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Abstract

The Dow Jones Industrial Average 30 (DJIA30) Index was analyzed to show that models based on the Fractal Market Hypothesis (FMH) are preferable to those based on the Efficient Market Hypothesis (EMH). In a first step, Rescaled Range Analysis was applied to search for long term dependence between index returns. The Hurst coefficient was computed as a measure of persistence in the trend of the observed time series. A Monte Carlo simulation based on both Geometric Brownian Motion (GBM) and Fractional Brownian Motion (FBM) models was used in the second step to investigate the forecasting ability of each model in a situation where information about future prices is lacking. In the third step, the volatility of the index returns obtained from the simulated GBM and FBM was considered together with that produced by a GARCH(1,1) model in order to determine the approach that minimizes the Value at Risk (VaR) and the

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Conditional Value at Risk (CVaR) of one asset portfolio where the DJIA30 index underlies an Exchange Traded Commodity (ETC). In the case observed returns could either follow a gaussian distribution or a Pareto distribution with a scale parameter equal to the inverse of the Hurst coefficient determined in the first step.

Keywords: *Fractal Analysis, Rescaled Range Analysis, Pareto distribution, Hurst coefficient, Geometric Brownian Motion, Fractional Brownian Motion, Value at Risk (VaR), Conditional Value at Risk (CVaR), Efficient Market Hypothesis, Fractal Market Hypothesis, Dow Jones Industrial Average Index.*

1 Introduction

There is a substantial amount of literature concerning the analysis of financial data based on the Efficient Market Hypothesis (EMH), which presumably traces back to the contributions of E. Fama [Fama (1970)]. In this framework, the works of Engle (1982) and Bollerslev (1986) are considered as landmarks in the analysis of time series volatility.

Since many modifications of the ARCH and GARCH models have been presented in the literature over the last 30 years, these approaches are based on EMH. The Efficient Market Hypothesis, EMH, is characterized by the fact that investors are considered rational and the market is deemed efficient. This means that price reflects true asset value. In other words, value comes from a large quantity of investor known information that ensures that it is fair. The EMH justifies the use of statistical instruments and probability distributions for its market analysis. It is usually assumed that observed returns and the error term of a specified model will follow a Gaussian distribution.

The empirical analysis of asset returns highlights issues on which simplifying assumptions waste important features of the Data Generating Process (DGP), which is formulated according to EMH. In this paper, the financial data was analyzed using a different perspective. Following Sheikh and Qiao (2009), it was assumed here that the DGP of observed returns would be based on a non-Gaussian distribution. This hypothesis required that specified models attend to extreme values observed sporadically.

Skewness, excess kurtosis, high tails, serial autocorrelation and heteroskedasticity are common problems that analysts are required to deal with these in these situations. This is the framework in which important contributions from Christoffersen et al. (1998), Nystrom and Skoglund (2006), Bacmann and Gawron (2004), Li, H et al. (2008), Jingzhi Huang, Li Xu (2014), Jingzhi Huang, Liuren Wu. (2004), Eraker, B et al. (2003) have been presented. Along the same lines, it is worth mentioning the work of Tversky (1990) on behavioral finance, where it is assumed that agents are not rational and that their behavior cannot be linked to the strong assumptions characterizing EMH. Last, this study reviews the concept of "mean-reverting," which is a stationary stochastic process typical of EMH and which focuses on non-stationary processes characterized by a long-term stochastic trend that influences the DGP in a persistent or non-persistent way.

It was precisely this setting that was used in Mandelbrot and Van Ness (1968) to introduce fractal geometry in finance. They assumed that a time series X is a fractal characterized by the important property of self-similarity, which implies that a time series X is exactly or approximately similar to a part of itself and that random variables composing a sequence of k self-similar time series $X_i (i = 1, \dots, k)$ are similar in distribution. This approach recalls the specification of a “biased random walk” process (Hurst, 1951) characterized by a Long Term Dependence (LRD) level measured by the Hurst coefficient H in the framework of the so-called “Rescaled Range Analysis” (R/S) (see also Alvarez-Ramirez et al. 2002, Cristescu et al. 2012, Morales et al. 2012, Serinaldi 2010,Turvey 2007).

Mandelbrot and Van Ness (1968) also introduced a slight but important change in the Geometric Brownian Motion (GBM) in order to obtain a biased distribution mimicking a stochastic process with jumps and a certain degree of cyclicality and long-term dependence. The resulting process has been called Fractal Brownian Motion (FBM) while a theory called “Fractal Market Hypothesis” (FMH) originated from this class of processes (see Peters, 1991). As well as being explained in Edgar Peters, 1991, FMH examines investor behavior throughout a market cycle including booms and busts. In this analysis, the most important problem is to decide the length of time to be examined: the basic element called “fractal”, which should be repeated in a market-leading projection. Investor behavior could be similar to a pattern that repeats itself on a daily, weekly, monthly, or even on a longer basis.

The aim of this paper is to analyze Dow Jones Industrial Average 30 (DJIA30) Index returns to show that the FMH based approach is preferable to the Efficient Market Hypothesis (EMH) based approach since the former leads to more accurate estimations and predictions. Since it is commonly assumed, in this framework, that the distribution of observed returns is non-gaussian we conjecture, as suggested in Mandelbrot and van Ness (1968) and in Peters (1991), that returns are distributed according to a Pareto or a fractal distribution because in this way it would be possible to pay more attention to risks related to observations located on the tails of the empirical distribution of returns.

In the first step, long term dependence among index returns was sought on the basis of the rescaled Range Statistics (R/S) in order to assess the average length of the market cycle through the V-statistic. Next, the Hurst coefficient was computed as a measure of persistence in the trend of the observed time series. Then a Monte Carlo simulation based on both GBM and FBM models was used in the second step in order to investigate which model is more efficient in gauging the future path that the observed index will cover in a situation where information about future prices is lacking. In the third step, the volatility of the index returns obtained from the simulated GBM and FBM was considered together with that produced by a GARCH(1,1) model in order to determine the approach that minimizes the Value at Risk (VaR) and the Conditional Value at Risk (CVaR) of one asset portfolio where the DJIA30 index underlies the Exchange Traded Commodity (ETC). In the case observed, alternatively, returns could either follow a gaussian distribution or a Pareto distribution with a scale parameter equal to the inverse of the Hurst coefficient determined in the first step.

The remainder of the paper is developed as follows. Section 2 deals with rescaled range analysis (RRA) and presents the results from the application of RRS to the Dow Jones index returns, wherein the Hurst coefficient H was obtained as a measure of long-term linear dependence in the data and as a stability test that was performed on it. Dow Jones index sample forecasts were computed in Section 3 starting from classical GBM and FBM, based on the H coefficient estimated in Section 2. Next, forecasts deriving from both models were compared with the original data and the most accurate model was determined based on some best-of-fit measures. Section 4 compares VaR and CVaR measures when either a gaussian distribution or a Pareto distribution was applied. Then three alternative risk factors are considered: the volatility of returns simulated in Section 3 through GBM and FBM, and those deriving from a simulated GARCH(1,1) model. The best approach was found using backtesting. Section 5 ends the paper with some concluding remarks.

2 Rescaled Range Analysis (RRA)

2.1 *Method*

Rescaled Range Analysis (RRA) is a statistical technique designed to assess the nature and magnitude of variability in data over time. In finance, RRA has been used to detect and evaluate the amount of persistence, randomness, or mean reversion in financial markets time series data. Insights into this kind of financial data naturally suggest investment strategies. The main steps of RRA, as presented in Peters (1994), can be summarized as follows:

1. A time series composed of M data points is converted into a time series of length $N = M - 1$ logarithmic rates:
$$N_i = \log\left(\frac{M_{i+1}}{M_i}\right) \qquad \text{with } i = 1, \dots, M-1$$
2. The N data points are divided into A consecutive sub-sequences, each composed of n observations ($n \cdot A = N$). Let $I_a = (r_{1a}, \dots, r_{na})$ that are a generic sub-sequence ($a = 1, \dots, A$) on which the average is computed using
$$e_a = \frac{1}{n}\sum_{k=1}^{n} r_{ka}$$
3. For each I_a, X_{ka} is computed as the cumulative sum of deviations from e_a:
$$X_{ka} = \sum_{i=0}^{k}(r_{ia} - e_a), \qquad (k = 0,1, \dots, n; a = 1,2, \dots, A)$$
together with the range R_{I_a} and the standard deviation S_{I_a}:

$$R_{I_a} = \max(X_{ka}) - \min(X_{ka}) \qquad S_{I_a} = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(r_{ia} - e_a)^2}$$

with $1 \le k \le n$ and $a = 1,2, \dots, A$.

4. The rescaled interval for a sub-period I_a is set to the normalized range R_{I_a}/S_{I_a} . Since in A consecutive sub-periods of length n are observed, the average value $(R/S)_n$ is defined as:
$$(R/S)_n = \frac{1}{A}\sum_{a=1}^{A}\left(\frac{R_{I_a}}{S_{I_a}}\right)$$
5. The length of the series n is incremented to include its first and last values and steps 1 to 6 are repeated until $n = N/2$. Once all the possible values of $(R/S)_n$ have been found, the length of the market cycle of the process must be determined[2]. Hurst (1951) introduced the V-statistic in order to compute the length of a cycle:
$$\mathrm{V}_n = \frac{(R/S)_n}{\sqrt{n}} \qquad \text{with } n = N/A$$
The sequential investigation of the relationship between the V-statistic and $\log(n)$ helps to understand the basic features of the observed process. A graphical inspection of this relation in 2 dimensions could reveal, for example, a straight line with no slope connecting the single points. This corresponds to the case with R/S statistics shifted to $\sqrt{n}$ as a common random walk observed according to EMH. Alternatively, a straight line with a positive (negative) slope is evidence of a persistent (anti-persistent) process in which R/S shifts to $\sqrt{n}$ at a higher (lower) rate corresponding to a value of the Hurst coefficient H, which is higher (lower) than 0.5. Thus, the sequential investigation allows us to understand the periods characterized by a break in the process. These were observed when a change in slope of the V-statistic occurred: if a positive (negative) slope corresponding to a persistent (anti-persistent) process turns suddenly to zero (i.e., a non-slope or a flat line) this would be evidence that a long memory component in the data disappeared and a cycle component in the process is over.
6. The investigation of the length of the market cycle is the core of the analysis since only the observations included in the market cycle were used to estimate the Hurst coefficient H, which is obtained from the following OLS regression model:
$$\log(R/S)_n = \log c + H \log n \qquad (1)$$

[2] This cycle should correspond to the economic cycle and thus it should present a growth period followed by a period of recession. It should also be characterized by an initial state wherein the cycle turns back after a new state has been observed; these mutations can occur at regular (periodic cycle) or irregular (non-periodic cycle) time periods.

7. Once the H coefficient has been estimated, it is possible to investigate it if its value changes over time in order to understand if the length of the market cycle is constant or not. This stability check is done starting from the original series of observed returns and by dividing it into disjointed sub-periods to which the R/S analysis as described in steps 1 to 6 above will be applied individually. For each estimated regression coefficient $\widehat{H}$, the null hypothesis $H = E(H)$ is tested against a two-tails alternative. $E(H)$ is obtained from a regression model equal to that specified in Eq.1 but replacing the observed values of $(R/S)_n$ with the theoretical ones, defined by Peters (1994) as:

$$E[(R/S)_n] = \frac{n - 0.5}{n} \sqrt{\frac{2}{n\pi} \sum_{i=1}^{n-1} \sqrt{\frac{n-1}{i}}}$$

It is assumed that the test statistic H has a Gaussian distribution. Hence, its standardized counterpart $\frac{H-E(H)}{\sqrt{Var(H)}}$ is a standardized normal distribution and the sample estimator of $Var(H)$ is $\frac{1}{An}$.

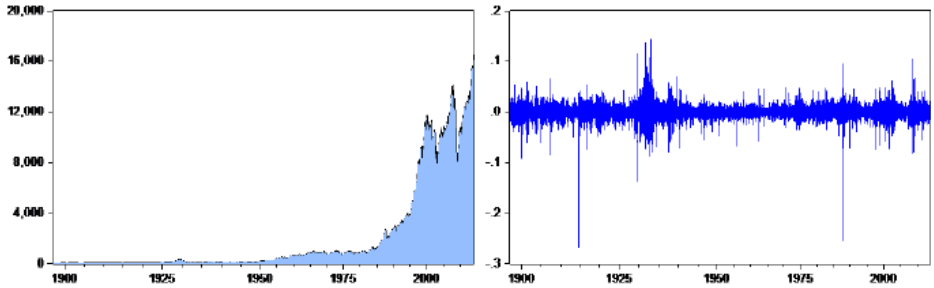
When estimating the Hurst coefficient H, we follow Peters (1994) and consider the following scenarios for the values of H:

a. $H = 1/2 \rightarrow$ the process is independent (past events are not correlated);

b. $0 < H < 1/2 \rightarrow$ the process is anti-persistent: it reverses to its trend more frequently than an independent process;

c. $1/2 < H < 1 \rightarrow$ the process is persistent and is characterized by LRD.

Figure 1: The DJIA 30 prices (left panel) and returns (right panel) from January 1, 1897, to January 1, 2014.



2.2 *Results of RRA*

Prices from the Dow Jones Industrial Average 30 (DJIA30) stock market index were downloaded from the economic data session of the website of the Federal Reserve of St. Louis[3]. These data are available on a daily basis from January 1, 1897 to January 1, 2014 (31,910 observations) and refer to trading days only.

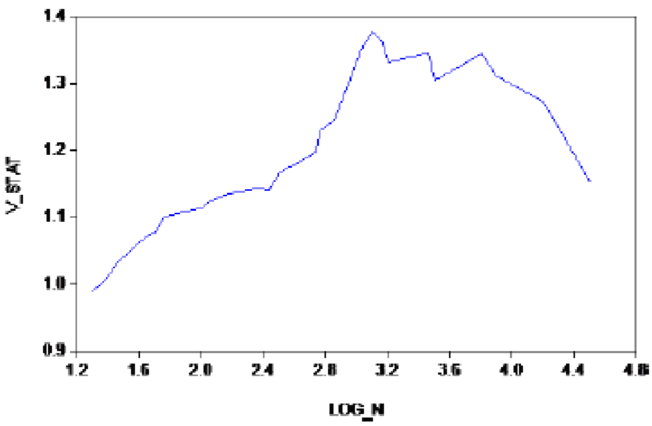
[3] https://www.stlouisfed.org/

The length of the observed time series is consistent with the approach suggested by Peters (1994), who recommends that long periods with a large number of subperiods, which are quite far one from one another, be considered and that a daily or weekly frequency be used in order to avoid problems leading to a biased estimation of the Hurst coefficient in the RRA. Usually, bias is due to: a) "undersampling", which typically characterizes low frequency data observed over a long period or data collected for a period that is too short with respect to the goal of the analysis; b) "oversampling", which refers to the situation of high frequency data observed over a short period. In this latter case, bias is caused by high serial correlation and high noise that typically lead to the Hurst coefficient being overestimated.

To apply RRA, log returns were computed, and the series of observed returns was filtered to avoid bias in the estimates of the Hurst coefficient caused by short-term autocorrelation and heteroscedasticity. DJIA30 prices and returns are in Figure 1.

RRA was then applied on the residuals of an ARMA-like model used to estimate the average level of observed returns plus a GARCH-like model capturing their volatility clustering structure (see Ruppert, 2011). In particular, MA(1)+ARCH(4) appeared as the most suitable specification of this class of models for the available data. In addition, the length of the residuals of the estimated model was reduced to 31,900 by discarding the 8 most recent observations. (2) In order to determine the Hurst index, we considered a smaller number of dividends than the 31,900 daily observations: precisely these, (1, 2, 4, 5, 10, 20, 25, 50, 100). Then we formed an identical number of sequences from the returns composed of continuous and non-overlapping observations with; ($n =$ 31,900, $n =$ 15,950, $n =$ 7,975, $n =$ 6,380, $n =$ 3,190, $n =$ 1,595, $n =$ 1276, $n =$ 638, $n =$ 319), respectively. This adjustment was necessary since RRA required the observed data to have a large, positive and finite number of factors that ensure robustness of the OLS estimates.

Figure 2: V-statistics for the DJIA 30 index from January 1, 1897, to January 1, 2014.



Notes: V_STAT is the V-statistic; LOG_N is the logarithm of the length of each sub-sequence I_a as defined in Section 2.1.

Steps 1 to 6 described in Section 2.1 were applied to the observed data. The results provide evidence that the features of the observed time series are far from being consistent with EMH. In particular, the estimated H for the DJIA30 returns equaled 0.5658. As for the V-statistics, Figure 2 shows that its maximum value was reached when $\log(n) = 3.1613$, which indicates that the average length of the market cycle for the 117 year period is equal to 4 years + 6 months corresponding to $10^{3.1613} = 1{,}450$ trading days.
When applying the OLS regression specified in Eq.1 to the whole series, observations related to the first 6 factors were discarded since the corresponding size of the data points included in a sub-period was too small ($n < 20$). Including these observations would have caused high variance of the V-statistics leading to overestimation of the Hurst coefficient. In view of that, the regression model was applied to only 23 observations. The estimated Hurst coefficient was equal to 0.5658. This value indicates that returns of the DJIA30 are not independent but are characterized by a long-term dependence that implies that positive returns observed in a period tend to be followed by other positive returns observed in another period, which is extremely far from the first one. Equivalently, the series is persistent ($0.5 \leq H \leq 1$) and thus it cannot be described by a standard random walk process. The reciprocal $1 - H = 0.4342$ refers to the coefficient obtainable by applying the same analysis on the series of observed volatilities which, as expected, is antipersistent.
Then, the stability test, described in step 7 of Section 2.1, was applied to 5 sub-periods each composed of 6,382 days. To obtain a consistent number of factors, the number of cases for each period was reduced to 6,370. After this adjustment, 23 factors were considered for each sub-period (i.e., 1,2,4,...,6,370) leading to consecutive but disjointed groups of size equal to 6,370, 3,185, 1,274, ..., 1, respectively. Consistent with the procedure used to estimate the Hurst coefficient on the whole series, for each sub-period observed, returns were filtered and the RRA was applied to the residuals of an ARMA(p,q)+GARCH(p,q) model, and an OLS regression was carried out on the individual sub-periods, each including at most 1,450 trading days. Again, periods associated with factors lower than 20 were discarded.
Results are summarized in Table 1. It is worth noting that the length of the different sub-periods differs consistently and thus the degree of anti-persistence is not constant. This was probably due to different reasons ascribable to different economic policies, social tensions, natural disasters, wars, economic boosts, etc. The first sub-period ($H = 0.59$) lasted about 4 years whereas in the second ($H = 0.58$), the length of the market cycle was about 9 years. The latter included events related to the Great Depression of 1929 up to World War II.

Table 1: Testing the mean of the Hurst coefficients for the returns of the DJIA30

Period	H	E(H)	Z	p-value
01/01/1897 - 01/01/2014	0.5658	0.5526	2.3475	0.0094
01/01/1897 - 09/09/1918	0.5923	0.5470	3.6176	0.0001
09/10/1918 – 01/29/1940	0.5805	0.5470	2.6734	0.0038
01/30/1940 – 07/16/1963	0.5610	0.5470	1.1183	0.1317
07/17/1963 – 12/02/1988	0.5511	0.5470	1.1305	0.1291
13/02/1988 – 12/31/2013	0.5144	0.5470	2.6029	0.0046

The third sub-period (H =0.56) again lasted about 4 years but was characterized by a large number of structural breaks. The fourth period ($H = 0.55$) was the closest to the random walk hypothesis since it was characterized by a cyclic nature that is not completely captured by the V-statistic. The last sub-period ($H = 0.51$) still presented a well-defined length of the market cycle (4.5 years). Summarizing, if we fix a significance level at 5% (or 10%) results reported in Table 1 suggest that the assumption of Gaussian distribution for the H coefficient would be rejected in all periods except in those from Feb. 1940 to July 1963 and from July 1963 to Feb. 1988, although in these periods the p-value was about 0.13.

3 Basic facts about Pareto distributions and estimation of the α parameter

Results obtained from the R/S analysis of DJIA30 returns highlight two main issues. First, the Hurst coefficient for the whole period was 0.5658 and thus the stochastic process could not be considered as a random walk but was characterized by a long-period trend component. Second, because the returns process was not Gaussian distributed it was necessary to assess a suitable theoretical distribution for it. Many solutions have been proposed for similar problems in the literature. For example, the approach of Sheikh and Qiao (2009) based on a combination of distributions aimed at mixing extreme value distributions for tails (to account for rare events) with other (more standard) distributions. In this paper, we resorted to the family of distributions used within the framework of fractal analysis, originally introduced by the Italian economist Vilfredo Pareto and used in finance by Mandelbrot (1968) and Peters (1994).
Pareto distributions are characterized by four parameters:

- the characteristic parameter $\alpha \in (0,2]$ controlling for kurtosis ($\alpha = 2$ in the case of a standard normal distribution); $\beta \in [-1,1]$ controlling for skewness;
- $\beta \in [-1,1]$ controlling for skewness;
- the dispersion parameter $\gamma \in (0, +\infty)$ measuring self-similarity of distributions and their riskiness;
- the location parameter $\delta \in (-\infty, +\infty)$.

This choice of the Pareto distribution for the DJIA30 returns series is motivated by the consideration that observed data seems consistent with the three most important characteristics of the Pareto (α-stable) distribution, namely:

- *Self-similarity*: the possibility to rescale the distribution based on the value specified for γ. This means that the probability distribution is stable when α and β are kept fixed while γ and δ can vary because of the stability or the "invariance in sum" property, which causes the characteristic function to be infinitely divisible. In view of that, Pareto distributions are also known as fractal. In other words, self-similarity and stability ensure that the distributions retain their shape (up to scale and shift) under addition.
- *Noah effect*: Mandelbrot (1968) refers to this effect for a stochastic process in which heavy tails in a fractal distribution depend on unexpected jumps characterizing the process. Despite the Gaussian distribution where a relevant structural change in the process derives from many small variations, in the case of the Pareto distribution such a structural change derives from a restricted number of large variations.
- *Infinite variance and asymptotic bias*: infinite variance means that sampling variance will not converge with the population variance, not even asymptotically. This lack of convergence implies the presence of a long memory effect in Pareto distributions arising when $\alpha < 2$. A similar characteristic is typical of the mean of a Pareto distribution when $\alpha < 1$: in this case the sample mean will never converge with the population mean, not even asymptotically.

Formally, stable Pareto distributions can be expressed by their characteristic function $\Phi(u) = E[\exp(i\mu X)] = \int_{-\infty}^{+\infty} \exp(i\mu X) dF(X)$ and the most common parameterization is that introduced by Mandelbrot (1968):

$$\Phi(t) = \exp^{i\delta X - |\gamma t|^{\alpha} \cdot [1 + i\beta \cdot sgn(t) \cdot \omega(t,a)]} \qquad (2)$$

or, in logarithmic form

$$\ln\Phi(t) = i\delta X - |\gamma t|^{\alpha} \cdot [1 + i\beta \cdot sgn(t) \cdot \omega(t,a)] \qquad (3)$$

where

$$i = \sqrt{-1} \qquad sgn(t) = \frac{|t|}{t} \qquad \omega(t,a) = \begin{cases} \tan\frac{\alpha\pi}{2} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi}\log|t| & \text{if } \alpha = 1 \end{cases}$$

Many methods have been used to estimate Pareto distribution parameters: among these, the Maximum Likelihood, the Fourier Transform and the Fast Fourier Transform (see Khindanova et al. 2001 for an overview) are worth mentioning. Although it is important to estimate all four parameters of the Pareto distribution, estimation of α has a primary role due to the unimodality of the distribution. The smaller α is, the stronger will be the leptokurtic feature of the distribution (the peak of the density becomes higher and the tails heavier), and vice versa. The two most reliable methods for estimating α are spectral analysis and R/S analysis. Consistent with the approach used for the estimation of the Hurst coefficient H, R/S is also used to estimate α for the distribution of the DJIA30 returns. Since fractals are self-similar and, consequently, scale-invariant under summation the scale parameter γ is the only one that can vary. Mandelbrot (1968) showed that α

identifies the fractal dimension of the observed probability distribution and that the sum of stable distributions is $n^{(1/\alpha)}$ times the original distributions, with n denoting sample size. Denoting with R_n the sum of stable distributions and with R_1 its first element the following relation holds:

$$R_n = R_1 \cdot n^{\frac{1}{\alpha}} \qquad (4)$$

Computing the logarithm of both members in Eq. 4 and solving for α we get:

$$\alpha = \frac{\log n}{\log(R_n) - \log(R_1)}$$

If $\log(R_n - R_1) \approx R_1 \cdot n^{\frac{1}{\alpha}}$ then it is possible to derive α from the Hurst coefficient H:

$$\alpha = \frac{1}{H}$$

To enforce the reliability of the estimation method described above, it is important to depurate the observed time series from its short memory component to avoid a possible bias in the estimated value of α.

4 Simulation of DJIA30 prices through Brownian Motion

In the following, results obtained for the estimation of the Hurst coefficient through R/S analysis were used to simulate a stochastic process, the so-called fractional Brownian motion (fBm), mimicking the most probable path for DIJA30 prices. The same process was compared with the realization of a simulated geometric Brownian motion (gBm) process. Observed returns were predicted by the two processes to assess the one providing the best fit.

The gBm originated from the stochastic process S_t satisfying the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dz \qquad (7)$$

with S_t denoting the returns of an asset A, with μ and σ being constants that identify drift and volatility, and dz being variations in S_t deriving from a Wiener process. Therefore, the analytic solution for Eq.7 obtained under Itô's interpretation is

$$S_t = S_{t-1} \cdot e^{[(\mu-\sigma^2/2)t+\sigma(W_t-W_{t-1})]} = S_{t-1} \cdot e^{[(\mu-\sigma^2/2)t+\sigma\epsilon\sqrt{1}]}, \quad t = \ldots, T \qquad (8)$$

In Eq.8, S_t is the price of the asset A at time t while μ and σ are the (constant) mean and standard deviation of observed returns S_t. The differences $(W_t - W_{t-1}) = \sigma\epsilon\sqrt{1}$ are the increments of the Wiener process obtained by multiplying a realization of a standard gaussian distribution by the square root of the time lag.

To simulate expected returns of the asset A, we need to fix the time horizon T and decompose it into sub-intervals of generic length Δt in order to generalize the second term of Eq.8 w.r.t. (Δt is the last term on the right in Eq.8). This specification is possible because $\ln\,(S_t/S_{t-1}) \sim N(\mu - \sigma^2/2t, \sigma^2 t)$.

The second simulated process is similar to a distorted random walk corresponding to a process with a long-memory component that alters the typical path followed by a standard random walk. A particular specification of such a process is the fractional Brownian motion (fBm) introduced by Manderlbrot and Van Ness (1968) and used, among others, in Dieker (2004) and in Kroeser and Botev (2013)

$$B_H(t) = B_H(0) + \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \times$$
$$\times \left\{ \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB(s) + \int_{0}^{t} (t-s)^{H-\frac{1}{2}} dB(s) \right\} \quad (9)$$

B_H at time t and H is the Hurst coefficient. $B_H(0) = 0$ indicates the first realization of the process. The main difference between fBm and regular Brownian motion is that while the increments in Brownian Motion are independent, increments for fBm are not. If $H > 1/2$, then there is a positive autocorrelation. If there is an increasing pattern in the previous steps, then it is likely that the current step will be increasing as well. If $H < 1/2$, then the autocorrelation will be negative. In view of that, B_H is a continuous-time Gaussian process with the autocovariance function $\frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. The fBm process has important properties that are considered in this study, namely:

- Self-similarity. For any constant a and a Gaussian process B_H: $B_H(a,t) \sim |a|^H B_H(t)$.
- Stationary increments: $B_H(t) - B_H(s) = B_H(t-s)$.
- Long range dependence. If $H > 1/2$: $\sum_n^\infty E\{B_H(1)[B_H(n+1) - B_H(n)]\} = \infty$.
- Integration: it is possible to define stochastic integrals for a fBm.

4.1 *Simulating a gBm process for the DJIA30*

To simulate the gBm we should consider the historical data concerning prices and returns of DJIA30 in the period between 1 January 1987 - 31 December 2000 and the simulation period covering the period between 1 January 2001 - 1 January 2014. To apply Eq.8 it is necessary to consider pseudo-random numbers extracted as realizations of a standard normal distribution. In particular, 330,200 (3,302 × 100) random numbers were generated since 3,302 is the number of trading days in the simulation period T. The sub-interval Δt is fixed at 1⁄240 = 0.0042 since 240 is the number of trading days in a year. The parameters μ and σ are equal to the expected mean and the standard deviation of DJIA30 returns, respectively. With these parameters, prices and returns of the DJIA30 were obtained for the simulation period by applying Eq.8 and averaging over the 100 simulations obtained for the 3,302 days to identify the most probable path the index is expected to follow.

4.2 *Simulating a fBm process for the DJIA30*

Moreover, asset returns are usually simulated through stable distributions with $1 < \alpha < 2$. Thus, in the specific case of DJIA30, it is possible to use the R/S outcomes and set $\alpha = \frac{1}{H} = \frac{1}{0.5658}$ =1.7674. As previously specified, the dispersion parameter γ measures the volatility of asset returns, which is such that a generic random variable X can be expressed as $X = \gamma X_0$, where X_0 has a unit scale parameter ($\gamma = 1$) but with the same values of X for the parameters α and β. For the specific case of the Pareto distribution, γ corresponds to the standard deviation while the variance is defined as $v_\alpha = \gamma^\alpha$. In this framework, if we consider a set of n stable and independent random variables $R_1, R_2, \ldots, R_n$ such that $R_i \sim S_\alpha(\beta_i, \gamma_i, \delta_i)$, the random variable $S_n = \sum_{i=1}^{n} \omega_i R_i$ is also stable with:

$$\alpha > 1; \quad \gamma = \sqrt[\alpha]{\sum_{i=1}^{n} (|\omega_i| \gamma_i)^\alpha}; \quad \beta = \frac{\sum_{i=1}^{n} \operatorname{sgn}(\omega_i) \beta_i (|\omega_i| \gamma_i)^\alpha}{\sum_{i=1}^{n} (|\omega_i| \gamma_i)^\alpha}; \quad \delta = \sum_{i=1}^{n} \omega_i \delta_i$$

It is worth noticing that the elicitation of the scale parameter is $\gamma = dt^{\frac{1}{\alpha}} = dt^H$. Within this framework, the continuous process of stable random variables was simulated assuming that: $\alpha =$1.7674; $\beta = \delta = 0$ and $\gamma = 1$. These assumptions led to the following values for the parameter of the Pareto distribution describing the returns of DJIA30 in the simulation period, obtained using the STABLE software implemented by Nolan ref: $\alpha =$1.7853; $\beta = -0.1233$; $\gamma = 0.0053$ and $\delta = 0.0059$.

4.3 *Simulation results*

We considered the results obtained by simulating both an fBm and a gBm in the period 1 January 2001 - 1 January 2014. The comparison was carried out by assessing the extent to which simulated processes were able to mimic the observed series.

The similarity between simulated and real data was at first evaluated graphically in Figure 3, which is a time series plot matrix that shows the observed prices, returns and volatility in the first column and the same data obtained through the simulation of an fBm (gBm) as described in Section 4.1 (4.2) in the second (third) column.

Plots in the first column of Figure 3 compare observed and simulated prices. It is worth noticing that gBm prices are increasing exponentially coherently with the mathematical properties of their DGP moving from 10,000 points to about 35,000 points. This representation is rather far from reality since it completely ignores periodic shocks causing trend inversions like those observed during the financial crises period (2007-2009) instead. Contrariwise, fBm simulated prices were more coherent with real prices due to the scale factor f_i used in the second step of the DGP (see Section 4.2). The simulated process was able to completely capture the shock caused by the financial crises in the 2007-2009 period (and partially capture the shock of the 2002-2004 period) as well as the subsequent trend inversions, although it was not able to properly describe the increasing trend observed in the 2012-2014 period.

The second and third column of the plot refer to returns and volatility. Returns generated from gBm were the result of a white noise process and reflect the EMH but the plot of the gBm simulated returns shows that gBm leads to less dispersed returns and is not able to detect any volatility clustering effect. Opposed conclusions can be drawn when inspecting the plots of fBm returns and volatility since, in this case, volatility clustering effects and volatility shocks were generally well captured by the process.

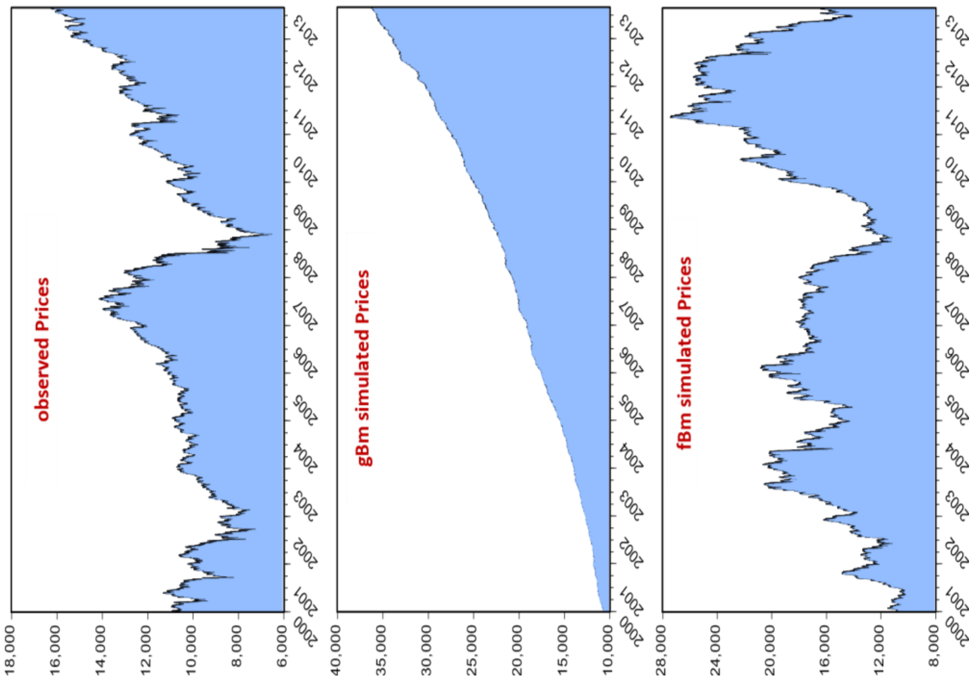
Besides the graphical representation of the observed and simulated processes, some of the forecasting ability of gBm and fBm was also evaluated through certain accuracy indexes. In particular, denoting with y_i the observed value, with $\hat{y}_i$ the predicted value, the following measures were considered for the simulation period composed of N time occasions:

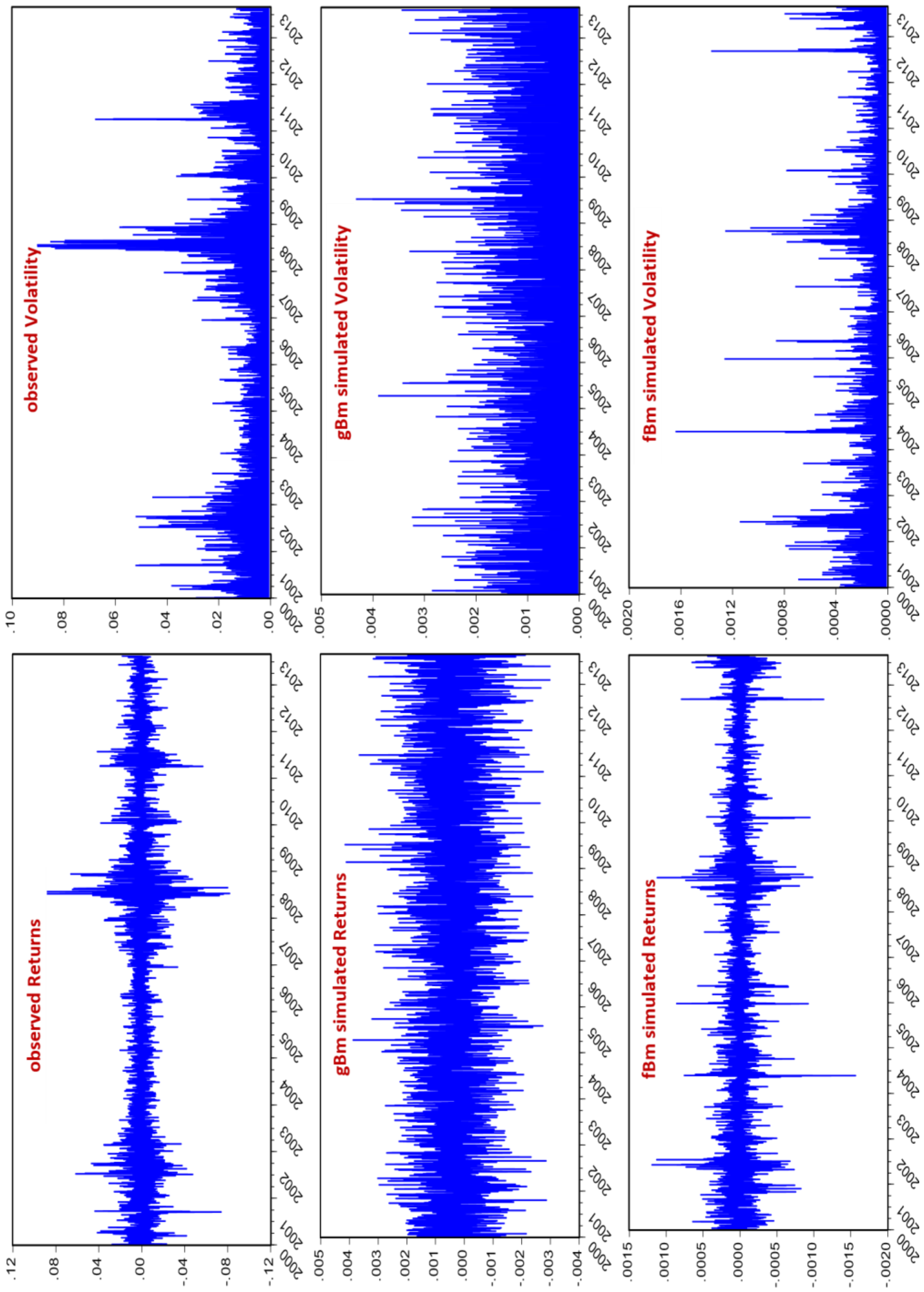
$$MPE = \frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)}{y_i} \cdot \frac{100}{N} \qquad MAPE = \sum_{i=1}^{N} \left| \frac{(y_i - \hat{y}_i)}{y_i} \right| \cdot \frac{100}{N}$$

$$U_1 = \frac{\sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N}}}{\sqrt{\frac{\sum_{i=1}^{N} y_i^2}{N}} + \sqrt{\frac{\sum_{i=1}^{N} \hat{y}_i^2}{N}}} \qquad U_2 = \frac{\sqrt{\frac{1}{N}\left(\frac{\sum_{i=1}^{N-1}(\hat{y}_{i+1} - y_{i+1})^2}{y_i}\right)}}{\sqrt{\frac{1}{N}\left(\sum_{i=1}^{N-1}\left(\frac{y_{i+1} - y_i}{y_i}\right)^2\right)}}$$

The Mean Absolute Percentage Error (MAPE) is the average of the absolute percentage errors (Makridakis, Wheelwright et al. 1999). As such, it is derived from the more rudimentary and less robust Mean Percentage Error (MPE). For both indexes, a value close to zero expresses a good forecasting accuracy of the process.

Figure 3: Observed and simulated time series for DJIA30 prices, returns and volatility from January 1, 2001 to January 1, 2014





The measures U_1 and U_2 are Theil's U coefficients (Theil, 1966). U_1 measures how much actual values and forecasts are closer to each other and is bounded between 0 and 1: values closer to 0 indicate greater forecasting accuracy. U_2 evaluates whether forecasts produced by a model perform better than the naïve forecasts, in this case represented by the ability of y_i in predicting y_{i+1}, which takes the value 1 under the naïve forecasting method. The more $U_2 > 1$, the better the forecasting

accuracy of the simulated process w.r.t. the naïve forecasts will be, and vice versa if $U_2 < 1$.
Results about forecasting accuracy of gBm and fBm are reported in Table 2. They show that, for all the forecasting indexes, fBm considerably outperformed gBm.

Table 2: Forecasting accuracy of gBm and fBm in the simulation period (1 January 2001 – 1 January 2014)

Index	Prices		Returns		Volatility	
	gBm	fBm	gBm	fBm	gBm	fBm
MPE	0.0258	0.0178	-0.0328	-0.0297	-0.0278	-0.0124
MAPE	0.0258	0.0179	0.0438	0.0308	0.0344	0.0294
U_1	0.3429	0.2483	0.9856	0.9248	0.8878	0.9781
U_2	54.3792	81.6609	0.9978	1.0877	0.8838	1.0017

5 Risk assessment of gBm and fBm portfolios

It is common opinion that the standard deviation is not the best measure of volatility and risk for financial asset portfolios. Following its definition, it could happen that positive deviations from the mean values lead to the illogical conclusion that a portfolio is very risky. To avoid this inconsistency, alternative (asymmetric) measures of risk defining a threshold value as an indicator of possible loss have been introduced in the literature. Among these there are Value at Risk (VaR) (Linsmeier and Person 2000) and Conditional Value at Risk (CVaR), (Rockafellar and Uryasev 2000), which are the most prominent examples.
VaR is the maximum expected loss for an asset (a portfolio) in a specific holding period. For its measurement, the elements that need to be considered are the holding period, the (estimated) risk factor, the time horizon, and the confidence level (usually 95% or 99%). If L is a random variable indicating a loss with cumulative distribution function F_L and l_i is one of its possible realizations, for a given confidence level α VaR is defined as $\mathrm{VaR}_\alpha(L) = \min\{l_i | F_L(l_i) \geq \alpha\}$ for $\alpha \in]0,1[$. The difficulties with controlling and optimizing VaR in non-normal portfolios have forced the search for similar percentile risk measures, which would also quantify downside risks and at the same time could be efficiently controlled and optimized. From this viewpoint, CVaR is a perfect candidate for conducting a VaR-style portfolio management. CVaR is defined as an average (expectation) of high losses residing in the α-tail of the loss distribution, or, equivalently, as a conditional expectation of losses exceeding the α-VaR level. The latter definition could be notationally indicated as: $\mathrm{CVaR}_\alpha(L) = E[L | L \geq \mathrm{VaR}_\alpha(L)]$.
In the following, a comparison between different models leading to measures of VaR and CVaR was carried out. Basically, outcomes obtained when assuming a

normal distribution of DJIA30 returns were compared with those obtained assuming a Pareto distribution of returns.

5.1 *VaR and CVaR estimation*

We initially assumed that the DJIA30 index could be bought through an asset that passively replicates the index, such as an ETF, with no transaction or tax costs. To compute VaR, the nonparametric historical simulation method and the stochastic Monte Carlo based simulation method were used. The crucial step in the VaR estimation process is the simulation of the risk factor, which, in the specific case of DJIA30, was the volatility of index returns in the period between 1 January 2001 - 1 January 2014. To simulate volatility, three different scenarios for the DGP were used:

- GARCH(1,1): this scenario assumes that returns are normally distributed in accordance with EMH. The GARCH(1,1) model was simulated 100 times for the period between 1 January 1987 - 31 December 2000, which is composed of 3,303 time occasions and the values of estimated volatility were those obtained by averaging values obtained from each individual simulation.
- gBm: this scenario still assumed the normality of returns and made use of the standard deviation of returns simulated through the Geometric Brownian motion described in Section 4.1.
- fBm: this scenario relied on the fractal market hypothesis and, consequently, on the Pareto distribution of index returns. In this case, the volatility was measured through a rescaled standard deviation obtained from the simulation of an fBm process.

Next, we assumed that one share of the portfolio was built along the simulation period and that the equivalent value of the portfolio was 1,000,000 USD. VaR was computed as the inverse of the cumulative distribution of the loss function under either the Gaussian or the Pareto DGP once a confidence level was set. We focused on 90%, 95% and 99% as possible confidence levels, which, in the case of a Gaussian (Pareto) distribution led to –1.282 (–2.058), –1.645 (–2.939) and –2.326 (–6.669) as reference quantiles, respectively. For the specific case of the Pareto distribution, we assumed that the distribution describing the returns of the DJIA30 index would be specified by the parameters $\alpha = 1.60$ and $\beta = 0.10$ only. Each time, the VaR-based theoretical loss was compared with the real (observed) loss. Exceptions were recorded when the real loss was greater than the theoretical loss. The total number of exceptions in comparison with the chosen confidence level allowed us to understand if the VaR was a realistic measure of potential losses. Next, the CVaR was computed as the expected number of exceptions.

5.2 *Results*

Comparing the various methods, we discovered remarkable differences that proved how, by choosing a Pareto probability distribution, a better impact, in the risk analysis, was given to us with respect to other distributions. In the following table, we indicate the different methods. The first column concerns the distribution types assumed for the yields: the Normal distribution (N) and the

Pareto (P) distribution (with the confidence levels of 90%, 95%, and 99%, respectively). The critical values are in brackets. For each of these, we listed the exception numbers coming from the distributions and the VAR back-testing (with their respective confidence level). For a standard normal distribution, the GARCH case (1.1) provides a good risk estimate (in fact in the VAR case, 90% of the times we had 263 exceptions against the maximum value allowed, which was 330). Assuming the Normality distribution hypothesis for the data, for an acceptable number of the 95% VaR loss, we had several exceptions equal to 165% against a maximum of 165.15%. The exceptions allowed this confidence level to be considered as a reliable risk measure. Under the same normality assumption, raising the acceptance number of the VaR from 95% to 99%, the situation was drastically reversed because we had a much lower number of exceptions, at 71%, than the maximum number allowed at 330.3%. Under the Pareto distribution hypothesis, we found the most important result of the research, which was that for all confidence levels, the GARCH(1.1), either 95% or 99%, can be considered a valid model to simulate the evolution of the return volatility of the DJIA30 index. As shown from the data, the volatility obtained by the MBG, did not seem to be capable of simulating such portfolio losses as to be considered a good predictor of the negative fluctuations of the market index. In fact, the number of exceptions was greater than 10%, 5 % and 1% out of 330.3, i.e., the number of days in which we were simulating the losses.

So, looking at the volatility obtained from the simulation of yields, through fractional Brownian motion, we got a smaller number of exceptions than MBG, in all cases and for all chosen confidence levels. However, on the other hand, we obtained worse results than we achieved when using the GARCH(1,1) model, which confirmed the quality of the model.

Table 3: Number of exceptions obtained from VaR in the simulation period (1 January 2001 - 1 January 2014)

$f(r_i)$	Simulation of volatility				$\max(L_i)$
	Garch(1,1)	gBm	fBm	$\text{fBm}^{0.3936}$	
Gaussian, $\alpha = 0.90$, $q_\alpha = -1.28$	263	1,354	346	23	303.30
Pareto, $\alpha = 0.90$, $q_\alpha = -2.06$	101	1,260	210	6	
Gaussian, $\alpha = 0.95$, $q_\alpha = -1.64$	165	1,309	266	9	165.15
Pareto, $\alpha = 0.95$, $q_\alpha = -2.06$	40	1,162	136	3	
Gaussian, $\alpha = 0.99$, $q_\alpha = -2.33$	71	1,230	184	5	33.03
Pareto, $\alpha = 0.99$, $q_\alpha = -6.67$	3	872	58	1	

Table 4: Average daily values of VaR for a replicant of DJIA30 with an equivalent value of 1,000,000 USD (1 January 2001 - 1 January 2014)

$f(r_i)$	Simulation of volatility			
	Garch(1,1)	gBm	fBm	$\text{fBm}^{0.3936}$
Gaussian $\alpha = 0.90$, $q_\alpha = -1.28$	-15,679.34 (-1.57%)	-1,070.45 (-0.11%)	-24,720.10 (-2.47%)	-52,860.04 (-5.29%)
Pareto $\alpha = 0.90$, $q_\alpha = -2.06$	-25,180.18 (-2.52%)	-1,719.09 (-0.17%)	-39,699.14 (-3.97%)	-84,890.37 (-8.49%)
Gaussian $\alpha = 0.95$, $q_\alpha = -2.65$	-20,124.20 (-2.01%)	-1,373.91 (-0.14%)	-31,727.87 (-3.17%)	-67,845.06 (-6.78%)
Pareto $\alpha = 0.95$, $q_\alpha = -2.94$	-35,961.25 (-3.60%)	-2,455.13 (-0.25%)	-56,696.61 (-5.67%)	-121,296.77 (-12.12%)
Gaussian $\alpha = 0.99$, $q_\alpha = -2.33$	-28,462.13 (-2.85%)	-1,943.16 (-0.19%)	-44,873.57 (-4.49%)	-95,954.96 (-9.60%)
Pareto $\alpha = 0.99$, $q_\alpha = -6.67$	-81,599.02 (-8.16%)	-5,570.90 (-0.56%)	-128,649.25 (-12.86%)	-275,096.18 (-27.51%)

Table 5: Average daily values of CVaR for a replication of DJIA30 with an equivalent value of 1,000,000 USD (1 January 2001 - 1 January 2014)

$f(r_i)$	Simulation of volatility			
	Garch(1,1)	gBm	fBm	$\text{fBm}^{0.3936}$
Gaussian $\alpha = 0.90$, $q_\alpha = -1.28$	-22,820.20 (-2.28%)	-9,335.44 (-0.93%)	-16,754.43 (-1.68%)	-24,173.70 (-2.42%)
Pareto $\alpha = 0.90$, $q_\alpha = -2.06$	-31,133.39 (-3.11%)	-9,875.19 (-0.99%)	-15,859.67 (-1.59%)	-29,460.04 (-2.95%)
Gaussian $\alpha = 0.95$, $q_\alpha = -2.65$	-20,781.90 (-2.68%)	-9,586.49 (-0.96%)	-16,234.65 (-1.62%)	-25,106.44 (-2.51%)
Pareto $\alpha = 0.95$, $q_\alpha = -2.94$	-39,885.02 (-3.99%)	-10,424.08 (-1.04%)	-14,282.34 (-1.43%)	-26,122.79 (-2.61%)
Gaussian $\alpha = 0.99$, $q_\alpha = -2.33$	-34,630.60 (-3.46%)	-10,065.21 (-1.01%)	-15,506.26 (-1.55%)	-28,677.38 (-2.87%)
Pareto $\alpha = 0.99$, $q_\alpha = -6.67$	-64,827.08 (-6.48%)	-12,190.64 (-1.22%)	-15,753.80 (-1.58%)	-40,049.44 (-4.00%)

From tables 4 and 5 the following considerations can be garnered: in table 4 the simulated volatility through MBG, was severely underestimated. This because it did not allow an average negative oscillation over 1% to be predicted in either VaR or CVaR. In table 5 this underestimation was confirmed by the very high number of exceptions between expected loss and actual loss. The MBG assumed a Normality of returns and all that follows. The result did not change if the VaR and the CVaR were calculated also using the critical values of the Pareto distribution.

- The second remark concerns the fact that the volatility simulated through the GARCH(1.1) provided much more realistic values of VaR and CVaR. The worst expected average daily loss was around –2.01.

However, if we consider the CVaR, which is the average of all those exceptional losses, while recording higher values for exceptional losses, they remained too close to those of VaR. This result is very different from the average values of VaR and CvaR estimated for US Large Cap equity of –7.75 and –9.95 respectively.

- For all confidence levels and for any type of hypothetical distribution in both the case of GARCH(1,1) and in the MBG, the average value of the VaR was lower than the average value of the CVaR. This was a positive figure since the average losses in the worst 10% of cases must have necessarily been higher than those relating to the central distribution area. This has a particular meaning in the Normal distribution hypothesis.
- Still, with reference to the MBF, we can observe that the exceptional average losses, expressed by the CVaR, contrary to the GARCH(1.1) and the MBG, were lower than those provided by the VaR. But it was also true that the probability that these might occur was much lower, so much so that the number of exceptions for the MBF was very low compared to the other two methods.
- From a risk management point of view, the above observations indicate that a forecast given by fractional Brownian motion of the risk factor volatility provided a much more reliable estimate concerning the GARCH(1,1) and the MBG methods. In fact, at first, we had a very small number of exceptions. Then, second, comparing the average VaR for the US Large Cap stocks (–7.75%) with that obtained for the DJIA 30 through MBF (–4.66%) we saw that this method was the only one that provided us with such a close estimate of similar assets in terms of characteristics and capitalization. Third, the fact that a lower CVaR compared to the VaR was obtained, again with reference to the MBF, was not such a bad thing. On the contrary, this meant that a large part of the risk was in the tail. It was the endogenies in the central part of the probability distribution both for the Normal distribution and even more in the case of the Pareto distribution, which allowed many less relevant, sudden, and unexpected real losses to be incurred.

6 Concluding remarks

The idea for this paper was born reading a study conducted by the J.P. Morgan and Chase Bank Asset Management departments on the non-normality of returns for financial assets. They simulated the price evolution of the 30 Dow Jones Industrial stock index as a stochastic process known as Brownian motion. In our formulations, we compared the Index values of DJIA30 such as prices, yields, and volatility, respectively found using the classic geometric Brownian motion and the fractional Brownian motion with the index prices of the real market to understand which one would be the best evaluation method. The data showed that the fractal Brownian motion gives a better estimation of the index price in contrast with the geometric Brownian Motion. It also showed that the hypothesis of the normal probability distribution is overcome by a Pareto distribution.

Our work confirms the thesis that the volatility concept deserves to be studied alone through fractal analysis and that the VAR estimation would be more precise if it were done using different risk factors. In conclusion, we believe that future research would be interesting if it were to adopt the same procedures mentioned above, utilizing more assets. In other words, the study should be applied to a portfolio of securities, to understand what consequences might be generated through the application of the geometric Brownian motion against the fractional Brownian motion. A better estimation of these indexes could be expressed through the application of both probability distributions: a Pareto distribution for the tails and a geometric Brownian Motion for the core. Furthermore, finding an efficient statistical method within the Pareto distribution regarding the analysis of a portfolio of securities would be interesting. Since we are discussing the context of an infinite variance, it is impossible to deal with the correlation among securities as an index for the creation of a portfolio containing diverse securities. For further modifications of the fractional Brownian model, we considered a new simulation Ex-Ante related to the timeline between 2014-2020, which showed the same robustness of the previously studied estimate. Specifically, DJIA30 reversal points were forecasted with high reliability.

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