Cost minimization analysis of a running firm with economic policy

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Abstract

In this paper the Cobb-Douglas production function is operated in a firm for the analysis of the cost minimization policies. In an economic world, gain of profit depends on the efficient use of raw materials and use of various techniques of the cost minimization. A firm's main target is to make maximum profit. Scientific based and efficient but minimum cost procedures will favor in this regard. To increase local and global demands, a firm of course develop production sector. An attempt has been taken in this study to minimize cost by considering four inputs, such as capital, labor, principal raw materials, and other inputs to form the economic model subject to a production constraint within the budget.

Keywords: Lagrange multiplier, minimum cost, Cobb-Douglas production function

1. Introduction

Cost minimization is a fundamental concept in business, economics, industry, factory, firm, and some other related fields [Samuelson, 1947]. It is a financial strategy and an economic tool that tries to reduce costs of products in a firm. Consequently, the firm can move to maximum profit during its total operations. To minimize cost, it does not support of reducing quality of products by using low cost raw materials, instead it tries continually to meet the customer needs [Carter, 2001; Mohajan et al., 2012; Wiese, 2021].

Cobb-Douglas production function is one of the most widely used production function in economics. In 1928, it is developed by the two American scholars, mathematician and economist Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976) [Cobb & Douglas, 1928]. The method of Lagrange multiplier is a very useful and powerful technique in multivariable calculus. It is a mathematical device that transfers a lower dimensional constrained problem to a higher dimensional unconstrained problem [Islam et al. 2010a, b; Mohajan, 2017a]. A firm takes rational decision to use various raw materials for minimizing the production cost [Roy et al., 2021].

We have tried to discuss sensitivity analysis of the inputs through the production of the firm with minimum cost. In a global competitive economy, sustainability of the firm is necessary and minimum cost is one the best two policies. Sensitivity analysis will help the firm to take correct decision for estimating appropriate quantity of inputs, such as capital, labor, principal raw materials, and other inputs [Mohajan, 2018b, 2021a].

In the study every step of calculation is given elaborately so that even a novice researcher can understand the mathematics and economic concepts easily, and we hope that s/he can take the procedures of the paper interestingly. This study also obeys the basic economic laws that help to interpret Cobb-Douglas production function [Mohajan, 2021b].

2. Literature Review

The literature review section is an introductory portion of research, which highlights the contributions of other scholars in the same field within the existing knowledge [Polit & Hungler, 2013]. Cobb-Douglas production function helps any firm to make rational decision on the quantity of each factor inputs to employ so as to minimize the production cost for its profit maximization [Cobb & Douglas, 1928]. John V. Baxley and John C. Moorhouse have discussed aspects of production functions by considering implicit functions with sufficient mathematical techniques [Baxley & Moorhouse, 1984].

In an economic article, Pahlaj Moolio and his coauthors have analytically discussed the optimization policy of output by using three variables as an explicit form of production function [Moolio et al., 2009]. Expert mathematician and physicist Jamal Nazrul Islam and his coworkers are very careful in their seminal works on multivariate calculus. They have applied necessary and sufficient conditions to analyze the maximum utility. They have worked in their paper with the light of famous paper "*Lagrange Multiplier Problems in Economics*" of John V. Baxley and John C. Moorhouse [Baxley & Moorhouse, 1984; Islam et al., 2010a, b]. Haradhan Kumar Mohajan and his coauthors have tried to work on the optimization for global social welfare [Mohajan et al., 2013]. In another paper he stresses on scientific method for sustainable production [Mohajan, 2021a]. He also applies statistical analysis in various returns scale [Mohajan, 2021b, c]. Earlier he has worked on three optimization mathematical models with necessary and sufficient conditions

[Mohajan, 2017a, b]. Lia Roy and her coauthors have also discussed the cost minimization policy and have shown that it is essential for the sustainable development of an industry [Roy et al., 2021].

3. Methodology of the Study

Methodology is the guideline of a research work. In any research it is an organized procedure that follows scientific methods efficiently [Kothari, 2008]. It tries to give a strategy for planning, arranging, designing and conducting a fruitful research confidently [Legesse, 2014]. In this study we have tried to discuss cost minimization with the help of Cobb-Douglas production function by showing mathematical calculations in some detail [Mohajan, 2021b]. We have started the model with four inputs, such as α amount of capital, β quantity of labor, γ quantity of principal raw materials, and δ quantity of other inputs. Here mathematical calculations provide that the Lagrange multiplier can be represented as the marginal cost function. Then the model is operated with the Cobb-Douglas production function. Here we have obtained the optimum values of α, β, γ, and δ. We have also found the optimum values of Lagrange multiplier θ and capital K. In the sensitivity analysis we have used 5×5 Hessian matrix to obtain minimum cost. Reliability and validity are wealth of a good research, and in our study we have tried to preserve them as far as possible [Mohajan, 2017b, 2022a,b].

In this study we have confined ourselves on the secondary data and mathematical analysis. The valuable data are collected from various published research papers, books of well-known authors, research reports, internet, etc. Throughout the article we have tried to provide the mathematical calculations with some details [Mohajan, 2018a, 2020; Mohajan & Mohajan, 2022a,b].

4. Objective of the Study

The principal objective of this research study is to analyze minimum cost of a firm. Other subsidiary objectives are as follows:

- to show mathematical calculations elaborately,
- to analyze Cobb-Douglas production function, and
- to provide sensitivity analysis for optimization.

5. Mathematical Discussion of the Model

Let us consider that a firm continually produces and sells X units of commodities. During the production period, it uses α amount of capital, β quantity of labor, γ quantity of principal raw materials, and δ quantity of other inputs [Mohajan et al., 2013; Mohajan, 2018b]. The firm maintains the optimum policy in every process and sector, such as in production, selling, assignment, and transformation to gain maximum profit for the sustainability in the global economic world. The firm wants to adjust minimization of cost in α, β, γ, and δ to minimize the cost function K as [Chiang, 1984; Mohajan et al., 2013; Roy et al., 2021];

$$K(\alpha,\beta,\gamma,\delta) = w\alpha + x\beta + y\gamma + z\delta\,, \qquad (1)$$

subject to the constrain of production function X with a suitable production function F as;

$$X = F(\alpha,\beta,\gamma,\delta)\,, \qquad (2)$$

where w is the services of capital α; x is the wage rate per unit of labor β; y is the cost of per unit of principal raw materials γ; and z is the cost of per unit of other inputs δ. Now we use Lagrange multiplier θ to combine economic constrained relations (1) and (2) with the Lagrangian function L, to produce a five-dimensional unconstrained relation as [Mohajan, 2015; Lia et al., 2021; Ferdous & Mohajan, 2022]:

$$L(\alpha,\beta,\gamma,\delta,\theta)=K(\alpha,\beta,\gamma,\delta)+\theta(X-F(\alpha,\beta,\gamma,\delta)). \quad (3)$$

We have considered that the firm minimizes costs in every step of production, so that it must minimize its total cost. In this study we will use asterisk to indicate the optimal values, i.e., we use α^*, β^*, γ^*, δ^*, and θ^* to indicate the optimal values of α, β, γ, δ, and θ respectively. We assume that these quantities must satisfy first order conditions, i.e., we can find desirable results by applying partial differentiation of the Lagrangian function (3) with respect to five variables, α, β, γ, δ, and θ, and considering them equal to zero we get [Mohajan, 2018b, 2021a],

$$L_\theta = X - F(\alpha,\beta,\gamma,\delta)=0, \quad (4a)$$

$$L_\alpha = K_\alpha - \theta F_\alpha = 0, \quad (4b)$$

$$L_\beta = K_\beta - \theta F_\beta = 0, \quad (4c)$$

$$L_\gamma = K_\gamma - \theta F_\gamma = 0, \quad (4d)$$

$$L_\delta = K_\delta - \theta F_\delta = 0, \quad (4e)$$

where $L_\delta = \frac{\partial L}{\partial \delta}$, $L_\theta = \frac{\partial L}{\partial \theta}$, etc. indicates the partial derivatives. Now incorporating the relations (4b-e) we can write the Lagrange multiplier θ as,

$$\theta = \frac{K_\alpha}{F_\alpha} = \frac{K_\beta}{F_\beta} = \frac{K_\gamma}{F_\gamma} = \frac{K_\delta}{F_\delta}. \quad (5)$$

Now we assume that the firm wants to increase quantities in all sectors for the continuous demand of the consumers. Let the firm increases its existing materials from α, β, γ, and δ to $\alpha + d\alpha$, $\beta + d\beta$, $\gamma + d\gamma$, and $\delta + d\delta$ respectively. Consequently, the corresponding changes of dX and dK can be found as,

$$dK = K_\alpha d\alpha + K_\beta d\beta + K_\gamma d\gamma + K_\delta d\delta, \quad (6)$$

$$dX = F_\alpha d\alpha + F_\beta d\beta + F_\gamma d\gamma + F_\delta d\delta. \quad (7)$$

Relations (6) and (7) provide,

$$\frac{dK}{dX}=\frac{K_{\alpha}d\alpha+K_{\beta}d\beta+K_{\gamma}d\gamma+K_{\delta}d\delta}{F_{\alpha}d\alpha+F_{\beta}d\beta+F_{\gamma}d\gamma+F_{\delta}d\delta}. \tag{8}$$

From (5) we get,

$$\theta=\frac{K_{\alpha}d\alpha}{F_{\alpha}d\alpha}=\frac{K_{\beta}d\beta}{F_{\beta}d\beta}=\frac{K_{\gamma}d\gamma}{F_{\gamma}d\gamma}=\frac{K_{\delta}d\delta}{F_{\delta}d\delta}=\frac{K_{\alpha}d\alpha+K_{\beta}d\beta+K_{\gamma}d\gamma+K_{\delta}d\delta}{F_{\alpha}d\alpha+F_{\beta}d\beta+F_{\gamma}d\gamma+F_{\delta}d\delta}. \tag{9}$$

Combining (8) and (9) we get,

$$\frac{dK}{dX}=\theta. \tag{10}$$

Hence, the Lagrange multiplier can be represented as the marginal cost function of the entire production process. The result displays that the total cost of the firm will be increased from the production of an additional unit of *X*. Equivalently, we can say that the total cost will be decreased similarly as the decreasing the production of one unit of *X* [Islam et al., 2011 a, b; Mohajan et al., 2013; Mohajan, 2021a].

6. Analysis with Cobb-Douglas Production Function

Now we introduce the Cobb-Douglas production function *F* as [Humphery, 1997; Roy et al., 2021; Mohajan, 2021b],

$$X=F\left(\alpha,\beta,\gamma,\delta\right)=A\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}, \tag{11}$$

where *A* is the efficiency parameter, which reflects the level of technology. Here *p*, *q*, *r*, and *s* are constants; *p* represents the elasticity of capital that measures the percentage change in *X* for 1% change in α when β, γ, and δ remain constants. Also *q*, *r*, and *s* represent similar properties. The constants *p*, *q*, *r*, and *s* must lie in the following intervals:

$$0<p<1, 0<q<1, \ 0<r<1, \text{ and } 0<s<1. \tag{12}$$

Further, if $\Sigma=p+q+r+s=1$, then Cobb-Douglas production function will provide constant returns to scale; $\Sigma=p+q+r+s>1$ displays increasing returns to scale, and $\Sigma = p + q + r + s < 1$ indicates decreasing returns to scale. Hence, using (11) relation (3) can be written as [Mohajan, 2018b],

$$L(\alpha, \beta, \gamma, \delta, \theta) = w\alpha + x\beta + y\gamma + z\delta + \theta\left(X - A\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}\right). \tag{13}$$

For optimization we can exhibit (13) as;

$$L_{\theta} = X - A\alpha^{p}\beta^{q}\gamma^{r}\delta^{s} = 0, \tag{14a}$$

$$L_{\alpha} = w - \theta Ap\alpha^{p-1}\beta^{q}\gamma^{r}\delta^{s} = 0, \tag{14b}$$

$$L_{\beta} = x - \theta Aq\alpha^{p}\beta^{q-1}\gamma^{r}\delta^{s} = 0, \tag{14c}$$

$$L_{\gamma} = y - \theta Ar\alpha^{p}\beta^{q}\gamma^{r-1}\delta^{s} = 0, \tag{14d}$$

$$L_{\delta} = z - \theta As\alpha^{p}\beta^{q}\gamma^{r}\delta^{s-1} = 0. \tag{14e}$$

From (14a) we can write,

$$\alpha^{p}\beta^{q}\gamma^{r}\delta^{s} = \frac{X}{A}. \tag{15}$$

Combining (14b) to (14e) we get,

$$\theta = \frac{w\alpha}{Ap\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}} = \frac{x\beta}{Aq\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}} = \frac{y\gamma}{Ar\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}} = \frac{z\delta}{As\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}} \tag{16}$$

$$\frac{w\alpha}{p} = \frac{x\beta}{q} = \frac{y\gamma}{r} = \frac{z\delta}{s}. \tag{17}$$

Again from (15) we can write,

$$\alpha = \frac{X^{\frac{1}{p}}}{A^{\frac{1}{p}}\beta^{\frac{q}{p}}\gamma^{\frac{r}{p}}\delta^{\frac{s}{p}}} \tag{18a}$$

$$\beta = \frac{X^{\frac{1}{q}}}{A^{\frac{1}{q}}\alpha^{\frac{p}{q}}\gamma^{\frac{r}{q}}\delta^{\frac{s}{q}}} \tag{18b}$$

$$\gamma = \frac{X^{\frac{1}{r}}}{A^{\frac{1}{r}}\alpha^{\frac{p}{r}}\beta^{\frac{q}{r}}\delta^{\frac{s}{r}}} \tag{18c}$$

$$\delta = \frac{X^{\frac{1}{s}}}{A^{\frac{1}{s}}\alpha^{\frac{p}{s}}\beta^{\frac{q}{s}}\gamma^{\frac{r}{s}}}. \tag{18d}$$

From (17) and (18d) we get,

$$\alpha = \frac{zp\delta}{ws} = \frac{zpX^{\frac{1}{s}}}{wsA^{\frac{1}{s}}\alpha^{\frac{p}{s}}\beta^{\frac{q}{s}}\gamma^{\frac{r}{s}}} \qquad \Rightarrow \alpha^{\left(\frac{s+p}{s}\right)} = \frac{zpX^{\frac{1}{s}}}{wsA^{\frac{1}{s}}\beta^{\frac{q}{s}}\gamma^{\frac{r}{s}}}$$

$$\alpha = \left(\frac{zpX^{\frac{1}{s}}}{wsA^{\frac{1}{s}}\beta^{\frac{q}{s}}\gamma^{\frac{r}{s}}}\right)^{\frac{s}{s+p}} = \frac{z^{\frac{s}{s+p}}p^{\frac{s}{s+p}}X^{\frac{1}{s+p}}}{w^{\frac{s}{s+p}}s^{\frac{s}{s+p}}A^{\frac{1}{s+p}}\beta^{\frac{q}{s+p}}\gamma^{\frac{r}{s+p}}}. \qquad (19)$$

Similarly, from (17) we get,

$$\gamma = \frac{wr\alpha}{py} \Rightarrow \frac{1}{\gamma^{\frac{r}{s+q}}} = \frac{y^{\frac{r}{s+q}}p^{\frac{r}{s+q}}}{w^{\frac{r}{s+q}}r^{\frac{r}{s+q}}\alpha^{\frac{r}{s+q}}}. \qquad (20)$$

Also from (17) we get,

$$\gamma = \frac{xr\beta}{yq} \Rightarrow \frac{1}{\gamma^{\frac{r}{s+p}}} = \frac{y^{\frac{r}{s+p}}q^{\frac{r}{s+p}}}{x^{\frac{r}{s+p}}r^{\frac{r}{s+p}}\beta^{\frac{r}{s+p}}}. \qquad (21)$$

Similarly from (17) and (18d) we get,

$$\beta = \frac{qz\delta}{sx} = \frac{qzX^{\frac{1}{s}}}{sxA^{\frac{1}{s}}\alpha^{\frac{p}{s}}\beta^{\frac{q}{s}}\gamma^{\frac{r}{s}}} \Rightarrow \beta^{\frac{s+q}{s}} = \frac{qzX^{\frac{1}{s}}}{sxA^{\frac{1}{s}}\alpha^{\frac{p}{s}}\gamma^{\frac{r}{s}}}$$

$$\beta = \frac{z^{\frac{s}{s+q}}q^{\frac{s}{s+q}}X^{\frac{1}{s+q}}}{x^{\frac{s}{s+q}}s^{\frac{s}{s+q}}A^{\frac{1}{s+q}}\alpha^{\frac{p}{s+q}}\gamma^{\frac{r}{s+q}}}. \qquad (22)$$

From (19) and (21) we get,

$$\alpha = \frac{z^{\frac{s}{s+p}}p^{\frac{s}{s+p}}X^{\frac{1}{s+p}}}{w^{\frac{s}{s+p}}s^{\frac{s}{s+p}}A^{\frac{1}{s+p}}\beta^{\frac{q}{s+p}}}.\frac{q^{\frac{r}{s+p}}y^{\frac{r}{s+p}}}{x^{\frac{r}{s+p}}r^{\frac{r}{s+p}}\beta^{\frac{r}{s+p}}} = \frac{y^{\frac{r}{s+p}}z^{\frac{s}{s+p}}p^{\frac{s}{s+p}}q^{\frac{r}{s+q}}X^{\frac{1}{s+p}}}{w^{\frac{s}{s+p}}x^{\frac{r}{s+p}}r^{\frac{r}{s+p}}s^{\frac{s}{s+p}}A^{\frac{1}{s+p}}\beta^{\frac{q+r}{s+p}}}. \qquad (23)$$

From (20) and (22) we get,

$$\beta = \frac{q^{\frac{s}{s+q}}z^{\frac{s}{s+q}}X^{\frac{1}{s+q}}}{s^{\frac{s}{s+q}}x^{\frac{s}{s+q}}A^{\frac{1}{s+q}}\alpha^{\frac{p}{s+q}}}.\frac{y^{\frac{r}{s+q}}p^{\frac{r}{s+q}}}{w^{\frac{r}{s+q}}r^{\frac{r}{s+q}}\alpha^{\frac{r}{s+q}}} = \frac{y^{\frac{r}{s+q}}z^{\frac{s}{s+q}}p^{\frac{r}{s+q}}q^{\frac{s}{s+q}}X^{\frac{1}{s+q}}}{w^{\frac{r}{s+q}}x^{\frac{s}{s+q}}r^{\frac{r}{s+q}}s^{\frac{s}{s+q}}\alpha^{\frac{p+r}{s+q}}A^{\frac{1}{s+q}}}. \qquad (24)$$

$$\frac{1}{\beta^{\frac{q+r}{s+p}}} = \left(\frac{w^{\frac{r}{s+q}}x^{\frac{s}{s+q}}r^{\frac{r}{s+q}}s^{\frac{s}{s+q}}\alpha^{\frac{p+r}{s+q}}A^{\frac{1}{s+q}}}{y^{\frac{r}{s+q}}z^{\frac{s}{s+q}}p^{\frac{r}{s+q}}q^{\frac{s}{s+q}}X^{\frac{1}{s+q}}}\right)^{\frac{q+r}{s+p}}$$

$$= \frac{w^{\frac{r(q+r)}{(s+p)(s+q)}} x^{\frac{s(q+r)}{(s+p)(s+q)}} r^{\frac{r(q+r)}{(s+p)(s+q)}} s^{\frac{s(q+r)}{(s+p)(s+q)}} \alpha^{\frac{(p+r)(q+r)}{(s+p)(s+q)}} A^{\frac{q+r}{(s+p)(s+q)}}}{y^{\frac{r(q+r)}{(s+p)(s+q)}} z^{\frac{s(q+r)}{(s+p)(s+q)}} p^{\frac{r(q+r)}{(s+p)(s+q)}} q^{\frac{s(q+r)}{(s+p)(s+q)}} X^{\frac{q+r}{(s+p)(s+q)}}}$$

$$\frac{1}{\beta^{\frac{q+r}{s+p}}} = \frac{w^{\frac{b}{a}} x^{\frac{d}{a}} r^{\frac{b}{a}} s^{\frac{d}{a}} \alpha^{\frac{c}{a}} A^{\frac{b}{ar}}}{y^{\frac{b}{a}} z^{\frac{d}{a}} p^{\frac{b}{a}} q^{\frac{d}{a}} X^{\frac{b}{ar}}}, \tag{25}$$

where $(s+p)(s+q) = a$, $r(q+r) = b$,

$(p+r)(q+r) = c$, $s(q+r) = d$ and $q+r = b/r$. (26)

From (23) and (25) we get,

$$\alpha = \frac{y^{\frac{r}{s+p}} z^{\frac{s}{s+p}} p^{\frac{s}{s+p}} q^{\frac{r}{s+q}} X^{\frac{1}{s+p}}}{w^{\frac{s}{s+p}} x^{\frac{r}{s+p}} r^{\frac{r}{s+p}} s^{\frac{s}{s+p}} A^{\frac{1}{s+p}}} \cdot \frac{w^{\frac{b}{a}} x^{\frac{d}{a}} r^{\frac{b}{a}} s^{\frac{d}{a}} \alpha^{\frac{c}{a}} A^{\frac{b}{ar}}}{y^{\frac{b}{a}} z^{\frac{d}{a}} p^{\frac{b}{a}} q^{\frac{d}{a}} X^{\frac{b}{ar}}} = \frac{w^{\frac{f}{a}} x^{\frac{f}{a}} r^{\frac{re}{a}} s^{\frac{se}{a}} \alpha^{\frac{c}{a}} A^{\frac{e}{a}}}{y^{\frac{re}{a}} z^{\frac{se}{a}} p^{\frac{f}{a}} q^{\frac{g}{a}} X^{\frac{e}{a}}}$$

where $r - s = e, f = (q - r - s)e,$ and $g = qs - pr$.

$$\alpha = \left[\frac{w^{\frac{f}{a}} x^{\frac{f}{a}} r^{\frac{re}{a}} s^{\frac{se}{a}} A^{\frac{e}{a}}}{y^{\frac{re}{a}} z^{\frac{se}{a}} p^{\frac{f}{a}} q^{\frac{g}{a}} X^{\frac{e}{a}}} \right]^{\frac{a}{a-c}} = \left[\frac{w^f x^f r^{re} s^{se} A^e}{y^{re} z^{se} p^f q^g X^e} \right]^{\frac{1}{a-c}}. \tag{27}$$

From (23) we get,

$$\frac{1}{\alpha^{\frac{p+r}{s+q}}} = \left[\frac{w^{\frac{s}{s+p}} x^{\frac{r}{s+p}} r^{\frac{r}{s+p}} s^{\frac{s}{s+p}} A^{\frac{1}{s+p}} \beta^{\frac{q+r}{s+p}}}{y^{\frac{r}{s+p}} z^{\frac{s}{s+p}} p^{\frac{s}{s+p}} q^{\frac{r}{s+q}} X^{\frac{1}{s+p}}} \right]^{\frac{p+r}{s+q}}$$

$$= \frac{w^{\frac{s(p+r)}{(s+p)(s+q)}} x^{\frac{r(p+r)}{(s+p)(s+q)}} r^{\frac{r(p+r)}{(s+p)(s+q)}} s^{\frac{s(p+r)}{(s+p)(s+q)}} \beta^{\frac{(q+r)(p+r)}{(s+p)(s+q)}} A^{\frac{p+r}{(s+p)(s+q)}}}{y^{\frac{r(p+r)}{(s+p)(s+q)}} z^{\frac{s(p+r)}{(s+p)(s+q)}} p^{\frac{s(p+r)}{(s+p)(s+q)}} q^{\frac{r(p+r)}{(s+p)(s+q)}} X^{\frac{p+r}{(s+p)(s+q)}}}$$

$$\frac{1}{\alpha^{\frac{p+r}{s+q}}} = \frac{w^{\frac{sh}{a}} x^{\frac{rh}{a}} r^{\frac{rh}{a}} s^{\frac{sh}{a}} \beta^{\frac{c}{a}} A^{\frac{h}{a}}}{y^{\frac{rh}{a}} z^{\frac{sh}{a}} p^{\frac{sh}{a}} q^{\frac{rh}{a}} X^{\frac{h}{a}}} \tag{28}$$

where $p + r = h$. Using (28) in (24) we can write,

$$\beta = \frac{y^{\frac{r}{s+q}} z^{\frac{s}{s+q}} p^{\frac{r}{s+q}} q^{\frac{s}{s+q}} X^{\frac{1}{s+q}}}{w^{\frac{r}{s+q}} x^{\frac{s}{s+q}} r^{\frac{r}{s+q}} s^{\frac{s}{s+q}} \alpha^{\frac{p+r}{s+q}} A^{\frac{1}{s+q}}} \cdot \frac{w^{\frac{sh}{a}} x^{\frac{rh}{a}} r^{\frac{rh}{a}} s^{\frac{sh}{a}} \beta^{\frac{c}{a}} A^{\frac{h}{a}}}{y^{\frac{rh}{a}} z^{\frac{sh}{a}} p^{\frac{sh}{a}} q^{\frac{rh}{a}} X^{\frac{h}{a}}}$$

$$\beta = \left[\frac{x^{\frac{i}{a}} r^{\frac{re}{a}} s^{\frac{se}{a}} p^{\frac{pe}{a}} A^{\frac{e}{a}}}{y^{\frac{re}{a}} z^{\frac{se}{a}} w^{\frac{pe}{a}} q^{\frac{j}{a}} X^{\frac{e}{a}}} \right]^{\frac{a}{a-c}}$$

where $i = (p - r - s)e$, and $j = (p + r + s)e$.

$$\beta = \left[\frac{x^{i} r^{re} s^{se} p^{pe} A^{e}}{y^{re} z^{se} w^{pe} q^{j} X^{e}} \right]^{\frac{1}{a-c}}. \qquad (29)$$

Using (27) in (17) we can write,

$$\gamma = \left[\frac{w^{a-c+f} x^{f} r^{a-c+re} s^{se} A^{e}}{y^{a-c+re} z^{se} p^{a-c+f} q^{g} X^{e}} \right]^{\frac{1}{a-c}} \qquad (30)$$

$$\delta = \left[\frac{w^{a-c+f} x^{f} r^{re} s^{a-c+se} A^{e}}{y^{re} z^{a-c+se} p^{a-c+f} q^{g} X^{e}} \right]^{\frac{1}{a-c}}. \qquad (31)$$

From (16) we get,

$$\theta = \frac{w}{Ap\alpha^{p-1}\beta^{q}\gamma^{r}\delta^{s}} = \frac{w}{Ap}\alpha^{1-p}\beta^{-q}\gamma^{-r}\delta^{-s}$$

$$\theta = \frac{w}{Ap}\left[\frac{w^{f} x^{f} r^{re} s^{se} A^{e}}{y^{re} z^{se} p^{f} q^{g} X^{e}} . \frac{x^{i} r^{re} s^{se} p^{pe} A^{e}}{y^{re} z^{se} w^{pe} q^{j} X^{e}} . \frac{w^{a-c+f} x^{f} r^{a-c+re} s^{se} A^{e}}{y^{a-c+re} z^{se} p^{a-c+f} q^{g} X^{e}} . \frac{w^{a-c+f} x^{f} r^{re} s^{a-c+se} A^{e}}{y^{re} z^{a-c+se} p^{a-c+f} q^{g} X^{e}} \right]^{\frac{1}{a-c}}$$

$$\theta = \left[\frac{w^{2a-2c-pe+3f} x^{3f+i} r^{a-c+4re} s^{a-c+4se} A^{c-a+3e}}{y^{a-c+4re} z^{a-c+4se} p^{2a-2c-pe+3f} q^{a-c+3g+j} X^{3e}} \right]^{\frac{1}{a-c}}. \qquad (32)$$

Now we can write minimum cost of the firm by using (34) (36) (38) and (40) in (1) as,

$$K = \left[\frac{w^{a-c+f} x^{f} r^{re} s^{se} A^{e}}{y^{re} z^{se} p^{f} q^{g} X^{e}} \right]^{\frac{1}{a-c}} + \left[\frac{x^{a-c+i} r^{re} s^{se} p^{pe} A^{e}}{y^{re} z^{se} w^{pe} q^{j} X^{e}} \right]^{\frac{1}{a-c}} + \left[\frac{w^{a-c+f} x^{f} r^{a-c+re} s^{se} A^{e}}{y^{re} z^{se} p^{a-c+f} q^{g} X^{e}} \right]^{\frac{1}{a-c}}$$

$$+ \left[\frac{w^{a-c+f} x^{f} r^{re} s^{a-c+se} A^{e}}{y^{re} z^{se} p^{a-c+f} q^{g} X^{e}} \right]^{\frac{1}{a-c}}. \qquad (33)$$

7. Sensitivity Analysis

Sensitivity analysis examines about the change of economic predictions. It is used within specific boundaries that depend on more input variables, such as capital, labor, principal raw materials, and other inputs. It deals with uncertainty, but tries to reduce the bad schemes [Herstein, 1953; Gowland, 1983; Dixit, 1990]. To minimize cost we consider the determinant of the 5×5 Hessian matrix [Mohajan, 2021a, b],

$$|H| = \begin{vmatrix} 0 & -X_\alpha & -X_\beta & -X_\gamma & -X_\delta \\ -X_\alpha & L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha\gamma} & L_{\alpha\delta} \\ -X_\beta & L_{\beta\alpha} & L_{\beta\beta} & L_{\beta\gamma} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\beta} & L_{\gamma\gamma} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\beta} & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} \tag{34}$$

$$= X_\alpha \begin{vmatrix} -X_\alpha & L_{\alpha\beta} & L_{\alpha\gamma} & L_{\alpha\delta} \\ -X_\beta & L_{\beta\beta} & L_{\beta\gamma} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\beta} & L_{\gamma\gamma} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\beta} & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} - X_\beta \begin{vmatrix} -X_\alpha & L_{\alpha\alpha} & L_{\alpha\gamma} & L_{\alpha\delta} \\ -X_\beta & L_{\beta\alpha} & L_{\beta\gamma} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\gamma} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix}$$

$$+ X_\gamma \begin{vmatrix} -X_\alpha & L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha\delta} \\ -X_\beta & L_{\beta\alpha} & L_{\beta\beta} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\beta} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\beta} & L_{\delta\delta} \end{vmatrix} - X_\delta \begin{vmatrix} -X_\alpha & L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha\gamma} \\ -X_\beta & L_{\beta\alpha} & L_{\beta\beta} & L_{\beta\gamma} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\beta} & L_{\gamma\gamma} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\beta} & L_{\delta\gamma} \end{vmatrix}$$

$$= X_\alpha \left\{ -X_\alpha \begin{vmatrix} L_{\beta\beta} & L_{\beta\gamma} & L_{\beta\delta} \\ L_{\gamma\beta} & L_{\gamma\gamma} & L_{\gamma\delta} \\ L_{\delta\beta} & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\beta} \begin{vmatrix} -X_\beta & L_{\beta\gamma} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\gamma} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} + L_{\alpha\gamma} \begin{vmatrix} -X_\beta & L_{\beta\beta} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\beta} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\beta} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\delta} \begin{vmatrix} -X_\beta & L_{\beta\beta} & L_{\beta\gamma} \\ -X_\gamma & L_{\gamma\beta} & L_{\gamma\gamma} \\ -X_\delta & L_{\delta\beta} & L_{\delta\gamma} \end{vmatrix} \right\}$$

$$- X_\beta \left\{ -X_\alpha \begin{vmatrix} L_{\beta\alpha} & L_{\beta\gamma} & L_{\beta\delta} \\ L_{\gamma\alpha} & L_{\gamma\gamma} & L_{\gamma\delta} \\ L_{\delta\alpha} & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\alpha} \begin{vmatrix} -X_\beta & L_{\beta\gamma} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\gamma} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\gamma} & L_{\delta\delta} \end{vmatrix} + L_{\alpha\gamma} \begin{vmatrix} -X_\beta & L_{\beta\alpha} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\delta} \begin{vmatrix} -X_\beta & L_{\beta\alpha} & L_{\beta\gamma} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\gamma} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\gamma} \end{vmatrix} \right\}$$

$$+ X_\gamma \left\{ -X_\alpha \begin{vmatrix} L_{\beta\alpha} & L_{\beta\beta} & L_{\beta\delta} \\ L_{\gamma\alpha} & L_{\gamma\beta} & L_{\gamma\delta} \\ L_{\delta\alpha} & L_{\delta\beta} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\alpha} \begin{vmatrix} -X_\beta & L_{\beta\beta} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\beta} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\beta} & L_{\delta\delta} \end{vmatrix} + L_{\alpha\beta} \begin{vmatrix} -X_\beta & L_{\beta\alpha} & L_{\beta\delta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\delta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\delta} \end{vmatrix} - L_{\alpha\delta} \begin{vmatrix} -X_\beta & L_{\beta\alpha} & L_{\beta\beta} \\ -X_\gamma & L_{\gamma\alpha} & L_{\gamma\beta} \\ -X_\delta & L_{\delta\alpha} & L_{\delta\beta} \end{vmatrix} \right\}$$

$$
-X_{\delta}\left\{-X_{\alpha}\begin{vmatrix} L_{\beta\alpha} & L_{\beta\beta} & L_{\beta\gamma} \\ L_{\gamma\alpha} & L_{\gamma\beta} & L_{\gamma\gamma} \\ L_{\delta\alpha} & L_{\delta\beta} & L_{\delta\gamma} \end{vmatrix} - L_{\alpha\alpha}\begin{vmatrix} -X_{\beta} & L_{\beta\beta} & L_{\beta\gamma} \\ -X_{\gamma} & L_{\gamma\beta} & L_{\gamma\gamma} \\ -X_{\delta} & L_{\delta\beta} & L_{\delta\gamma} \end{vmatrix} + L_{\alpha\beta}\begin{vmatrix} -X_{\beta} & L_{\beta\alpha} & L_{\beta\gamma} \\ -X_{\gamma} & L_{\gamma\alpha} & L_{\gamma\gamma} \\ -X_{\delta} & L_{\delta\alpha} & L_{\delta\gamma} \end{vmatrix} - L_{\alpha\gamma}\begin{vmatrix} -X_{\beta} & L_{\beta\alpha} & L_{\beta\beta} \\ -X_{\gamma} & L_{\gamma\alpha} & L_{\gamma\beta} \\ -X_{\delta} & L_{\delta\alpha} & L_{\delta\beta} \end{vmatrix}\right\}
$$

$$
\begin{aligned}
&= -X_{\alpha}^{2}\left\{L_{\beta\beta}\left(L_{\gamma\gamma}L_{\delta\delta} - L_{\delta\gamma}L_{\gamma\delta}\right) + L_{\beta\gamma}\left(L_{\delta\beta}L_{\gamma\delta} - L_{\gamma\beta}L_{\delta\delta}\right) + L_{\beta\delta}\left(L_{\gamma\beta}L_{\delta\gamma} - L_{\delta\beta}L_{\gamma\gamma}\right)\right\} \\
&- X_{\alpha}L_{\alpha\beta}\left\{-X_{\beta}\left(L_{\gamma\gamma}L_{\delta\delta} - L_{\delta\gamma}L_{\gamma\delta}\right) + L_{\beta\gamma}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\gamma} + X_{\delta}L_{\gamma\gamma}\right)\right\} \\
&+ X_{\alpha}L_{\alpha\gamma}\left\{-X_{\beta}\left(L_{\gamma\beta}L_{\delta\delta} - L_{\delta\beta}L_{\gamma\delta}\right) + L_{\beta\beta}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\beta} + X_{\delta}L_{\gamma\beta}\right)\right\} \\
&- X_{\alpha}L_{\alpha\delta}\left\{-X_{\beta}\left(L_{\gamma\beta}L_{\delta\gamma} - L_{\delta\beta}L_{\gamma\gamma}\right) + L_{\beta\beta}\left(-X_{\delta}L_{\gamma\gamma} + X_{\gamma}L_{\delta\gamma}\right) + L_{\beta\gamma}\left(-X_{\gamma}L_{\delta\beta} + X_{\delta}L_{\gamma\beta}\right)\right\} \\
&+ X_{\alpha}X_{\beta}\left\{L_{\beta\alpha}\left(L_{\gamma\gamma}L_{\delta\delta} - L_{\delta\gamma}L_{\gamma\delta}\right) + L_{\beta\gamma}\left(L_{\delta\alpha}L_{\gamma\delta} - L_{\gamma\alpha}L_{\delta\delta}\right) + L_{\beta\delta}\left(L_{\gamma\alpha}L_{\delta\gamma} - L_{\delta\alpha}L_{\gamma\gamma}\right)\right\} \\
&+ X_{\beta}L_{\alpha\alpha}\left\{-X_{\beta}\left(L_{\gamma\gamma}L_{\delta\delta} - L_{\delta\gamma}L_{\gamma\delta}\right) + L_{\beta\gamma}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\gamma} + X_{\delta}L_{\gamma\gamma}\right)\right\} \\
&- X_{\beta}L_{\alpha\gamma}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\delta} - L_{\delta\alpha}L_{\gamma\delta}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\} \\
&+ X_{\beta}L_{\alpha\delta}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\gamma} - L_{\delta\alpha}L_{\gamma\gamma}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\gamma} + X_{\gamma}L_{\delta\gamma}\right) + L_{\beta\gamma}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\} \\
&- X_{\gamma}X_{\alpha}\left\{L_{\beta\alpha}\left(L_{\gamma\beta}L_{\delta\delta} - L_{\delta\beta}L_{\gamma\delta}\right) + L_{\beta\beta}\left(L_{\delta\alpha}L_{\gamma\delta} - L_{\gamma\alpha}L_{\delta\delta}\right) + L_{\beta\delta}\left(L_{\gamma\alpha}L_{\delta\beta} - L_{\delta\alpha}L_{\gamma\beta}\right)\right\} \\
&- X_{\gamma}L_{\alpha\alpha}\left\{-X_{\beta}\left(L_{\gamma\beta}L_{\delta\delta} - L_{\delta\beta}L_{\gamma\delta}\right) + L_{\beta\beta}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\beta} + X_{\delta}L_{\gamma\beta}\right)\right\} \\
&+ X_{\gamma}L_{\alpha\beta}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\delta} - L_{\delta\alpha}L_{\gamma\delta}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\delta} + X_{\gamma}L_{\delta\delta}\right) + L_{\beta\delta}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\} \\
&- X_{\gamma}L_{\alpha\delta}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\beta} - L_{\delta\alpha}L_{\gamma\beta}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\beta} + X_{\gamma}L_{\delta\beta}\right) + L_{\beta\beta}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\} \\
&+ X_{\alpha}X_{\delta}\left\{L_{\beta\alpha}\left(L_{\gamma\beta}L_{\delta\gamma} - L_{\delta\beta}L_{\gamma\gamma}\right) + L_{\beta\beta}\left(L_{\delta\alpha}L_{\gamma\gamma} - L_{\gamma\alpha}L_{\delta\gamma}\right) + L_{\beta\gamma}\left(L_{\gamma\alpha}L_{\delta\beta} - L_{\delta\alpha}L_{\gamma\beta}\right)\right\} \\
&+ X_{\delta}L_{\alpha\alpha}\left\{-X_{\beta}\left(L_{\gamma\beta}L_{\delta\gamma} - L_{\delta\beta}L_{\gamma\gamma}\right) + L_{\beta\beta}\left(-X_{\delta}L_{\gamma\gamma} + X_{\gamma}L_{\delta\gamma}\right) + L_{\beta\gamma}\left(-X_{\gamma}L_{\delta\beta} + X_{\delta}L_{\gamma\beta}\right)\right\} \\
&- X_{\delta}L_{\alpha\beta}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\gamma} - L_{\delta\alpha}L_{\gamma\gamma}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\gamma} + X_{\gamma}L_{\delta\gamma}\right) + L_{\beta\gamma}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\} \\
&+ X_{\delta}L_{\alpha\gamma}\left\{-X_{\beta}\left(L_{\gamma\alpha}L_{\delta\beta} - L_{\delta\alpha}L_{\gamma\beta}\right) + L_{\beta\alpha}\left(-X_{\delta}L_{\gamma\beta} + X_{\gamma}L_{\delta\beta}\right) + L_{\beta\beta}\left(-X_{\gamma}L_{\delta\alpha} + X_{\delta}L_{\gamma\alpha}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -X_{\alpha}^{2}L_{\beta\beta}L_{\gamma\gamma}L_{\delta\delta} + X_{\alpha}^{2}L_{\beta\beta}L_{\delta\gamma}L_{\gamma\delta} - X_{\alpha}^{2}L_{\beta\gamma}L_{\delta\beta}L_{\gamma\delta} + X_{\alpha}^{2}L_{\beta\gamma}L_{\gamma\beta}L_{\delta\delta} - X_{\alpha}^{2}L_{\beta\delta}L_{\gamma\beta}L_{\delta\gamma} \\
&+ X_{\alpha}^{2}L_{\beta\delta}L_{\delta\beta}L_{\gamma\gamma} + X_{\alpha}X_{\beta}L_{\alpha\beta}L_{\gamma\gamma}L_{\delta\delta} - X_{\alpha}X_{\beta}L_{\alpha\beta}L_{\delta\gamma}L_{\gamma\delta} + X_{\alpha}X_{\delta}L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\delta} - X_{\alpha}X_{\gamma}L_{\alpha\beta}L_{\beta\gamma}L_{\delta\delta} \\
&+ X_{\alpha}X_{\gamma}L_{\alpha\beta}L_{\beta\delta}L_{\delta\gamma} - X_{\alpha}X_{\delta}L_{\alpha\beta}L_{\beta\delta}L_{\gamma\gamma} - X_{\alpha}X_{\beta}L_{\alpha\gamma}L_{\gamma\beta}L_{\delta\delta} + X_{\alpha}X_{\beta}L_{\alpha\gamma}L_{\delta\beta}L_{\gamma\delta} - X_{\alpha}X_{\delta}L_{\alpha\gamma}L_{\beta\beta}L_{\gamma\delta} \\
&+ X_{\alpha}X_{\gamma}L_{\alpha\gamma}L_{\beta\beta}L_{\delta\delta} - X_{\alpha}X_{\gamma}L_{\alpha\gamma}L_{\beta\delta}L_{\delta\beta} + X_{\alpha}X_{\delta}L_{\alpha\gamma}L_{\beta\delta}L_{\gamma\beta} + X_{\alpha}X_{\beta}L_{\alpha\delta}L_{\gamma\beta}L_{\delta\gamma} - X_{\alpha}X_{\beta}L_{\alpha\delta}L_{\delta\beta}L_{\gamma\gamma} \\
&+ X_{\alpha}X_{\delta}L_{\alpha\delta}L_{\beta\beta}L_{\gamma\gamma} - X_{\alpha}X_{\gamma}L_{\alpha\delta}L_{\beta\beta}L_{\delta\gamma} + X_{\alpha}X_{\gamma}L_{\alpha\delta}L_{\beta\gamma}L_{\delta\beta} - X_{\alpha}X_{\delta}L_{\alpha\delta}L_{\beta\gamma}L_{\gamma\beta} + X_{\alpha}X_{\beta}L_{\beta\alpha}L_{\gamma\gamma}L_{\delta\delta} \\
&- X_{\alpha}X_{\beta}L_{\beta\alpha}L_{\delta\gamma}L_{\gamma\delta} + X_{\alpha}X_{\beta}L_{\beta\gamma}L_{\delta\alpha}L_{\gamma\delta} - X_{\alpha}X_{\beta}L_{\beta\gamma}L_{\gamma\alpha}L_{\delta\delta} + X_{\alpha}X_{\beta}L_{\beta\delta}L_{\gamma\alpha}L_{\delta\gamma} - X_{\alpha}X_{\beta}L_{\beta\delta}L_{\delta\alpha}L_{\gamma\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&- X_{\beta}^{2}L_{\alpha\alpha}L_{\gamma\gamma}L_{\delta\delta} + X_{\beta}^{2}L_{\alpha\alpha}L_{\delta\gamma}L_{\gamma\delta} - X_{\beta}X_{\delta}L_{\alpha\alpha}L_{\beta\gamma}L_{\gamma\delta} + X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\beta\gamma}L_{\delta\delta} - X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\beta\delta}L_{\delta\gamma} \\
&+ X_{\beta}X_{\delta}L_{\alpha\alpha}L_{\beta\delta}L_{\gamma\gamma} + X_{\beta}^{2}L_{\alpha\gamma}L_{\gamma\alpha}L_{\delta\delta} - X_{\beta}^{2}L_{\alpha\gamma}L_{\delta\alpha}L_{\gamma\delta} + X_{\beta}X_{\delta}L_{\alpha\gamma}L_{\beta\alpha}L_{\gamma\delta} - X_{\beta}X_{\gamma}L_{\alpha\gamma}L_{\beta\alpha}L_{\delta\delta} \\
&+ X_{\beta}X_{\gamma}L_{\alpha\gamma}L_{\beta\delta}L_{\delta\alpha} - X_{\beta}X_{\delta}L_{\alpha\gamma}L_{\beta\delta}L_{\gamma\alpha} - X_{\beta}^{2}L_{\alpha\delta}L_{\gamma\alpha}L_{\delta\gamma} + X_{\beta}^{2}L_{\alpha\delta}L_{\delta\alpha}L_{\gamma\gamma} - X_{\beta}X_{\delta}L_{\alpha\delta}L_{\beta\alpha}L_{\gamma\gamma}
\end{aligned}
$$

$$+X_{\beta}X_{\gamma}L_{\alpha\delta}L_{\beta\alpha}L_{\delta\gamma}-X_{\beta}X_{\gamma}L_{\alpha\delta}L_{\beta\gamma}L_{\delta\alpha}+X_{\beta}X_{\delta}L_{\alpha\delta}L_{\beta\gamma}L_{\gamma\alpha}-X_{\gamma}X_{\alpha}L_{\beta\alpha}L_{\gamma\beta}L_{\delta\delta}+X_{\gamma}X_{\alpha}L_{\beta\alpha}L_{\delta\beta}L_{\gamma\delta}$$
$$-X_{\gamma}X_{\alpha}L_{\beta\beta}L_{\delta\alpha}L_{\gamma\delta}+X_{\gamma}X_{\alpha}L_{\beta\beta}L_{\gamma\alpha}L_{\delta\delta}-X_{\gamma}X_{\alpha}L_{\beta\delta}L_{\gamma\alpha}L_{\delta\beta}+X_{\gamma}X_{\alpha}L_{\beta\delta}L_{\delta\alpha}L_{\gamma\beta}+X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\gamma\beta}L_{\delta\delta}$$
$$-X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\delta\beta}L_{\gamma\delta}+X_{\gamma}X_{\delta}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\delta}-X_{\gamma}^{2}L_{\alpha\alpha}L_{\beta\beta}L_{\delta\delta}+X_{\gamma}^{2}L_{\alpha\alpha}L_{\beta\delta}L_{\delta\beta}-X_{\gamma}X_{\delta}L_{\alpha\alpha}L_{\beta\delta}L_{\gamma\beta}$$
$$-X_{\beta}X_{\gamma}L_{\alpha\beta}L_{\gamma\alpha}L_{\delta\delta}+X_{\beta}X_{\gamma}L_{\alpha\beta}L_{\delta\alpha}L_{\gamma\delta}-X_{\gamma}X_{\delta}L_{\alpha\beta}L_{\beta\alpha}L_{\gamma\delta}+X_{\gamma}^{2}L_{\alpha\beta}L_{\beta\alpha}L_{\delta\delta}-X_{\gamma}^{2}L_{\alpha\beta}L_{\beta\delta}L_{\delta\alpha}$$
$$+X_{\gamma}X_{\delta}L_{\alpha\beta}L_{\beta\delta}L_{\gamma\alpha}+X_{\gamma}X_{\beta}L_{\alpha\delta}L_{\gamma\alpha}L_{\delta\beta}-X_{\gamma}X_{\beta}L_{\alpha\delta}L_{\delta\alpha}L_{\gamma\beta}+X_{\gamma}X_{\delta}L_{\alpha\delta}L_{\beta\alpha}L_{\gamma\beta}-X_{\gamma}^{2}L_{\alpha\delta}L_{\beta\alpha}L_{\delta\beta}$$
$$+X_{\gamma}^{2}L_{\alpha\delta}L_{\beta\beta}L_{\delta\alpha}-X_{\gamma}X_{\delta}L_{\alpha\delta}L_{\beta\beta}L_{\gamma\alpha}+X_{\alpha}X_{\delta}L_{\beta\alpha}L_{\gamma\beta}L_{\delta\gamma}-X_{\alpha}X_{\delta}L_{\beta\alpha}L_{\delta\beta}L_{\gamma\gamma}+X_{\alpha}X_{\delta}L_{\beta\beta}L_{\delta\alpha}L_{\gamma\gamma}$$
$$-X_{\alpha}X_{\delta}L_{\beta\beta}L_{\gamma\alpha}L_{\delta\gamma}+X_{\alpha}X_{\delta}L_{\beta\gamma}L_{\gamma\alpha}L_{\delta\beta}-X_{\alpha}X_{\delta}L_{\beta\gamma}L_{\delta\alpha}L_{\gamma\beta}-X_{\delta}X_{\beta}L_{\alpha\alpha}L_{\gamma\beta}L_{\delta\gamma}+X_{\delta}X_{\beta}L_{\alpha\alpha}L_{\delta\beta}L_{\gamma\gamma}$$
$$-X_{\delta}^{2}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\gamma}+X_{\delta}X_{\gamma}L_{\alpha\alpha}L_{\beta\beta}L_{\delta\gamma}-X_{\delta}X_{\gamma}L_{\alpha\alpha}L_{\beta\gamma}L_{\delta\beta}+X_{\delta}^{2}L_{\alpha\alpha}L_{\beta\gamma}L_{\gamma\beta}+X_{\delta}X_{\beta}L_{\alpha\beta}L_{\gamma\alpha}L_{\delta\gamma}$$
$$-X_{\delta}X_{\beta}L_{\alpha\beta}L_{\delta\alpha}L_{\gamma\gamma}+X_{\delta}^{2}L_{\alpha\beta}L_{\beta\alpha}L_{\gamma\gamma}-X_{\delta}X_{\gamma}L_{\alpha\beta}L_{\beta\alpha}L_{\delta\gamma}+X_{\delta}X_{\gamma}L_{\alpha\beta}L_{\beta\gamma}L_{\delta\alpha}-X_{\delta}^{2}L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}$$
$$-X_{\delta}X_{\beta}L_{\alpha\gamma}L_{\gamma\alpha}L_{\delta\beta}+X_{\delta}X_{\beta}L_{\alpha\gamma}L_{\delta\alpha}L_{\gamma\beta}-X_{\delta}^{2}L_{\alpha\gamma}L_{\beta\alpha}L_{\gamma\beta}+X_{\delta}X_{\gamma}L_{\alpha\gamma}L_{\beta\alpha}L_{\delta\beta}-X_{\delta}X_{\gamma}L_{\alpha\gamma}L_{\beta\beta}L_{\delta\alpha}$$
$$+X_{\delta}^{2}L_{\alpha\gamma}L_{\beta\beta}L_{\gamma\alpha}\,.$$

$$=-X_{\alpha}^{2}L_{\beta\beta}L_{\gamma\gamma}L_{\delta\delta}+X_{\alpha}^{2}L_{\beta\beta}L_{\gamma\delta}^{2}-X_{\alpha}^{2}L_{\beta\gamma}L_{\delta\beta}L_{\gamma\delta}+X_{\alpha}^{2}L_{\beta\gamma}^{2}L_{\delta\delta}-X_{\alpha}^{2}L_{\beta\delta}L_{\gamma\beta}L_{\delta\gamma}+X_{\alpha}^{2}L_{\beta\delta}^{2}L_{\gamma\gamma}$$

$$+X_{\alpha}X_{\beta}L_{\alpha\beta}L_{\gamma\gamma}L_{\delta\delta}-X_{\alpha}X_{\beta}L_{\alpha\beta}L_{\gamma\delta}^{2}+X_{\alpha}X_{\delta}L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\delta}-X_{\alpha}X_{\gamma}L_{\alpha\beta}L_{\beta\gamma}L_{\delta\delta}+X_{\alpha}X_{\gamma}L_{\alpha\beta}L_{\beta\delta}L_{\delta\gamma}$$
$$-X_{\alpha}X_{\delta}L_{\alpha\beta}L_{\beta\delta}L_{\gamma\gamma}-X_{\alpha}X_{\beta}L_{\alpha\gamma}L_{\gamma\beta}L_{\delta\delta}+X_{\alpha}X_{\beta}L_{\alpha\gamma}L_{\delta\beta}L_{\gamma\delta}-X_{\alpha}X_{\delta}L_{\alpha\gamma}L_{\beta\beta}L_{\gamma\delta}+X_{\alpha}X_{\gamma}L_{\alpha\gamma}L_{\beta\beta}L_{\delta\delta}$$
$$-X_{\alpha}X_{\gamma}L_{\alpha\gamma}L_{\beta\delta}^{2}+X_{\alpha}X_{\delta}L_{\alpha\gamma}L_{\beta\delta}L_{\gamma\beta}+X_{\alpha}X_{\beta}L_{\alpha\delta}L_{\gamma\beta}L_{\delta\gamma}-X_{\alpha}X_{\beta}L_{\alpha\delta}L_{\delta\beta}L_{\gamma\gamma}+X_{\alpha}X_{\delta}L_{\alpha\delta}L_{\beta\beta}L_{\gamma\gamma}$$
$$-X_{\alpha}X_{\gamma}L_{\alpha\delta}L_{\beta\beta}L_{\delta\gamma}+X_{\alpha}X_{\gamma}L_{\alpha\delta}L_{\beta\gamma}L_{\delta\beta}-X_{\alpha}X_{\delta}L_{\alpha\delta}L_{\beta\gamma}^{2}+X_{\alpha}X_{\beta}L_{\beta\alpha}L_{\gamma\gamma}L_{\delta\delta}-X_{\alpha}X_{\beta}L_{\beta\alpha}L_{\gamma\delta}^{2}$$
$$+X_{\alpha}X_{\beta}L_{\beta\gamma}L_{\delta\alpha}L_{\gamma\delta}-X_{\alpha}X_{\beta}L_{\beta\gamma}L_{\gamma\alpha}L_{\delta\delta}+X_{\alpha}X_{\beta}L_{\beta\delta}L_{\gamma\alpha}L_{\delta\gamma}-X_{\alpha}X_{\beta}L_{\beta\delta}L_{\delta\alpha}L_{\gamma\gamma}-X_{\beta}^{2}L_{\alpha\alpha}L_{\gamma\gamma}L_{\delta\delta}$$
$$+X_{\beta}^{2}L_{\alpha\alpha}L_{\gamma\delta}^{2}-X_{\beta}X_{\delta}L_{\alpha\alpha}L_{\beta\gamma}L_{\gamma\delta}+X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\beta\gamma}L_{\delta\delta}-X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\beta\delta}L_{\delta\gamma}+X_{\beta}X_{\delta}L_{\alpha\alpha}L_{\beta\delta}L_{\gamma\gamma}$$

$$+X_{\beta}^{2}L_{\alpha\gamma}^{2}L_{\delta\delta}-X_{\beta}^{2}L_{\alpha\gamma}L_{\delta\alpha}L_{\gamma\delta}+X_{\beta}X_{\delta}L_{\alpha\gamma}L_{\beta\alpha}L_{\gamma\delta}-X_{\beta}X_{\gamma}L_{\alpha\gamma}L_{\beta\alpha}L_{\delta\delta}+X_{\beta}X_{\gamma}L_{\alpha\gamma}L_{\beta\delta}L_{\delta\alpha}$$
$$-X_{\beta}X_{\delta}L_{\alpha\gamma}^{2}L_{\beta\delta}-X_{\beta}^{2}L_{\alpha\delta}L_{\gamma\alpha}L_{\delta\gamma}+X_{\beta}^{2}L_{\alpha\delta}^{2}L_{\gamma\gamma}-X_{\beta}X_{\delta}L_{\alpha\delta}L_{\beta\alpha}L_{\gamma\gamma}+X_{\beta}X_{\gamma}L_{\alpha\delta}L_{\beta\alpha}L_{\delta\gamma}$$
$$-X_{\beta}X_{\gamma}L_{\alpha\delta}^{2}L_{\beta\gamma}+X_{\beta}X_{\delta}L_{\alpha\delta}L_{\beta\gamma}L_{\gamma\alpha}-X_{\gamma}X_{\alpha}L_{\beta\alpha}L_{\gamma\beta}L_{\delta\delta}+X_{\gamma}X_{\alpha}L_{\beta\alpha}L_{\delta\beta}L_{\gamma\delta}-X_{\gamma}X_{\alpha}L_{\beta\beta}L_{\delta\alpha}L_{\gamma\delta}$$
$$+X_{\gamma}X_{\alpha}L_{\beta\beta}L_{\gamma\alpha}L_{\delta\delta}-X_{\gamma}X_{\alpha}L_{\beta\delta}^{2}L_{\gamma\alpha}+X_{\gamma}X_{\alpha}L_{\beta\delta}L_{\delta\alpha}L_{\gamma\beta}+X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\gamma\beta}L_{\delta\delta}-X_{\beta}X_{\gamma}L_{\alpha\alpha}L_{\delta\beta}L_{\gamma\delta}$$
$$+X_{\gamma}X_{\delta}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\delta}-X_{\gamma}^{2}L_{\alpha\alpha}L_{\beta\beta}L_{\delta\delta}+X_{\gamma}^{2}L_{\alpha\alpha}L_{\beta\delta}^{2}-X_{\gamma}X_{\delta}L_{\alpha\alpha}L_{\beta\delta}L_{\gamma\beta}-X_{\beta}X_{\gamma}L_{\alpha\beta}L_{\gamma\alpha}L_{\delta\delta}$$
$$+X_{\beta}X_{\gamma}L_{\alpha\beta}L_{\delta\alpha}L_{\gamma\delta}-X_{\gamma}X_{\delta}L_{\alpha\beta}^{2}L_{\gamma\delta}+X_{\gamma}^{2}L_{\alpha\beta}^{2}L_{\delta\delta}-X_{\gamma}^{2}L_{\alpha\beta}L_{\beta\delta}L_{\delta\alpha}+X_{\gamma}X_{\delta}L_{\alpha\beta}L_{\beta\delta}L_{\gamma\alpha}$$
$$+X_{\gamma}X_{\beta}L_{\alpha\delta}L_{\gamma\alpha}L_{\delta\beta}-X_{\gamma}X_{\beta}L_{\alpha\delta}^{2}L_{\gamma\beta}+X_{\gamma}X_{\delta}L_{\alpha\delta}L_{\beta\alpha}L_{\gamma\beta}-X_{\gamma}^{2}L_{\alpha\delta}L_{\beta\alpha}L_{\delta\beta}+X_{\gamma}^{2}L_{\alpha\delta}^{2}L_{\beta\beta}$$
$$-X_{\gamma}X_{\delta}L_{\alpha\delta}L_{\beta\beta}L_{\gamma\alpha}+X_{\alpha}X_{\delta}L_{\beta\alpha}L_{\gamma\beta}L_{\delta\gamma}-X_{\alpha}X_{\delta}L_{\beta\alpha}L_{\delta\beta}L_{\gamma\gamma}+X_{\alpha}X_{\delta}L_{\beta\beta}L_{\delta\alpha}L_{\gamma\gamma}-X_{\alpha}X_{\delta}L_{\beta\beta}L_{\gamma\alpha}L_{\delta\gamma}$$
$$+X_{\alpha}X_{\delta}L_{\beta\gamma}L_{\gamma\alpha}L_{\delta\beta}-X_{\alpha}X_{\delta}L_{\beta\gamma}^{2}L_{\delta\alpha}-X_{\delta}X_{\beta}L_{\alpha\alpha}L_{\gamma\beta}L_{\delta\gamma}+X_{\delta}X_{\beta}L_{\alpha\alpha}L_{\delta\beta}L_{\gamma\gamma}-X_{\delta}^{2}L_{\alpha\alpha}L_{\beta\beta}L_{\gamma\gamma}$$
$$+X_{\delta}X_{\gamma}L_{\alpha\alpha}L_{\beta\beta}L_{\delta\gamma}-X_{\delta}X_{\gamma}L_{\alpha\alpha}L_{\beta\gamma}L_{\delta\beta}+X_{\delta}^{2}L_{\alpha\alpha}L_{\beta\gamma}^{2}+X_{\delta}X_{\beta}L_{\alpha\beta}L_{\gamma\alpha}L_{\delta\gamma}-X_{\delta}X_{\beta}L_{\alpha\beta}L_{\delta\alpha}L_{\gamma\gamma}$$

$$+X_{\delta}^{2}L_{\alpha\beta}^{2}L_{\gamma\gamma}-X_{\delta}X_{\gamma}L_{\alpha\beta}L_{\beta\alpha}L_{\delta\gamma}+X_{\delta}X_{\gamma}L_{\alpha\beta}L_{\beta\gamma}L_{\delta\alpha}-X_{\delta}^{2}L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}-X_{\delta}X_{\beta}L_{\alpha\gamma}^{2}L_{\delta\beta}$$
$$+X_{\delta}X_{\beta}L_{\alpha\gamma}L_{\delta\alpha}L_{\gamma\beta}-X_{\delta}^{2}L_{\alpha\gamma}L_{\beta\alpha}L_{\gamma\beta}+X_{\delta}X_{\gamma}L_{\alpha\gamma}L_{\beta\alpha}L_{\delta\beta}-X_{\delta}X_{\gamma}L_{\alpha\gamma}L_{\beta\beta}L_{\delta\alpha}+X_{\delta}^{2}L_{\alpha\gamma}^{2}L_{\beta\beta}\,. \quad (35)$$

Taking first order partial derivatives of (11) we get,

$$X_\alpha = pA\alpha^{p-1}\beta^q\gamma^r\delta^s,\ X_\beta = qA\alpha^p\beta^{q-1}\gamma^r\delta^s,\ X_\gamma = rA\alpha^p\beta^q\gamma^{r-1}\delta^s,$$

$$X_\delta = sA\alpha^p\beta^q\gamma^r\delta^{s-1}. \tag{36}$$

Taking second order partial derivatives of (13) we get,

$$L_{\alpha\alpha} = \theta Ap(p-1)\alpha^{p-2}\beta^q\gamma^r\delta^s,\ L_{\alpha\beta} = L_{\beta\alpha} = \theta Apq\alpha^{p-1}\beta^{q-1}\gamma^r\delta^s,\ L_{\alpha\gamma} = L_{\gamma\alpha} = \theta Apr\alpha^{p-1}\beta^q\gamma^{r-1}\delta^s,$$

$$L_{\alpha\delta} = L_{\delta\alpha} = \theta Aps\alpha^{p-1}\beta^q\gamma^r\delta^{s-1},\ L_{\beta\beta} = \theta Aq(q-1)\alpha^p\beta^{q-2}\gamma^r\delta^s,\ L_{\beta\gamma} = L_{\gamma\beta} = \theta Aqr\alpha^p\beta^{q-1}\gamma^{r-1}\delta^s,$$

$$L_{\beta\delta} = L_{\delta\beta} = \theta Aqs\alpha^p\beta^{q-1}\gamma^r\delta^{s-1},\ L_{\gamma\gamma} = \theta Ar(r-1)\alpha^p\beta^q\gamma^{r-2}\delta^s,$$

$$L_{\gamma\delta} = L_{\delta\gamma} = \theta Ars\alpha^p\beta^q\gamma^{r-1}\delta^{s-1}, L_{\delta\delta} = \theta As(s-1)\alpha^p\beta^q\gamma^r\delta^{s-2}. \tag{37}$$

Using (36) and (37) in (35) we get,

$$\begin{aligned}
|H| = \theta^3A^5\alpha^{5p-2}\beta^{5q-2}\gamma^{5r-2}\delta^{5s-2}\{& -p^2q(q-1)r(r-1)s(s-1) + p^2q(q-1)r^2s^2 - p^2q^2r^2s^2 \\
&+ p^2q^2r^2s(s-1) - p^2q^2r^2s^2 + p^2q^2r(r-1)s^2 + p^2q^2r(r-1)s(s-1) - p^2q^2r^2s^2 + p^2q^2r^2s^2 \\
&- p^2q^2r^2s(s-1) + p^2q^2r^2s^2 - p^2q^2r(r-1)s^2 - p^2q^2r^2s(s-1) + p^2q^2r^2s^2 - p^2q(q-1)r^2s^2 \\
&+ p^2q(q-1)r^2s(s-1) - p^2q^2r^2s^2 + p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r(r-1)s^2 + p^2q(q-1)r(r-1)s^2 \\
&- p^2q(q-1)r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r(r-1)s(s-1) - p^2q^2r^2s^2 + p^2q^2r^2s^2 \\
&- p^2q^2r^2s(s-1) + p^2q^2r^2s^2 - p^2q^2r(r-1)s^2 - p(p-1)q^2r(r-1)s(s-1) + p(p-1)q^2r^2s^2 \\
&- p(p-1)q^2r^2s^2 + p(p-1)q^2r^2s(s-1) - p(p-1)q^2r^2s^2 + p(p-1)q^2r^2s(s-1) + p^2q^2r^2s(s-1) \\
&- p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s(s-1) + p^2q^2r^2s^2 - p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r(r-1)s^2 \\
&- p^2q^2r(r-1)s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s(s-1) + p^2q^2r^2s^2 \\
&- p^2q(q-1)r^2s^2 + p^2q(q-1)r^2s(s-1) - p^2q^2r^2s^2 + p^2q^2r^2s^2 + p(p-1)q^2r^2s(s-1) \\
&- p(p-1)q^2r^2s^2 + p(p-1)q(q-1)r^2s^2 - p(p-1)q(q-1)r^2s(s-1) + p(p-1)q^2r^2s^2 \\
&- p(p-1)q^2r^2s^2 - p^2q^2r^2s(s-1) + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r^2s(s-1) - p^2q^2r^2s^2 \\
&+ p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q(q-1)r^2s^2 - p^2q(q-1)r^2s^2 \\
&+ p^2q^2r^2s^2 - p^2q^2r(r-1)s^2 + p^2q(q-1)r(r-1)s^2 - p^2q(q-1)r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 \\
&- p(p-1)q^2r^2s^2 + p(p-1)q^2r(r-1)s^2 - p(p-1)q(q-1)r(r-1)s^2 + p(p-1)q(q-1)r^2s^2 \\
&- p(p-1)q^2r^2s^2 + p(p-1)q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r(r-1)s^2 + p^2q^2r(r-1)s^2 - p^2q^2r^2s^2 \\
&+ p^2q^2r^2s^2 - p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q^2r^2s^2 + p^2q^2r^2s^2 - p^2q(q-1)r^2s^2 \\
&+ p^2q(q-1)r^2s^2\}.
\end{aligned}$$

$$|H| = pqrs\theta^3A^5\alpha^{5p-2}\beta^{5q-2}\gamma^{5r-2}\delta^{5s-2}\{p(r-s)(q-1)+\Sigma\} \tag{38}$$

In this model, $A > 0$, $\alpha, \beta, \gamma, \delta > 0$; and *p*, *q*, *r*, and *s* are the rate of inputs and hence these will never be negative. Here $|H|$ is a determinant of 5×5 Hessian matrix. Hence, for the correct economic analysis, $|H|>0$, as required, for cost to be minimized [Moolio et al., 2009]. From (12) we have, $0<p<1$, so that, $p-1<0$; and hence, equation (38) provides $|H|>0$ only if, $s>r$, so that, $\Sigma > p(r-s)(q-1)$. Let, $\{q(s-r)(p-1)+\Sigma\}=\Delta>0$, then (38) becomes,

$$|H| = pqrs\theta^3 A^5 \alpha^{5p-2}\beta^{5q-2}\gamma^{5r-2}\delta^{5s-2}\Delta . \tag{39}$$

Now using the values of $\alpha,\beta,\gamma,\delta$, and θ in (39) we get,

$$|H| = pqrsA^5\Delta\left[\frac{w^{2a-2c-pe+3f}x^{3f+i}r^{a-c+4re}s^{a-c+4se}A^{c-a+3e}}{y^{a-c+4re}z^{a-c+4se}p^{2a-2c-pe+3f}q^{a-c+3g+j}X^{3e}}\right]^{\frac{3}{a-c}}\left[\frac{w^f x^f r^{re}s^{se}A^e}{y^{re}z^{se}p^f q^g X^e}\right]^{\frac{5p-2}{a-c}}$$
$$\left[\frac{x^i r^{re}s^{se}p^{pe}A^e}{y^{re}z^{se}w^{pe}q^j X^e}\right]^{\frac{2q-2}{a-c}}\left[\frac{w^{a-c+f}x^f r^{a-c+re}s^{se}A^e}{y^{a-c+re}z^{se}p^{a-c+f}q^g X^e}\right]^{\frac{5r-2}{a-c}}\left[\frac{w^{a-c+f}x^f r^{re}s^{a-c+se}A^e}{y^{re}z^{a-c+se}p^{a-c+f}q^g X^e}\right]^{\frac{5s-2}{a-c}}. \tag{40}$$

Equation (40) indicates that $|H|>0$ and consequently, cost is in minimization.

8. Conclusion and Recommendation

In this study we have used Lagrange multiplier method to demonstrate cost minimization analysis of a running firm. We have also used Cobb-Douglas production function to show more clearly the economic analysis of the firm. In this study we have applied Lagrange multipliers method to minimize cost function subject to constraint of production. Sensitivity analysis is given for the prediction of future production of the firm, and in the analysis we have obtained that cost of the running firm is of course minimized. Throughout the article we have tried to show mathematical calculations in some details.

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