A new characterization of consumer heterogeneity in a growing economy

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Abstract

Caselli and Ventura (2000) introduced various sources of consumer heterogeneity, while studying changes in consumption, assets, income distributions, and preferences for public services and focusing on their average values to determine the features of the representative economic behavior. Based on this model, we propose a new approach that features the economic behavior of heterogeneous consumers. We introduce the view of direction vectors representing the principal components derived from the main sources of consumer heterogeneity rather than considering their average values. We expect that our progressive approach will provide us with further insight into the essence of what we have been hitherto considering from an average perspective.

1 Introduction

Caselli and Ventura (2000) introduced various sources of consumer heterogeneity into a representative consumer growth model and provided some tools for studying changes in consumption, assets, and income distributions. Their study focused on the average values and described the applicability of their technique to the Ramsey–Cass–Koopmans and Arrow–Romer models.

Building on their studies, we propose a new approach to consider consumer heterogeneity by borrowing the idea of principal component analysis. We introduce directional vectors that capture the main characteristics of the economy and the evaluation determined by these vectors based on preferences for the main components of the economy, namely, assets, wages, consumption, and public services. Accordingly, our approach differs from that of Caselli and Ventura (2000) and other economists who have focused on the average values to consider the representative features (behavior) of the economy.

Herein, we explain our new insights that characterize the economy by direction vectors and evaluation quantities rather than by the average values. For simplicity, let us consider the case of two heterogeneous consumers in one period. We denote assets, wages, after-tax consumption, and public service preferences by $A(a_1, a_2)$, $W(\omega_1, \omega_2)$, $C(c_1, c_2)$, and $B(b_1, b_2)$, respectively.

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When considering the characteristics of each of these quantities (e.g., attributes a_1 and a_2), one of the most natural approaches is to assess their average values (e.g., $a_1/2 + a_2/2$). Although this approach is somewhat reasonable, it is inadequate because it considers assets, wages, consumption, and preference for public services separately.

However, in practice, these four types of quantities can be related to one another. For two heterogeneous consumers, if we assume that these quantities are related, we should consider the linear combination of the amounts (e.g., $p_1 a_1 + p_2 a_2$ with $p_1 + p_2 = 1$) as a generalization of the average. In other words, it is necessary to determine the coefficients appropriately, say p_1 and p_2, from the relationship between them, thereby establishing the characteristics of the economy.

Additionally, various quantities, such as assets, are usually measured (determined) based on their origin. Measuring distances from the new origin $O' = (o'_1, o'_2)$ with $o'_j = (a_j + \omega_j + c_j + b_j)/4$ $(j = 1, 2)$ is rather tractable because point O' can be regarded as the center of four essential indicators of the economy (i.e., assets, wages, consumption, and preferences for public services). From this perspective, we consider $p_1\tilde{a_1} + p_2\tilde{a_2}, p_1\tilde{\omega_1} + p_2\tilde{\omega_2}, p_1\tilde{c_1} + p_2\tilde{c_2}, p_1\tilde{b_1} + p_2\tilde{b_2}$ instead of $p_1 a_1 + p_2 a_2, p_1\omega_1 + p_2\omega_2, p_1 c_1 + p_2 c_2, p_1 b_1 + p_2 b_2$, where $\tilde{a_j} = a_j - o'_j, \tilde{\omega_j} = \omega_j - o'_j, \tilde{c_j} = c_j - o'_j, \tilde{b_j} = b_j - o'_j$ for $j = 1, 2$.

Thus, how should the values of p_1 and p_2 be determined? The method proposed herein entails determining p_1 and p_2 by the straight line $L^{(1)}$ (or a direction vector (p_1, p_2)) presented as follows and observing these four quantities (see Figure 1).

1. Let $L^{(1)}$ be a straight line passing through O' and let points $A^{(1)}$, $W^{(1)}$, $C^{(1)}$, and $B^{(1)}$ be the feet of perpendicular from the points A, W, C, and B to the line $L^{(1)}$, respectively.

2. We find that the distances from O' to points $A^{(1)}$, $W^{(1)}$, $C^{(1)}$, and $B^{(1)}$ are $p_1\tilde{a_1} + p_2\tilde{a_2}$, $p_1\tilde{\omega_1} + p_2\tilde{\omega_2}$, $p_1\tilde{c_1} + p_2\tilde{c_2}$, and $p_1\tilde{b_1} + p_2\tilde{b_2}$, respectively (see Section 3 for details).

3. We can uniquely determine the straight line $L^{(1)}$ such that the variance of these values is maximal. Coefficients p_1 and p_2 are obtained because determining p_1 and p_2 is identical to determining $L^{(1)}$.

The reasons for considering such a straight line are as follows (more details are provided in Section 3):

1. Our approach determines p_1 and p_2 to maximize the variance of four values ($p_1\tilde{a_1} + p_2\tilde{a_2}$, $p_1\tilde{\omega_1} + p_2\tilde{\omega_2}$, $p_1\tilde{c_1} + p_2\tilde{c_2}$, and $p_1\tilde{b_1} + p_2\tilde{b_2}$). Therefore, we can see the differences between assets, wages, consumption, and preference for public services more clearly than for any other p_i's.

2. Considering the straight line $L^{(2)}$ perpendicular to $L^{(1)}$ and employing the same approach to find $L^{(1)}$, we can recover the original information fully.

3. We also find the contribution of $p_1\tilde{a_1} + p_2\tilde{a_2}$, $p_1\tilde{\omega_1} + p_2\tilde{\omega_2}$, $p_1\tilde{c_1} + p_2\tilde{c_2}$, and $p_1\tilde{b_1} + p_2\tilde{b_2}$ from the original information.

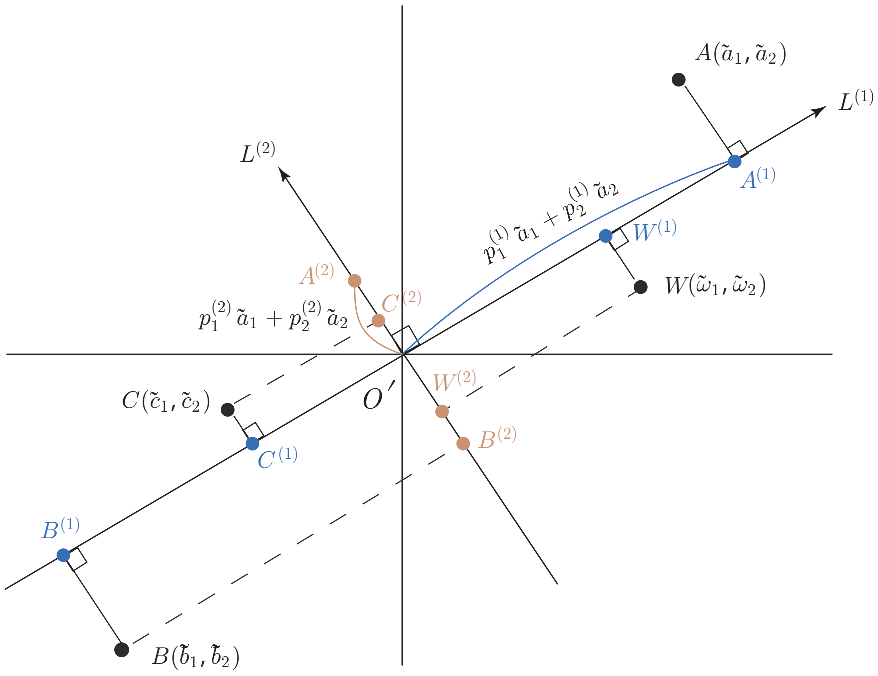

Figure 1: Illustration of our new approach

Note: In Section 1, we regard p_j as $p_j^{(1)}$ in Figure 1 for simplicity. Other terms that are not explained in Section 1 but are presented in Figure 1 are explained in Section 3.

2 A growing economy with heterogeneous households in Caselli and Ventura (2000)

This section provides an overview of Caselli and Ventura's (2000) model. We borrow the heterogeneous household model developed by Caselli and Ventura (2000) because it is introduced in a familiar textbook, Barro and Sala-i-Martin (2004), in the literature on economic growth.

Consider an economy with infinitely lived heterogeneous households, indexed by $j = 1, ..., J$. We assume that J is large and each household is so small that the choices of each household have a negligible effect on aggregate quantities and prices. Let $\boldsymbol{a}(t) = (a_1(t), \ldots, a_J(t)) \in \mathbb{R}^J$ be the vector of the stock of financial assets of consumers, $\boldsymbol{c}(t) = (c_1(t), \ldots, c_J(t)) \in \mathbb{R}^J$ the vector of after-tax consumption, $(\beta_1, \ldots, \beta_J) \in \mathbb{R}^J$ the vector of the value attached to the average public goods $g(t)$, $\boldsymbol{b}(t) = (b_1(t), \ldots, b_J(t)) = (\beta_1 g(t), \ldots, \beta_J g(t)) \in \mathbb{R}^J$ the vector of the value of the average publicly provided goods received by the household in terms of the private consumption good, $(\pi_1, \ldots, \pi_J) \in \mathbb{R}^J$ the vector of skill level of the agent, and $\boldsymbol{\omega}(t) = (\omega_1(t), \ldots, \omega_J(t)) = (\pi_1 w(t), \ldots, \pi_J w(t)) \in \mathbb{R}^J$ the vector of the average after-tax wage rate, where $w(t)$ is the wage rate. Without loss of generality, we assume that $(1/J)\sum_j \beta_j = 1$ and $(1/J)\sum_j \pi_j = 1$.

The utility of household j is given by $U_j = \int_0^\infty u_j(c_j(t), b_j(t))e^{-\rho t}dt$, where $u_j(c_j(t), b_j(t)) = \left((c_j(t) + b_j(t))^{1-\theta} - 1\right)/(1-\theta)$. Here, $\theta > 0$ and $\rho > 0$ are the inter-temporal elasticity of substitution and subjective discount rate, respectively. We assume that each household j supplies one unit of labor inelastically. Thus, household j faces the following budget constraint:

$$\dot{a}_j(t) = r(t)a_j(t) + \omega_j(t) - \frac{c_j(t)}{p}, \tag{1}$$

where $r(t)$, $\tau_c \in (0, 1)$, and $p = 1 - \tau_c$ are the interest rate, consumption tax rate, and ratio between after-tax and before-tax consumption, respectively. Let $C_j(t)$ be before-tax consumption. Then, $c_j(t) = (1 - \tau_c)C_j(t)$. We assume that both τ_c and p are constant over time as in Caselli and Ventura (2000).

Household j maximize utility U_j subject to (1), by taking $r(t)$, $\omega_j(t)$, $a_j(0)$, $b_j(t)$ and constant parameters (i.e., ρ, θ, and p) as given. The first order condition is:

$$\dot{c}_j(t) = \frac{r(t) - \rho}{\theta}(c_j(t) + b_j(t)) - \dot{b}_j(t) \tag{2}$$

and the usual transversality condition: $\lim_{s\to\infty} a_j(t) \exp\left(\int_t^s -r(v)dv\right) \geq 0$.[1]

We define the average after-tax consumption as $c(t) = (1/J)\sum_j c_j(t)$. The average public consumption $g(t)$ is financed by revenue from consumption tax $\sum_j \tau_c C_j(t)$:

$$g(t) = \frac{1}{J}\sum_j \tau_c C_j(t) = \frac{1-p}{p}\frac{1}{J}\sum_j c_j(t), \tag{3}$$

There are many identical competitive firms. Each firm produces a single final good $y(t)$ using the average stock of capital $k(t)$ and labor $l(t)$ according to a constant-returns-to-scale technology

$$y(t) = f(k(t), l(t)), \tag{4}$$

which is twice differentiable in $k(t)$ and $l(t)$ and satisfies $f_k(k,l) = \partial f(k,l)/\partial k > 0$, $f_l(k,l) = \partial f(k,l)/\partial l > 0$, $\partial^2 f(k,l)/\partial k^2 < 0$, and $\partial^2 f(k,l)/\partial l^2 < 0$. Profit maximization yields the interest and wage rates as $r(t) = f_k(k(t), l(t))$ and $w(t) = f_l(k(t), l(t))$, respectively, which together with the definition of $\omega_j(t)$ leads to $\omega_j(t) = \pi_j f_l(k(t), l(t))$.

A competitive equilibrium of this economy consists of paths of $\{c_j(t), a_j(t), b_j(t), \omega_j(t), r(t), c(t), k(t)\}_{t=0}^{\infty}$ such that each household j maximizes its utility given the initial asset holding $a_j(0)$, while taking the time path of prices $\{w(t), r(t)\}_{t=0}^{\infty}$ as given. firms maximize their profits by taking $\{w(t), r(t)\}_{t=0}^{\infty}$ as given, and $\{w(t), r(t)\}_{t=0}^{\infty}$ are such that all markets (labor, asset, and final goods markets) clear as $l = 1$, $(1/J)\sum_j a_j(t) = k(t)$, and

$$\dot{k}(t) = f(k(t), 1) - \frac{c(t)}{p}. \tag{5}$$

Here, (5) is obtained by averaging (1) and using (3), (4), $(1/J)\sum_j a_j(t) = k(t)$, $r(t) = f_k(k(t), 1)$, $\omega_j(t) = \pi_j f_l(k(t), 1)$, and $(1/J)\sum_j \pi_j = 1$. Averaging (2) and considering that $(1/J)\sum_j b_j(t) =$

[1]Integrating (1) and (2) yields $a_j(t) = \int_t^\infty \left(\frac{c_j(s)}{p} - \omega_G(s)\right) \exp\left(-\int_t^s r(v)dv\right) ds$ and

$$\begin{aligned} &c_j(t) + b_j(t) \\ &= \left[\int_t^\infty \frac{\exp(\int_t^s \frac{(1-\theta)r(v)-\rho}{\theta} dv)}{p} ds\right]^{-1} \left[a_j(t) + \int_t^\infty \left(\omega_j(s) + \frac{b_j(s)}{p}\right) \exp\left(-\int_t^s r(v)dv\right) ds\right]. \end{aligned}$$

$g(t)$, (3), and $r(t) = f_k(k(t), l(t))$ is derived as

$$\frac{\dot{c}(t)}{c(t)} = \frac{f_k(k(t), 1) - \rho}{\theta}. \tag{6}$$

The uniqueness of $\{a_j(t), c_j(t), b_j(t), \omega_j(t), r(t), k(t), c(t)\}$ for all t is shown by Caselli and Ventura (2000). Using this, we propose a new approach to characterize the economy with heterogeneous agents by the "direction vectors" and those evaluation quantities rather than by the "average values."

3 A new approach to characterize a (growing) economy with heterogeneous agents

In Section 1, we discuss the importance of adequately providing the coefficients $p_1 x_1 + p_2 x_2$ $(x_j \in \{\tilde{a}_j, \tilde{\omega}_j, \tilde{c}_j, \tilde{b}_j\}, j = 1, 2)$ instead of considering the average to understand the characteristics of assets, wages, after-tax consumption, and public service preferences. In this section, we explain the following in more generalized situations (J heterogeneous consumers at time t):

1. The coefficients $p_1^{(i)}, \ldots, p_J^{(i)}$ of $\sum_j p_j^{(i)} x_j$ $(x_j \in \{\tilde{a}_j, \tilde{\omega}_j, \tilde{c}_j, \tilde{b}_j\}, i, j = 1, \ldots, J)$ should be determined by the eigenvectors of the distribution matrix S defined later.
2. We can obtain the contribution ratios, that is, what percentages of information can be known by $\sum_j p_j^{(i)} x_j$, from the eigenvalues.

Let

$$\begin{aligned} o'_j(t) &= \frac{a_j(t) + \omega_j(t) + c_j(t) + b_j(t)}{4} \quad (j = 1, \ldots, J), \\ O'(t) &= (o'_1(t), \ldots, o'_J(t)). \end{aligned}$$

Here, consider a straight line $L^{(i)}(t)$ passing through $O'(t)$ drawn on the J-dimensional space, and the foot of the perpendicular from the four-point

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - O'(t) \quad (\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\})$$

to $L^{(i)}(t)$ defined as $A^{(i)}(t)$, $W^{(i)}(t)$, $C^{(i)}(t)$, and $B^{(i)}(t)$, respectively. Let $\boldsymbol{p}^{(i)} = \boldsymbol{p}^{(i)}(t) = (p_1^{(i)}(t), \ldots, p_J^{(i)}(t)) \in \mathbb{R}^J$ be the unit direction vector of line $L^{(i)}(t)$. Note that $\sum_j p_j^{(i)} x_j = \tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(i)}$, and that $\boldsymbol{p}^{(i)}$ depends on t. However, we often omit it for notational simplicity and the same shall apply to other variables.

First, let us consider $L^{(1)}(t)$ which maximizes the variance of the distances from the origin to points $A^{(1)}(t)$, $C^{(1)}(t)$,$B^{(1)}(t)$, and $W^{(1)}(t)$. Second, take $L^{(2)}(t)$ perpendicular to $L^{(1)}(t)$, which maximizes the variance of the distances from the origin to points $A^{(2)}(t)$, $W^{(2)}(t)$, $C^{(2)}(t)$, and $B^{(2)}(t)$. Third, take $L^{(3)}(t)$ perpendicular to $L^{(1)}(t)$ and $L^{(2)}(t)$, which maximizes the variance of the distances from the origin to points $A^{(3)}(t)$, $W^{(3)}(t)$, $C^{(3)}(t)$, and $B^{(3)}(t)$. Similarly, we consider the other lines $L^{(4)}(t), \ldots, L^{(J)}(t)$.

The position vectors of feet $A^{(1)}(t)$, $W^{(1)}(t)$, $C^{(1)}(t)$, and $B^{(1)}(t)$ become

$$\left(\tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(1)}\right) \boldsymbol{p}^{(1)} \quad (\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}).$$

Note that the positions of $A^{(1)}(t)$, $W^{(1)}(t)$, $C^{(1)}(t)$, and $B^{(1)}(t)$ depend on $L^{(1)}(t)$. Similarly, the position vectors of feet $A^{(2)}(t)$, $W^{(2)}(t)$, $C^{(2)}(t)$, and $B^{(2)}(t)$ become

$$\left[\left(\tilde{\boldsymbol{x}} - \left(\tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(1)}\right)\boldsymbol{p}^{(1)}\right) \cdot \boldsymbol{p}^{(2)}\right]\boldsymbol{p}^{(2)} = \left(\tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(2)}\right)\boldsymbol{p}^{(2)} \quad (\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}),$$

and so on. Thus, as the vectors of feet $A^{(i)}(t)$, $W^{(i)}(t)$, $C^{(i)}(t)$, and $B^{(i)}(t)$, we put

$$\tilde{\boldsymbol{x}^{(i)}} = \left(\tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(i)}\right)\boldsymbol{p}^{(i)} = \left(\boldsymbol{p}^{(i)}\tilde{\boldsymbol{x}}^T\right)\boldsymbol{p}^{(i)} \quad (\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}).$$

Therefore, variance the $V^{(i)}(t)$, more precisely the variance of "$\boldsymbol{p}^{(i)}\tilde{\boldsymbol{a}}(t)^T$, $\boldsymbol{p}^{(i)}\tilde{\boldsymbol{c}}(t)^T$, $\boldsymbol{p}^{(i)}\tilde{\boldsymbol{b}}(t)^T$, and $\boldsymbol{p}^{(i)}\tilde{\boldsymbol{\omega}}(t)^T$," is given by

$$V^{(i)}(t) = \frac{1}{4}\sum_{\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}} \left(\boldsymbol{p}^{(i)}(t)\boldsymbol{x}^T\right)^2,$$

where $\boldsymbol{v}^T$ is the transpose of vector $\boldsymbol{v}$. Next, we define the covariance functions

$$s_{j,j'}(t) = \tilde{a_j}(t)\tilde{a_{j'}}(t) + \tilde{c_j}(t)\tilde{c_{j'}}(t) + \tilde{b_j}(t)\tilde{b_{j'}}(t) + \tilde{\omega_j}(t)\tilde{\omega_{j'}}(t)$$

for $j, j' = 1, \ldots, J$, and put

$$S(t) = \begin{pmatrix} s_{1,1}(t) & s_{1,2}(t) & \cdots & \cdots & s_{1,J}(t) \\ s_{2,1}(t) & s_{2,2}(t) & & & s_{2,J}(t) \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ s_{J,1}(t) & \cdots & \cdots & \cdots & s_{J,J}(t) \end{pmatrix}.$$

Note that $S(t)$ is a symmetric matrix. Additionally, $\left(\boldsymbol{p}^{(i)}\boldsymbol{x}^T\right)^2 = \boldsymbol{p}^{(i)}\left(\boldsymbol{x}\boldsymbol{x}^T\right)\left(\boldsymbol{p}^{(i)}\right)^T$ for $\boldsymbol{x} = (x_1, \ldots, x_J)$ and $S = \sum_{\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t), \tilde{\boldsymbol{\omega}}(t)\}} \boldsymbol{x}\boldsymbol{x}^T$ as

$$\begin{aligned} \left(\boldsymbol{p}^{(i)}\boldsymbol{x}^T\right)^2 &= \left((p_1^{(i)}, \ldots, p_J^{(i)})(x_1, \ldots, x_J)^T\right)^2 \\ &= (p_1^{(i)}, \ldots, p_J^{(i)})\left((x_1, \ldots, x_J)^T(x_1, \ldots, x_J)\right)(p_1^{(i)}, \ldots, p_J^{(i)})^T \\ &= \boldsymbol{p}^{(i)}\left(\boldsymbol{x}^T\boldsymbol{x}\right)\left(\boldsymbol{p}^{(i)}\right)^T \end{aligned}$$

and

$$\begin{aligned} S(t) &= \tilde{\boldsymbol{a}}(t)^T\tilde{\boldsymbol{a}}(t) + \tilde{\boldsymbol{\omega}}(t)^T\tilde{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{c}}(t)^T\tilde{\boldsymbol{c}}(t) + \tilde{\boldsymbol{b}}(t)^T\tilde{\boldsymbol{b}}(t) \\ &= \sum_{\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}} \boldsymbol{x}^T\boldsymbol{x}. \end{aligned}$$

Then, we have

$$\begin{aligned} V^{(i)}(t) &= \frac{1}{4}\sum_{\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}} \left(\boldsymbol{p}^{(i)}x^T\right)^2 \\ &= \frac{1}{4}\sum_{\boldsymbol{x} \in \{\tilde{\boldsymbol{a}}(t), \tilde{\boldsymbol{\omega}}(t), \tilde{\boldsymbol{c}}(t), \tilde{\boldsymbol{b}}(t)\}} \boldsymbol{p}^{(i)}\left(\boldsymbol{x}^T\boldsymbol{x}\right)\left(\boldsymbol{p}^{(i)}\right)^T \\ &= \frac{1}{4}\left(\boldsymbol{p}^{(i)}(t)\right)S(t)\left(\boldsymbol{p}^{(i)}(t)\right)^T. \end{aligned}$$

To find the maximal value of $V^{(i)}(t)$ under the constraint $\left|\boldsymbol{p}^{(i)}\right| = \sum_{j=1}^{J}\left(p_j^{(i)}\right)^2 = 1$, let us consider the Lagrangian function. Here, we shall fix the parameter t and proceed with the discussion, and then vary t.

$$\mathcal{L}(\boldsymbol{p}^{(i)}; \lambda^{(i)}) = \frac{1}{4}\boldsymbol{p}^{(i)} S \left(\boldsymbol{p}^{(i)}\right)^T + \lambda^{(i)} \left(1 - \sum_{j=1}^{J}\left(p_j^{(i)}\right)^2\right).$$

From the Leibniz rule, it follows that

$$\begin{aligned}
\frac{\partial}{\partial p_j^{(i)}}\mathcal{L}(\boldsymbol{p}^{(i)}; \lambda^{(i)}) &= \frac{1}{4}\left(\frac{\partial}{\partial p_j^{(i)}}\left(\boldsymbol{p}^{(i)}S\right)\right)\left(\boldsymbol{p}^{(i)}\right)^T + \frac{1}{4}\boldsymbol{p}^{(i)}S\left(\frac{\partial}{\partial p_j^{(i)}}\left(\boldsymbol{p}^{(i)}\right)^T\right) - 2p_j^{(i)}\lambda^{(i)} \\
&= \frac{1}{4}\left((0,\dots,0,1,0,\dots,0)S\right)\left(\boldsymbol{p}^{(i)}\right)^T + \frac{1}{4}\boldsymbol{p}^{(i)}S(0,\dots,0,1,0,\dots,0)^T - 2p_j^{(i)}\lambda^{(i)} \\
&= \frac{1}{4}\left((s_{j,1},\dots,s_{j,J})\left(\boldsymbol{p}^{(i)}\right)^T + \boldsymbol{p}^{(i)}(s_{1,j},\dots,s_{J,j})^T\right) - 2p_j^{(i)}\lambda^{(i)} \\
&= \frac{1}{2}\left(\sum_{j'=1}^{J} p_{j'}^{(i)} s_{j,j'} - 4p_j^{(i)}\lambda^{(i)}\right).
\end{aligned}$$

Thus, the condition $\partial\mathcal{L}(\boldsymbol{p}^{(i)}; \lambda^{(i)})/\partial p_j^{(i)} = 0$ is equivalent to $\sum_{j'=1}^{J} p_{j'}^{(i)} s_{j,j'} = 4p_j^{(i)}\lambda^{(i)}$. Considering all the situations for $j = 1, \dots, J$, we obtain the following equation:

$$S\left(\boldsymbol{p}^{(i)}\right)^T = 4\lambda^{(i)}\left(\boldsymbol{p}^{(i)}\right)^T \qquad (i = 1, \dots, J). \tag{7}$$

From (7), the eigenvalues and corresponding eigenvectors of matrix S are $4\lambda^{(i)}$ and $\left(\boldsymbol{p}^{(i)}\right)^T$, respectively. Because matrix S is a real symmetric matrix (or a symmetric matrix with a real-valued function represented by t), eigenvalues become real (or a real-valued function of t), and eigenvectors for different eigenvalues are orthogonal to each other (see, Axler (2015) for example). By multiplying $\boldsymbol{p}^{(i)}$ from the left to (7), we also find $V^{(i)}(t) = \lambda^{(i)}(t)$.

In summary, we demonstrate the following essential points. First, because $\boldsymbol{p}^{(1)}, \dots, \boldsymbol{p}^{(J)}$ satisfy $V^{(1)} \geq \cdots \geq V^{(J)}$, the coefficient $\boldsymbol{p}^{(1)}$ of $\sum_j p_j^{(1)} x_j (= \tilde{\boldsymbol{x}} \cdot \boldsymbol{p}^{(1)})$ is the most important among $\boldsymbol{p}^{(1)}, \dots, \boldsymbol{p}^{(J)}$, and the importance decreases in order of $\boldsymbol{p}^{(2)}$, $\boldsymbol{p}^{(3)}$, and so on. Second, each $\boldsymbol{p}^{(i)}$ is orthogonal to one another, and therefore, $\sum_j p_j^{(i)} x_j$ is an economic indicator with characteristics of various variables. Furthermore, because $\sum_j p_j^{(i)} x_j$ is continuous in t, it represents the essential economic indicators in each period t. Finally, the eigenvalues $\lambda^{(i)}$'s of S equal to $V^{(i)}$'s, indicating that if the contribution ratio of each $\sum_j p_j^{(i)} x_j$ is defined as $V^{(i)}/(V^{(1)} + \cdots + V^{(J)})$, the ratio is also obtained by $\lambda^{(i)}$'s.

4 Conclusion

In this study, we introduce new J quantities $\sum_j p_j^{(i)} x_j$ $(i = 1, \dots, J)$ that capture the characteristics of the economy, which differ from the averages. These J quantities represent the properties of the economy with contributions $\lambda^{(i)}/(\lambda^{(1)} + \cdots + \lambda^{(J)})$. Our approach can be applied to a growing economy in heterogeneous countries (e.g., Ventura (1997)) and growth models with heterogeneous preferences for leisure (e.g., García-Peñalosa and Turnovsky (2008)).

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