On blocking mechanisms in economies with club goods

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Abstract

The paper investigates some classical results concerning the core and competitive equilibria in an economy consisting of both private and club goods, where club goods are treated as articles of choice just like private goods. Clubs in this framework are described by the characteristic of their members and the local project the club endorses. The space of economic agents in our economy is described by a measure space of agents which includes both negligible and non-negligible agents. The competitive equilibria notion in our setup is known as the club equilibria. In this paper, we establish three results: (i) equivalence between the core (resp. club equilibria) of a mixed economy with the core (resp. club equilibria) of its associated continuum economy, which extends the classical core-equivalence theorem of Ellickson et al. [11] to the case of a mixed economy; (ii) extensions of Schmeidler's theorem (refer to [24]) and Vind's theorem (refer to [29]) to a club economy with a mixed measure space of agents, which provides a sharper characterization of club equilibrium states; and (iii) characterizations of club equilibria in a mixed economy by considering the veto power of the grand coalition in infinitely many economies obtained by perturbation of initial endowments in the original economy.

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1 Introduction

Clubs are discretionary groups set up by individuals to support social consumption. In recent years the term "clubs" has been broadened to incorporate organizations and projects such as toll roads, private parks, satellites, television network connections, golf clubs, etc. Club goods stand roughly in the middle of the spectrum that ranges from pure public to pure private goods. Club goods in sharp distinction from public goods are exclusive to only the members of the club. Individual members often willingly or unwillingly impose congestion costs on other members thereby adding to their disutility. In such settings, the club literature has mainly emphasized that individuals calculate the marginal utility or disutility of joining a club, and such optimizing behavior on behalf of individuals in terms determines the optimal sizes of the clubs in equilibrium. Seminal works in club literature ranging from that by Tiebout [28], Wiseman [30], Buchanan [8] and others aimed at finding "optimal club sizes". However, such findings based on individual choices were possible because of the finite number of individuals. This coupled with the core "indivisible" nature of club projects rendered models devoid of "perfect competition".

Ellickson et al. [11] first introduced a framework that tackled issues on both fronts. They adapt the continuum framework introduced by Aumann [1] by introducing clubs and club memberships in parallel to a continuum of agents. Club sizes are limited in our framework, so any particular club of a certain type (gyms, swimming pools, libraries, etc.) can only have a finite number of members, although a number of clubs for a certain type may be large enough. Thus, clubs are significant concerning individuals but infinitesimal concerning the market. Each club is identified through the non-Samuelson public project it provides and the characteristics of the members of the club. Individuals act as members of the clubs and are bestowed with some external characteristics upon them. These characteristics are external since they are not only observable to other agents but also affect other agents' utilities. As clubs have limited size compared to the market the externalities arising from such member characteristics are internalized within the clubs. However, trading of private goods is not restricted to clubs as the number of private goods are more than one and individuals are members of more than one club. The model becomes further robust from the parallel treatment of private goods and club memberships. Every membership embodies in itself a description of all relevant aspects such as the profile of characteristics of other members, members in question, the purpose of the club, and resources necessary to form the club. The price for these memberships is contingent upon the characteristics of the member, the characteristic of other members of the club, and the club projects. Thus, prices for club memberships reflect the externalities within clubs.

Despite such parallel treatment, there exist some fundamental differences between club memberships and private goods. Prices of public goods are always positive but that of memberships can be positive, negative, or even zero. The other obvious difference arises from the indivisible nature of club memberships as opposed to private goods which are purely divisible in nature. The main difference between the two lies however in the feasibility condition. The feasibility of private goods implies equality of demand and supply. For club memberships, feasibility entails that given a particular proportion of members for a particular club type, the number of clubs of the given type should be such that the proportion remains intact in the aggregate. They not only establish that competitive equilibrium exists in their setup but also can be decentralized using core allocations under some reasonable assumptions.

Since its inception, general equilibrium studies have focused mainly on two avenues of studying equilibrium in the economy. One is through the market economy and such allocations are referred to as Walrasian equilibrium allocations. The other avenue is that of the set of allocations arrived through cooperative behavior amongst the agents. We term a such set of allocations as the core allocations. Edgeworth [10] in his seminal work conjectured that as the number of agents tends to infinity one can expect the set of core allocations to merge with the set of Walrasian equilibrium allocations and was later validated by Aumann in his paper Aumann [1]. Aumann claimed that with an infinite number of agents the number of possible coalitions increases and also market power of individual agents becomes negligible which guarantees equivalence. Shitovitz [27] in 1973 presented opposition to Aumann's framework and claimed that no market is entirely competitive. To that extent, he introduced a market with large traders and called such economies "mixed economies". Shitovitz in his paper showed that unless and until there exist at least two large traders with similar initial endowments and preferences the set of competitive equilibrium fails to converge to the set of core allocations. Greenberg and Shitovitz [17] later introduced associated continuum economies to mixed economies. Alongside establishing an equivalence between the core of these two economies, they show that the set of Walrasian equilibrium allocations is the same for the two economies. We define mixed club economies in our setup and following the lines of Greenberg and Shitovitz [17] introduce associated continuum economies to our mixed club economies. Although establishing an equivalence between the club equilibrium allocations was immediate, the same cannot be said for the equivalence between the cores. To work around the indivisible nature of club goods we work with a certain subclass of allocations. The average club membership consumption for each such allocation belongs to $Lists_a$ for all $a \in A$. Following in lines of Greenberg and Shitovitz [17] we establish the equivalence between the cores of the mixed economy and associated continuum economy under additional assumptions. To this end, we adopt an assumption from Basile et al. [2] which says in a coalition containing all large agents, there exists a positive measurable subset of small agents of that coalition who mimics the characteristics and choices of the large agents. Finally, in virtue of the above two results, we can guarantee an extension of the core equivalence theorem in Shitovitz's seminal work [27] in the case of a mixed economy with impure public goods.

However, forming coalitions to block require communication between individuals forming a coalition, which however at times may be quite costly. So characterizing core allocations concerning coalitions of certain sizes has been studied quite extensively in the literature. Vind [29] added a very insightful remark to Aumann's [1] core-equivalence theorem. He claimed that by restricting blocking to coalitions of a fixed size lying between zero and the grand coalition one can guarantee the equivalence between the set of equilibrium and core allocations. The finite-dimensional version of this theorem is guaranteed by the application of Lyapnov's convexity theorem. However, an immediate extension of this result does not follow for infinite dimensional spaces. Work by Beloso et al. [20] extends Vind's theorem for economies where agents have myopic utility functions and the commodity space is the space of bounded sequences l_∞. Bhowmik and Cao [5] provide an extension where the commodity space is an ordered Banach space with a non-empty positive cone. Later, Bhowmik and Cao [5] provided a characterization of Walrasian equilibrium in line with Vind where commodity space is a Banach lattice with an empty interior. Bhowmik and Graziano [7] provided an extension of Vind's theorem for economies with large agents. Their work was the first ever in this direction. They considered blocking by generalized coalitions, thereby allowing agents to participate in coalitions with a fraction of their endowments. It is important to highlight that Vind's work provides a sharp extension to that of Schmeidler [24]. Schmeidler remarked that to block coalitions outside the set of equilibrium allocations it is sufficient to consider blocking by arbitrary "small" coalitions only. In our paper, we attempt an extension of Schmeidler's theorem for mixed club economies. Although, we restrict ourselves to allocations that give constant utility to all atoms, and the average club membership for atoms in a blocking coalition (if any) are defined, i.e. they are integer consistent. Under such assumptions for mixed club economies given an ε, we guarantee a weaker version of Schmeidler's theorem where the measure of the blocking coalition is less than or equal to ε. However, in the case of an atomless economy with no such additional assumptions, we can find a blocking coalition whose measure is exactly equal to ε, thereby establishing an exact version of Schmeidler's theorem. We also provide an extension of Vind's theorem, but having already formulated Schmeidler's theorem, given a blocking coalition S, we restrict our attention to coalition with measure ε, where $\lambda(S) < \varepsilon < \lambda(A)$. As in the case of Schmeidler's theorem, we not only confine ourselves to allocations that provide the same utilities to all atoms, the utility level for these allocations equals the utility achieved from the average consumption bundle of such allocations. The existence of such a class of allocations is warranted if non-core allocations can never be blocked by coalitions containing exactly one atom. Our extension is two folds, one is where the blocking coalition S includes all atoms of the economy and the second case is when S does not contain all the atoms. In both scenarios, given any $\varepsilon > 0$, we construct a blocking coalition with a measure greater or equal to ε. Thus, we are enabled to guarantee a somewhat weaker version of Vind's theorem. Note that, for both Schmeidler and Vind's theorem our conjectures require the number of atoms in the economy to be infinite. We cannot make any comments on our results when the number of atoms is finite. Also, the blocking in our case is with normal coalitions and not generalized coalitions. The treatment of non-core allocations by normal coalitions of fixed sizes to obtain Vind's theorem for mixed economies is a first in the literature.

One special characterization of Walrasian equilibrium allocations for an atomless economy was posited by Hervés-Beloso and Moreno- García [19]. Instead of exercising the veto power of infinitely many coalitions in a single economy, they exercised the veto power of the grand coalition in infinitely many perturbed economies. Such economies were constructed by perturbing the initial endowments of a coalition of agents. The choice of the size of such coalitions may be arbitrarily large, arbitrarily small, or maybe of a fixed given size. Hervés-Beloso and Moreno- García showed that the set of Walrasian equilibrium allocations is equivalent to those that are non-dominated in any of the perturbed economies. They referred to them as "Robustly Efficient" set of allocations. Later, Bhowmik and Cao [6] in their paper developed the notion of robustly efficient allocations for a mixed economy with an infinite dimension commodity space. Graziano and Romaniello [16] in their paper showed that for an economy with infinitely many public goods, characterizing linear cost share equilibrium in terms of non-dominated allocations in infinitely many perturbed economies does away with the dependency on the cost distribution scheme, cost share function, unlike core. This followed the fact that the grand coalition always contributed a share of one to the formation of public projects. Hervés-Beloso and Moreno- García in their seminal work showed that the second welfare theorem follows directly from their main result and that the second welfare theorem fails to hold in clubs framework has been established in Ellickson et al. [11]. Thus, characterizing club equilibrium in terms of robustly efficient allocation for club economies is not possible. Bhowmik and Kaur [3] attempt to find an approximation for robustly efficient allocations by assuming that the net trade in club memberships belongs to the class of consistent club memberships. They show that set of club equilibrium can be characterized using such approximate robustly efficient allocations under some stringent conditions. One question that automatically arises out of their work is whether club equilibrium can be characterized by means of robust efficient allocations. We partially answer the question by finding an approximation for which we can show that the set of club equilibrium is a subset of them without any such stringent conditions. The major difference in our approximate notion with that of Bhowmik and Kaur [3] is that we do away with the assumption on net trades of club membership. Furthermore, compared to Hervés-Beloso and Moreno- García for an allocation to be dominated, it needs to be dominated in a sequence of economies and not just one.

The paper is organized as follows. Section 2 is attributed to describing the economic model. Section 3 talks about mixed economies and their associated continuum economies alongside the different equivalence theorems between the two economies. Section 4 contains a discussion of Schmeidler and Vind's theorem for both atomless and club economies with atoms. Section 5 presents our notion of ϵ-robust efficiency and section 6 concludes.

2 Economic Model

We assume that the space of agents for our economy is a complete, finite and positive measure space. We denote it by (A, Σ, λ) with A being the set of agents and Σ as the corresponding *σ-algebra* whose economic weights on the market are given by the measure λ. We decompose A into two parts: one of them is denoted by A_0 which is the atomless part or the set of small agents in the economy, and the other part is denoted by A_1 which is the union of all atoms or large agents. Without any confusion, we also denote by A_1 the collection of all large agents. The economy is said to be **atomless**or **continuum** if $A_1 = \emptyset$. Define

$$\mathscr{T} := \{S \in \Sigma : A_1 \subseteq S\}$$

as the set of all coalitions containing all large agents. Now let N denote the set of private commodities. We assume that the commodities are perfectly divisible[1]. Thus, the space of private goods is described as the N-dimensional Euclidean space $\mathbb{R}^N$. The consumption set of private commodities for each agent is encompassed by the non-negative orthant $\mathbb{R}^N_+$. Furthermore, let $\mathbb{R}^N_{++}$ denote the strictly positive elements of $\mathbb{R}^N$. For any two commodity bundles $x, y \in \mathbb{R}^N_+$, $x \geq y$ implies $x_i \geq y_i$ for all $i \in N$; $x > y$ implies that $x \geq y$, however $x \neq y$; and $x \gg y$ implies that $x_i > y_i$ for each $i \in N$. We denote $\|x\|_1 := \sum_{n=1}^{N} x_n$.

2.1 Clubs

Each potential member of a club, as in Ellickson et al [11] is bestowed with some external characteristics. These characteristics are external to the extent that they are observable to other members and also create externality within clubs. Examples of such characteristics can be sex, appearance, religion, etc. To capture such externalities, we define a broad set of finite characteristics from which an agent may accrue. Let Ω denote the set of such **external characteristics**. An element $\omega \in \Omega$ denotes the characteristic of an individual agent relevant to other members. Each club can be characterized by the composition of its members where the composition is defined as the number of individuals for each characteristic. For that we define, a map $\pi : \Omega \to \mathbb{Z}_+$, $\mathbb{Z}_+$ being the set of non-negative integers. We identify the composition of a club with such a map and term it as **profile** of a club. Thus, for any $\omega \in \Omega$ the number $\pi(\omega)$ denotes the number of individuals having characteristic ω. Therefore, for a profile π of a club, the total number of members is $\|\pi\|_1 := \sum_{\omega \in \Omega} \pi(\omega)$.

Each club endorses a public project (local to the club) which is termed as **activities**. Such activities are part of a finite set of club activities available to the profile of agents. Activities are part of an abstract set as in Mas-Colell [22]. The set is abstract in the sense that there does not exist a common pre-defined ordering over this set of activities and ranking is entirely subjective to individual members. We denote the set of such activities by Γ. Activities are not traded and ranking amongst them may be influenced by private goods consumption. We define a **club type** as a pair (π, γ), where π denotes the profile of the club and $\gamma \in \Gamma$ denotes the activity it endorses. In our economy, there exist only a finite set of possible club types, denoted by $Clubs := \{(\pi, \gamma)\}$. Now club projects are to be financed by members of the clubs only. In absence of the notion of money in our model, projects are to be financed collaboratively by members. Thus, the requirement of inputs for a club type, denoted by $inp\,(\pi, \gamma)$, is a vector of $\mathbb{R}^N_+$.

[1] Without loss of generality we assume that N also denotes the cardinality for the set commodities.

The next concept we define is pertaining to club memberships. Memberships, in general grants the right of admission to individuals for clubs. An agent of external characteristic $\omega \in \Omega$ can become a member of club type (π, γ) if and only if the description of the club type allows membership for individuals with characteristic ω, i.e., $\pi(\omega) \geq 1$. A **club membership** is thus a triplet $m = (\omega, \pi, \gamma)$, where $(\pi, \gamma) \in$ *Clubs* and $\pi(\omega) \geq 1$. The set of all club memberships is denoted by $\mathscr{M}$. An agent may subscribe to many clubs or none and also can purchase multiple memberships of one particular club type. We define a map $\mathscr{L} : \mathscr{M} \to \{0, 1, 2, \dots\}$, where $\mathscr{L}(\omega, \pi, \gamma)$ of a membership $m = (\omega, \pi, \gamma)$ denotes the number of that membership being bought. We term the above-defined map as *list*. Now we denote the set of all such possible *list* by the following notation:

$$Lists = \{\mathscr{L} : \mathscr{L} \text{ is a } list\}.$$

Letting $\mathscr{R}^{\mathscr{M}}$ be the set of all mappings from the set $\mathscr{M}$ to the real line, we can frequently view *Lists* as a subset of $\mathscr{R}^{\mathscr{M}}$. Throughout the rest of the paper, we also assume that there is an exogenously given upper bound M on the number of memberships an individual may choose.

2.2 Club Economy

A club economy is composed of agents along with their external characteristics, choice sets, initial endowments, and utility functions. These components of the club economy are defined as follows:

(1) Each agent a is associated with some characteristics $\omega_a \in \Omega$.

(2) The choice set of an agent a, denoted by X_a, specifies the set of all possible pairs of private goods and club membership consumption. Thus, $X_a \subset \mathbb{R}^N \times Lists$. We restrict our attention to only non-negative bundles of private goods. Moreover, club memberships embody a notion of excludability in themselves. Thus, the feasible consumption set for agent a is $X_a = \mathbb{R}^N_+ \times Lists_a$, where $Lists_a \subseteq Lists$ denotes the set of feasible lists for $a \in A$.

(3) The initial endowment of agent a is denoted by e_a, which is a member of the space of private goods.

(4) The utility function of agent a is $u_a : X_a \to \mathbb{R}$.

Definition 2.1. A **club economy** $\mathscr{E}$ is a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$, satisfying the following conditions:

(i) The mapping $a \mapsto \omega_a$ is measurable for all $a \in A$;

(ii) The correspondence $a \mapsto X_a$ is a measurable for all $a \in A$;

(iii) The mapping $a \mapsto e_a$ is an integrable with $\int_A e_a d\lambda \in \mathbb{R}^N_{++}$ for all $a \in A$;

(iv) The mapping $(a, x, l) \mapsto u_a(x, l)$ is jointly measurable with $u_a(\cdot, l)$ is continuous and strongly monotonic for all $a \in A$; and

(v) The utility function $u_a : X_a \to \mathbb{R}$ is quasi-concave for all $a \in A_1$[2].

Given a club economy $\mathscr{E}$, we call two agents $a, b \in A$ are of **same type** if $(\omega_a, X_a, e_a, u_a) = (\omega_b, X_b, e_b, u_b)$. Throughout, we assume that all agents in A_1 are of the same type if $A_1 \neq \emptyset$.

2.3 Club Consistency and Feasible States

In everyday life, club memberships are indivisible and hence the need for a consistency requirement. Clubs in our framework are such that their sizes are limited and they have no market power. Therefore, juxtaposed to the continuum of agents in our model, the above requirement translates to club sizes being finite. Since clubs are composed of members, individual memberships to clubs must be bounded and finite.[3] All these makes clubs infinitesimal relative to the society. Also, external characteristics as stated earlier inflicts externalities, but such externalities are confined within the clubs, thereby enabling the model to remain competitive.

A **state** of $\mathscr{E}$ is basically a measurable mapping $(x, l) : A \to \mathbb{R}^N_+ \times \mathbb{R}^{\mathcal{M}}$, which specifies for any agent $a \in A$ the amount of private good consumption x_a and the club membership vector l_a. It is said to be **individually feasible** if $(x_a, \gamma_a) \in X_a$ λ-a.e. In a standard general equilibrium model, social feasibility just requires market clearance for private goods. However, in this framework, an additional condition of consistency of matching of agents is required. To this end, we introduce the concept of a consistent membership vector.

[2]Here by quasi-concavity of u_a, we mean that there exists an extension of u_a, $\tilde{u}_a$ where $\tilde{u}_a : co(X_a) \to \mathbb{R}$ and $co(X_a)$ denote the convex hull of X_a. It is worthwhile to point out that the quasi-concavity plays a role only in the proof of Lemma 3.1 where we shall restrict ourselves to situations under which the convex combinations belong to X_a.

[3]All these restrictions on club sizes and individual memberships to be bounded along with a finite number of public goods makes the choices finite-dimensional as pointed by [11]

Definition 2.2. Given a membership vector $\overline{\mu} \in \mathscr{R}^{\mathscr{M}}$, if for each club type $(\pi, \gamma) \in$ *Clubs*, there exists a number $\psi(\pi, \gamma) \in \mathbb{R}_{+} \setminus \{0\}$ such that

$$\overline{\mu}(\omega, \pi, \gamma) = \psi(\pi, \gamma)\, \pi(\omega)$$

for all $\omega \in \Omega$, then we call such a membership vector $\overline{\mu}$ **consistent**.

In the above definition, the number $\psi(\pi, \gamma)$ may be interpreted as the *number* of clubs of type (π, γ) accounted for in $\overline{\mu}$. Define

$$\mathscr{C}ons := \left\{ \overline{\mu} \in \mathscr{R}^{\mathscr{M}} : \overline{\mu} \text{ is consistent} \right\}.$$

Recognized that $\mathscr{C}ons$ is a vector subspace of $\mathscr{R}^{\mathscr{M}}$.

Next, for any coalition B, a choice function $\mu : B \to Lists$ is **consistent for *B*** if the corresponding aggregate membership vector $\overline{\mu}_B = \int_B \mu_a d\lambda$ is consistent. Notice that the co-ordinate $\overline{\mu}(\omega, \pi, \gamma)$ of $\overline{\mu}$ specifies the total number of memberships chosen by the members in B with external characteristics ω for club type (π, γ). Consistency is the requirement that these numbers are in the same proportion as in the club type.

Now, we will define conditions under which a state is feasible to society as a whole. Over and above the already defined conditions of consistency and individual feasibility, private goods need to achieve clearance. This is guaranteed by material balance. One part of the material balance is input to club activities or projects for each agent $a \in A$. We define the allocation rule as in [11] and it takes the form given by $\frac{1}{|\pi|} inp(\pi, \gamma)$. Thus, an agent a with her club membership choice l_a is required to contribute the following amount of input:

$$\tau(l_a) := \sum_{(\omega, \pi, \gamma)} \frac{1}{|\pi|} inp(\pi, \gamma)\, l_a(\omega, \pi, \gamma).$$

Definition 2.3. A state (x, l) is **feasible for a coalition *B*** if it abides by the following conditions:

- **Individual Feasibility**: $(x_a, l_a) \in X_a$ λ-a.e. on B;
- **Material Balance**: $\int_B x_a d\lambda + \int_B \tau(l_a) d\lambda = \int_B e_a d\lambda$; and
- **Consistency**: $\bar{l}_B$ is consistent.

For $B = A$ then we simply call it **feasible**.

2.4 Equilibrium and Optimality

In this subsection, we shall lay out some definitions. As in every general equilibrium model, we shall begin with Pareto optimality and then outline the cooperative behavior of the individuals. While doing so, we shall resort to both strong and weak notions of such concepts.

Definition 2.4. A **club equilibrium** of $\mathscr{E}$ consists of a feasible state (x, l) and a price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, such that:

- **Budget Feasibility** : For λ-a.e. on A, we have $(p, q) \cdot (x_a, l_a) = p \cdot x_a + q \cdot l_a \leq p \cdot e_a$;
- **Optimization**: For λ-a.e. on A, we have $(y_a, l'_a) \in X_a$ and $u_a(y_a, l'_a) > u_a(x_a, \gamma_a)$ together imply $p \cdot y'_a + q \cdot l'_a > p.e_a$.
- **Budget Balance for Club types** : For each $(\pi, \gamma) \in \mathcal{C}lubs$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p.inp(\pi, \gamma).$$

A **club quasi-equilibrium** of $\mathscr{E}$ also consists of a feasible state (x, l) and a price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, satisfies the first and third conditions in the definition of a club equilibrium, but instead of the second condition it satisfies:

- **Quasi-optimization**: For λ-a.e. on A, we have $(y_a, l'_a) \in X_a$ and $u_a(y_a, l'_a) > u_a(x_a, l_a)$ together imply $p \cdot y_a + q \cdot l'_a \geq p.e_a$.

To introduce our next concept, we define a **coalition** as a measurable subset B of A whose measure is positive. Furthermore, a **sub-coalition** of a coalition B is a coalition B' such that $B' \subseteq B$.

Definition 2.5. A feasible state (x, l) of the economy $\mathscr{E}$ is said to be **objected** if there exists some coalition B and a state (y, l') feasible for B such that $u_a(y_a, l'_a) > u_a(x_a, l_a)$ λ-a.e. on B. The **core** of $\mathscr{E}$, denoted by $\mathscr{C}(\mathscr{E})$, is the set of all feasible states that are not objected.

3 Interpretation via continuum economy

Before we prove our next theorem we need to introduce the associated continuum economy $\mathscr{E}^*$ to our mixed economy $\mathscr{E}$. We associate with the set of large agents A_1, an atomless positive measure space $\left(A_1^*, \Sigma_{A_1}^*, \lambda_{A_1}^*\right)$ such that $A_0 \cap A_1^* = \emptyset$ and $\lambda(A_1) =$

$\lambda^*(A_1^*)$. For every large agent T_n there exists a one-to-one correspondence with an atomless measurable set T_n^* such that $\lambda(T_n) = \lambda^*(T_n^*)$. Thus $A_1^* = \bigcup\{T_n^* : n \geq 1\}$. We basically identify the interval $[\lambda(A_0), \lambda(A)]$ with A_1^*, which is union of countably many disjoint intervals T_n^*, where

$$T_1^* = (\lambda(A_0), \lambda(A_0) + \lambda(T_1))$$

and for any $n \geq 2$,

$$T_n^* = \left[\lambda(A_0) + \lambda\left(\bigcup_{i=1}^{n-1} T_i\right), \lambda(A_0) + \lambda\left(\bigcup_{i=1}^{n} T_i\right)\right].$$

We then define the measure space $(A^*, \Sigma^*, \lambda^*)$, where $A^* = A_0 \cup A_1^*$, the associated σ-algebra Σ^* is the direct sum of the two σ-algebras i.e.

$$\Sigma^* = \left\{C \cup D : C \cap D = \emptyset, C \in \Sigma_{A_0}, D \in \Sigma_{A_1}^*\right\}$$

and the associated measure $\lambda^* : \Sigma^* \to \mathbb{R}_+$ is defined as

$$\lambda^*(C^*) = \lambda_{A_0}(C \cap A_0) + \lambda_{A_1}^*(C \cap A_1^*)$$

for any $C \in \Sigma^*$. The commodity space of $\mathscr{E}^*$ is taken to be the same as in $\mathscr{E}$. Therefore, the associated continuum economy $\mathscr{E}^*$ is defined as $a \mapsto (\omega_a^*, X_a^*, e_a^*, u_a^*)$, where

$$\omega_a^* := \begin{cases} \omega_a, & \text{if } a \in A_0; \\ \omega_{T_n}, & \text{if } a \in T_n^* \text{ and } n \geq 1, \end{cases}$$

$$X_a^* := \begin{cases} X_a, & \text{if } a \in A_0; \\ X_{T_n}, & \text{if } a \in T_n^* \text{ and } n \geq 1, \end{cases}$$

$$e_a^* := \begin{cases} e_a, & \text{if } a \in A_0; \\ e_{T_n}, & \text{if } a \in T_n^* \text{ and } n \geq 1, \end{cases}$$

and

$$u_a^* := \begin{cases} u_a, & \text{if } a \in A_0; \\ u_{T_n}, & \text{if } a \in T_n^* \text{ and } n \geq 1. \end{cases}$$

Next, we compare the allocations between these two economies. Given an allocation (f, l) in $\mathscr{E}$, we define an allocation $(f^*, l^*) := \Xi(f, l)$ in $\mathscr{E}^*$ as

$$(f_a^*, l_a^*) := \begin{cases} (f_a, l_a), & \text{if } a \in A_0 \text{ ;} \\ (f_{T_n}, l_{T_n}), & \text{if } a \in T_n^* \text{ and } n \geq 1. \end{cases}$$

Similarly, given an allocation (f^*, l^*) in $\mathscr{E}^*$, we define the corresponding allocation $(f, l) := \varphi(f^*, l^*)$ in $\mathscr{E}$ as

$$(f_a, l_a) := \begin{cases} (f_a^*, l_a^*), & \text{if } a \in A_0 \text{ ;} \\ \left(\frac{1}{\lambda^*(T_n^*)} \int_{T_n^*} f_a^* d\lambda^*, \frac{1}{\lambda^*(T_n^*)} \int_{T_n^*} l_a^* d\lambda^*\right), & \text{if } a = T_n \text{ and } n \geq 1. \end{cases}$$

The next lemma claims that a state in the core of the economy $\mathscr{E}$ assigns equivalent consumption bundles to agents that are of the same type. This result plays a crucial role in the proof of our main results.

Lemma 3.1. *Let $\mathscr{E}$ be a mixed club economy. Suppose that $R \in \mathscr{T}$ is a coalition such that all members of R are of the same type. Assume further that (f, l) is a state of $\mathscr{E}$ that cannot be blocked by a coalition S such that $\lambda(S \cap R) > 0$ and $|\{n \in \mathbb{N} : T_n \subseteq S\}| \leq 1$ if $A_1 \neq \emptyset$. If* [4]

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda,\ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in \mathbb{R}_+^N \times \mathbb{Z}_+^{\mathscr{M}},$$

then $u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R)$ λ-a.e on R.

Proof. We first assume that u_R and $e(R)$ the common values of u_a and $e(a)$, respectively. Define

$$B := \left\{ a \in R : u_R(\widetilde{f}_R, \widetilde{l}_R) > u_R(f_a, l_a) \right\}$$

and

$$C := \left\{ a \in R : u_R(\widetilde{f}_R, \widetilde{l}_R) < u_R(f_a, l_a) \right\}.$$

Recognized that B and C are Σ-measurable sets. We shall complete the proof by showing that none of these sets has a positive measure. To this end, we first assume that $\lambda(B) > 0$. By the continuity of u_R, we can choose a sub-coalition D of B and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that

(i) D is an atom of B if $B \cap A_1 \neq \emptyset$; and

(ii) $u_R(\widetilde{f}_R - z, \widetilde{l}_R) > u_R(f_a, l_a)$ for all $a \in D$.

Define $r_0 \in (0, 1]$ by letting $r_0 := \frac{\lambda(D)}{\lambda(R)}$. By the Lyapunov convexity theorem, there exists a sub-coalition E of $A \setminus R$ such that

(iii) $\int_E (f_a - e_a) d\lambda = r_0 \int_{A \setminus R} (f_a - e_a) d\lambda$; and

[4]Notice that $\widetilde{f}_R$ always belongs to $\mathbb{R}_+^N$, but $\widetilde{l}_R$ may not belong to $\mathbb{Z}_+^{\mathscr{M}}$.

(iv) $\int_E l_a d\lambda = r_0 \int_{A\setminus R} l_a d\lambda$.

Let $S := D \cup E$. Define $g : A \to \mathbb{R}^N_+$ by

$$g_a := \begin{cases} \widetilde{f}_R - z, \ a \in D; \\ \\ f_a + \frac{z\lambda(D)}{\lambda(E)}, \text{ otherwise.} \end{cases}$$

It is claimed that (f, l) is blocked by S via (g, l). To see this, first, note that

$$\int_S l_a d\lambda = r_0 \int_A l_a d\lambda \in \mathscr{C}ons.$$

As a consequence of this, we have

$$\int_S \tau(l_a) d\lambda = r_0 \int_A \tau(l_a) d\lambda.$$

As

$$\int_D (g_a - e_a) d\lambda = r_0 \int_R (f_a - e_a) d\lambda - z\lambda(D),$$

it is just routine to verify that

$$\int_S (g_a + \tau(l_a) - e_a) d\lambda = r_0 \int_A (f_a + \tau(l_a) - e_a) d\lambda = 0.$$

Therefore, (f, l) is blocked by the coalition S, which violates the hypothesis of our lemma. This leads to a contradiction. Thus, we conclude that $\lambda(B) = 0$. We now assume that $\lambda(C) > 0$. By the continuity of u_R, we can find some sub-coalition G of C such that $u_R(\alpha f_a, l_a) > u_R(\widetilde{f}_R, \widetilde{l}_R)$ λ-a.e. on G. Define $h : A \to \mathbb{R}^N_+$ by setting $h_a := \alpha f_a$ if $a \in G$; and $h_a := f_a$, otherwise. By Jensen's inequality, we obtain

$$u_R\left(\frac{1}{\lambda(R)} \int_R h_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda\right) \geq u_R(\widetilde{f}_R, \widetilde{l}_R).$$

In the light of the strong monotonicity of u_R, we have

$$\begin{aligned} u_R(\widetilde{f}_R, \widetilde{l}_R) &> u_R\left(\frac{1}{\lambda(R)} \int_R h_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda\right) \\ &\geq u_R(\widetilde{f}_R, \widetilde{l}_R), \end{aligned}$$

which is a contradiction. Therefore, we have $\lambda(C) = 0$. Hence, we conclude that $u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R)$ λ-a.e on R. □

We now present a very crucial lemma of this section, which is an extension of Proposition 6 of Basile et al. [2] to the case of impure public goods. This shall later facilitate in achieving an extension of the main result in Greenberg and Shitovitz [17] to the case of impure public goods. To do this, we assumed that there exists a coalition of negligible agents such that the type of each of its members is the same as that of a large agent.

Proposition 3.2. *Let $\mathscr{E}$ be a mixed club economy. Suppose that R is a coalition in the atomless economy $\mathscr{E}^*$ such that agents in R are of same type, $A_1^* \subseteq R$, and $\lambda^*(R \setminus A_1^*) > 0$. Assume that (f^*, l^*) is a feasible state of $\mathscr{E}^*$ such that $(f^*, l^*) \notin \mathscr{C}(\mathscr{E}^*)$ and*

$$(\widetilde{f}_R^*, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in \mathbb{R}_+^N \times \mathbb{Z}_+^{\mathscr{M}}.$$

Then (f^*, l^*) can be blocked by a coalition contained in A_0.

Proof. Since $(f^*, l^*) \notin \mathscr{C}(\mathscr{E}^*)$, there exists a coalition $S \in \Sigma^*$ and a state (g, ν) such that $u_a(g_a, \nu_a) > u_a(f_a, l_a)$ λ^*-a.e. on S;

$$\int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) \, d\lambda + \int_{S \cap A_1^*} (g_a - e_a^* + \tau(\nu_a)) \, d\lambda^* = 0; \tag{3.1}$$

and

$$\int_{S \cap A_0} \nu_a d\lambda + \int_{S \cap A_1^*} \nu_a^* d\lambda^* \in \mathscr{C}ons. \tag{3.2}$$

If $\lambda^*(S \cap A_1^*) = 0$ then we are done. So, we assume that $\lambda^*(S \cap A_1^*) > 0$, and notice that ν only takes finitely many values.[5] Let the range of ν be $\{\nu^1, \cdots, \nu^l\}$. For each $1 \leq j \leq l$, define $K_j := \{a \in A^* : \nu_a = \nu^j\}$. Notice that K_j is Σ^*-measurable for all $1 \leq j \leq l$. Define

$$\mathbb{J} := \{j : 1 \leq j \leq l \text{ and } \lambda^*(S \cap A_1^* \cap K_j) > 0\}.$$

For each $j \in \mathbb{J}$, denote $F_j := S \cap A_1^* \cap K_j$. Consequently, we have

$$S \cap A_1^* = \bigcup \{F_j : j \in \mathbb{J}\}.$$

Thus, from Equation (3.1) and Equation (3.2), we have

$$\int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) \, d\lambda + \sum_{j \in \mathbb{J}} \lambda^*(F_j) \left(g^j - e^j + \tau(\nu^j) \right) = 0 \tag{3.3}$$

[5]It follows from the fact that there is a uniform upper bound M on the number of club memberships that each agent can have.

and

$$\int_{S\cap A_0} \nu_a d\lambda + \sum_{j\in\mathbb{J}} \lambda^*(F_j)\,\nu^j \in \mathscr{C}ons, \tag{3.4}$$

where $e^j := e^*_a$ for $a \in F_j$ and $j \in \mathbb{J}$, and

$$g^j := \frac{1}{\lambda^*(F_j)} \int_{F_j} g_a d\lambda^*$$

for all $j \in \mathbb{J}$. We denote by $G \subseteq A_0$ the coalition $R \setminus A^*_1$.

Case 1. $\lambda^*(S \cap A^*_1) \le \lambda(G)$. In this case, we can choose a sub-coalition G_1 of G such that $\lambda(G_1) = \lambda^*(F_1)$. Since $\lambda^*(R \cap A^*_1 \setminus F_1) \le \lambda(G \setminus G_1)$, we can analogously choose another sub-coalition $G_2 \subseteq G \setminus G_1$ such that $\lambda(G_2) = \lambda^*(F_2)$. Continuing this way, for each $j \ge 2$, we can choose a sub-coalition $G_j \subseteq G \setminus (G_1 \cup G_2, \cdots \cup G_{j-1})$ such that $\lambda(G_j) = \lambda^*(F_j)$. Thus, Equation (3.3) boils down to

$$\int_{S\cap A_0} (g_a - e_a + \tau(\nu_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(G_j)\left(g^j - e^j + \tau\left(\nu^j\right)\right) = 0 \tag{3.5}$$

and

$$\int_{S\cap A_0} \nu_a d\lambda + \sum_{j\in\mathbb{J}} \lambda(G_j)\,\nu^j \in \mathscr{C}ons. \tag{3.6}$$

Case 2. $\lambda^*(S \cap A^*_1) > \lambda(G)$. In this case, we first choose $\alpha \in (0, 1]$ such that $\lambda(G) = \alpha\lambda^*(S \cap A^*_1)$. As in **Case 1**, there is a partition $\{\widehat{G}_j : j \in \mathbb{J}\}$ of G such that $\lambda(\widehat{G}_j) = \alpha\lambda^*(F_j)$ for all $j \in \mathbb{J}$. Applying the Lyapunov convexity theorem, we can find a sub-coalition E of $S \cap A_0$ such that

$$\int_E (g_a - e_a + \tau(\nu_a))\, d\lambda = \alpha \int_{S\cap A_0} (g_a - e_a + \tau(\nu_a))\, d\lambda$$

and

$$\int_E \nu_a d\lambda = \alpha \int_{S\cap A_0} \nu_a d\lambda.$$

Thus, it follows from Equation (3.3) that

$$\int_E (g_a - e_a + \tau(\nu_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widehat{G}_j)\left(g^j - e^j + \tau\left(\nu^j\right)\right) d\lambda = 0 \tag{3.7}$$

and

$$\int_E \nu_a d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widehat{G}_j)\nu^j \in \mathscr{C}ons. \tag{3.8}$$

Therefore, by Equations (3.5)-(3.8), we conclude that there exist a coalition $B_0 \subseteq S \cap A_0$ and a sequence $\{\widetilde{G}_j : j \in \mathbb{J}\} \subseteq \Sigma_G$ of pairwise disjoint coalitions such that

$$\int_{B_0} (g_a - e_a + \tau(\nu_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widetilde{G}_j)\left(g^j - e^j + \tau\left(\nu^j\right)\right) d\lambda = 0. \tag{3.9}$$

and

$$\int_{B_0} \nu_a d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widetilde{G}_j)\nu^j \in \mathscr{C}ons. \tag{3.10}$$

Define $G_0 := \bigcup\{\widetilde{G}_j : j \in \mathbb{J}\}$ and $(\widetilde{g}, \widetilde{\nu}_a) : G_0 \to R^N_+ \times \mathbb{R}^{\mathcal{M}}_+$ by letting $(\widetilde{g}_a, \widetilde{\nu}_a) := (g^j, \nu^j)$, if $a \in \widetilde{G}_j$. By Lemma 3.1, it follows that $u_a(f^*_a, l^*_a)$ is constant λ^*-a.e. on R. As a consequence, we have $u_a(\widetilde{g}_a, \widetilde{\nu}_a) > u_a(f^*_a, l^*_a)$ λ-a.e. on G_0. If $\lambda(B_0 \cap G_0) = 0$, then it is evident from Equation (3.9) that $B_0 \cup G_0$ blocks (f, l) via $(h, \hat{l})$, which is defined as

$$(h_a, \hat{l}_a) := \begin{cases} (\widetilde{g}_a, \widetilde{\nu}_a), & \text{if } a \in G_0; \\ \\ (g_a, \nu_a), & \text{otherwise.} \end{cases}$$

This is a contradiction. Thus, we assume that $\lambda(B_0 \cap G_0) \neq 0$. We define a measurable set $C := (B_0 \setminus G_0) \cup (G_0 \setminus B_0)$. By the Lyapunov convexity theorem, there is some $C_0 \in \Sigma_C$ such that

$$\int_{C_0} (h_a - e_a + \tau(\hat{l}_a))d\lambda = \frac{1}{2}\int_C (h_a - e_a + \tau(\hat{l}_a))d\lambda$$

and

$$\int_{C_0} \hat{l}_a d\lambda = \frac{1}{2}\int_C \hat{l}_a d\lambda.$$

By Lemma 4.2, there a state (φ, l'') such that

$$\int_{B_0\cap G_0} \varphi_a d\lambda = \frac{1}{2}\int_{B_0\cap G_0} (g_a + \widetilde{g}_a)d\lambda$$

and

$$\int_{B_0\cap G_0} l''_a d\lambda = \frac{1}{2}\int_{B_0\cap G_0} (\nu_a + \widetilde{\nu}_a)d\lambda.$$

It follows that

$$\int_{B_0 \cap G_0} \tau(l''_a) d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (\tau(\nu_a) + \tau(\widetilde{\nu}_a)) d\lambda.$$

Therefore, we conclude that

$$\int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l''_a)) \, d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (g_a + \widetilde{g}_a - 2e_a + \tau(\nu_a) + \tau(\widetilde{\nu}_a))) \, d\lambda.$$

In view of above, we can re-write Equation (3.9) and Equation (3.10) as follows

$$\int_{C_0} (h_a - e_a + \tau(\hat{l}_a)) d\lambda + \int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l''_a)) \, d\lambda = 0$$

and

$$\int_{C_0} \hat{l}_a d\lambda + \int_{B_0 \cap G_0} l''_a d\lambda \in \mathscr{C}ons.$$

Therefore, (f, l) is blocked by $C_0 \cup (B_0 \cap G_0) \in \Sigma_{A_0}$ via the state (y, ξ), where (y, ξ) is defined by

$$(y_a, \xi_a) := \begin{cases} \left(h_a, \hat{l}_a\right), & \text{if } a \in C_0; \\ \\ (\varphi_a, l''_a), & \text{otherwise.} \end{cases}$$

This completes the proof. □

In light of the above lemma, we can show that if an allocation belongs to the core of a mixed economy then it also belongs to the core of the associated continuum economy and vice versa. Hence, the next two theorems establish our extension of the main result in Greenberg and Shitovitz [17].

Theorem 3.3. *Let $\mathscr{E}$ be a mixed club economy. Suppose that $R \in \mathscr{T}$ is a coalition of agents of same type, and $\lambda(R \setminus A_1) > 0$. Assume that (f, l) is a feasible state of $\mathscr{E}$ such that*

$$(\widetilde{f}^*_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \; \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in \mathbb{R}^N_+ \times \mathbb{Z}^{\mathscr{M}}_+.$$

Then $(f, l) \in \mathscr{C}(\mathscr{E}) \Rightarrow (f^*, l^*) := \Xi(f, l) \in \mathscr{C}(\mathscr{E}^*)$.

Proof. The theorem directly follows from Proposition 3.2. □

Theorem 3.4. *Let $\mathscr{E}$ be a mixed club economy and (f^*, l^*) be a state of the associated continuum economy $\mathscr{E}^*$ such that $\varphi(f^*, l^*)$ exists. Then given that $(f^*, l^*) \in \mathscr{C}(\mathscr{E}^*)$ we have $(f, l) := \varphi(f^*, l^*) \in \mathscr{C}(\mathscr{E})$.*

Proof. Let $(f^*, l^*) \in \mathscr{C}(\mathscr{E}^*)$. On the contrary, let us assume that $(f, l) := \varphi(f^*, l^*) \notin \mathscr{C}(\mathscr{E})$. Then there exists a coalition S and a state (y, μ) such that:

(i) $\int_S y_a d\lambda + \int_S \tau(\mu_a) d\lambda = \int_S e_a d\lambda$;

(ii) $u_a(y_a, \mu_a) > u_a(f_a, l_a)$ for each $a \in S$; and

(iii) $\int_S \mu_a d\lambda \in \mathscr{C}ons$.

We define $(y^*, \mu^*) := \Xi(y, \mu)$ and $S^* := (S \cap A_0) \cup \bigcup\{T_i^* : T_i \subseteq S\}$. It is claimed that (f^*, l^*) is blocked by S^* via (y^*, μ^*). Indeed, for all $a \in T_i^*$ with $T_i \subseteq S$, we have

$$u_a(y_a^*, \mu_a^*) = u_{T_i}(y_{T_i}, \mu_{T_i}) > u_{T_i}(f_{T_i}, l_{T_i}) \geq \frac{1}{\lambda^*(T_i^*)} \int_{T_i^*} u_{T_i}(f_a^*, l_a^*) d\lambda^*.$$

Pick an arbitrary i with $T_i \subseteq S$. Applying Lemma 3.1 to an atomless economy we can conclude that $u_a(f_a^*, l_a^*)$ is λ^*-a.e. constant on T_i^*. This implies that $u_a(y_a^*, \mu_a^*) > u_a(f_a^*, l_a^*)$ λ^*-a.e. on T_i^*. Therefore, (f^*, l^*) is blocked by S^* via (y^*, μ^*), which leads to a contradiction. Hence, our supposition was wrong and (f, l) belongs to $\mathscr{C}(\mathscr{E})$. □

Now we shall replicate the core equivalence theorem for the mixed economy $\mathscr{E}$. But, before that, we shall show the set of equilibrium allocations are equivalent. The following theorem summarizes that.

Theorem 3.5. *$\mathscr{W}(\mathscr{E}^*)$ is equivalent to $\mathscr{W}(\mathscr{E})$ i.e.*

(i) $(f, l) \in \mathscr{W}(\mathscr{E}) \Rightarrow (f^*, l^*) := \Xi(f, l) \in \mathscr{W}(\mathscr{E}^*)$;

(ii) $(f^*, l^*) \in \mathscr{W}(\mathscr{E}^*) \implies (f, l) := \varphi(f^*, l^*) \in \mathscr{W}(\mathscr{E})$.

Proof. The proof of (i) is trivial. For the proof of (ii), let $(f^*, l^*) \in \mathscr{W}(\mathscr{E}^*)$. Suppose that the corresponding equilibrium price is (p, q), which implies that (f_a^*, l_a^*) is the maximal element in individual a's budget set for λ-a.e. on A^*. For each $n \geq 1$ and for all $a, b \in T_n^*$, the budget sets of agents a and b are the same, which implies

$$p \cdot f_{T_n} + q \cdot l_{T_n} \leq p \cdot e_{T_n}.$$

It follows from the fact $u_a(f_a^*, l_a^*) = u_b(f_b^*, l_b^*)$ for all $t, s \in T_n^*$ and the quasi-concavity of the utility function that $u_{T_n}(f_{T_n}, l_{T_n}) \geq u_a^*(f_a^*, l_a^*)$ for all $a \in T_n^*$. Consequently, $(f, l) \in \mathscr{W}(\mathscr{E})$. □

Combining Theorem 3.3 and Theorem 3.5, one can establish the equivalence between the core and the set of club equilibrium states in the mixed economy $\mathscr{E}$.

Theorem 3.6. *Let $\mathscr{E}$ be a mixed club economy. Suppose that $R \in \mathscr{T}$ is a coalition of agents having the same type, and that $\lambda(R \setminus A_1) > 0$. Assume that (f, l) is a state of $\mathscr{E}$ such that*

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \; \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in \mathbb{R}^N_+ \times \mathbb{Z}^{\mathscr{M}}_+.$$

Then $(f, l) \in \mathscr{W}(\mathscr{E})$ if and only if $(f, l) \in \mathscr{C}(\mathscr{E})$.

Proof. The implication $(f, l) \in \mathscr{W}(\mathscr{E}) \Rightarrow (f, l) \in \mathscr{C}(\mathscr{E})$ is immediate. For the other direction, let $(f, l) \in \mathscr{C}(\mathscr{E})$. Then from Theorem 3.3, it follows that $(f^*, l^*) := \Xi(f, l) \in \mathscr{C}(\mathscr{E}^*)$. By Theorem 5.1 in Ellickson et al. [11], we can infer that (f^*, l^*) belongs to $\mathscr{W}(\mathscr{E}^*)$. Hence, in view of Theorem 3.5, we have $(f, l) := \varphi(f^*, l^*)$ belongs to $\mathscr{W}(\mathscr{E})$. This completes the proof. □

4 The Size of Blocking Coalitions

In this section, we investigate the size of blocking coalitions in the sense of Schmeidler [24] and Vind [29] in an atomless economy or an economy comprised of small as well as infinitely many large agents. It is worthwhile pointing out that our results in a mixed economy involve blocking by ordinary coalitions and are the first attempt towards this direction as the related literature only covers the case of Aubin coalitions, see Bhowmik and Graziano [7].

Extending the Schmeidler Theorem: The core of an economy $\mathscr{E}$ depends on the blocking power of an arbitrary coalition. However, information transmission within coalitions is costly: the larger the coalition, the more difficult it is to communicate among its members. Therefore, it is reasonable to consider only small coalitions. Schmeidler considers the above-mentioned issue in [24] and showed that if a state is blocked by a coalition S then it can be blocked by a coalition of any given measure less than the measure of S. In this paper, we prove a generalized version of this result to a club economy with a mixed measure space of agents. An immediate implication of this theorem is that the core (and hence the set of club equilibrium states) can be implemented by the formation of small coalitions not only in an atomless economy but also in an economy involving atoms. To this end, we define

$$\mathcal{A} := \{(f, l) : (f, l) \text{ is a state and } u_a(f_a, l_a) \text{ is constant for all } a \in A_1\}.$$

Recognized that, if an economy is atomless then any state belongs to $\mathcal{A}$. For an economy with atoms, a state (f, l) belongs to $\mathcal{A}$ if one of the following conditions is satisfied: (i) $(f,l)|_{A_1}: A_1 \to \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}_+$ is constant; and (ii) if (f,l) cannot be blocked by a coalition in $\mathscr{T}$ and

$$(\widetilde{f}_{A_1}, \widetilde{l}_{A_1}) := \left(\frac{1}{\lambda(A_1)} \int_{A_1} f_a d\lambda,\ \frac{1}{\lambda(A_1)} \int_{A_1} l_a d\lambda \right) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}_+$$

then, by Lemma 3.1, we have $(f,l) \in \mathcal{A}$.

Theorem 4.1. *Assume that $\mathscr{E}$ is a mixed club economy with infinitely many large agents. Let (f,l) be a state belonging in $\mathcal{A}$. Suppose that (f,l) is blocked by a coalition S via a state (g,l') in $\mathscr{E}$, where S satisfies one of the following conditions:*

(i) $S \in \Sigma_{A_0}$.

(ii) $S \notin \Sigma_{A_0}$ *and* $\frac{1}{\lambda(S \cap A_1)} \int_{S \cap A_1} l'_a d\lambda \in \mathbb{Z}^{\mathcal{M}}_+$.

Then, for each $0 < \varepsilon < \lambda(S)$, there exist a sub-coalition R of coalition S such that (f,l) is blocked by R and $\lambda(R) \leq \varepsilon$ with the equality occurs when the economy is atomless.

Proof. Since (f,l) is blocked by a coalition S via (g,l') such that

(1) $u_a(g_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S;

(2) $\int_S g_a d\lambda + \int_S \tau(l'_a) d\lambda = \int_S e_a d\lambda$; and

(3) $\int_S l'_a d\lambda \in \mathscr{C}ons$.

Firstly, assume (i), i.e., $S \in \Sigma_{A_0}$. Take any $\varepsilon \in (0, \lambda(S))$. Choose some $\alpha \in (0,1)$ such that $\varepsilon = \alpha\lambda(S)$. By the Lyapunov convexity theorem, there exists a sub-coalition R of S such that $\lambda(R) = \alpha\lambda(S)$ and

$$\int_R [g_a + \tau(l'_a) - e_a] d\lambda = \alpha \int_S [g_a + \tau(l'_a) - e_a] d\lambda = 0$$

and

$$\int_R l'_a d\lambda = \alpha \int_S l'_a d\lambda \in \mathscr{C}ons.$$

Consequently, (f,l) is blocked by R whose measure is ε. We now assume that condition (ii), i.e., S satisfies $S \notin \Sigma_{A_0}$ and

$$\frac{1}{\lambda(S \cap A_1)} \int_{S \cap A_1} l'_a d\lambda \in \mathbb{Z}^{\mathscr{M}}_+.$$

Choose any $\varepsilon \in (0, \lambda(S))$. Since $\lambda(A) = \lambda(A_0) + \sum_{n=1}^{\infty} \lambda(T_n) < \infty$, there is some $m \geq 1$ with

$$\lambda(T_m) \leq \frac{\varepsilon\lambda(S \cap A_1)}{\lambda(S)}.$$

Therefore, there must exist some $\delta \in (0, 1]$ such that

$$\lambda(T_m) = \frac{\varepsilon\delta\lambda(S \cap A_1)}{\lambda(S)}.$$

Define

$$\widetilde{g} := \frac{1}{\lambda(S \cap A_1)} \int_{S \cap A_1} g_a d\lambda \text{ and } \widetilde{l'} := \frac{1}{\lambda(S \cap A_1)} \int_{S \cap A_1} l'_a d\lambda.$$

In view of the fact that $(f, l) \in \mathcal{A}$, we conclude that (f, l) is blocked by S via (h, μ), where

$$h_a = \begin{cases} \widetilde{g}, & \text{if } a \in A_1; \\ g_a, & \text{otherwise,} \end{cases}$$

and

$$\mu_a = \begin{cases} \widetilde{l'}, & \text{if } a \in A_1; \\ l'_a, & \text{otherwise.} \end{cases}$$

Let $\alpha \in (0, 1)$ be such that $\varepsilon\delta = \alpha\lambda(S)$. This implies that $\lambda(T_m) = \alpha\lambda(S \cap A_1)$. It then follows from the definition of (h, μ) that

$$\int_{T_m} [h_a + \tau(\mu_a) - e_a] \, d\lambda = \alpha \int_{S \cap A_1} [h_a + \tau(\mu_a) - e_a] \, d\lambda$$

and

$$\int_{T_m} \mu_a d\lambda = \alpha \int_{S \cap A_1} \mu_a d\lambda.$$

By the Lyapunov convexity theorem, there exists a sub-coalition B of $S \cap A_0$ such that

(4) $\lambda(B) = \alpha\lambda(S \cap A_0)$;

(5) $\int_B [h_a + \tau(\mu_a) - e_a] \, d\lambda = \alpha \int_{S \cap A_0} [h_a + \tau(\mu_a) - e_a] \, d\lambda$; and

(6) $\int_B \mu_a d\lambda = \alpha \int_{S \cap A_0} \mu_a d\lambda$.

Thus, $E := B \cup T_m$ satisfies $\lambda(E) = \alpha\lambda(S) = \varepsilon\delta \leq \varepsilon$ and

$$\int_E \mu_a d\lambda = \alpha \int_S \mu_a d\lambda = \alpha \int_S l'_a d\lambda \in \mathscr{C}ons.$$

Furthermore, it can be readily verified that

$$\int_E [h_a + \tau(\mu_a) - e_a]\, d\lambda = \alpha \int_S [g_a + \tau(l'_a) - e_a]\, d\lambda = 0.$$

This completes the proof. □

Extending the Vind Theorem: Vind [29] established a sharper characterization of the core of an atomless economy in the sense that any non-core state can be blocked by a coalition of any given size less than that of the grand coalition. Thus, arbitrary large-sized coalitions are entitled to block each feasible state outside the core, which means the core can be seen as a solution supported by an arbitrary large majority of agents. In what follows, we formulate two different approximate versions of this theorem in a mixed economy with club goods. The first one of these yields the exact version of Vind's theorem in an atomless economy as a simple corollary. To prove our results, we first obtain the following lemma.

Lemma 4.2. *Suppose that $\mathscr{E}$ is an atomless economy. Let (f, l) and (g, l') be two states of $\mathscr{E}$ such that $u_a(g_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S for some coalition S. Then given any $0 < \alpha < 1$ there is a state (h, l'') such that*

(i) $u_a(h_a, l''_a) > u_a(f_a, l_a)$ *λ-a.e. on S;*

(ii) $\int_S h_a d\lambda = \int_S (\alpha g_a + (1-\alpha) f_a) d\lambda$; *and*

(iii) $\int_S l''_a d\lambda = \int_S (\alpha l'_a + (1-\alpha) l_a) d\lambda$.

Proof. Consider a vector measure $\upsilon : \Sigma_S \to \mathbb{R}^{N+1} \times \mathbb{R}^{\mathscr{M}}$ such that

$$\upsilon(B) := \left\{ \left(\lambda(B), \int_B (g_a - f_a) d\lambda, \int_B (l'_a - l_a) d\lambda \right) : B \in \Sigma_S \right\}.$$

Pick any $\alpha \in (0, 1)$. In view of Lyapunov's Convexity theorem, there exists a sub-coalition R of S such that $\upsilon(R) = \alpha \upsilon(S)$, which implies

(a) $\lambda(R) = \alpha \lambda(S)$;

(b) $\int_R (g_a - f_a) d\lambda = \alpha \int_S (g_a - f_a) d\lambda$; and

(c) $\int_R (l'_a - l_a)\, d\lambda = \alpha \int_S (l'_a - l_a) d\lambda$.

It follows from (iv) in Definition 2.1 that there exists a function $\widetilde{g} : R \to \mathbb{R}^N_+$ and some $z \in \mathbb{R}^N_+ \setminus \{0\}$ such that $u_a(\widetilde{g}_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on R and

$$\int_R \widetilde{g}_a d\lambda = \int_R g_a d\lambda - z.$$

Finally, define $h : S \to \mathbb{R}^N_+$ and $l'' : S \to \mathbb{R}^{\mathscr{M}}$ by letting

$$h_a := \begin{cases} \widetilde{g}_a & \text{if } a \in R\,; \\ f_a + \frac{z}{\lambda(S\setminus R)} & \text{if } a \in S \setminus R\,, \end{cases}$$

and

$$l''_a := \begin{cases} l'_a & \text{if } a \in R\,; \\ l_a & \text{if } a \in S \setminus R\,. \end{cases}$$

From the strong monotonicity of utility functions, it follows directly that $u_a(h_a, l''_a) > u_a(f_a, l_a)$ λ-a.e. on S. Further, in the presence of (b) and (c), we can readily verify that

$$\int_S h_a d\lambda = \int_S (\alpha g_a + (1-\alpha) f_a)\, d\lambda$$

and

$$\int_S l''_a d\lambda = \int_S (\alpha l'_a + (1-\alpha) l_a)\, d\lambda.$$

This completes the proof. □

The following theorem is an extension of the main result in Vind [29] to our framework. It states that, if a feasible state is blocked by a coalition S containing all large agents then, for any $0 < \varepsilon < \lambda(A)$, it can be blocked by a coalition $E \supseteq S$ whose measure is not less than ε. In a particular case of an atomless economy, the coalition E can be chosen so that its measure is exactly equal to ε, which is the exact version of Vind's theorem to our framework. To state and prove our first result, we define the following: for $A_1 \neq \emptyset$ and any state (f, l), we define

$$\widetilde{\mathscr{T}}(f,l) := \left\{ R \in \mathscr{T} : \text{ members of } R \text{ are of the same type and } (\widetilde{f}_R, \widetilde{l}_R) \in \mathbb{R}^N_+ \times \mathbb{Z}^{\mathscr{M}}_+ \right\},$$

where

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda,\ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right).$$

Therefore, if $(\widetilde{f}_{A_1}, \widetilde{l}_{A_1}) \in \mathbb{R}^N_+ \times \mathbb{Z}^{\mathscr{M}}_+$ then $A_1 \in \widetilde{\mathscr{T}}(f,l)$. We now define a sub-collection $\mathscr{B}$ of $\mathcal{A}$ by letting $\mathscr{B} := \mathcal{A}$ if $A_1 = \emptyset$; and

$$\mathscr{B} := \left\{ (f,l) \in \mathcal{A} : u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R) \ \lambda\text{-a.e. on } R \text{ and for some } R \in \widetilde{\mathscr{T}}(f,l) \right\},$$

if $A_1 \neq \emptyset$. Notice that if the economy is atomless then $\mathcal{A} = \mathscr{B}$ is just *the set of all states.* For $A_1 \neq \emptyset$, for any state (f, l) and any $R \in \widetilde{\mathscr{T}}(f, l)$, if (f, l) is not blocked by any coalition S satisfying $\lambda(S \cap R) > 0$ and $|\{n \in \mathbb{N} : T_n \subseteq S\}| \leq 1$ then, by Lemma 3.1, we have $(f, l) \in \mathscr{B}$.

Theorem 4.3. *Let $\mathscr{E}$ be an economy with either no large agents or it contains infinitely many large agents. Suppose that (f, l) is a feasible state in $\mathscr{B}$ and $S \in \mathscr{T}$ is a coalition such that (f, l) is blocked by S with $\lambda(S) < \lambda(A)$ via a state (g, μ) such that*

$$\widetilde{\mu}_{A_1} := \frac{1}{\lambda(A_1)} \int_{A_1} \mu_a d\lambda \in \mathbb{Z}_+^{\mathscr{M}}.$$

Then, for any $0 < \varepsilon < \lambda(A)$, there exists a coalition E such that $S \subseteq E$, $\lambda(E) \geq \varepsilon$ and (f, l) is blocked by E.

Proof. Given that (f, l) is a feasible state in $\mathscr{B}$ and $S \in \mathscr{T}$ is a coalition such that (f, l) is blocked by S via a state (g, μ). Thus, if $A_1 \neq \emptyset$ then there is a coalition $R \in \widetilde{\mathscr{T}}(f, l)$ such that $u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R)$ λ-a.e on R. Define a state $(\widehat{f}, \widehat{l})$ by $(\widehat{f}_a, \widehat{l}_a) := (f_a, l_a)$ if $a \in A$ and $A_1 = \emptyset$; and

$$(\widehat{f}_a, \widehat{l}_a) := \begin{cases} (f_a, l_a), & \text{if } a \in A_0 \setminus R; \\ (\widetilde{f}_R, \widetilde{l}_R), & \text{otherwise,} \end{cases}$$

if $A_1 \neq \emptyset$. It follows that $(\widehat{f}, \widehat{l})$ is blocked by S via the state (g, μ), which means

(i) $u_a(g_a, \mu_a) > u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on S;

(ii) $\int_S g_a d\lambda + \int_S \tau(\mu_a) d\lambda = \int_S e_a d\lambda$; and

(iii) $\int_S \mu_a d\lambda \in \mathscr{C}ons$.

Without any loss of generality, take any ε satisfying $\lambda(S) < \varepsilon < \lambda(A)$. Let $\delta \in (0, 1)$ be an element such that

$$\delta = 1 - \frac{\varepsilon - \lambda(S)}{\lambda(A \setminus S)}.$$

If $A_1 \neq \emptyset$ then choose some $m \geq 1$ such that $\lambda(T_m) \leq \delta\lambda(A_1)$. Let δ_0 be such that $\delta_0 \leq \delta$ and $\lambda(T_m) = \delta_0\lambda(A_1)$. Define

$$\alpha := \begin{cases} \delta, & \text{if } A_1 = \emptyset; \\ \delta_0, & \text{otherwise.} \end{cases}$$

By (iv) in Definition 2.1, we can choose a function $\xi : S \to \mathbb{R}^N_+$ such that $u_a(\xi_a, l'_a) > u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on S and

$$\int_S \xi_a d\lambda = \int_S g_a d\lambda - z.$$

By Lemma 4.2, there exists some state (x, l') such that

(iv) $u_a(x_a, l'_a) > u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on $S \cap A_0$;

(v) $\int_{S\cap A_0} x_a d\lambda = \int_{S\cap A_0} (\alpha\xi_a + (1-\alpha)\widehat{f}_a) d\lambda$; and

(vi) $\int_{S\cap A_0} l'_a d\lambda = \int_{S\cap A_0} (\alpha\mu_a + (1-\alpha)\widehat{l}_a) d\lambda$.

If $A_1 \neq \emptyset$ then we define (y, l'') such that

$$(y_{T_n}, l''_{T_n}) := \begin{cases} \left(\widetilde{\xi}_{A_1}, \widetilde{\mu}_{A_1}\right), & \text{if } n = m; \\ (\widehat{f}_{T_n}, \widehat{l}_{T_n}), & \text{otherwise,} \end{cases}$$

where

$$\widetilde{\xi}_{A_1} := \frac{1}{\lambda(A_1)} \int_{A_1} \xi_a d\lambda.$$

It follows from the fact[6] $u_R(\xi_a, \mu_a) > u_R(\widetilde{f}_R, \widetilde{l}_R)$ for all $a \in A_1$ and (iv) and (v) in Definition 2.1 that $u_R\left(\widetilde{\xi}_{A_1}, \widetilde{\mu}_{A_1}\right) > u_R(\widetilde{f}_R, \widetilde{l}_R)$. Furthermore, it can be readily verified that:

$$\begin{aligned}
\int_{A_1} y_a d\lambda &= \tilde{\xi}_{A_1}\lambda(T_m) + \sum_{n=1, n\neq m}^{\infty} \hat{f}_{T_n}\lambda(T_n); \\
&= \tilde{\xi}_{A_1}\lambda(T_m) + \sum_{n=1, n\neq m}^{\infty} \hat{f}_{T_n}\lambda(T_m) - \hat{f}_{T_m}\lambda(T_m); \\
&= (\tilde{\xi}_{A_1} - \hat{f}_{T_n})\lambda(T_m) + \sum_{n=1}^{\infty} \hat{f}_{T_n}\lambda(T_n); \\
&= \alpha(\tilde{\xi}_{A_1} - \hat{f}_{T_n})\lambda(A_1) + \sum_{n=1}^{\infty} \hat{f}_{T_n}\lambda(T_n); \\
&= \alpha\int_{A_1} (\xi - \hat{f}) d\lambda + \int_{A_1} \hat{f} d\lambda; \\
&= \int_{A_1} (\alpha\xi + (1-\alpha)\hat{f}) d\lambda.
\end{aligned}$$

[6]As the utility function u_a is same for all $a \in R$, we simply denote it by u_R.

Similarly, it can be shown that

$$\int_{A_1} l''_a d\lambda = \int_{A_1} (\alpha\mu_a + (1-\alpha)\,\widehat{l}_a) d\lambda$$

Consider a state (h, ν), defined by

$$(h_a, \nu_a) = \begin{cases} (y_a, l''_a), & \text{if } a = T_n \text{ and } n \geq 1; \\ (x_a, l'_a), & \text{otherwise.} \end{cases}$$

It follows that

(vii) $u_a(h_a, \nu_a) > u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on S;

(viii) $\int_S h_a d\lambda = \int_S (\alpha\xi_a + (1-\alpha)\,\widehat{f}_a) d\lambda$; and

(ix) $\int_S \nu_a d\lambda = \int_S (\alpha\mu_a + (1-\alpha)\,\widehat{l}_a) d\lambda$.

Another use of Lyapunov's convexity theorem ensures the existence of a sub-coalition C of $A \setminus S$ such that

(x) $\lambda(C) = (1-\alpha)\lambda(A \setminus S)$;

(xi) $\int_C [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda = (1-\alpha) \int_{A \setminus S} [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda$; and

(xii) $\int_C \widehat{l}_a d\lambda = (1-\alpha) \int_{A \setminus S} \widehat{l}_a d\lambda$.

Lastly, let us define $E := S \cup C$ and a state $(\psi, \iota) : A \to \mathbb{R}_+^N \times \mathbb{Z}_+^{\mathscr{M}}$ such that

$$(\psi_a, \iota_a) := \begin{cases} (h_a, \nu_a), & \text{if } a \in S; \\ \left(\widehat{f}_a + \frac{z\alpha}{\lambda(C)}, \widehat{l}_a\right), & \text{otherwise.} \end{cases}$$

It follows that $\lambda(E) \geq \varepsilon$. By the strong monotonicity of utility functions, we have $u_a(\psi_a, \iota_a) > u_a(f_a, l_a)$ λ-a.e. on E. In the presence of (ix) and (xii), it can readily verified that

$$\int_E \iota_a d\lambda(a) = \alpha \int_S \mu_a d\lambda + (1-\alpha) \int_A l_a d\lambda.$$

Since $\int_S \mu_a d\lambda \in \mathscr{C}ons$ and $\int_A l_a d\lambda \in \mathscr{C}ons$, we have $\int_E \iota_a d\lambda(a) \in \mathscr{C}ons$. Furthermore, using (ii), (viii), (xii) and the feasibility of (f, l), we deduce that

$$\int_E (\psi_a + \tau(\iota_a) - e_a) d\lambda = 0.$$

Therefore, we have a coalition E that blocks the state (f, l) through the state (ψ, ι). $\square$

Corollary 4.4. *Let $\mathscr{E}$ be an economy containing infinitely many large agents. Suppose that (f, l) is a feasible state and $S \in \mathscr{T}$ is a coalition such that (f, l) is blocked by S with $\lambda(S) < \lambda(A)$ via a state (g, μ) such that*

$$\widetilde{\mu}_{A_1} := \frac{1}{\lambda(A_1)} \int_{A_1} \mu_a d\lambda \in \mathbb{Z}_+^{\mathscr{M}}.$$

Assume further that (f, l) cannot be blocked by a coalition containing exactly one large agent. If

$$(\widetilde{f}_{A_1}, \widetilde{l}_{A_1}) := \left(\frac{1}{\lambda(A_1)} \int_{A_1} f_a d\lambda, \ \frac{1}{\lambda(A_1)} \int_{A_1} l_a d\lambda \right) \in \mathbb{R}_+^N \times \mathbb{R}_+^{\mathscr{M}}$$

then, by Lemma 3.1, we have $(f, l) \in \mathscr{B}$. Then, from Theorem 4.3, we have for any $0 < \varepsilon < \lambda(A)$, there exists a coalition E such that $S \subseteq E$, $\lambda(E) \geq \varepsilon$ and (f, l) is blocked by E. In an economy with only private commodities, as $\widetilde{f}_{A_1}$ always belongs to $\mathbb{R}_+^N$, we can conclude that our theorem gives a nice extension of Vind's theorem to a mixed economy under standard assumptions in the sense that any feasible state outside the core can be blocked by large coalitions.

Corollary 4.5. *If the economy $\mathscr{E}$ is atomless then, as any state belongs to $\mathscr{B}$, from Theorem 4.3, we conclude that if a feasible state (f, l) is blocked by a coalition S with $\lambda(S) < \lambda(A)$ then for any $0 < \varepsilon < \lambda(A)$ there is a coalition E such that $\lambda(E) \geq \varepsilon$ and (f, l) is blocked by E. It follows from Theorem 4.1 that there is a coalition B blocking (f, l) and such that $\lambda(B) = \varepsilon$. Furthermore, if $\lambda(S) = \lambda(A)$ then again, by Theorem 4.1, we conclude that there is a coalition C which blocks (f, l) and satisfies $\lambda(C) = \varepsilon$. Therefore, Theorem 4.3 along with Theorem 4.1 can be considered as a generalized version of Vind's theorem to a mixed economy.*

The next version of Vind's theorem differs from the earlier one in the sense that it covers the case when the blocking coalition S may not contain all large agents. To tackle this situation, we assume that there is a coalition of small agents having types similar to those of large agents.

Theorem 4.6. *Let $\mathscr{E}$ be an economy with infinitely many large agents. Assume that (f, l) is feasible state and $R \in \mathscr{T}$ is a coalition such that $\lambda(R \setminus A_1) > 0$, all members of R are of the same type, and*

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in \mathbb{R}_+^N \times \mathbb{R}_+^{\mathcal{M}}.$$

Let S be a coalition such that (f, l) is blocked by S. We further assume that it is not blocked by any coalition containing at most one large agent. Then for any $0 < \varepsilon < \lambda(A)$, there exists a coalition E such that $\lambda(E) \geq \varepsilon$ and (f, l) is blocked by E.

Proof. Firstly, by Lemma 3.1, we can assume that S is contained in A_0. Since (f, l) is blocked by S, there exists a state (g, l') such that

(i) $u_a(g_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S;

(ii) $\int_S g_a d\lambda + \int_S \tau(l'_a) d\lambda = \int_S e_a d\lambda$; and

(iii) $\int_S l'_a d\lambda \in \mathscr{C}ons$.

It is enough to choose ε so that $\lambda(S) < \varepsilon < \lambda(A)$. Let $\delta \in (0, 1)$ be an element such that

$$\delta = 1 - \frac{\varepsilon - \lambda(S)}{\lambda(A \setminus S)}.$$

Choose some $m \geq 1$ such that $\lambda(T_m) \leq \delta\lambda(A_1)$. Let δ_0 be such that $\delta_0 \leq \delta$ and $\lambda(T_m) = \delta_0\lambda(A_1)$. By (iv) in Definition 2.1, we can choose a function $\xi : S \to \mathbb{R}^N_+$ such that $u_a(\xi_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S and

$$\int_S \xi_a d\lambda = \int_S g_a d\lambda - z.$$

We define a state $(\widehat{f}, \widehat{l})$ such that

$$(\widehat{f}_a, \widehat{l}_a) := \begin{cases} (f_a, l_a), & \text{if } a \in A_0 \setminus R; \\ (\widetilde{f}_R, \widetilde{l}_R), & \text{otherwise.} \end{cases}$$

By Lemma 3.1, we have $u_a(f_a, l_a) = u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on A. In view of Lemma 4.2, we can find some state (h, μ) such that

(iv) $u_a(h_a, \mu_a) > u_a(\widehat{f}_a, \widehat{l}_a)$ λ-a.e. on S;

(v) $\int_S h_a d\lambda = \int_S (\delta_0 \xi_a + (1 - \delta_0)\widehat{f}_a) d\lambda$; and

(vi) $\int_S \mu_a d\lambda = \int_S (\delta_0 l'_a + (1 - \delta_0)\widehat{l}_a) d\lambda$.

By the Lyapunov convexity theorem, there is a sub-coalition B of $A_0 \setminus S$ such that

(vii) $\lambda(B) = (1 - \delta_0)\lambda(A_0 \setminus S)$;

(viii) $\int_B [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda = (1 - \delta_0) \int_{A_0 \setminus S} [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda$; and

(ix) $\int_B \widehat{l}_a d\lambda = (1 - \delta_0) \int_{A_0 \setminus S} \widehat{l}_a d\lambda$.

Notice that

(x) $\lambda(A_1 \setminus T_m) = (1 - \delta_0)\lambda(A_1)$;

(x) $\int_{A_1 \setminus T_m} [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda = (1 - \delta_0) \int_{A_1} [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda$; and

(xi) $\int_{A_1 \setminus T_m} \widehat{l}_a d\lambda = (1 - \delta_0) \int_{A_1} \widehat{l}_a d\lambda$.

Let $C := B \cup (A_1 \setminus T_m)$. It follows that

(xii) $\lambda(C) = (1 - \delta_0)\lambda(A \setminus S)$;

(xiii) $\int_C [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda = (1 - \delta_0) \int_{A \setminus S} [\widehat{f}_a + \tau(\widehat{l}_a) - e_a] d\lambda$; and

(xiv) $\int_C \widehat{l}_a d\lambda = (1 - \delta_0) \int_{A \setminus S} \widehat{l}_a d\lambda$.

We now define $E := S \cup C$ and a state $(\psi, \nu) : A \to \mathbb{R}^N_+ \times \mathscr{R}^{\mathscr{M}}$ such that

$$(\psi_a, \nu_a) = \begin{cases} (h_a, \mu_a), & \text{if } a \in S; \\ \left(\widehat{f}_a + \frac{z\delta_0}{\lambda(C)}, \widehat{l}_a\right), & \text{otherwise.} \end{cases}$$

Notice that $\lambda(E) \geq \varepsilon$. The strong monotonicity of utility functions guarantees that

$$u_a(\psi_a, \nu_a) > u_a(\widehat{f}_a, \widehat{l}_a) = u_a(f_a, l_a)$$

λ-a.e. on E. Since $\int_E \widehat{f} d\lambda = \int_E f d\lambda$ and $\int_E \widehat{l} d\lambda = \int_R l d\lambda$, similar to Theorem 4.6, we can show that the coalition E blocks the state (f, l) through the state (ψ, ν). □

5 Approximate Robust Efficiency

In this section, we introduce the concept of (approximate) robust efficiency and establish that any club equilibrium state is approximately robustly efficient. Although we have not been able to show that any club equilibrium state is robustly efficient (a question that is proposed by Bhowmik and Kaur [3]), we provide a partial answer. To this end, we start with the definition of an (approximately) robustly efficient state.

Definition 5.1. A state (f, l) is said to be **dominated** in an economy $\mathscr{E}(S, B, f, l, \alpha)$ if there exists a state (g, γ) such that

(i) $u_a(g_a, \gamma_a) > u_a(f_a, l_a)\, \mu$ a.e. on A;

(ii) $\int_A g_a d\lambda + \int_A \tau(\gamma_a) d\lambda = \int_A e(S, f, \alpha) d\lambda + \int_B \tau(l_a) d\lambda$; and

(iii) $\int_A \gamma_a d\lambda,\ \int_B l_a d\lambda \in \mathcal{C}ons$.

A state (f,l) is termed as **robustly efficient** if it is not dominated.

Definition 5.2. A state (f,l) is said to be **sequentially ε-dominated** if there exist a sequences $\{\mathscr{E}(S_n, B_n, f, l, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(g^n, \gamma^n) : n \geq 1\}$ of states such that (f,l) is dominated by (g^n, γ^n) in $\mathscr{E}(S_n, B_n, f, l, \alpha_n)$ and the following conditions are satisfied:

(i) there is a coalition R such that $u_a(h_a^n, \gamma_a^n) > u_a(f_a, l_a)$ for all $h_a^n \in g_a^n + \mathbb{B}(0,\varepsilon)$ with $a \in R$ and $n \geq 1$; and

(ii) $\mathbb{I}_{B_n} = \mathbb{I}_{S_n}$ and $\lambda(B_n^i) \geq \alpha_n.\lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and

(iii) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0,0)$.

A state (f,l) is called **ε-robustly efficient** if it is not sequentially ε-dominated. Furthermore, a state (f,l) is said to be **approximate robustly efficient** if it is ε-robustly efficient for all $\varepsilon > 0$.

If we denote by $\mathrm{RE}_\varepsilon(\mathscr{E})$ the set of ε-robustly efficient states and $\widetilde{\mathrm{RE}}(\mathscr{E})$ the set of approximate robustly efficient states then $\{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}$ is accending sequence and satisfying

$$\widetilde{\mathrm{RE}}(\mathscr{E}) = \bigcap \{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}.$$

Theorem 5.3. *Let (f,l) be a feasible allocation. Then (f,l) is a club equilibrium allocation if and only if it is approximate robustly efficient allocation, where utility from any allocation pair of the form $(0_a, \gamma_a) \in X_a$ is assumed to be zero, for all $a \in A$.*

Proof. Assume that (f,l) is a club equilibrium allocation. Let (p,q) be a corresponding equilibrium price. Without loss of generality, we assume that $\|p\|_1 = 1$. Suppose by the way of contradiction that (f,l) is not an ε-robustly efficient allocation for some $\varepsilon > 0$. This implies that there exist there exist a sequences $\{\mathscr{E}(S_n, B_n, f, l, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(g^n, \gamma^n) : n \geq 1\}$ of allocations such that (f,l) is dominated by (g^n, γ^n) in $\mathscr{E}(S_n, B_n, f, l, \alpha_n)$, which means

(i) $u_a(g_a^n, \gamma_a^n) > u_a(f_a, l_a)\,\mu$ a.e. on A;

(ii) $\int_A g_a^n d\lambda + \int_A \tau(\gamma_a^n)\, d\lambda = \int_A e(S_n, f, \alpha_n)\, d\lambda + \int_{B_n} \tau(l_a)\, d\lambda$; and

(iii) $\int_A \gamma_a^n d\lambda,\ \int_B l_a d\lambda \in \mathcal{C}ons$.

In addition, the following conditions are satisfied:

(iv) there is a coalition R such that $u_a(h_a^n, \gamma_a^n) > u_a(f_a, l_a)$ for all $h_a^n \in g_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and

(v) $\mathbb{I}_{B_n} = \mathbb{I}_{S_n}$ and $\lambda(B_n^i) \geq \alpha_n \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and

(vi) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

For each $i \in \mathbb{I}_S$, there is a sub-coalition C_n of B_n such that $\lambda(C_n^i) = \alpha_n \lambda(S_n^i)$ for all $n \geq 1$. Thus, we have

$$\int_{B_n} l_a d\lambda - \alpha_n \int_{S_n} l_a d\lambda = \int_{B_n \setminus C_n} l_a d\lambda.$$

Since $\{\lambda(B_n) : n \geq 1\}$ converges to 0, we have $\{q \cdot \int_{B_n \setminus C_n} l_a d\lambda : n \geq 1\}$ converges to 0. Let $n_0 \geq 1$ be an integer such that

$$q \cdot \int_{B_{n_0} \setminus C_{n_0}} l_a d\lambda < \frac{\varepsilon \lambda(R)}{2N}.$$

Letting

$$\delta := \frac{2q}{\lambda(R)} \int_{B_{n_0} \setminus C_{n_0}} l_a d\lambda,$$

we note that $\delta < \frac{\varepsilon}{2N}$. It follows that $z_0 := (\delta, \cdots, \delta) \in \mathbb{B}(0, \varepsilon)$. Thus we consider $\widetilde{h} : A \to \mathbb{R}_+^N$ such that

$$\widetilde{h}_a = \begin{cases} g_a^{n_0} - z_0, & \text{if } a \in R; \\ \\ g_a^{n_0}; & \text{otherwise .} \end{cases}$$

As a consequence, we have

$$\int_A \widetilde{h}_a d\lambda = \int_A g_a^{n_0} d\lambda - \lambda(R) z_0.$$

It follows from (i) and (iv) that

$$p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0} > p \cdot e_a \geq p \cdot f_a + q \cdot l_a$$

λ-a.e. on A. Consequently,

$$\int_{S_{n_0}} \left(p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0} \right) d\lambda > \int_{S_{n_0}} p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda,$$

which further implies that

$$\int_A \left(p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, f, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda.$$

This immediately yields that

$$\int_A \left(p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}\right) d\lambda - \lambda(R)\delta > \int_A p \cdot e\left(S_{n_0}, f, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda,$$

which is equivalent to

$$\int_A \left(p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, f, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda + \lambda(R)\delta.$$

Thus we have that

$$\int_A \left(p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, f, \alpha_{n_0}\right) d\lambda + \int_{B_{n_0}} q \cdot l_a d\lambda. \tag{5.1}$$

Given (iii), we have

$$\int_A p \cdot [\tau(\gamma_a^{n_0}) - \tau(l_a)] d\lambda = \int_A q \cdot [\gamma_a^{n_0} - l_a] d\lambda.$$

Thus, it follows from (ii) that

$$\int_A \left(p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}\right) d\lambda = \int_A p \cdot e\left(S_{n_0}, f, \alpha_{n_0}\right) d\lambda + \int_{B_{n_0}} q \cdot l_a d\lambda.$$

This contradicts (5.1). □

Remark 5.4. The converse of the above theorem however fails to hold. As already pointed out earlier dominating an allocation requires it to be dominated in a sequence of economies compared to only one in Hervés-Beloso and Moreno-García. Thus the notion of blocking is much weaker in our case compared to the original definition of robust efficiency. Thus, the set of allocations that can be blocked reduces yielding that our class of approximate robustly efficient allocations is a superset of the set of robustly efficient set of allocations and hence a super-set of the club equilibrium allocations.

Remark 5.5. As the sequence $\{\alpha_n\}_{n\geq 1}$ tends towards zero, the set of agents over which the initial club consumption is assumed to be consistent gets smaller and smaller. Thus, asymptotically again our consistency condition for blocking kind of tends towards where only the final consumption of club membership needs to be consistent as the one defined in Ellickson et al. [11].

6 Conclusion

We provide some further concluding remarks to our analysis done in this paper and also posit a few possible extensions to our work in this section.

Remark 6.1. Shitovitz [27] conjectured that the core of an economy with a large number of agents coincides with the set of competitive equilibrium states when there exist at least two agents of similar types. In our framework, even with such a set of a large number of agents, the equivalence result is obtained only for a special class of states. Given an associated continuum economy to our mixed economy, we restrict ourselves to states in the continuum economy for which the average consumption bundle is defined. We show that, for these restricted states, the set of club equilibrium states in the continuum economy is equivalent to the set of club equilibrium states of the mixed economy. We also establish the main result in Greenberg and Shitovitz [17] in our model under the assumption that there is a coalition of small agents having types similar to those of large agents. In the proof of this result, the finite set of possible club types helps us in partitioning agents in such a way that memberships are constant over each member of the partition which further enables defining average consumption. Thus, given a feasible state in the core of the mixed economy, by our Theorem 3.3, we can claim that it belongs to the core of the continuum economy. By the core equivalence theorem of Ellickson et al. [11], one can infer that such a state is a club equilibrium state of the continuum economy. Finally, Theorem 3.5 enables us to conclude that the associated state in the mixed economy is a club equilibrium state in the mixed economy. Therefore, we generalized the core-equivalence theorem of Ellickson et al. [11] to a mixed economy to a certain extent.

Remark 6.2. The paper by Ellickson et al. [11] was one of the seminal works in club literature that focused on building a competitive model of club economy and not remaining restricted to determining the optimal club sizes. They extend Aumann's [1] result of the classical core equivalence theorem to their setting. Vind [29] later characterized the core of a continuum economy as in Aumann's framework by restricting the size of coalitions to any arbitrary size greater than zero and less than the grand coalition. In Section 4, we provide a similar characterization of the core in line with Vind. This further strengthens the decentralization of equilibrium further in two ways. First, forming large coalitions can be costly as it requires establishing communication between a large number of agents. Thus even if one concentrates in such cases on coalitions of small sizes one can still guarantee the equivalence result in Ellickson et al. [11]. Secondly, by concentrating on a class of coalitions $\left\{D \subset A : \lambda(D) > \frac{\lambda(A)}{2}\right\}$, it can be inferred that core allocations can also be characterized as outcomes from a majority voting rule.

Remark 6.3. Bhowmik and Kaur [3] in their work provided the first-ever characterization of club equilibrium in terms of *approximately* robustly efficient states, a concept originally introduced by Hervés-Beloso and Moreno-García [19] in an exchange economy without club goods. Since the set of core states is equivalent to the set of club equilibrium states, the set of (*approximately*) robustly efficient states provided another characterization of the core. However, Bhowmik and Kaur [3] fails to establish a similar result without taking an approximate version of robust efficiency. As noted by Hervés-Beloso and Moreno-García [19] that the equivalence between the set of club equilibrium states and the set of robustly efficient states yield the welfare theorems as simple corollaries. However, as pointed out by Ellickson et al. [11] that the welfare theorem fails to hold in a club economy. Thus, one should not expect the characterization of the club equilibria in terms of the set of robustly efficient states. Thus, Bhowmik and Kaur [3] adopted the notion of robust efficiency by taking the aggregate net trade of club consumption as consistent rather than the aggregate final consumption of club goods as consistent. We depart from any such stringent assumption and establish that for a weaker version of robustly efficient states, namely "ε-robustly efficient allocations" for each $\varepsilon > 0$, each club equilibrium state can be supported as an ε-robustly efficient allocation. However, as emphasized in Remark 5.4, our notion of blocking is much weaker compared to that of Hervés-Beloso and Moreno-García [19] and therefore, it is unclear to us that whether the reverse inclusion stands not true in our framework. Thus, one important extension to our work can be finding a notion of robust efficiency weaker than our notion of robust efficiency such that the equivalence result of Hervés-Beloso and Moreno-García [19] can be established.

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