}(-B_dU_{cd}+B_cU_{dd})+U_{bd}(-B_cU_{da}+B_dU_{ca})\}\\
&-U_{ad}\{-B_b(U_{ca}U_{dc}-U_{da}U_{cc})+U_{ba}(-B_dU_{cc}+B_cU_{dc})+U_{bc}(-B_cU_{da}+B_dU_{ca})\}\Big]\\
&-\frac{\lambda^*}{|J|}\Big[B_a\{-B_b(U_{cc}U_{dd}-U_{dc}U_{cd})+U_{bc}(-B_dU_{cd}+B_cU_{dd})+U_{bd}(-B_cU_{dc}+B_dU_{cc})\}\\
&-B_c\{-B_b(U_{ca}U_{dd}-U_{da}U_{cd})+U_{ba}(-B_dU_{cd}+B_cU_{dd})+U_{bd}(-B_cU_{da}+B_dU_{ca})\}\\
&+B_d\{-B_b(U_{ca}U_{dc}-U_{da}U_{cc})+U_{ba}(-B_dU_{cc}+B_cU_{dc})+U_{bc}(-B_cU_{da}+B_dU_{ca})\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a^*}{|J|}\{B_aU_{ba}U_{cc}U_{dd} \; -B_aU_{ba}U_{dc}U_{cd}+B_aU_{bc}U_{da}U_{cd}-B_aU_{bc}U_{ca}U_{dd}+B_aU_{bd}U_{ca}U_{dc}\\
&-B_aU_{bd}U_{da}U_{cc}-B_bU_{aa}U_{cc}U_{dd}+B_bU_{aa}U_{dc}U_{cd}-B_dU_{aa}U_{bc}U_{cd}+B_cU_{aa}U_{bc}U_{dd}-B_cU_{aa}U_{bd}U_{dc}\\
&+B_dU_{aa}U_{bd}U_{cc}+B_bU_{ac}U_{ca}U_{dd}-B_bU_{ac}U_{da}U_{cd}+B_dU_{ac}U_{ba}U_{cd}-B_cU_{ac}U_{ba}U_{dd}+B_cU_{ac}U_{bd}U_{da}\\
&-B_dU_{ac}U_{bd}U_{ca}-B_bU_{ad}U_{ca}U_{dc}+B_bU_{ad}U_{da}U_{cc}-B_dU_{ad}U_{ba}U_{cc}+B_cU_{ad}U_{ba}U_{dc}-B_cU_{ad}U_{bc}U_{da}\\
&+B_dU_{ad}U_{bc}U_{ca}\}+\frac{\lambda^*}{|J|}\{B_aB_bU_{cc}U_{dd} \; -B_aB_bU_{cd}^2+B_aB_dU_{bc}U_{cd}-B_aB_cU_{bc}U_{dd}+B_aB_cU_{bd}U_{dc}\\
&-B_aB_dU_{bd}U_{cc}+B_bB_cU_{ca}U_{dd}-B_bB_cU_{da}U_{cd}+B_cB_dU_{ba}U_{cd}-B_c^2U_{ba}U_{dd}+B_c^2U_{bd}U_{da}-B_cB_dU_{bd}U_{ca}\\
&-B_bB_dU_{ca}U_{dc}+B_bB_dU_{da}U_{cc}-B_d^2U_{ba}U_{cc}+B_cB_dU_{ba}U_{dc}-B_cB_dU_{bc}U_{da}+B_d^2U_{bc}U_{ca}\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a^*}{|J|}\{B_aU_{ba}U_{cc}U_{dd} \; -B_aU_{ba}U_{cd}^2+B_aU_{bc}U_{da}U_{cd}-B_aU_{bc}U_{ca}U_{dd}+B_aU_{bd}U_{ca}U_{dc}-B_aU_{bd}U_{da}U_{cc}\\
&-B_bU_{aa}U_{cc}U_{dd}+B_bU_{aa}U_{cd}^2-B_dU_{aa}U_{bc}U_{cd}+B_cU_{aa}U_{bc}U_{dd}-B_cU_{aa}U_{bd}U_{dc}+B_dU_{aa}U_{bd}U_{cc}\\
&+B_bU_{ac}^2U_{dd}-B_bU_{ac}U_{da}U_{cd}+B_dU_{ac}U_{ba}U_{cd}-B_cU_{ac}U_{ba}U_{dd}+B_cU_{ac}U_{bd}U_{da}-B_dU_{ac}^2U_{bd}\\
&-B_bU_{ad}U_{ca}U_{dc}+B_bU_{ad}^2U_{cc}-B_dU_{ad}U_{ba}U_{cc}+B_cU_{ad}U_{ba}U_{dc}-B_cU_{ad}^2U_{bc}+B_dU_{ad}U_{bc}U_{ca}\}\\
&+\frac{\lambda^*}{|J|}\{B_aB_bU_{cc}U_{dd}-B_aB_bU_{cd}^2+B_aB_dU_{bc}U_{cd}-B_aB_cU_{bc}U_{dd}+B_aB_cU_{bd}U_{dc}-B_aB_dU_{bd}U_{cc}\\
&+B_bB_cU_{ca}U_{dd}-B_bB_cU_{da}U_{cd}+B_cB_dU_{ba}U_{cd}-B_c^2U_{ba}U_{dd}+B_c^2U_{bd}U_{da}-B_cB_dU_{bd}U_{ca}-B_bB_dU_{ca}U_{dc}\\
&+B_bB_dU_{da}U_{cc}-B_d^2U_{ba}U_{cc}+B_cB_dU_{ba}U_{dc}-B_cB_dU_{bc}U_{da}+B_d^2U_{bc}U_{ca}\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a^*}{|J|}\frac{A^3 a^{3x} b^{3y} c^{3z} d^{3w}}{a^2 b^2 c^2 d^2}\ \{kabxyz(z-1)w(w-1) - kabxyz^2 w^2 + kabxyz^2 w^2 - kabxyz^2 w(w-1) \\
&+ kabxyz^2 w^2 - kabxyz(z-1)w^2 - lb^2 x(x-1)z(z-1)w(w-1) + lb^2 x(x-1)z^2 w^2 - nbdx(x-1)yz^2 w \\
&+ mbcx(x-1)yzw(w-1) - mbcx(x-1)yzw^2 + nbdx(x-1)yz(z-1)w + lb^2 dx^2 z^2 w(w-1) - lb^2 x^2 z^2 w^2 \\
&+ nbdx^2 yz^2 w - mbcx^2 yzw(w-1) + mbcx^2 yzw^2 - nbdx^2 yz^2 w - lb^2 x^2 z^2 w^2 + lb^2 x^2 z(z-1)w^2 \\
&- nbdx^2 yz(z-1)w + mbcx^2 yzw^2 - mbcx^2 yzw^2 + nbdx^2 yz^2 w\} + \frac{\lambda^*}{|J|}\frac{A^2 a^{2x} b^{2y} c^{2z} d^{2w}}{a^2 b^2 c^2 d^2} \\
&\{kla^2 b^2 z(z-1)w(w-1) - kla^2 b^2 z^2 w^2 + kna^2 bdyz^2 w - kma^2 bcyzw(w-1) + kma^2 bcyzw^2 \\
&- kna^2 bdyz(z-1)w + lmab^2 cxzw(w-1) - lmab^2 cxzw^2 + mnabcdxyzw - m^2 abc^2 xyw(w-1) \\
&+ m^2 c^2 xyw^2 - mnabcdxyzw - nlab^2 dxz^2 w + nlab^2 dxz(z-1)w - n^2 abd^2 xyz(z-1) + mnabcdxyzw \\
&- mnabcdxyzw + n^2 abd^2 xyz^2\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a^*}{|J|}\frac{A^3 xyzwa^{3x} b^{3y} c^{3z} d^{3w}}{a^2 bc^2 d^2}\ \{-ka(z-1)w + y^{-1} lb(1-w-z) + mc + nd\} + \frac{\lambda^*}{|J|}\frac{A^2 zwa^{2x} b^{2y} c^{2z} d^{2w}}{abc^2 d^2} \\
&\{klab(1-z-w) + knady + kmacy - lmbcx + z^{-1} m^2 c^2 xy - nlbdx + w^{-1} n^2 d^2 xyz\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{|J|}\frac{A^3 xyzwa^{3x} b^{3y} c^{3z} d^{3w}}{abc^2 d^2}\left\{\frac{B}{\Omega}(1-w-z) + \frac{zB}{\Omega} + \frac{wB}{\Omega}\right\} + \frac{1}{|J|}\frac{A^3 xyzwa^{2x} b^{2y} c^{2z} d^{2w}}{abc^2 d^2}\frac{Aa^x b^y c^z d^w \Omega}{B} \\
&\qquad \times\left\{\frac{B^2}{\Omega^2}(1-z-w) + \frac{zB^2}{\Omega^2} + \frac{wB^2}{\Omega^2}\right\}
\end{aligned}
$$

$$
= -\frac{1}{|J|}\frac{A^3 xyzwa^{3x} b^{3y} c^{3z} d^{3w}}{abc^2 d^2}\left\{\frac{B}{\Omega} - \frac{B}{\Omega}\right\} = 0. \qquad (37)
$$

Equation (37) notices that there will be no effect on the level of labor *b*, if the interest rate of capital *a* increases, i.e., there is no relation between labor *b* and capital *a* in the production procedure of the industry (Islam et al., 2010a,b; Mohajan, 2017a). We think for an industry, capital and budget are essential items. Now we consider the analysis for $\frac{\partial a^*}{\partial B}$. In the following Theorem 5 we will try to find a relationship between capital and budget to obtain a maximum profit during sustainable production procedures.

Theorem 5: If the budget size of the industry increases, the level of input of capital a must be increased and the maximization of production will provide maximization of profit, i.e., $\frac{\partial a^*}{\partial B} > 0$.

Proof: If the industry makes a change in budget B, it will effect on a, b, c, and d. If the industry wants to increase its production in parallel to its maximization of profit, it must try to increase its existing budget (Islam et al., 2010a, b; Mohajan et al., 2013).

$$\frac{\partial a^*}{\partial B} = -\frac{1}{|J|}\text{Cofactor of } C_{12}$$

$$= \frac{1}{|J|}\begin{vmatrix} -B_a & U_{ab} & U_{ac} & U_{ad} \\ -B_b & U_{bb} & U_{bc} & U_{bd} \\ -B_c & U_{cb} & U_{cc} & U_{cd} \\ -B_d & U_{db} & U_{dc} & U_{dd} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{ -B_a\begin{vmatrix} U_{bb} & U_{bc} & U_{bd} \\ U_{cb} & U_{cc} & U_{cd} \\ U_{db} & U_{dc} & U_{dd} \end{vmatrix} - U_{ab}\begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} + U_{ac}\begin{vmatrix} -B_b & U_{bb} & U_{bd} \\ -B_c & U_{cb} & U_{cd} \\ -B_d & U_{db} & U_{dd} \end{vmatrix} - U_{ad}\begin{vmatrix} -B_b & U_{bb} & U_{bc} \\ -B_c & U_{cb} & U_{cc} \\ -B_d & U_{db} & U_{dc} \end{vmatrix}\right\}$$

$$\begin{aligned} &= \frac{1}{|J|}\left[-B_a\{U_{bb}(U_{cc}U_{dd} - U_{dc}U_{cd}) + U_{bc}(U_{db}U_{cd} - U_{cb}U_{dd}) + U_{bd}(U_{cb}U_{dc} - U_{db}U_{cc})\}\right. \\ &-U_{ab}\{-B_b(U_{cc}U_{dd} - U_{dc}U_{cd}) + U_{bc}(-B_dU_{cd} + B_cU_{dd}) + U_{bd}(-B_cU_{dc} + B_dU_{cc})\} \\ &+U_{ac}\{-B_b(U_{cb}U_{dd} - U_{db}U_{cd}) + U_{bb}(-B_dU_{cd} + B_cU_{dd}) + U_{bd}(-B_cU_{db} + B_dU_{cb})\} \\ &\left.-U_{ad}\{-B_b(U_{cb}U_{dc} - U_{db}U_{cc}) + U_{bb}(-B_dU_{cc} + B_cU_{dc}) + U_{bc}(-B_cU_{db} + B_dU_{cb})\}\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{|J|}\{-B_aU_{bb}U_{cc}U_{dd} + B_aU_{bb}U_{dc}U_{cd} - B_aU_{bc}U_{db}U_{cd} + B_aU_{bc}U_{cb}U_{dd} - B_aU_{bd}U_{cb}U_{dc} + B_aU_{bd}U_{db}U_{cc} \\ &+ B_bU_{ab}U_{cc}U_{dd} - B_bU_{ab}U_{dc}U_{cd} + B_dU_{ab}U_{bc}U_{cd} - B_cU_{ab}U_{bc}U_{dd} + B_cU_{ab}U_{bd}U_{dc} - B_dU_{ab}U_{bd}U_{cc} \\ &- B_bU_{ac}U_{cb}U_{dd} + B_bU_{ac}U_{db}U_{cd} - B_dU_{ac}U_{bb}U_{cd} + B_cU_{ac}U_{bb}U_{dd} - B_cU_{ac}U_{bd}U_{db} + B_dU_{ac}U_{bd}U_{cb} \\ &+ B_bU_{ad}U_{cb}U_{dc} - B_bU_{ad}U_{db}U_{cc} + B_dU_{ad}U_{bb}U_{cc} - B_cU_{ad}U_{bb}U_{dc} + B_cU_{ad}U_{bc}U_{db} - B_dU_{ad}U_{bc}U_{cb}\} \end{aligned}$$

$$= \frac{1}{|J|}\{-B_aU_{bb}U_{cc}U_{dd} + B_aU_{bb}U_{cd}^2 - B_aU_{bc}U_{db}U_{cd} + B_aU_{bc}^2U_{dd} - B_aU_{bd}U_{cb}U_{dc} + B_aU_{bd}^2U_{cc}$$
$$+ B_bU_{ab}U_{cc}U_{dd} - B_bU_{ab}U_{cd}^2 + B_dU_{ab}U_{bc}U_{cd} - B_cU_{ab}U_{bc}U_{dd} + B_cU_{ab}U_{bd}U_{dc} - B_dU_{ab}U_{bd}U_{cc}$$
$$- B_bU_{ac}U_{cb}U_{dd} + B_bU_{ac}U_{db}U_{cd} - B_dU_{ac}U_{bb}U_{cd} + B_cU_{ac}U_{bb}U_{dd} - B_cU_{ac}U_{bd}^2 + B_dU_{ac}U_{bd}U_{cb}$$
$$+ B_bU_{ad}U_{cb}U_{dc} - B_bU_{ad}U_{db}U_{cc} + B_dU_{ad}U_{bb}U_{cc} - B_cU_{ad}U_{bb}U_{dc} + B_cU_{ad}U_{bc}U_{db} - B_dU_{ad}U_{bc}^2\}$$

$$= \frac{1}{|J|}\,\frac{A^3a^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\,\{-ka^2y(y-1)z(z-1)w(w-1) + ka^2y(y-1)z^2w^2 - ka^2y^2z^2w^2$$
$$+ ka^2y^2z^2w(w-1) - ka^2y^2z^2w^2 + ka^2y^2z(z-1)w^2 + labxyz(z-1)w(w-1) - labxyz^2w^2$$
$$+ nadxy^2z^2w - macxy^2zw(w-1) + macxy^2zw^2 - nadxy^2z(z-1)w - labxyz^2w(w-1) + labxyz^2w^2$$
$$- nadxy(y-1)z^2w + macxy(y-1)zw(w-1) - macxy^2zw^2 + nadxy^2z^2w + labxyz^2w^2$$
$$- labxyz(z-1)w^2 + nadxy(y-1)z(z-1)w - macxy(y-1)zw^2 + macxy^2zw^2 - nadxy^2z^2w\}$$

$$= \frac{1}{|J|}\,\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{a^2b^2c^2d^2}\,\{-ka^2x^{-1}(y-1)(z-1)(w-1) + ka^2x^{-1}(y-1)zw - ka^2x^{-1}yzw$$
$$+ ka^2x^{-1}yz(w-1) - ka^2x^{-1}yzw + ka^2x^{-1}y(z-1)w + lab(z-1)(w-1) - labzw + nadyz - macy(w-1)$$
$$+ macyw - nady(z-1) - labz(w-1) + labzw - nad(y-1)z + mac(y-1)(w-1) - macyw + nadyz$$
$$+ labzw - lab(z-1)w + nad(y-1)(z-1) - mac(y-1)w + macyw - nadyz\}$$

$$= \frac{1}{|J|}\,\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\,\frac{B}{\Omega}(1-\Omega) + \frac{B}{\Omega}(x+y+z+w)$$

$$= \frac{1}{|J|}\,\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\,\frac{B}{\Omega} > 0\,. \tag{38}$$

The inequality (38) indicates that when the budget size of the industry increases the level of input of capital a must be increased for increasing the production to obtain maximum profit.

Theorem 6: If the interest rate of the capital a increases, the industry may decrease the level of input capital a for its production to maintain profit maximization strategy, i.e., $\frac{\partial a^*}{\partial k} < 0$.

Proof: Now we find out the effect on capital a when its interest rate, k increases. From the equation (36), we find that (Mohajan, 2021a, b; Wiese, 2021),

$$\frac{\partial a^*}{\partial k} = -\frac{1}{|J|}\left[-a^* C_{12} - \lambda^* C_{22}\right]$$

$$= +\frac{a^*}{|J|}\left[C_{12}\right] + \frac{\lambda^*}{|J|}\left[C_{22}\right]$$

$$= +\frac{a^*}{|J|}\text{Cofactor of } C_{12} + \frac{\lambda^*}{|J|}\text{Cofactor of } \mathrm{C}_{22}$$

$$= -\frac{a^*}{|J|}\begin{vmatrix} -B_a & U_{ab} & U_{ac} & U_{ad} \\ -B_b & U_{bb} & U_{bc} & U_{bd} \\ -B_c & U_{cb} & U_{cc} & U_{cd} \\ -B_d & U_{db} & U_{dc} & U_{dd} \end{vmatrix} + \frac{\lambda^*}{|J|}\begin{vmatrix} 0 & -B_b & -B_c & -B_d \\ -B_b & U_{bb} & U_{bc} & U_{bd} \\ -B_c & U_{cb} & U_{cc} & U_{cd} \\ -B_d & U_{db} & U_{dc} & U_{dd} \end{vmatrix}$$

$$= -\frac{a^*}{|J|}\left\{-B_a\begin{vmatrix} U_{bb} & U_{bc} & U_{bd} \\ U_{cb} & U_{cc} & U_{cd} \\ U_{db} & U_{dc} & U_{dd} \end{vmatrix} - U_{ab}\begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} + U_{ac}\begin{vmatrix} -B_b & U_{bb} & U_{bd} \\ -B_c & U_{cb} & U_{cd} \\ -B_d & U_{db} & U_{dd} \end{vmatrix} - U_{ad}\begin{vmatrix} -B_b & U_{bb} & U_{bc} \\ -B_c & U_{cb} & U_{cc} \\ -B_d & U_{db} & U_{dc} \end{vmatrix}\right\}$$

$$+\frac{\lambda^*}{|J|}\left\{B_b\begin{vmatrix} -B_b & U_{bc} & U_{bd} \\ -B_c & U_{cc} & U_{cd} \\ -B_d & U_{dc} & U_{dd} \end{vmatrix} - B_c\begin{vmatrix} -B_b & U_{bb} & U_{bd} \\ -B_c & U_{cb} & U_{cd} \\ -B_d & U_{db} & U_{dd} \end{vmatrix} + B_d\begin{vmatrix} -B_b & U_{bb} & U_{bc} \\ -B_c & U_{cb} & U_{cc} \\ -B_d & U_{db} & U_{dc} \end{vmatrix}\right\}$$

$$= -\frac{a^*}{|J|}\left[-B_a\left\{U_{bb}\left(U_{cc}U_{dd} - U_{dc}U_{cd}\right) + U_{bc}\left(U_{db}U_{cd} - U_{cb}U_{dd}\right) + U_{bd}\left(U_{cb}U_{dc} - U_{db}U_{cc}\right)\right\}\right.$$

$$-U_{ab}\left\{-B_b\left(U_{cc}U_{dd} - U_{dc}U_{cd}\right) + U_{bc}\left(-B_dU_{cd} + B_cU_{dd}\right) + U_{bd}\left(-B_cU_{dc} + B_dU_{cc}\right)\right\}$$

$$
\begin{aligned}
&+U_{ac}\left\{-B_b\left(U_{cb}U_{dd}-U_{db}U_{cd}\right)+U_{bb}\left(-B_dU_{cd}+B_cU_{dd}\right)+U_{bd}\left(-B_cU_{db}+B_dU_{cb}\right)\right\}\\
&-U_{ad}\left\{-B_b\left(U_{cb}U_{dc}-U_{db}U_{cc}\right)+U_{bb}\left(-B_dU_{cc}+B_cU_{dc}\right)+U_{bc}\left(-B_cU_{db}+B_dU_{cb}\right)\right\}\Big]\\
&+\frac{\lambda^*}{|J|}\Big[B_b\left\{-B_b\left(U_{cc}U_{dd}-U_{dc}U_{cd}\right)+U_{bc}\left(-B_dU_{cd}+B_cU_{dd}\right)+U_{bd}\left(-B_cU_{dc}+B_dU_{cc}\right)\right\}\\
&-B_c\left\{-B_b\left(U_{cb}U_{dd}-U_{db}U_{cd}\right)+U_{bb}\left(-B_dU_{cd}+B_cU_{dd}\right)+U_{bd}\left(-B_cU_{db}+B_dU_{cb}\right)\right\}\\
&+B_d\left\{-B_b\left(U_{cb}U_{dc}-U_{db}U_{cc}\right)+U_{bb}\left(-B_dU_{cc}+B_cU_{dc}\right)+U_{bc}\left(-B_cU_{db}+B_dU_{cb}\right)\right\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&=-\frac{a^*}{|J|}\Big\{-B_aU_{bb}U_{cc}U_{dd}+B_aU_{bb}U_{dc}U_{cd}-B_aU_{bc}U_{db}U_{cd}+B_aU_{bc}U_{cb}U_{dd}-B_aU_{bd}U_{cb}U_{dc}+B_aU_{bd}U_{db}U_{cc}\\
&+B_bU_{ab}U_{cc}U_{dd}-B_bU_{ab}U_{dc}U_{cd}+B_dU_{ab}U_{bc}U_{cd}-B_cU_{ab}U_{bc}U_{dd}+B_cU_{ab}U_{bd}U_{dc}-B_dU_{ab}U_{bd}U_{cc}\\
&-B_bU_{ac}U_{cb}U_{dd}+B_bU_{ac}U_{db}U_{cd}-B_dU_{ac}U_{bb}U_{cd}+B_cU_{ac}U_{bb}U_{dd}-B_cU_{ac}U_{bd}U_{db}+B_dU_{ac}U_{bd}U_{cb}\\
&+B_bU_{ad}U_{cb}U_{dc}-B_bU_{ad}U_{db}U_{cc}+B_dU_{ad}U_{bb}U_{cc}-B_cU_{ad}U_{bb}U_{dc}+B_cU_{ad}U_{bc}U_{db}-B_dU_{ad}U_{bc}U_{cb}\Big\}\\
&+\frac{\lambda^*}{|J|}\Big\{-B_b^2U_{cc}U_{dd}+B_b^2U_{dc}U_{cd}-B_bB_dU_{bc}U_{cd}+B_bB_cU_{bc}U_{dd}-B_bB_cU_{bd}U_{dc}+B_bB_dU_{bd}U_{cc}\\
&+B_bB_cU_{cb}U_{dd}-B_bB_cU_{db}U_{cd}+B_cB_dU_{bb}U_{cd}-B_c^2U_{bb}U_{dd}+B_c^2U_{bd}U_{db}-B_cB_dU_{bd}U_{cb}\\
&-B_bB_dU_{cb}U_{dc}+B_bB_dU_{db}U_{cc}-B_d^2U_{bb}U_{cc}+B_cB_dU_{bb}U_{dc}-B_cB_dU_{bc}U_{db}+B_d^2U_{bc}U_{cb}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&=-\frac{a^*}{|J|}\Big\{-B_aU_{bb}U_{cc}U_{dd}+B_aU_{bb}U_{cd}^2-B_aU_{bc}U_{db}U_{cd}+B_aU_{bc}^2U_{dd}-B_aU_{bd}U_{cb}U_{dc}+B_aU_{bd}^2U_{cc}\\
&+B_bU_{ab}U_{cc}U_{dd}-B_bU_{ab}U_{cd}^2+B_dU_{ab}U_{bc}U_{cd}-B_cU_{ab}U_{bc}U_{dd}+B_cU_{ab}U_{bd}U_{dc}-B_dU_{ab}U_{bd}U_{cc}\\
&-B_bU_{ac}U_{cb}U_{dd}+B_bU_{ac}U_{db}U_{cd}-B_dU_{ac}U_{bb}U_{cd}+B_cU_{ac}U_{bb}U_{dd}-B_cU_{ac}U_{bd}^2+B_dU_{ac}U_{bd}U_{cb}\\
&+B_bU_{ad}U_{cb}U_{dc}-B_bU_{ad}U_{db}U_{cc}+B_dU_{ad}U_{bb}U_{cc}-B_cU_{ad}U_{bb}U_{dc}+B_cU_{ad}U_{bc}U_{db}-B_dU_{ad}U_{bc}^2\Big\}\\
&+\frac{\lambda^*}{|J|}\Big\{-B_b^2U_{cc}U_{dd}+B_b^2U_{cd}^2-B_bB_dU_{bc}U_{cd}+B_bB_cU_{bc}U_{dd}-B_bB_cU_{bd}U_{dc}+B_bB_dU_{bd}U_{cc}\\
&+B_bB_cU_{cb}U_{dd}-B_bB_cU_{db}U_{cd}+B_cB_dU_{bb}U_{cd}-B_c^2U_{bb}U_{dd}+B_c^2U_{bd}^2-B_cB_dU_{bd}U_{cb}\\
&-B_bB_dU_{cb}U_{dc}+B_bB_dU_{db}U_{cc}-B_d^2U_{bb}U_{cc}+B_cB_dU_{bb}U_{dc}-B_cB_dU_{bc}U_{db}+B_d^2U_{bc}^2\Big\}
\end{aligned}
$$

$$
=-\frac{1}{|J|}\frac{A^3a^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\{-kay(y-1)z(z-1)w(w-1)+kay(y-1)z^2w^2-kay^2z^2w^2+kay^2z^2w(w-1)
$$
$$
-kay^2z^2w^2+kay^2z(z-1)w^2+lbxyz(z-1)w(w-1)-lbxyz^2w^2+ndxy^2z^2w-mcxy^2zw(w-1)
$$
$$
+mcxy^2zw^2-ndxy^2z(z-1)w-lbxyz^2w(w-1)+lbxyz^2w^2-ndxy(y-1)z^2w+mcxy(y-1)zw(w-1)
$$
$$
-mcxy^2zw^2+ndxy^2z^2w+lbxyz^2w^2-lbxyz(z-1)w^2+ndxy(y-1)z(z-1)w-mcxy(y-1)zw^2
$$
$$
+mcxy^2zw^2-ndxy^2z^2w\}+\frac{\lambda^*}{|J|}\frac{A^2a^{2x}b^{2y}c^{2z}d^{2w}}{b^2c^2d^2}\{-l^2b^2z(z-1)w(w-1)+l^2b^2z^2w^2-nlbdyz^2w
$$
$$
+lmbcyzw(w-1)-lmbcyzw^2+nlbdyz(z-1)w+lmbcyzw(w-1)-lmbcyzw^2+nlcdy(y-1)zw
$$
$$
-m^2c^2y(y-1)w(w-1)+m^2c^2y^2w^2-nlcdy^2zw-nlbdyz^2w+nlbdyz(z-1)w-n^2d^2y(y-1)z(z-1)
$$
$$
+mncdy(y-1)zw-mncdy^2zw+n^2d^2y^2z^2\}
$$

$$
=-\frac{1}{|J|}\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\{-kax^{-1}(y-1)(z-1)(w-1)+kax^{-1}(y-1)zw-kax^{-1}yzw+kax^{-1}yz(w-1)
$$
$$
-kax^{-1}yzw+kax^{-1}y(z-1)w+lb(z-1)(w-1)-lbzw+ndyz-mcy(w-1)+mcyw-ndy(z-1)
$$
$$
-lbz(w-1)+lbzw-nd(y-1)z+mc(y-1)(w-1)-mcyw+ndyz+lbzw-lb(z-1)w
$$
$$
+nd(y-1)(z-1)-mc(y-1)w+mcyw-ndyz\}
$$
$$
+\frac{\lambda^*}{|J|}\frac{A^2yzwa^{2x}b^{2y}c^{2z}d^{2w}}{b^2c^2d^2}\{-l^2b^2y^{-1}(z-1)(w-1)+y^{-1}l^2b^2zw-z^{-1}m^2c^2(y-1)(w-1)+z^{-1}m^2c^2yw
$$
$$
-w^{-1}n^2d^2(y-1)(z-1)+w^{-1}n^2d^2yz\}
$$

$$
=-\frac{1}{|J|}\frac{A^3xyzwa^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\frac{B}{\Omega}+\frac{1}{|J|}\frac{A^2yzwa^{2x}b^{2y}c^{2z}d^{2w}}{b^2c^2d^2}\frac{Aa^xb^yc^zd^w\Omega}{B}\frac{B^2}{\Omega^2}(y+z+w)
$$

$$
=-\frac{1}{|J|}\frac{A^3yzwa^{3x}b^{3y}c^{3z}d^{3w}}{b^2c^2d^2}\frac{B}{\Omega}(2x-\Omega)<0 \tag{39}
$$

where $(2x-\Omega)>0$. Equation (39) provides, $\frac{\partial a^*}{\partial k}<0$. Equation (39) indicates that if the interest rate of the capital a increases, the industry may decrease the level of input capital a for the sustainability of its production and also for profit maximization (Moolio et al., 2009; Mohajan et al., 2013).

9. Conclusions

In this study we have tried to discuss profit maximization policy of a running industry and ultimately we have realized that of course profit is in maximization. We have taken help of Lagrange multiplier method in the mathematical analysis procedures. We have also analyzed the Cobb-Douglas production function with the subject to constraint of budget. We have successfully verified that the profit is obviously maximum by the use of mathematical economics model. We have provided the prediction of future production for maximum profit with the use of comparative statics by applying the implicit function theorem. Throughout the paper we have tried to show mathematical calculations in some details.

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