A Theory of Convex Differentials in Economics

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A Theory of *Convex Differentials* in Economics:

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Abstract: *There are proposed convex and linear functions as convex differentials among subsets of commodity and price spaces in a convex and Euclidean space linearity. These subsets are tied together in the whole commodity and price space through fixed-points' equilibrium structure in the form of nonnegative price functions.*

Definition1: *A convex differential D ∈ ℝ^n, of the usual form dy = (y' dx) as a convex linear approximation, is proposed as a Euclidean distance, a linear function, between any two points as d(p , q) for all points p and q ∈ ℝ^n.*

Proposition1: *As in Definition1, there exists a Euclidean distance as a convex differential D = || d - d+ ||, given a d+(p+ , q+) and d(p , q) for all points p, q and p+ and q+ ∈ ℝ^n.*

Lemma1: *Take two subsets X, Y ∈ ℝ^n where X and Y are the consumption and production sets respectively then there are x ∈ X and y ∈ Y such that f(x) : X → ℝ and f(y) : Y → ℝ*

A lift h : X →Y , and
lift g : Y → X through ℝ^n
Also,
f : X → X ∈ ℝ^n and
f : Y → Y ∈ ℝ^n

Take a convex economy set E ∈ ℝ^n with subsets X and Y ⊂ E. The lifts h and g ∈ E imply a unique clearing price vector as an existence condition of equilibrium in E of the form such that there exists at least one fixed-point f(a) = a for an a ∈ E. This implies convexity of E. (Brouwer's Fixed-point Theorem)

Remark1: *There can be direct fixed-point functions between sets X and Y ∈ E ∈ ℝ^n. These maps can be conceived as convex linear differentials as a Euclidean distance D.*

Theorem1: *A subset $M \subset E$ is and the subset $T \subset E$ are proposed where M is the market development level and T as technology set; both with the lifts,*

$$h : X \to M \text{ and } g : Y \to M \in E$$

And similarly for T the lifts,

$$p : X \to T \text{ and } q : Y \to T$$

h, g, p, and q can be conceived as convex differentials D between M and $T \subset E \in \mathbb{R}^n$. (The motivation for it will become clear in the following)

Statement: *There are mutual fixed-points among X, Y, M and $T \subset E \in \mathbb{R}^n$*

Lemma2: *Let's have n-number of open exchange economies/sectors/industries trading with each other as Ei, where i = 1…n.*

Take two economies Ei and Ej with subsets Xi, Yi, Mi, and Ti and Xj, Yj, Mj and Tj respectively. Then there can be En economies with Xn, Yn, Mn, and Tn subsets $\in$ En. Similarly exports set An and Imports set Bn $\in$ En are proposed.

Then there are n-number of functions fn among all subsets of Ei and Ej and then similarly in the subsets $\subset$ En producing a unique, or a unique set of, fixed-points through the mutual convex differentials Dn among all subsets $\subset$ En.

Remark2: *The inter-economies dynamics have a real-world continuity and existence in the form of the principle of Purchasing Power Parity and the Law of One Price $\Leftrightarrow Xn \cap Yn \cap Mn \cap Tn \cap An \cap Bn$, all $\subset En \cap \mathbb{R}^n+ \neq \varnothing$.*

Corollary1: *Let there be exports and imports subsets An and Bn $\subset$ En respectively with their mutual convex differentials Dn. Given the subsets Xn, Yn, Mn, Tn, An and Bn $\subset$ En, there is an existence of a unique equilibrium in intra-economies and inter-economies space En $\in \mathbb{R}^n$.*

Corollary2: *An increase in, for instance, an $Ai \subset An \subset En \in \mathbb{R}^n$, must be implied by the lifts through Mn, Xn, Tn, Yn, Tn and Bn, all $\in \mathbb{R}^n$.*

Proof1:

(Brouwer's Fixed-point Theorem)

(via No-Retraction Theorem)

A convex set E ∈ ℝ^n and a subset T ⊂ E (or the exports subset A ⊂ E) while C is the boundary of E. (Where C can also be taken as a vector T+ of technology on the boundary of economy E and a vector A+ of exports on the boundary of E is also an equivalent of C.)

There does not exist a retraction r : E → C on the unit disk E ∈ ℝ^2 of the form r(a) = a for all a ∈ C.

Proof: *Take r : E →T+ be a retraction to boundary of unit disk or unit sphere in ℝ^n for producing a contradiction. Remove points a, b ∈ T+ where a, b can be taken as some technologies as some new machine with a related new labor skill both complementing each other in E. Now there are two disjoint open arcs T+\{a,b}. Let A= r^-1(a) and B = r^-1(b) and a ∈ A and b ∈ B, where A and B are some arbitrary sets. Now A and B intersect T+ but still given the r is a continuous function and{a, b}, and therefore A and B, are closed. Only at points a and b sets A and B can intersect T+ because only a and b, from A and B, are in T+. The closure of (T+\{a, b}) = T+. There is a subset of closure of (E\(A ∪ B) ⊃ T+ = P while the set P is open and path-connected. P is the set of price functions. A closed arc of T+ is T+a which contains a. T+a has endpoints, x_a and y_a, in the closure of P and the closure of P is path-connected but given E\(A ∪ B), the path connecting x_a and y_a cannot intersect A or B due to, for instance, an absence of the needed M+ (where M ⊂ E, is the market development set). P implies any set from within E, not from the boundary T+, therefore any path within E is connected. So the path x_a and y_a when unioned with T+\{a, b} is another connected set. Retraction image r^-1 of the union path is T+\{a, b} because the path circumvented A and B sets which is impossible. "The image of a connected set under a continuous function cannot be disconnected". [1]*

As r(x)⇔r^-1(x) so r cannot be.

Brouwer's Fixed-point Theorem:

Definition: A vector field on B^n in ℝ^n is an ordered pair (x, v(x)) where x ∈ B^n and v is a continuous map of B^n into ℝ^n. A vector field is non-vanishing if v(x) ≠ 0 for all x ∈ B^n. [2]

Theorem: Given a nonvanishing vector field on B^2 unit disk (or alternatively a ball) there exists a point of boundary S^1, the non-retracting non-deforming unit circle, where the vector field directly points inward and then a point in S^1 where the vector field directly points outward.

Remark3: *It is impossible for a continuous function to not have a fixed-point.[1]*

Remark3.1: *If Farkas' lemma holds for a price space $P \in E \in \mathbb{R}^n$, then there exists a real-valued correspondence $\varphi(P): P \to E \in \mathbb{R}^n$, where $\varphi(P)$ must be continuous; alongside the usual weaker versions of continuity.*

Remark4: *The nonvanishing $v(x) \neq 0$ condition will imply Farkas' lemma for the fixed-point existence for the economic case with nonnegative-price feasibility condition.*

Economic Proof via Brouwer's Fixed-Point Theorem[2]: If $f : B^n \to B^n$ is a continuous function then there exists a point $x \in B^n$ such that $f(x) = x$.

Proof2: *For obtaining a contradiction, let's assume $f(x) \neq x$ for every $x \in B^n$. Assume that x is a technology t+ on the frontier of the economy while the economy is taken as a unit ball $B^n \in \mathbb{R}^n$. And f(x) is the market development m+ of t+. There must be a point in B^n where $f(x) = x$. Let's define vector field v(x) as a nonvanishing vector field which is p+ price system map or a profitability map which corresponds to p+ = (m+ - t+) as $v(x) = (f(x) - x)$, while p+ or v(x) being a nonvanishing vector field p+ ≠ 0, the non-zero condition, in B^n. There must be a point that violates the non-zero condition and that is S^1 boundary at the unit circle or unit ball with an x on the boundary where the vector field v(x) must point directly outward with no corresponding f(x) or with a t+ with no corresponding m+. Let's say v(x) is p+ the nonvanishing vector field as price maps of new technology t+ with market development m+. Take $v(x) = ax$ where $a > 0$ and $v(x) = (f(x) - x) = ax$ and then $f(x) = (1 + a)x$, the additive scaling at the boundary, but it violates v(x)'s definition as (f(x) - x) or violates market development m+ always staying greater than the t+ for the non-zero condition. So $f(x) = (1 + a)x$ implies that $f(x) \notin B^n$ anymore: which is a contradiction. So $f(x) = x$ for some $x \in B^n$ and has a fixed-point.*

Brouwer fixed point theorem: Let $f : K \to K$ be a continuous function from compact convex set K to itself. Then f has a fixed point.[3]

Lemma3: *Let D be a nonempty closed technology subset of a convex metric space X as a market development level space, if it is apt to consider economy as a convex metric space. The inward set at x is defined as $I_D(x) = \{w \in X : w = x \text{ or } y = W(w, x, 1/\pi)$ for some technology $y \in D$ and $\pi \geq 1\}$. Where $w \in X$ is a pricing corresponding to y.*
An element $x \in X$ is called the fixed-point of the multivalued mapping $T : X \to CB(X)$ if $x \in T(x)$. A multivalued mapping T is said to be weakly inward on D if $Tx \subseteq$ the closure of $I_D(x)$ for $x \in D$ while T is here taken as a functional on market development level mapping it up to D. An

element $y \in D$ is called an 'element of best approximation' of $x \in X$ (by the elements of the set D) if we have $d(x,y) = d(x, D)$.[4]

Proposition2: *The $d(x, y) = d(x, D)$, above in Lemma3, can be conceived of as a convex differential as a Euclidean Distance $D = || d+ - d ||$, for some $d(p, q)$ and $d+(p+, q+)$, with possible fixed-points between subsets of an economy.*

Sperner's Lemma: *Given a homeomorphism assumption between unit disk and polytope simplex as a triangle, in a Sperner-labeled triangle further triangulated into smaller triangles there exists an 'odd number' of smaller triangles with the same vertices labeling as of the original big triangle. This is like a micro patch model inside the original model polytope of the triangle here. This implies a fixed-point such that this micro patch of the original model is the equilibrium point of the structure. [5]*

Remark5: *The triangle polytope is an interesting case for economic application. Consider from Sperner's lemma:*

"(Gale-Nikaido-Debreu GND lemma: strong version) Let Δ be the unit-simplex of $\mathbb{R}$^N . Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into R

N . Suppose ζ satisfies the following condition:

$\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0$.

Then there exists a $p \in \Delta$ such that $\zeta(p) \cap \mathbb{R}$^N- $\neq \varnothing$".

And then again as a restatement of GND: "(Gale-Nikaido-Debreu) Let S denote the unit-sphere, for the norm $||\cdot||2$

of $\mathbb{R}$^N. Let ζ be an upper semicontinuous correspondence from $S \cap \mathbb{R}$^N+ in $\mathbb{R}$^N which satisfies $\forall q \in S \cap \mathbb{R}$^N+,

$\forall z \in \zeta(q), q \cdot z \leq 0$.

Then,

$\exists\ q \in S \cap \mathbb{R}$^N+ ,

such that $\zeta(q) \cap \mathbb{R}$^N− $\neq \varnothing$". [6]

This latter "$q \cdot z \leq 0$" implies the nonvanishing vector field 'economic proof' in Proof2 above. Similarly it connects with economic proof in Proof3 below.

Proof3: *Sperner's via No-Retraction [5][7]: Given a $\Delta \in \mathbb{R}$^2 with its boundary as $d\Delta$, a continuous function as retraction $r : \Delta \rightarrow d\Delta$ is not possible.*

Proof: There is an f which maps vertices of each small triangle, in the big triangle which is labeled on vertices as {1, 2, 3}. Suppose this f is a retraction, for contradiction.

$|f(a) - f(b)| < 1\ \forall\ a, b \in \Delta$

and |a - b| < p where p is such that every small triangle, in triangulation of the big triangle, has length < p. The f maps each small triangle so that they are each within 1 of each other. (To be continued)

Remark6: *It corresponds with Remark5 and Proof2. Take the f to be a continuous price map in the commodity space a, b…, and then allow f to retract to the boundary of the convex simplical assumption of the market, the f must get negative (which here means the violation of Sperner labeling) for every a or b in Δ: f cannot be negative as price cannot because it violates the free disposal assumption in Economic theory.*

(Proof continues):
But the side length < p for each small Δ. So no small triangle can have a Sperner labeling with vertices as {1, 2, 3} which gives us the contradiction.

Nonnegative Price Theorem: *Given a free disposal assumption, a technology T+ on the boundary of an economy E and a continuous price correspondence P ∈ E,*
P(T+) = 0

Proof4:
"Farkas Lemma: If C is a closed convex cone, then for any b ∈ ℝ^n \ C there is n ∈ C such that (n, b) < 0. Where C* is the polar dual of C." [8] (to be continued)*

Discussion:
Schauder's fixed-point theorem extends the Brouwer's fixed-point theorem into Banach spaces while Zorn's lemma (also implying Well-ordering Theorem and Axiom of Choice) is traditionally used to prove Hahn-Banach Theorem in Banach spaces. Given that, the limit of a convergent subsequence, which "depends upon axiom of choice", is a fixed-point because of continuity. [8]

(Proof continues)
Proposition3: *Zorn's lemma through Well-ordering theorem, due to the 'nonnegative price' economic implication, implies Farkas' lemma at least for the case of assumption of a feasibly convex economy.*

Well-Ordering Theorem (in place of Zorn's Lemma):

Zermelo: For every set X there is a well-ordering on X.
Proof: Given a partial order on X if every subset chain x ∈ X has a least element then X is well-ordered.

Remark7: *Farkas lemma implied through a nonnegative price condition provides the 'no-negative price least-element condition' for economy (because price space is the dual of the commodity space).*

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