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Abstract

This paper begins with a survey of the literature on the political economy approaches to labor income taxation. We focus on recent progress made by examining in detail the specific properties of non-linear taxes derived in the context of voting. Next, we present new results on the existence of majority voting equilibrium that unify work in the standard framework. Finally, we discuss how recent theoretical results help us uncover empirical patterns from the last 50 years in the US tax system, namely a sharp decrease in top marginal tax rates, the rise of the Earned Income Tax Credit (EITC), and increased progressivity in the middle of the income distribution.

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1 Introduction

Motivation for this survey. The politics of income taxation is an important topic, both in terms of normative and positive content. Can we predict future tax reforms? Are the taxes generated by political processes efficient? What taxes are optimal subject to political constraints? How is income redistributed by the political process? Recently, there has been much literature generated in this area, with a focus on policy prescriptions. No up to date survey is available, rendering entry into this literature costly. A consequence is that new lessons are not integrated well into the academic and policy spheres.

There are three reasons why this has happened. First, there are barriers to entry not only to the topic of this survey, but even to basic optimal income taxation without political economy considerations. The classical papers in this area require specialized training as a prerequisite, so it's very hard for graduate students to pick up the basics. Second, the Condorcet paradox creates huge problems in the literature we survey. The space of alternatives over which voting occurs, namely tax functions, is generally of high dimension and in many cases is infinite dimensional. As we shall explain below, this causes problems with existence of a voting equilibrium. Only recently has progress been made on this issue. Third, it is often the case that restrictions on tax functions are imposed in order to generate a majority rule equilibrium. The restrictions sometimes seem arbitrary, and taxes in the restricted class are often Pareto dominated by tax systems not in the class. An obvious example is linear taxes, that are generally Pareto dominated by nonlinear ones. From an empirical perspective, the politics of top marginal tax rates or the Earned Income Tax Credit (EITC) – i.e. a work subsidy program where marginal tax rates are negative at some income level before becoming positive at a higher income level – cannot be fully captured by linear tax schedules.

Punch line. Any political economy approach to non-linear income taxation faces the difficulty that the set of non-linear tax schedules is a multi-dimensional policy space. With such a policy space, the existence of a Condorcet winner is not to be expected. Many political economy approaches to redistributive taxation deal with this complication by restricting attention to a subset of tax systems in which a Condorcet winner can be found. The advantage of this approach is that it allows for clear-cut political economy predictions. A disadvantage is that the restricted set of tax systems may become too small for an analysis that is empirically appealing. Another approach taken by the literature is to use a solution concept different from majority rule equilibrium or Condorcet winner.

Recent papers have overcome the issue of restricting of tax systems by applying mechanism design techniques to deal with the multidimensional policy domain, or shifted their attention to *reforms* of unrestricted tax systems instead of the design of tax systems from scratch.

Labor income taxation in the US: the last 50 years. Three major trends in the US labor income tax system in the last 50 years have presented challenges to the traditional normative and positive approaches to labor income taxation: (i) a sharp decrease in the top marginal tax rate; (ii) the introduction and extension of the Earned Income Tax Credit; (iii) sharp progressivity in the middle of the income distribution. In the concluding remarks section we review these trends and document these patterns empirically. We then show how the recent conceptual reframing of tax reforms can help us understand these trends from a political economy perspective.

Recent surveys. The only recent survey is by Bellani and Ursprung (2019); their focus is different from ours: they review the literature on the public choice analysis of redistribution policies and do not review recent papers on non-linear taxes and voting at the core of this survey.

Outline. Section 2 will review the literature on models with exogenous incomes and no public good, but where redistribution is a major concern. The literature on voting over income tax schedules with *ex ante* restrictions on their functional form in the context of endogenous income is discussed in Section 3. In Section 4, we explore the relationship between models with exogenous income and those with endogenous income, and examine how the analysis of models with exogenous income can serve as a precursor to and aid in the analysis of those with endogenous income. The literature on voting over income tax schedules in the context of endogenous income, but with restrictions on tax functions not explicitly limiting their functional form, is reviewed in Section 5. In Section 6, we introduce our new theoretical results, generalizing those found in the literature, by further reliance on the government budget constraint. Section 8 returns to the three stylized facts listed above and concludes with musings about how future work should proceed. An appendix contains the longer proofs.

2 Exogenous income and the political economy of income taxation: Pork-barrel spending and the divide the dollar game

We first review the class of models of political competition in which a policy proposal specifies how a homogeneous cake of a given size – i.e. exogenous income – should be distributed among voters. Two strands of literature characterize the political outcome of these games.

The first strand makes use of the probabilistic voting formulation of the problem (Hinich, 1977; Coughlin and Nitzan, 1981; Ledyard, 1981, 1984; see Banks and Duggan, 2005, for a unifying framework). Applications of probabilistic voting to income taxation can be found in papers by Lindbeck and Weibull (1987), Cox and McCubbins (1986), and Dixit and Londregan (1996). They consider two-party (or two-candidate) competition over redistribution policy. The basic assumption is that voters derive utility from the tax and transfer scheme chosen by politicians, but also from policies that are not related to the redistribution scheme chosen. Thus, one component of every voter's utility depends on the policy through its effects on her consumption. This component is known by everyone. The other component is derived from other policies in the parties' political programs, or from personal attributes of the candidates (often called valence). It is only imperfectly observed by voters, though Dixit and Londregan (1998) extend the setup to account for possibility that a party can change their ideology. Therefore, these models assign probability distributions to individuals' party preferences.

Such restrictions permit characterization of the redistributive equilibrium. The main advantage of this approach is that a rich set of potential policies (i.e. a multi-dimensional policy domain) does not preclude the existence of the equilibrium. The main weaknesses of these approach are (i) the focus on a type of tax and transfer system that cannot be connected to important parts of actual tax systems based on non-linear tax functions, in particular distortionary taxes; (ii) a lack of connection with the normative analysis of tax systems based on social welfare maximization subject to informational and budget constraints following Mirrlees (1971); (iii) it is impossible to distill the pure force of redistribution since other dimensions (e.g. ideology) play a role; and (iv) the Nash equilibrium tax systems of these games can generally be defeated, after all uncertainty is resolved, by another tax system that is preferred by a majority of voters.

The pure force of redistribution appears in the analysis of Myerson (1993): all voters are identical to begin and do not have any ideological affinities, unlike the models above. All that matters to them is the tax they pay or the transfer they receive. This game is solved by importing techniques from the military game called Colonel Blotto, pioneered by Borel (1921). The Colonel Blotto game is a two-player resource allocation game in which each player is endowed with a level of a resource (number of soldiers) to allocate across a set of battlefields. Within each battlefield, the player that allocates the higher level of resource wins the battlefield, and each player's payoff is the sum of the valuations of the battlefields won (see the generalization by Kovenock and Roberson, 2020). Myerson (1993) defined a battlefield as a voter, and solved the game for a continuum of voters. Whereas the equilibrium results provide interesting insights on the role of electoral rules for the generation of inequality (or of a favored minority) in an initially homogenous electorate, the analysis does not connect to actual tax systems or to the normative analysis of taxation; taxes remain non-distortionary.

Crutzen and Sahuguet (2009) take on this last point and, using insight from Lizzeri and Persico (2001), add to the paper by Myerson (1993) the possibility that the tax on initial endowments is distortionary. This brings the analysis closer to the literature on normative tax where taxes are distortionary, although in this setup the distortion does not come from incentive compatibility constraints. Their main result is to prove that politicians continue to use taxes and transfers even in the presence of distortions, and that the political equilibrium generates inefficient taxation.

Carbonell-Nicolau and Ok (2007) attempt to explain why statutory income tax schedules in practice have progressive marginal tax rates. In an endowment economy, they consider political competition over arbitrary tax functions. They show that mixed strategy Nash equilibria exist and identify certain cases in which marginal-rate progressive taxes are chosen almost surely by two competing parties. However, without restrictions on the tax policy space, the support of at least one equilibrium in mixed strategies cannot be contained within the set of marginal-rate progressive taxes.

3 Endogenous income and voting over tax functions restricted by functional form

3.1 Linear tax systems

Pioneering contributions by Romer (1975) and Roberts (1977) have pinned down the outcome of majority voting in a model with an endogenous labor supply reaction to the taxation of labor income.[1] The main restriction used to make progress is that the tax function is linear: the tax and transfer system is a combination of a uniform lump sum redistribution and a constant marginal tax rate. Given a government budget constraint, the essence of this model is a one-dimensional conflict between voters. It has a counterpart in the optimal linear income taxation literature, using a social welfare function in place of a voting mechanism, in the setup considered by Sheshinski (1972). These papers tackle an important issue: whereas individual preferences over tax schedules are not single peaked in the space of tax and transfer systems, they are single crossing. This property allows the ordering of individuals with respect to redistribution alternatives and allows a median voter result, namely that the tax and transfer system is preferred by a majority of the population if and only if it is preferred by the median individual. Gans and Smart (1996) present a generalization of the approach by showing that the existence of a majority voting equilibrium on one-dimensional choice domains depends on order restrictions on voter preferences which imply or are equivalent to a general ordinal version of the single-crossing condition. Weymark (1984) considers both majority and Pareto improving directions for local reform of a linear income tax.

Median voter theorems for linear income taxation have been widely used. A prominent example is the prediction due to Meltzer and Richard (1981) that tax rates are an increasing function of the difference between median and average income. The explanatory power of this framework was found to be limited (see, for instance, the review in Acemoglu, Naidu, Restrepo and Robinson, 2015) and has led to analyses in which the preferences for redistributive tax policies are also shaped by prospects for upward mobility or a desire for a fair distribution of incomes; see Piketty (1995), Bénabou and Ok (2001), Alesina and Angeletos (2005), Bénabou and Tirole (2006), and Alesina, Stantcheva and Teso (2018).

[1] These papers are also reviewed in Drazen (2000) and Persson and Tabellini (2000).

3.2 Other parametric tax systems

From the 1970's on, removing the linearity restriction on taxes has been a target of scholars interested in the study of positive theories of tax systems. A stream of literature imposed quadratic tax structures (Roemer, 1999; De Donder and Hindriks, 2003) or put restrictions directly on the tax rates (Casamatta, Cremer and De Donder, 2010).

Another important development was to restrict tax systems to those with a constant rate of progressivity (Bénabou, 2000; 2002). This class has a long tradition in public finance starting with Musgrave and Thin (1948); see also Feldstein (1969). The assumption is becoming more and more popular (especially in macroeconomics), as Heathcote, Storesletten and Violante (2017) show that one can fit the US tax and transfer system very well and have meaningful discussions about optimal progressivity of taxation within this class of functions. A key advantage is that quadratic tax functions can simultaneously allow discussion about the top tax rate (for individuals with the highest income) as well as discussion of negative tax rates at the bottom of the distribution. Combined with additional assumptions about the economy, Heathcote et al. (2017) show that the median voter theorem applies to their economy because preferences are single peaked in the degree of progressivity of the tax system.

4 The Relationship Between Exogenous and Endogenous Income Models

4.1 The General Relationship

As we will detail with notation next, the relationship between the exogenous income tax models and the endogenous income tax models is one of implementation, in the sense of mechanism design. Let us begin with the exogenous income model. Denote type, or income, or marginal productivity by w, and the tax on income by $g(w)$. For the exogenous income model, type w of an individual is interpreted as income and is common knowledge; this is natural, as there is no signal about w to use. Then after tax income is given by $w - g(w)$; this is also taken to be the utility level of type w. Turning to the endogenous income model, gross income is given by $y(w) = w \cdot l(w)$, where $l(w)$ is the labor choice of individual w. Individual type w is not observable to anyone but the individual, though $y(w)$ is observed by all. Thus, type w is inferred.

An income tax, which is an indirect mechanism, is given by $\tau(y)$. A direct mechanism in this context would be a tax that is a function of w directly. In that case, it would again be denoted by $g(w)$, *where a person's type is known only to themselves.* The Revelation and Taxation Principles provide conditions under which the indirect mechanism (as a function of y) can be represented by a direct mechanism (as a function of w) and vice-versa.

More precisely, let the utility function of type w be denoted by $u(c, l, w)$, where c is consumption. Suppose that private good consumption (or after tax income) is *essential*, i.e. for all w, y and y', $u(0, y/w, w) < u(y' - \tau, y'/w, w)$ if $y' - \tau > 0$. Say that a direct allocation mechanism $(y(\cdot), g(\cdot))$ is *reasonable* if $y(w) - g(w) > 0$, for all w. If private good is essential, the Taxation Principle states that a reasonable allocation (namely, a direct mechanism) is resource feasible and incentive compatible if and only if it is decentralizable with an income tax (see Guesnerie, 1995, for a formal proof).

With this minimal notation, we can discuss the general relationship between exogenous and endogenous income models of taxation. Given an income tax with endogenous income, $\tau(y)$, we can allow taxpayers of type w to choose labor supply $l(w)$ and thus $y(w)$. So we can define $\tau(y(w)) \equiv g(w)$. Thus, for every tax with endogenous income, a corresponding exogenous income tax can be defined. Conversely, under certain conditions on $g(w)$, notably monotonicity, an incentive compatible tax on endogenous income $\tau(\cdot)$ can be constructed. In fact, there is a continuum of such taxes, but there is a best tax among these according to the Pareto criterion.

When working with this structure, it is often convenient to find a direct mechanism first, and then apply the taxation principle to obtain an income tax. It is sometimes convenient to prove results first in the context of exogenous incomes and then bring the results to the endogenous (Mirrlees) income tax context. This can be achieved either through using implementation results such as Berliant and Page (1996), or by directly extending the more elementary proof with exogenous incomes to the more complex environment with endogenous incomes and incentives. We shall illustrate this below in sections 6.2 and 6.3.

4.2 The Relationship in the Context of Voting

We begin our analysis of voting, and in particular single crossing of tax schedules in the exogenous and endogenous income models, with an observation. First, it is worth noting that Pareto domination of one tax system over another in the exogenous income model, meaning that two tax systems do not cross,[2] is equivalent to Pareto domination of one tax system over another in the endogenous income model. Second, single crossing in the exogenous income model is equivalent to single crossing in the endogenous income model. The latter is not obvious, but is proved in Berliant and Gouveia (2020) by using implementation results.

The main task of this subsection is to examine in detail the important work of Hemming and Keen (1983) and Gans and Smart (1996). Hemming and Keen (1983) consider an exogenous income model with no information asymmetries. We say that two tax systems are *comparable* if one tax system Lorenz dominates another for all pre-tax income distributions that generate the same revenue for both tax systems. Their key result is contained in Proposition 1 of their paper, stating that two tax systems are comparable if and only if the post tax income functions (as a function of pre-tax income) are single crossing.

Gans and Smart (1996) consider a Mirrlees endogenous income model with standard regularity assumptions on utility functions. They use the Hemming and Keen result to prove their Proposition 2, stating that if any two tax systems from an admissible set are comparable in the sense of Hemming and Keen, then a majority voting equilibrium exists. We wish to clarify the interpretation of this result. Let us be very specific.

In the case of endogenous choice of labor supply and consequently gross or pre-tax income, the pre-tax income schedule is induced by the distribution of types *and the tax schedule itself.* In this case, the endogenously generated distribution of pre-tax income may differ for two tax systems, even when they generate the same total revenue. Consequently, the pre-tax income distribution for one tax system will never be observed when the other tax system is imposed. So a condition that holds for every pre-tax income distribution (generating the same revenue) makes no sense. There is generally only one pre-tax income distribution generated for each tax, and this distribution generally differs across taxes. But what happens if the distribution of types is allowed to vary?

This is actually a very subtle issue in Gans and Smart (1996). In their Proposition 2, the condition of comparability of tax systems in Hemming and Keen (1983) is independent of incentive compatibility. That is, it applies to pre-tax income distributions that might not be incentive compatible when

[2]Here, we use no crossing in a weak sense, in that one tax system has values at least as large as the other for all types.

consumers can choose their labor supply. Using this, we can go further. Fix an income tax. *If the distribution of types is allowed to vary*, some pre-tax income distributions might not be attainable under incentive compatibility for any distribution of types.

For example, for an arbitrary tax, we might wish to generate a trivial or degenerate pre-tax distribution of income where all consumers are in a gap in the net income schedule. (A gap is a place in the net income schedule where no consumers locate.) That can be made concrete by selecting a type and letting the net income function be tangent to that type's indifference curve in exactly two places, and selecting a pre-tax income that is between the two tangencies. Another possibility for generating a degenerate pre-tax income distribution that cannot be achieved in an incentive compatible manner is to choose an income level that is above the maximal gross income level that would be acceptable to consumers under this tax (namely, beyond the tangency for the top type). Then ask that the income level be the only one in the pre-tax distribution of incomes.

The Hemming and Keen (1983) condition does not place any restrictions on the pre-tax income distribution (aside from the fact that it generates the same revenue for both taxes). It is intended to apply to lump-sum taxes. Gans and Smart (1996) do not translate the Hemming and Keen (1983) condition to the context of incentive compatibility. *So the pre-tax income distributions that must be generated for comparison according to Hemming and Keen (1983) might not be achievable under incentive compatibility in Gans and Smart (1996).*

5 Endogenous income and voting over tax functions unrestricted by functional form: Political economy and Mirrleesian taxation

Mirrlees (1971) is the workhorse model for non-linear income taxation under adverse selection. It's the normative benchmark.[3] The formulation of the optimal tax problem is one of mechanism design. The social planner maximizes a social welfare function, often utilitarian, subject to the revenue constraint and incentive constraints on taxpayers. The strength of this approach is that it pro-

[3]Vickrey (1945) is an earlier attempt to formulate the problem. Piketty and Saez (2013) provides a recent survey of the normative literature.

vides a rigorous justification for the constraints that a social welfare maximizer faces. Technologies, endowments, and revenue required give rise to resource constraints, whereas privately held information gives rise to incentive compatibility constraints. Hence, Mirrlees (1971) characterizes a welfare-maximizing income tax with no a priori assumption on the functional form of the tax function. The path breaking contribution of Mirrlees (1971) was initially followed up mostly by theoretical work, but the insights obtained when rewriting the optimal tax formula with sufficient statistics provided fertile ground for public economists connecting normative guidance with data analysis. Contributions in this line of research include Piketty (1997), Diamond (1998), Saez (2001), Golosov, Tsyvinski and Werquin (2014), and Jacquet and Lehmann (2020); see the surveys of sufficient statistics approaches by Chetty (2009) and Kleven (2021). Access to administrative tax return data, the development of microsimulation models, and progress in estimating relevant elasticities created the foundations for providing policy recommendations. Diamond and Saez (2011) is an example of such recommendations based on the connection between normative and empirical approaches. We return to this empirical work in the Section 7 below where we connect theoretical insights with recent empirical analyses of tax systems.

It is important to bear in mind three considerations before digging into our literature review below. First, the progress in pinning down political equilibria rested on formulating the problem of finding political equilibrium *allocations*, namely direct mechanisms, instead of finding political equilibrium *tax functions*, that are indirect mechanisms. Thus, we consider politicians who compete over such allocations, then use the *Taxation Principle*, once equilibrium allocations are characterized, to obtain the equilibrium tax functions. This was an important first step. In contrast, the papers reviewed above formulated political competition directly on tax functions.

Second, further progress was made by studying the classical political economy models (Downsian, probabilistic voting, citizen-candidate, legislative bargaining, partisan politics) in the context of incentive compatible taxation and by making specific functional form assumptions on utility functions. From classical results in social choice and voting theory, we cannot hope for a general normative model of universal applicability. Of the various restrictions on utility functions generally employed, the assumption that utility is quasi-linear in consumption good and some restrictions on indifference curves (e.g. single-crossing conditions) are often made. Such assumptions mirror those often used in the theory of optimal taxation.

Third, two variations of the political game were formulated. The first has political forces introduced as a new constraint of political sustainability. In this case, the objective function of the mechanism designer is not very different than in the normative Mirrlees approach, but a new constraint is added on top of resource and incentive compatibility constraints (e.g., Acemoglu, Golosov and Tsyvinski, 2008). The set of allocations is then more constrained than in the second-best approach. A second formulation was one of mechanism designers who directly compete subject to the same constraints as a welfare maximizer. In this case, the set of feasible allocations for each competitor remain the same as those available in the Mirrlees approach, but the allocation choice is the result of political competition (e.g., Bierbrauer and Boyer, 2016).

Some early attempts at integrating a model of political economy with a Mirrlees framework were given in Snyder and Kramer (1988), Berliant and Gouveia (2020), and Röell (2012).[4] Snyder and Kramer (1988) leverages an underground sector that is untaxed. Berliant and Gouveia (2020) exploits the idea that the draw of types (productivity or wage) is unknown to the mechanism designer, so the budget constraint must be met for every possible draw. This reduces the dimension of the set of alternatives to a tractable one. Röell (2012) will be discussed shortly below.

Stiglitz (1982) provides a two-type version of the normative Mirrlees (1971) setup without political considerations. One advantage of this reformulation, on top of the more intuitive understanding of which incentive compatibility constraints are relevant, is to characterize the entire set of second best Pareto efficient allocations. Blomquist and Christiansen (1999), Roemer (2012), and Bierbrauer and Boyer (2013) are attempts to characterize the political equilibria in this two-type setup.[5] Tracing out the whole Pareto frontier turns out be important for political economy considerations; whereas higher welfare weights for low-skilled individuals relative to high-skilled ones makes the incentive compatibility constraints of high-skilled workers relevant (i.e. binding) for a social welfare maximizer, it is not clear why the outcome of political competition would lead to any particular welfare weights for workers.

[4] The first drafts of the latter two papers were circulated in the late 1980's.

[5] Blomquist and Christiansen (1999) focus on the question of whether the public provision of private goods can emerge in a political equilibrium. They assume that voters not only care about themselves but also about the well-being of others. Roemer (2012) uses a solution concept known as Party Unanimity Nash Equilibrium (see Roemer, 1999), different from a majority rule based solution concept.

Röell (2012), Bohn and Stuart (2013) and Brett and Weymark (2017, 2018, 2020) study non-linear taxes in the citizen-candidate framework, initiated by Osborne and Slivinski (1996) and Besley and Coate (1997): citizens compete for office and lack powers of commitment. An elected candidate will therefore implement her own preferred policy. Voting over candidates is equivalent to voting over the non-linear tax policies that the different candidates would select if they could dictate tax policy. This is the main restriction on feasible tax schedules, and it implies that only schedules that are selfishly-optimal can be feasible, and thus emerge as a majority voting equilibrium. Brett and Weymark (2016, 2017) provide a characterization of the tax schedule that the median voter would choose if she could dictate tax policy. Specifically, they show that the median voter's preferred schedule coincides with the one maximizing the Rawlsian social welfare function for incomes above the median, and with the one maximizing the max social welfare function (the opposite of the Rawlsian one) for incomes below the median. In between is a region of transition that gives rise to bunching. Interestingly, this work shows how the pattern of incentive compatibility constraints that emerges from the political process may be very different from the one arising from social welfare maximization, where no bunching is either assumed or it is an outcome of incentive compatibility plus other postulates; for the latter, see Jacquet and Lehmann (2020, Assumption 2). A criticism of this line of research is that an incentive compatible tax schedule that is not individually optimal for any taxpayer, but that favors the extremes of the distribution of worker types, might majority rule dominate the tax schedule most preferred by the median voter. Indirectly, this criticism asks why this particular game form is privileged.

There is a line of research where governments remain welfare maximizers but are still subject to political conflicts. For example, political competition from foreign governments in the context of the European Union, or other states in the context of the United States, can shape equilibrium; see the literature on migration and taxes (Bierbrauer, Brett and Weymark, 2013; Morelli, Yang and Ye, 2012; Lehmann, Simula and Trannoy, 2014). Competition can also derive from governments that have different preferences for redistribution. This is the case in Martimort (2001), who studies strategic budget deficits and optimal taxation in a model with partisan politics.

Acemoglu, Golosov and Tsyvinski (2008, 2010) relate dynamic problems of optimal taxation to problems of political agency as in Barro (1973) and Ferejohn (1986). Politicians have no commitment power and can even deviate from their within-period commitments, but they are subject to electoral accountability: if they pursue policies not in line with the expectations of the electorate, they can be punished by being removed from office. Acemoglu et al. (2008) develop a benchmark framework for the analysis of government policy in the context of a dynamic game between a self-interested government and citizens, but focus on situations in which there are no restrictions on tax policies. Acemoglu et al. (2010) use this framework for the analysis of the political economy of taxation and dynamic Mirrlees economies.

Bierbrauer and Boyer (2016) use a fully-fledged game-theoretic analysis to characterize equilibrium outcomes. The centerpiece of this analysis is pure competition between office motivated politicians in the tradition of Downs (1957). The "purity" has two dimensions: First, there is no *a priori* restriction on the set of admissible policies, so politicians can propose any policy that respects the economy's information structure and the economy's resource constraint. Second, competition is pure in that neither politicians nor voters have ideological biases nor partisan motives. Also, there is no incumbency advantage or any other difference in valence. Pork is an important component of the model, in that there are private favors offered by the politicians to the various types of voters in addition to a public program. The distribution of pork is independent of type and random, and the favor is observed by the voter prior to voting for one of the two candidates. The main insight is that equilibrium policies are Pareto-efficient in a first best sense, even though voters have private information about their preferences over mechanisms for public good provision and income tax schedules. By the *Taxation Principle*, this finding admits a different interpretation. The incentive compatibility constraints which emerge in a private information environment are equivalent to the implementability constraints which emerge in a decentralized economic system, i.e. a system where individuals make choices subject to constraints that are affected by government policy. To give examples of such policies, think of households that choose labor supply and consumption expenditures subject to a potentially non-linear budget constraint that is shaped by an income tax function. Hence, the main result is relevant for a society in which individuals are free to choose both economically and politically. According to the main result, political equilibria in such a free society give rise to surplus-maximizing outcomes. This result is akin to the first welfare theorem, where the political equilibrium allocation plays the role of the competitive equilibrium allocation. As in other contexts, symmetry can be a powerful tool to drive Nash equilibrium toward efficiency, outside of Prisoners' Dilemma scenarios.

There is, however, no counterpart to the second welfare theorem. Political equilibria do not give rise to (utilitarian) welfare-maximizing outcomes beyond the surplus maximizing one, and this may be interpreted as a political failure. For a model of redistributive income taxation, in which welfare rather than surplus is the standard policy objective, the results imply that political equilibria give rise to an undesirable laissez-faire outcome.

To the best of our knowledge, there is no paper studying what non-linear tax policies would result from the legislative bargaining political process *à la* Baron and Ferejohn (1989). Closest to this exercise is Battaglini and Coate (2008). They present a dynamic theory of public spending, taxation, and debt to study optimal (linear) taxation and debt finance in a federal system using the model of legislative bargaining.

Saez and Stantcheva (2016) examine generalized welfare functions with weights that may reflect political equilibrium outcomes. Starting from the premise that observed tax policies are not entirely driven by welfare considerations, but also by non-welfarist value judgments or political economy forces, they propose generalized social welfare functions.

There is an interesting relationship between Saez and Stantcheva (2016) and Bierbrauer, Tsyvinski and Werquin (2022). Bierbrauer et al. (2022) develop a probabilistic voting model of political competition in which the parties' platform choices (tax policies) and the voters' participation in elections (turnout) are jointly determined in equilibrium. Traditionally, equilibria in probabilistic voting models are akin to maximizing welfare functions with weights reflecting the potential for a party to attract a segment of the electorate (e.g. how many swing voters are present in a segment of the electorate). Generalizing this insight, Bierbrauer et al. (2022) show that political economy forces can be captured by specific generalized social welfare weights that emerge in the political process. Interestingly, they show that their political equilibrium outcomes may be incompatible with the maximization of a standard concave social welfare function; the generalized social welfare weights of Saez and Stantcheva (2016) are therefore needed.

Probabilistic voting has also been used to study non-linear labor and capital income taxation in dynamic settings in Fahri and Werning (2008), Sleet and Yeltekin (2008), and Farhi, Sleet, Werning and Yeltekin. (2012). The main purpose of these papers is to provide a political economy model that addresses the progressivity of capital taxation. Their main result is that progressive taxation of capital emerges naturally in this setting, contrasting sharply with the normative benchmark result of zero capital taxes where the government is free of political constraints; see, e.g., Judd (1985) and Chamley (1986). Building on Farhi et al. (2012)'s setup, Scheuer and Wolitzky (2016) study dynamic taxation under the assumption that a policy is sustainable if and only if it maintains the support of a large enough political coalition over time. The key feature of the model is that the players anticipate that some reform threat will arise in a later period, and that the government's proposed capital tax policy will be implemented if and only if the reform is defeated in terms of popular support. They show that optimal marginal capital taxes are either progressive or U-shaped, so that savings are subsidized for the poor and/or the middle class but are taxed for the rich.

The approach in the papers reviewed above is fundamentally different from the research that is laid out Bierbrauer, Boyer and Peichl (2021). This last paper examines the political support for *tax reforms* in contrast with tax systems themselves.[6] Therefore, specific assumptions about the political process game form and the rationality of political parties are not invoked (How many parties do compete? What are the party objectives? In what order do the parties announce their platforms?). Instead, the set of politically feasible reforms is characterized without deriving specific predictions about which of these reforms will be taken up in the political process. A theory of tax reforms that are politically feasible in the sense that a majority of individuals prefers the reform over the given status quo is developed. The theory gets traction from focusing on monotonic tax reforms, i.e. reforms so that changes in the tax burden are a monotonic function of income.[7] One advantage of this approach is that it allows for an easy connection between a normative perspective and a political economy perspective on tax reforms. Normative analyses frequently examine the welfare implications of raising or lowering the marginal tax rates in a narrow bracket of incomes. These tax perturbations satisfy the monotonicity assumption on which the political economy analysis is based. Thus, this technology can also analyze whether a given tax system can be reformed

[6] The tax reform approaches were developed in the 1970s (see Feldstein, 1976). Guesnerie (1995) surveys the literature on tax reforms and contains an analysis of tax reforms that emphasizes political economy forces, formalized as the requirement that political equilibrium tax schedules be coalition proof.

[7] Examples of monotonic tax reforms are (i) a reform that involves tax cuts for all incomes, with larger cuts for larger incomes, and (ii) a reform that involves higher taxes for everyone, with increases that are a larger for the rich.

in a way that is both politically feasible (for example, preferred by a majority) and/or welfare-improving, or whether the scope for politically feasible reforms has been exhausted.

Their first theorem is a median voter result for monotonic tax reforms: a monotonic tax reform is supported by a majority of the population if and only if the person with median income is among the beneficiaries. They then study the extent to which reforms of the federal income tax in the US were monotonic reforms and find that the tax reforms were, by and large, monotonic, with monotonic tax cuts – i.e. larger tax cuts for richer taxpayers – being the most prevalent reform type. In the second part of the paper, they provide a characterization of political feasibility: if the status quo is a Pareto-efficient tax system, tax cuts for below median incomes and tax increases for above median incomes are politically feasible. If tax rates on high incomes are revenue-maximizing in the *status quo*, only tax cuts below the median are politically feasible. An implication of this result is that a sequence of politically feasible reforms should lead to lower and lower marginal tax rates below the median and, possibly, to higher and higher marginal tax rates above the median. Moreover, such a sequence should give rise to an income range with a pronounced progression of marginal tax rates that connects the low rates below the median with the high rates above the median. This is very similar to the conclusions of Brett and Weymark (2016, 2017) derived from a very different model (we return to this comparison below in Corollary 3). Bierbrauer et al. (2021) provide an empirical analysis of US tax reforms, motivated by these theoretical implications, that we summarize in Section 7 below.

6 A Canonical Model

The previous literature has a focus on consumer preference over tax systems in conjunction with restrictions on the shape of tax schedules, for example using single crossing conditions. Here we shall leverage, in addition to consumer preference and the shape of tax schedules, the revenue constraint. This allows us to weaken the restrictions on consumer preference and shape of the tax schedules, since some tax systems do not generate enough revenue to be feasible.

6.1 The model with exogenous income

6.1.1 Notation and Basic Results

A consumer's endowment, which is also her type, is described by $w \in [\underline{w}, \overline{w}]$, where $[\underline{w}, \overline{w}] \subseteq \mathbb{R}_+$. Let F be a distribution function on $[\underline{w}, \overline{w}]$, the possible types. Where we need it, its density will be called f, and in that case we are implicitly assuming that F has a density. In general, we will allow atoms in F (except where noted), so models with a finite number of agents are subsumed.

The *aggregate revenue* required from the economy is called $R \in \mathbb{R}$.

A *lump sum tax function* is a function $g : W \to \mathbb{R}$ that takes w to tax liability. In an economy without a labor/leisure decision, this is a lump sum tax function on endowments or on exogenous income. Type is known to everyone. Utility for type $w \in [\underline{w}, \overline{w}]$ in this simple economy is just net after tax income, namely $w - g(w)$.

Definition 1: Define the *total variation of a function* g as:

$$TV(g) \equiv \sup \left\{ \sum_{k=1}^{K} |g(w_k) - g(w_{k-1})| \, |K \geq 1, \, w_k \in [\underline{w}, \overline{w}], \, w_0 < w_1 < \cdots < w_K \right\}$$

Fix $c > 0$, $c' \in \mathbb{R}$ and let

$$G = \{g : [\underline{w}, \overline{w}] \to \mathbb{R} \mid TV(g) \leq c \text{ and for all } w \in [\underline{w}, \overline{w}], \, c' \leq g(w) \leq w\}$$

Functions of bounded total variation include classes of continuous or monotonic functions. The last condition in the definition of G implies that the lump sum tax does not bankrupt any type. The revenue constraint will be imposed soon.

The obvious next step is to look at the best tax functions for the median voter, where the median is defined according to the distribution F. Let w^* be the median type under the distribution F.

Proposition 1: If there is $\widehat{g} \in G$ such that $\int_{\underline{w}}^{\overline{w}} \widehat{g}(w) dF(w) \geq R$, then there exists $g^* \in G$ that maximizes $w^* - g(w^*)$ over $g \in G$ subject to $\int_{\underline{w}}^{\overline{w}} g(w) dF(w) \geq R$.

Proposition 1 follows from Helly's theorem and Lebesgue's dominated convergence theorem. It means that the set of favorite lump sum tax functions for the median type that satisfies the revenue constraint is not empty. Notice that g^* is generally not unique.

*Definition 2***:** A collection $\mathbf{E}$ of functions mapping $[\underline{w}, \overline{w}]$ into $\mathbb{R}$ is called *strongly single crossing* if for any pair $g, g' \in \mathbf{E}$, there exist $\widetilde{w}, \widetilde{w}' \in [\underline{w}, \overline{w}]$, $\widetilde{w} < \widetilde{w}'$ such that $g(\underline{w}) > g'(\underline{w})$ implies $g(w) > g'(w)$ for all $w \in [\underline{w}, \widetilde{w})$, $g(w) = g'(w)$ for all $w \in [\widetilde{w}, \widetilde{w}']$ and $g(w) < g'(w)$ for all $w \in (\widetilde{w}', \overline{w}]$.

6.1.2 The main results

Definition 3: Fix $G' \subseteq G$. Then $g \in G'$ is called a *majority voting equilibrium* if $\int_{\underline{w}}^{\overline{w}} g(w)dF(w) \geq R$ and there is no $g' \in G'$ with $\int_{\underline{w}}^{\overline{w}} g'(w)dF(w) \geq R$ and measurable $W \subseteq [\underline{w}, \overline{w}]$ with $g'(w) < g(w)$ $\forall w \in W$ and $\int_W dF(w) > \frac{1}{2}$.[8]

The condition we focus on is the following.

Definition 4: $G' \subseteq G$ is said to satisfy *Condition P* if:

$$\text{For all } g, g' \in G', \int_{\{w \in [\underline{w}, \overline{w}] | g'(w) < g(w)\}} dF(w) > \frac{1}{2} \implies \tag{P}$$
$$g'(w^*) < g(w^*) \text{ or } \int_{\underline{w}}^{\overline{w}} g'(w)dF(w) < \int_{\underline{w}}^{\overline{w}} g(w)dF(w)$$

Definition 5: $G' \subseteq G$ is said to satisfy *Condition M* if:

$$\text{There exists } \widehat{g} \in G' \text{ with } \int_{\underline{w}}^{\overline{w}} \widehat{g}(w)dF(w) \geq R, \tag{M}$$
$$\text{and for all } g \in G' \text{ with } \int_{\underline{w}}^{\overline{w}} g(w)dF(w) > R \implies$$
$$\text{there exists } g' \in G' \text{ with } g'(w) \leq g(w) \ \forall w \in [\underline{w}, \overline{w}]$$
$$\text{and } \int_{\underline{w}}^{\overline{w}} g'(w)dF(w) = R$$

Remarks:

1. Condition (P) means that if a majority strictly prefers one of the two tax systems, then either the median tax or the mean tax must be lower.

[8] Since we employ functions of bounded variation, we don't need to worry about sets of zero measure.

2. Notice that if g is a majority voting equilibrium, then under condition (M) $\int_{\underline{w}}^{\overline{w}} g(w)dF(w) = R$, for otherwise g can be Pareto dominated.

3. Notice also that a slightly weaker condition than condition (P) is necessary at a majority voting equilibrium $g \in G'$, namely: For any other $g' \in G'$,

$$\int_{\{w\in[\underline{w},\overline{w}]|g'(w)<g(w)\}} dF(w) > \frac{1}{2} \implies$$
$$g'(w^*) < g(w^*)$$
$$\text{or } \int_{\underline{w}}^{\overline{w}} g'(w)dF(w) < \int_{\underline{w}}^{\overline{w}} g(w)dF(w) = R$$

   In this case, the second of the two consequents always holds.

4. Of course, the single crossing property implies Condition (P). Employ our version, that we call strongly single crossing.

   Assume that $\int_{\{w\in[\underline{w},\overline{w}]|g'(w)<g(w)\}} dF(w) > \frac{1}{2}$. Under strong single crossing, $w^* \in \{w \in [\underline{w}, \overline{w}] \mid g'(w) < g(w)\}$.

5. Evidently, Pareto domination of one tax system g' over another g, namely $g'(w) \leq g(w)$ for all $w \in [\underline{w}, \overline{w}]$ with strict inequality for a set of positive measure, in turn implies a (rather obvious) single crossing property weaker than strong single crossing.

Theorem 1: If $G' \subseteq G$ satisfies Conditions (P) and (M) and is closed in the topology of pointwise convergence, then there exists a majority rule equilibrium, and it is among those satisfying the government budget constraint most preferred by the median voter.

Proof: Find $g \in G'$ that minimizes $\widehat{g}(w^*)$ subject to $\int_{\underline{w}}^{\overline{w}} \widehat{g}(w)dF(w) \geq R$; such exists by the first part of condition (M), Helly's theorem, and Lebesgue's dominated convergence theorem. By the second part of condition (M), we can take $\int_{\underline{w}}^{\overline{w}} g(w)dF(w) = R$. Now suppose that there is $g' \in G'$ such that $\int_{\{w\in[\underline{w},\overline{w}]|g'(w)<g(w)\}} dF(w) > \frac{1}{2}$ and $\int_{\underline{w}}^{\overline{w}} g'(w)dF(w) \geq R$. Under condition (P) it follows that

$$g'(w^*) < g(w^*) \text{ or } \int_{\underline{w}}^{\overline{w}} g'(w)dF(w) < \int_{\underline{w}}^{\overline{w}} g(w)dF(w) = R$$

It is impossible that the first condition is satisfied. The second is a contradiction.

Next we provide an example (of G') that satisfies Condition (P) but not single crossing.

Definition 6: For any Lebesgue measurable subset A of $[\underline{w}, \overline{w}]$, define $1_A(w)$ to be its indicator function, namely $1_A(w) = 1$ if $w \in A$, $1_A(w) = 0$ otherwise.

Example 1: Let $G'' \equiv \{g \in G \mid g = 1_A(w) \text{ for some measurable } A\}$, and let F be nonatomic. Notice that A need not be connected (but it can't be too wild, since the total variation of its indicator function is bounded). Each element of G'' is a $0-1$ tax, but the idea can be extended in obvious ways. This example is mainly for intuition, and is not intended to be realistic. Let $R \leq \int_{\underline{w}}^{\overline{w}} w dF(w)$. Now take $g, g' \in G''$ and suppose that

$$\int_{\{w \in [\underline{w}, \overline{w}] \mid g'(w) < g(w)\}} dF(w) > \frac{1}{2}.$$

Define the set $B = \{w \in [\underline{w}, \overline{w}] \mid g'(w) < g(w)\}$. So $g'(w) = 0$ for $w \in B$, $g(w) = 1$ for $w \in B$. Hence,

$$\int_{\underline{w}}^{\overline{w}} g'(w) dF(w) = \int_{B^c} dF(w) < \frac{1}{2} < \int_B g(w) dF(w) \leq \int_{\underline{w}}^{\overline{w}} g(w) dF(w)$$

and the second consequent of condition (P) is satisfied. Condition (M) follows from Lyapunov's theorem.

Definition 7: $G'' \subseteq G$ is said to be *zero ordered* if for all $g \in G''$, g is C^0 and for all $g, g' \in G''$, either $g(w) \geq g'(w)$ for all $w \in [\underline{w}, \overline{w}]$, or $g'(w) \geq g(w)$ for all $w \in [\underline{w}, \overline{w}]$.

Definition 8: $G'' \subseteq G$ is said to be *first ordered* if for all $g \in G''$, g is C^1 and for all $g, g' \in G''$, either $\frac{dg(w)}{dw} \geq \frac{dg'(w)}{dw}$ for all $w \in [\underline{w}, \overline{w}]$, or $\frac{dg'(w)}{dw} \geq \frac{dg(w)}{dw}$ for all $w \in [\underline{w}, \overline{w}]$.

Definition 9: $G'' \subseteq G$ is said to be *second ordered* if for all $g \in G''$, g is C^2 and there exist constants $k > 0$ and $\theta > 0$ such that for all $g, g' \in G''$, either $\frac{d^2g(w)}{dw^2} - \frac{d^2g'(w)}{dw^2} = 0$ for all $w \in [\underline{w}, \overline{w}]$, or $\frac{d^2g(w)}{dw^2} - \frac{d^2g'(w)}{dw^2} \geq k/f(w)$ for all $w \in [\underline{w}, \overline{w}]$, or $\frac{d^2g'(w)}{dw^2} - \frac{d^2g(w)}{dw^2} \geq k/f(w)$ for all $w \in [\underline{w}, \overline{w}]$. Moreover, $\left|\frac{dg'(w_L)}{dw} - \frac{dg(w_L)}{dw}\right| \leq \theta/f(w_L)$ where $w_L = \min_{g(w)=g'(w)} w$. Finally, $\overline{w} - \underline{w} \geq \frac{(2+2\sqrt[3]{2})\theta}{k}$.

Remark 2: Zero ordered is the Pareto preorder. *First ordered* is a strengthening of single crossing. *Second ordered* is a strengthening of double crossing.[9] Notice that, for $g, g' \in G$, if g Pareto dominates g', then g single crosses g'. Notice also that if g single crosses g', g double crosses g'. One could go on to define third ordered and so forth. For zero and first ordered, the revenue part of condition (P) won't come into play. It will for second ordered. For concreteness in the interpretation of second ordered, think of $f(w)$ as the uniform density.

It is important to note that another parameter can be introduced to weaken the assumptions used in the definition of second ordered, namely population. Thus far, it has been normalized to one. The joint constraints on $\overline{w} - \underline{w}$, θ, k, and f might be tight, since f integrates to 1. However, setting $\underline{w} = 0$, we can introduce a population parameter $\phi \geq 1$ and define $f_\phi(w) \equiv f(\frac{w}{\phi})$ with support $[0, \phi\overline{w}]$, that will loosen the requirements for second ordered, in particular the very last condition. In that case, R should be interpreted as per capita revenue, where ϕR is total revenue. Since ϕ will appear on both sides of inequalities and equalities (sometimes via change of variable), there is no change in the expressions above. This additional parameter will be exploited in the model with endogenous income, where the arguments are more complicated.

Corollary 1: If $G'' \subseteq G$ is second ordered and closed in the topology of pointwise convergence, $R \leq \int_{\underline{w}}^{\overline{w}} w dF(w)$, and

$$g \in G'' \text{ and } \int_{\underline{w}}^{\overline{w}} g(w) dF(w) > R \Longrightarrow \exists \lambda > 0 \text{ with } g'(w) = g(w) - \lambda \in G'',$$

then there is a majority rule equilibrium.

Example 2: Fix $k, \theta > 0$.

$$\begin{aligned} G'' \equiv & \Big\{ g \in G \mid g(w) = \beta w^2 + \rho w + \gamma; \beta = i \cdot \frac{k}{2 \inf_{w \in [\underline{w}, \overline{w}]} f(w)}, i = 0, \pm 1, \pm 2, ..., \pm \bar{i}; \\ & |2\beta w + \rho| \leq \frac{\theta}{2 \sup_{w \in [\underline{w}, \overline{w}]} f(w)} \text{ for all } w \in [\underline{w}, \overline{w}] \Big\} \end{aligned}$$

Then G'' satisfies the conditions of Corollary 1 with, say, a bounded Pareto distribution of income:

$$\begin{aligned} f(w) &= \frac{\alpha \underline{w}^\alpha w^{-\alpha-1}}{1 - \left(\frac{\underline{w}}{\overline{w}}\right)} \\ \text{where } \alpha &> 0 \text{ is a parameter} \end{aligned}$$

[9] Although we have not defined double crossing formally, its definition should be apparent. We will use this concept only in remarks.

and with appropriate c, c', R, $\underline{w}$ and $\overline{w}$ ($R \leq \underline{w}$).

Proof of Corollary 1: See Appendix.

6.2 The model with endogenous income

We now turn to the voting model with incentives based on Mirrlees (1971).

6.2.1 Notation

There are two goods in the model. There is a consumption good, where quantity consumed is denoted by $c \geq 0$, and labor, where labor supply is denoted by l, $0 \leq l \leq 1$. Consumers have an endowment of 1 unit of *labor/leisure* and no consumption good. Let $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ be the utility function of the agents, writing $u(c, l)$ as the utility function of every type w. We further assume that u is differentiable. The parameter w, an agent's type, is now to be interpreted as the wage rate or productivity of an agent. Thus w is the value of an agent of type w's endowment of labor. The gross income earned by an agent of type w is $y = w \cdot l$, and it is equal to consumption $c \geq 0$ when there are no taxes.

A *tax system* is a function $\tau : \mathbb{R} \to \mathbb{R}$ that takes y to tax liability.

First we discuss the typical consumer's problem under the premise that the consumer does not lie about its type, and later turn to incentive problems. A consumer of type $w \in [\underline{w}, \overline{w}]$ is confronted with the following maximization problem in this model:

$$\begin{aligned} \max_{c,l} u(c, l) \text{ subject to } w \cdot l - \tau(w \cdot l) &\geq c \text{ with } \tau \text{ given,} \\ \text{and subject to } c &\geq 0,\, l \geq 0,\, l \leq 1. \end{aligned}$$

For fixed τ, we call arguments that solve this optimization problem $c(w)$ and $l(w)$ (omitting τ) as is common in the literature. Define $y(w) \equiv w \cdot l(w)$. We call y the gross income function corresponding to a tax τ and y' the gross income function corresponding to a tax τ'.

The basic set of tax functions for the optimal income tax model is defined as:

$$T \equiv \{\tau : \mathbb{R}_+ \to \mathbb{R} \mid \tau \text{ is measurable and for all } y \in \mathbb{R}_+ \; \tau(y) \leq y\}$$

Remark 3: In an economy without a labor/leisure decision, g is a lump sum tax function on endowments or exogenous income. In an economy with a labor/leisure decision, it is defined by $g(w) \equiv \tau(y(w))$. Given a g, there are sufficient conditions (e.g. g is nondecreasing) for implementing it in terms of τ and y.[10] Given a τ, there is a resulting $y(w)$ from consumer optimization, and thus a resulting $g(w)$. Thus, existence of g is necessary and sufficient (under some weak conditions) for an incentive compatible tax function τ.

Next we recall some assumptions from Berliant and Page (2001) specialized to our setting:

Definition 10: A utility function u is called *well-behaved* if it is continuous and for all $l \in [0, 1]$ it is strictly increasing in consumption c.

Definition 11: A utility function u is said to satisfy the *boundary condition* if for all $0 \leq c \leq w$, $u(0, 0) > u(c, 1)$.

The boundary condition simply says that leisure is essential.

Definition 12: A collection $T' \subseteq T$ of functions mapping $\mathbb{R}_+$ into $\mathbb{R}$ is called *strongly single crossing* if for all $\tau, \tau' \in T'$, letting $y(\cdot), y'(\cdot)$ be the gross income functions associated with τ and τ', respectively, for incomes $y_1, y_2, y_3 \in y([\underline{w}, \overline{w}]) \cap y'([\underline{w}, \overline{w}])$, $y_1 < y_2 < y_3$, $\tau(y_3) < \tau'(y_3)$ and $\tau(y_2) > \tau'(y_2)$ implies $\tau(y_1) \geq \tau'(y_1)$.

Although we have used the same terminology for single crossing in the cases of exogenous and endogenous incomes, it should be clear from the context (and the domain of the functions) which definition applies.

6.2.2 The main results

Definition 13: Fix $T' \subseteq T$. Then $\tau \in T'$ is called a *majority voting equilibrium* if $\int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w) \geq R$ and there is no $\tau' \in T'$ with $\int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) \geq R$ and measurable $W \subseteq [\underline{w}, \overline{w}]$ with

$$u\left(y(w) - \tau(y(w)), \frac{y(w)}{w}\right) < u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right)$$

$a.s.(w \in W)$ and $\int_W dF(w) > \frac{1}{2}$.

[10]See Berliant and Gouveia (2001) and Berliant and Page (1996).

Definition 14: $T' \subseteq T$ is said to satisfy *Condition P'* if:

$$\text{For all } \tau, \tau' \in T', \tag{P'}$$
$$\int_{\left\{w \in [\underline{w}, \overline{w}] \mid u\left(y(w) - \tau(y(w)), \frac{y(w)}{w}\right) < u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right)\right\}} dF(w) > \frac{1}{2} \implies$$
$$u\left(y(w^*) - \tau(y(w^*)), \frac{y(w^*)}{w^*}\right) < u\left(y'(w^*) - \tau'(y'(w^*)), \frac{y'(w^*)}{w^*}\right)$$
$$\text{or } \int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) < \int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w)$$

Definition 15: $T' \subseteq T$ is said to satisfy *Condition M'* if:

$$\text{There exists } \widehat{\tau} \in T' \text{ with } \int_{\underline{w}}^{\overline{w}} \widehat{\tau}(\widehat{y}(w))dF(w) \geq R, \tag{M'}$$
$$\text{and for all } \tau \in T' \text{ with } \int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w) > R \implies$$
$$\text{there exists } \tau' \in T' \text{ with } \tau'(y'(w)) \leq \tau(y(w)) \ a.s.(F),$$
$$\tau'(y'(w^*)) \leq \tau(y(w^*)),^{11}$$
$$\text{and } \int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) = R$$

Remark 3: If F is atomless and the class of tax functions T' is sufficiently broad, Theorem 2 of Berliant and Page (2006) implies that condition (M') is satisfied.

Theorem 2: If utility is well-behaved and satisfies the boundary condition, and $T' \subseteq T$ satisfies Conditions (P') and (M') and is closed in the topology of pointwise convergence, then there exists a majority voting equilibrium, and it is among those satisfying the government budget constraint most preferred by the median voter.

Proof: These assumptions on the utility function are stronger than those used in Berliant and Page (2001). There, the measure representing the social welfare weights in the Mirrlees context is called μ. Set the measure μ of consumers used there to be an atom with measure 1 at w^*, zero elsewhere.[12]

[12] There is a technical issue just here that must be addressed. Berliant and Page (2001) use an additional assumption, that F and μ are *equivalent* in the sense that they have the

Berliant and Page (2001, Theorem 1) shows that there is a $\tau^* \in T'$ that maximizes $u\left(y(w^*) - \tau(y(w^*)), \frac{y(w^*)}{w^*}\right)$ subject to $\int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w) \geq R$. However, this optimum is unconstrained by T'. Given Theorem 2 of that paper on the equivalence of the μ-tax design problem and the μ-menu design problem, T' closed in the topology of pointwise convergence implies existence of an optimum for the μ-menu design problem constrained to menus corresponding to taxes in T', implying a solution to the μ-tax design problem constrained to T'.

In fact under condition (M') we can take $\int_{\underline{w}}^{\overline{w}} \tau^*(y^*(w))dF(w) = R$. Now suppose that there is $\tau' \in T'$ such that

$$\int_{\left\{w \in [\underline{w},\overline{w}] \mid u\left(y^*(w) - \tau^*(y^*(w)), \frac{y^*(w)}{w}\right) < u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right)\right\}} dF(w) > \frac{1}{2}$$

and

$$\int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) \geq R.$$

Under condition (P') it follows that

$$\begin{aligned} u\left(y(w^*) - \tau(y(w^*)), \frac{y(w^*)}{w^*}\right) \quad &< \quad u\left(y'(w^*) - \tau'(y'(w^*)), \frac{y'(w^*)}{w^*}\right) \\ \text{or} \int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) \quad &< \quad \int_{\underline{w}}^{\overline{w}} \tau^*(y^*(w))dF(w) = R \end{aligned}$$

Either condition results in a contradiction.

Evidently, Pareto domination of one tax system τ' over another τ, namely $\tau'(y) \leq \tau(y)$ for all $y \in [0, \overline{w}]$, implies the strongly single crossing property. Notice that if T' is strongly single crossing, then it satisfies condition (P'). It is an extreme example since it always satisfies the first consequent. Next we provide another extreme example that always satisfies the second consequent.

Example 3: Let $T'' \equiv \{\tau \in T \mid \tau(y) = 1_A(y)$ for some measurable $A \subseteq \mathbb{R}_+\}$,[13] and let F be nonatomic. Notice that A need not be connected. Each element of T'' is a $0-1$ tax, but the idea can be extended in obvious ways. Let $R \leq \underline{w}$.

same sets of measure zero. That obviously doesn't apply directly here, since all weight is given to the median voter. With a finite number of voters, weight 1 could simply be assigned to the median voter. The easy way to address this with a continuum of voters, preserving atomlessness, is to create a measure space of agents with a continuum having type w^*, and give them weight 1.

[13] Here we use an indicator function on gross income. We have not defined that, but the definition is obvious. This is the only place we use it.

Now take $\tau, \tau' \in T''$ and suppose that

$$\int_{\left\{w\in[\underline{w},\overline{w}]|u\left(y(w)-\tau(y(w)),\frac{y(w)}{w}\right)<u\left(y'(w)-\tau'(y'(w)),\frac{y'(w)}{w}\right)\right\}} dF(w) > \frac{1}{2}.$$

We claim that:

$$\begin{aligned} &\left\{w \in [\underline{w}, \overline{w}] \mid u\left(y(w) - \tau(y(w)), \frac{y(w)}{w}\right) < u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right)\right\} \\ \subseteq \;& \left\{w \in [\underline{w}, \overline{w}] \mid \tau'(y'(w)) < \tau(y'(w))\right\}. \end{aligned}$$

For if this were not true for some w, then $\tau'(y'(w)) \geq \tau(y'(w))$, so this type can choose $y'(w)$ under tax system τ, resulting in

$$u\left(y'(w) - \tau(y'(w)), \frac{y'(w)}{w}\right) \geq u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right) > u\left(y(w) - \tau(y(w)), \frac{y(w)}{w}\right)$$

contradicting the definition of $y(w)$.

Hence,

$$\int_{\{w\in[\underline{w},\overline{w}]|\tau'(y'(w))<\tau(y'(w))\}} dF(w) > \frac{1}{2}.$$

Define $g'(w) \equiv \tau'(y'(w))$, $g(w) \equiv \tau(y'(w))$, and the set $B = \{w \in [\underline{w}, \overline{w}] \mid g'(w) < g(w)\}$. So $g'(w) = 0$ for $w \in B$, $g(w) = 1$ for $w \in B$. Hence,

$$\int_{\underline{w}}^{\overline{w}} g'(w)dF(w) = \int_{B^c} dF(w) < \frac{1}{2} < \int_B g(w)dF(w) \leq \int_{\underline{w}}^{\overline{w}} g(w)dF(w)$$

and the second consequent of condition (P') is satisfied. Condition (M') follows from Lyapunov's theorem.

Definition 16: $T'' \subseteq T$ is said to be *zero ordered* if for all $\tau \in T''$, τ is C^0 and for all $\tau, \tau' \in T''$, either $\tau(y) \geq \tau'(y)$ for all $y \in [0, \overline{w}]$, or $\tau'(y) \geq \tau(y)$ for all $y \in [0, \overline{w}]$.

Definition 17: $T'' \subseteq T$ is said to be *first ordered* if for all $\tau \in T''$, τ is C^1 and for all $\tau, \tau' \in T''$, either $\frac{d\tau(y)}{dy} \geq \frac{d\tau'(y)}{dy}$ for all $y \in [0, \overline{w}]$, or $\frac{d\tau'(y)}{dy} \geq \frac{d\tau(y)}{dy}$ for all $y \in [0, \overline{w}]$.

Recall that $\phi \geq 1$ is the population parameter.

Definition 18: $T'' \subseteq T$ is said to be *second ordered* if for all $\tau \in T''$, τ is C^2 and there are constants $k', \theta', \eta > 0$ such that for all $\tau, \tau' \in T''$, either $\frac{d^2\tau(y)}{dy^2} - \frac{d^2\tau'(y)}{dy^2} = 0$ for all $y \in [0, \overline{w}]$, or $\frac{d^2\tau(y)}{dy^2} - \frac{d^2\tau'(y)}{dy^2} \geq \frac{k'}{\inf_{w\in[\underline{w},\overline{w}]} f(w)} \equiv k$ for all $y \in [0, \overline{w}]$, or $\frac{d^2\tau'(y)}{dy^2} - \frac{d^2\tau(y)}{dy^2} \geq \frac{k'}{\inf_{w\in[\underline{w},\overline{w}]} f(w)} \equiv k$ for all $y \in [0, \overline{w}]$. Moreover, $\left|\frac{d\tau'(y_L)}{dy} - \frac{d\tau(y_L)}{dy}\right| \leq \frac{\theta'}{\sup_{w\in[\underline{w},\overline{w}]} f(w)} \equiv \theta$ for $y_L = \min_{\tau'(y)=\tau(y)} y$. For all $\tau \in T''$, for all $y \in [0, \overline{w}]$, $\left(1 - \frac{d\tau}{dy}\right)^2 \leq \eta$, where $\eta > k$. Finally, there is a linear (or constant) tax in T''.

Remark 4: Zero ordered is the Pareto preorder. *First ordered* is a strengthening of single crossing. *Second ordered* is a strengthening of double crossing. Notice that, for $\tau, \tau' \in T$, if τ Pareto dominates τ', then τ single crosses τ'. Notice also that if τ single crosses τ', τ double crosses τ'. One could go on to define third ordered and so forth. For zero and first ordered, the revenue part of condition (P') doesn't come into play. It does for second ordered. For concreteness, think of $f(w)$ as the uniform density.

Definition 19: A utility function u is called *log-linear* if $u(c, l) = c + \ln(1 - l)$. For density f and $\phi \geq 1$, define $f_\phi(w) = f(\frac{w}{\phi})$, where f_ϕ has support $[\phi \underline{w}, \phi \overline{w}]$. Let F_ϕ be the associated distribution function.

Corollary 2: Suppose that $\underline{w} = 0$, u is log-linear and $R \leq \int_{\underline{w}}^{\overline{w}} w dF(w)$, and that $T'' \subseteq T$ is second ordered and closed in the topology of pointwise convergence, and

$$\tau \in T'' \text{ and } \int_0^{\overline{w}} \tau(y(w)) dF(w) > R \Longrightarrow \exists \lambda > 0 \text{ with } \tau'(y'(w)) = \tau(y(w)) - \lambda \in T''.$$

Then fixing the other parameters, for all sufficiently large ϕ, there is a majority rule equilibrium.

Example 4: Fix $k', \theta' > 0$ so that, defining k and θ as in the definition of second ordered, it's possible to take $\eta > k$ such that $\sqrt{\eta} + 1 \geq \theta/2$:

$$\begin{aligned} T'' \equiv & \left\{ \tau \in T \mid \tau(y) = \beta y^2 + \rho y + \gamma; \beta = i \cdot \frac{k'}{2 \inf_{w \in [\underline{w}, \overline{w}]} f(w)}, i = 0, \pm 1, \pm 2, ..., \pm \bar{i}; \right. \\ & \left. |2\beta y + \rho| \leq \frac{\theta'}{2 \sup_{w \in [\underline{w}, \overline{w}]} f(w)} = \theta/2 \leq \sqrt{\eta} + 1 \text{ for } y \in [0, \overline{w}] \right\} \end{aligned}$$

Then T'' satisfies the conditions of Corollary 2 with, say, a bounded Pareto distribution of types:

$$\begin{aligned} f(w) &= \frac{\alpha(w+1)^{-\alpha-1}}{1 - \left(\frac{1}{\overline{w}+1}\right)} \\ \text{where } \alpha &> 0 \text{ is a parameter} \end{aligned}$$

and with appropriate R and $\overline{w}$, and ϕ large enough.

Proof of Corollary 2: See Appendix.

Remark 5: Although the assumptions of Corollary 2 are restrictive and its proof is messy, Corollary 2 gives an idea of the critical assumptions, conclusions and arguments as well as how they might be extended. It is likely that extensions of the corollary will involve taking the upper bound of the distribution of types and population large enough, as we have done using the parameter ϕ. One key argument, particularly relative to the exogenous income model, is the bounding of $\frac{dw}{dy}$ above and below. This relies crucially on the utility function, but likely can be generalized using the same arguments as in our proof.

6.3 Tax reform

Until now, we have considered majority voting equilibria in the context of a static model. That is, we have examined the existence and characterization of majority voting equilibrium (or a related concept in the survey portion of this essay) within a given set of tax systems. In our specific framework, it is easy and useful to convert our framework into one capable of examining tax reforms following Bierbrauer et al. (2021).

For simplicity and specificity, take the context of Theorem 2. Take a *status quo* tax $\tau \in T$. The set of potential tax reforms $T'(\tau)$ satisfies the following conditions:

Definition 20: $T'(\tau) \subseteq T$ is said to satisfy *Condition P'(τ)* if:

$$\tau \in T'(\tau) \text{ and for all } \tau' \in T'(\tau), \tag{P'(t)}$$

$$\int_{\left\{w\in[\underline{w},\overline{w}] | u\left(y(w)-\tau(y(w)),\frac{y(w)}{w}\right)<u\left(y'(w)-\tau'(y'(w)),\frac{y'(w)}{w}\right)\right\}} dF(w) > \frac{1}{2} \implies$$

$$u\left(y(w^*) - \tau(y(w^*)), \frac{y(w^*)}{w^*}\right) < u\left(y'(w^*) - \tau'(y'(w^*)), \frac{y'(w^*)}{w^*}\right)$$

$$\text{or} \int_{\underline{w}}^{\overline{w}} \tau'(y'(w))dF(w) < \int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w)$$

This will form a *larger set* than monotonic reforms, which correspond to single crossing ones in the static case. For example, we could look at second ordered reforms, where τ' is *second ordered* relative to τ, but not necessarily monotonic relative to τ; in other words $\tau' - \tau$ is not necessarily a monotone function, and in fact τ' and τ could cross twice. That would be a special case of $T'(\tau)$ that is different from monotonic tax reforms.

Corollary 3: Fix a status quo tax $\tau \in T$, and set revenue $R = \int_{\underline{w}}^{\overline{w}} \tau(y(w))dF(w)$. If $T'(\tau) \subseteq T$ satisfies Condition (P'(t)), then either τ is a majority voting equilibrium in $T'(\tau)$, or there is $\tau' \in T'(\tau)$ that majority defeats τ, where τ' increases utility over τ for the median type w^* and generates at least as much revenue as τ.[14]

Thus, tax reform can proceed sequentially, moving from a *status quo* tax $\tau_0 \in T$ to a tax $\tau_1 \in T'(\tau_0)$ that majority dominates τ_0 but generates at least as much revenue as τ_0, and then to a tax $\tau_2 \in T'(\tau_1)$, and so forth. Here, we are assuming all agents are myopic. Since τ_i could be one of many possible taxes that majority defeats τ_{i-1}, there is *path dependence* in the sequence of tax reforms. We can refine it a bit by taking a tax $\tau_i \in T'(\tau_{i-1})$ that majority defeats τ_{i-1} and delivers the highest utility for the median voter; if utility is well-behaved and satisfies the boundary condition, and $T'(\tau) \subseteq T$ is closed under pointwise convergence, then such a tax exists. The reform process stops when no tax in $T'(\tau_i)$ majority defeats τ_i itself.

Since the utility of the median voter rises monotonically in this process, if $T'(\cdot)$ is a correspondence that is lower hemicontinuous in the pointwise convergence topology, then existence of a limit to the tax reform process follows from an argument similar to that in the proof of Theorem 2 above: For $\{\tau_i\}_{i=0}^{\infty}$, take a corresponding sequence of menus, extract a converging subsequence of menus and a limit menu, and then convert the limit menu to a tax function. Call this tax function τ'. If a tax schedule $\tau'' \in T'(\tau')$ majority rule beats τ', then the median voter utility under τ'' is higher than under τ', and by lower hemicontinuity there is a sequence $\tau_i'' \in T'(\tau_i)$ that will both majority rule beat $\tau_i \in T'(\tau_i)$ (due to Lebesgue's dominated convergence theorem applied to $\int_{\left\{w\in[\underline{w},\overline{w}] | u\left(y_i(w)-\tau_i(y_i(w)), \frac{y_i(w)}{w}\right) < u\left(y_i''(w)-\tau_i''(y_i''(w)), \frac{y_i''(w)}{w}\right)\right\}} dF(w)$) and yield higher utility to the median voter than τ_{i+1}. That is a contradiction.

Since the median voter, in a sense, controls the tax reform process, Corollary 3 implies that a sequence of politically feasible tax reforms should push tax rates in the direction of the second best efficiency lower bound for incomes below the median and, possibly, in the direction of the second best efficiency upper bound for incomes above the median.[15] Mechanically, this should lead to more pronounced progression over an intermediate range of incomes. There is a connection between this result and the main result in Brett and Weymark (2016, 2017): whereas the restriction on the policy domains are

[14]Notice that τ' itself might not be a majority voting equilibrium in $T'(\tau)$.

[15]Implicit in this discussion are the incentive constraints; they imply a second best context.

different (we impose Condition (P'(t)), whereas they restrict their attention to selfishly-optimal tax schemes), the equilibrium outcome shares the characteristic of sharp progressivity in the middle of the income distribution. An implication is that it might not be possible to distinguish the two varieties of models empirically.

7 Concluding remarks and open challenges

We conclude by summarizing the main takeaways, linking the theoretical results to empirical observations and then suggest avenues for future research.

7.1 Summary of the theoretical literature

This paper reviews the literature on voting over income taxes in the following sequence. First we examine models where citizens have exogenous income. Second we turn to models where citizens have endogenous income. Finally, we explore the connections between the two. At the core of our survey is the literature on voting over income tax schedules in the context of endogenous income without *a priori* restrictions on the functional form of taxes, in the tradition of the normative literature following Mirrlees (1971). New theoretical results generalizing those found in the literature are derived.

7.2 Policy-making and income tax schedules in the US: Three empirical patterns

We discuss three important empirical patterns that are characteristic of the US income tax system. They are puzzling when viewed from a normative or optimal income tax perspective, or when compared to the predictions of prominent political economy models. Instead, they can be better understood using a *tax reform* approach to voting over income taxes. This section builds on Bierbrauer et al. (2021). Our purpose here is to examine the implications of condition (P'(t)) for real tax reform in the US.

7.2.1 The sharp decrease in the top marginal tax rate over time

The statutory top marginal tax rate is an object of contention in politics. The last 50 years have witnessed a sharp overall decrease in this rate, a decrease that has slowed down or been reversed slightly under some administrations (Bush, Clinton, Obama). Few presidential elections are fought without proposals to change this rate (see the review of recent campaign proposals in Bierbrauer et al., 2021).[16]

In order to discuss the political forces that might affect this rate, flexibility in a non-linear tax schedule is clearly required. It is hard to rationalize this empirical pattern with existing models.

The insight offered by the tax reform approach is as follows. Begin with a Pareto efficient tax system.[17] A top marginal tax rate decrease *alone* would not be politically feasible. But if marginal tax rates are decreased at the top of the income distribution at the same time as marginal tax rates for taxpayers from the top down to the median voter are decreased, then tax reforms decreasing the top tax rate could be supported politically. The key is that the decrease in tax liability for these taxpayers could be compensated by an increase in tax rates for taxpayers below the median. The revenue implications of the reform are captured by the elasticities of taxable income over the range of taxpayers. This is entirely consistent with condition (P'(t)). Examples include the tax reforms of the Reagan, Bush and Trump presidencies that can be made politically feasible for elasticities of taxable income sufficiently high. Often such level of elasticity was considered plausible at the time of the reforms.

7.2.2 Introduction and extension of the Earned Income Tax Credit

The Earned Income Tax Credit (EITC) provides a refundable tax credit to lower income working families. Expansions of the credit since its introduction in 1975 have made the EITC a central element of the US tax and transfer system.[18] In 2019, 25 million eligible workers and families received about $63 billion from the EITC. The average amount of EITC received was about $2,476.[19] The EITC schedule has been expanded several times since its inception in 1975. In today's tax-and-transfer schedule, negative effective and

[16]An important issue related to this discussion is the difference between the top statutory marginal tax rate and the top effective marginal tax rate. The latter might differ by taxpayer, for example due to the alternative minimum tax. The top statutory marginal tax rate might not actually be faced by any taxpayer, but might be purely of symbolic value. Effective marginal tax rates using the NBER TAXSIM microsimulation model of recent major reforms can be found in Figure E.4 of Bierbrauer et al. (2021).

[17]As in Diamond (1998) and Diamond and Saez (2011), the top marginal tax rate might not be zero.

[18]For surveys, see Moffitt (2003), Hotz and Scholz (2003), Nichols and Rothstein (2015), and Hoynes (2019).

[19]These statistics come from the Internal Review Service EITC Fast Facts.

marginal tax rates at low earned incomes are a consequence of the EITC.[20]

Rationales for negative marginal tax rates in the normative literature, that would be the outcome of the maximization of a utilitarian social welfare function, are not obvious. Mirrlees (1971) stipulates non-negative marginal tax rates for all levels of income.[21] Thus, the EITC is a challenge for the theory of optimal taxation. In response to that challenge, Saez (2002) suggested the use of an extended version of the Mirrlees model that includes fixed costs of labor market participation and gives rise to behavioral responses both at the intensive and the extensive margin. With such a framework, the EITC can be justified as being part of a policy that is, in a utilitarian sense, optimal.[22] With a tax reform perspective, Bierbrauer et al. (2020) find that the EITC can be rationalized under weaker conditions than using an optimal tax perspective. First, they find that the introduction of the EITC was Pareto-improving, and not just utilitarian-welfare-improving. Second, when exploring alternative assumptions about intensive and extensive margin elasticities, they find that the EITC was Pareto-improving even without behavioral responses at the extensive margin. Thus, the introduction of the EITC was a good idea – even under the behavioral assumptions of the basic Mirrlees model.[23]

To the best of our knowledge, the first paper bringing a political economy perspective to the EITC is Bierbrauer et al. (2021). The key insight (Theorem 2 in Bierbrauer et al., 2021) is that within second best Pareto bounds, a small tax reform pushing marginal tax rates into the negative region for a bracket of income is politically feasible. Intuitively, a reform proposing negative marginal tax rates in a segment of income favors the individuals that get a reduction of the tax liability, i.e. all individuals that have income weakly above this segment. Once the loss of revenue for individuals above the segment is out-

[20]Whereas we present evidence only for the US, negative income tax rates are present in other countries, e.g. the United Kingdom (due to the working tax credit) or France (due to the Prime d'activité).

[21]Negative marginal tax rates can be rationalized in the basic version of the Mirrlees model only with a welfare function that has non-monotonic welfare weights, e.g., one that assigns higher weights to people with low or middle income than to people with no income; see Stiglitz (1982), Choné and Laroque (2011) or Brett and Weymark (2017).

[22]Follow-up papers include Jacquet, Lehmann and Van der Linden (2013) and Hansen (2021). Saez and Stantcheva (2016) enrich the traditional welfarist approach to account for current tax policy debates while maintaining the desirability of Pareto efficiency.

[23]Kleven (2020) recently suggests that previous estimates of extensive margin elasticities were too high. While this debate has a bearing on the desirability of the EITC from an optimal tax perspective, it is of no consequence for the conclusion of Bierbrauer et al. (2020) that the introduction of the EITC was Pareto-improving.

weighed by the reduction of their tax burden, they benefit from such a reform. If a majority of taxpayers is in this range of income, the reform is preferred by a majority.[24]

The major problem faced by the theoretical analysis of negative marginal tax rates is that, as Mirrlees and others realized early on, the second order conditions for incentive compatibility will be violated, so the first order approach to incentive compatibility will not function properly. Taxpayers will be at local minima subject to the budget constraint modified for taxes, and type will decrease with earned income when the first order approach is applied; see Figure 3 and footnote 17 of Berliant and Gouveia (2001). This is still consistent with the single crossing property and concavity of the utility function. However, the second order conditions are irrelevant for the theory we have proposed in the preceding sections, which does not rely on the first order approach to incentive constraints.[25]

7.2.3 Sharp progressivity around the median of the income distribution

Corollary 3 implies that a sequence of politically feasible tax reforms should push tax rates in the direction of the lower second best efficiency bound for below median incomes and, possibly, in the direction of the upper second best efficiency bound for above median incomes. Mechanically, this should lead to more pronounced progression over an intermediate range of incomes.[26] Indeed, negative marginal tax rates have to be phased out quickly over a range of incomes above median income, leading to a large increase in marginal tax rates in this middle segment of the income distribution.

To see this pattern for the US, we document the evolution of effective marginal tax rates in Figure 1 by plotting the pre- and the post-reform values.

[24] The political economy of more sophisticated reforms involving two brackets is studied in Bierbrauer et al. (2020).

[25] Additional caveats are in order. The sufficient statistics approach to the Mirrlees model relies on the first order approach to incentive compatibility, and verification of the second order conditions typically relies in turn on assumptions about endogenous objects. In addition, in the context of social welfare maximization, the second order conditions for the objective function, often utilitarian, are not verified. Thus, local minima, local maxima, and inflection points might be characterized. Finally, the second best utility possibility set might not be convex, so not all welfare optima can be found using a utilitarian objective.

[26] Whereas we present suggestive evidence of this pattern only for the US, similar observations can be made for Germany (where it is referred to as the "Mittelstandsbauch" or middle class belly), the Netherlands (see Jacobs, Jongen and Zoutman, 2017), and France.

Here we employ standard acronyms and year codes for tax bills in the US. The transition from RA78 to ATRA12 reveals that there was indeed a lowering of marginal tax rates for low incomes and increased progression for incomes that were somewhat higher. These changes are associated with the introduction and then the expansion of the EITC. The EITC led to lower, in fact negative, marginal tax rates for the working poor. Low-income households with children were the main recipients of these earnings subsidies. The negative marginal tax rates were phased out over a range of higher incomes, beginning with the income level qualifying for the maximal credit. This led to a big increase in marginal tax rates in the next higher segment of the income distribution, including a transition from negative to positive marginal tax rates.

In contrast, Figure 1 does not reveal a strong tendency towards higher marginal tax rates above the median. The sufficient statistics presented in Bierbrauer et al. (2020) provide a possible explanation: the conclusion that there was room to reduce marginal tax rates for the poor is robust to alternative assumptions about the elasticity of taxable income (ETI). This is not true for higher taxes on the rich. With an ETI around 1, which has been considered plausible by scholars since the 1990s, such tax increases appear to be Pareto inferior to the *status quo*. However, this analysis begs the question of revenue neutrality. In political terms, is the government budget constraint binding for tax reform?

7.3 Open challenges

The dynamics of tax reforms is an important area for future research. True dynamics is complicated, as all agents (including the government) have foresight. Moreover, there are issues of whether or not the government can commit to future tax regimes, as well as whether the government can remember type revelation by taxpayers from previous periods, if that is useful to it. For example, if consumers have foresight and are patient, then they may pool early on so as not to reveal their types, separating only later. If they are myopic, then they might separate immediately, revealing their types, so that type-specific lump sum taxes can be used in subsequent periods; see Berliant and Ledyard (2014).

The standard techniques for handling these problems is to have types or productivity change randomly from period to period (see Fernandes and Phelan, 2000), and use the first order approach to the incentive constraints; the

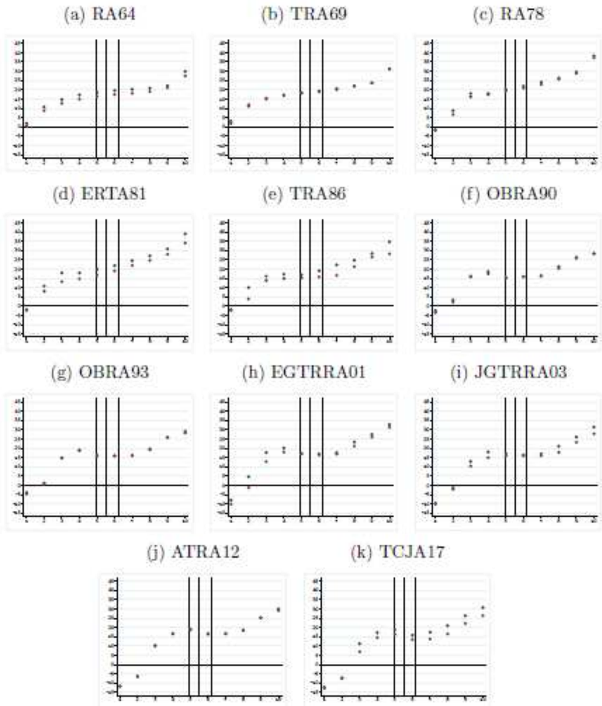

Figure 1: Effective marginal tax rates by decile before and after each reform

Notes: Figure 1 shows, separately for each decile, the effective marginal tax rates (EMTRs) before (blue) and after (red) major reforms of the US federal personal income tax (see Table H.1 in Bierbrauer et al. (2021) for details). Deciles are computed based on pre-tax income without capital gains while tax base includes capital gains. All computations are on the individual level. For this, the income of couples filing jointly is allocated equally to each spouse. In order to simulate counterfactual tax payments income from year 0 are inflated to year 1 using the CPI-U-RS deflator as uprating factor. The vertical lines show different locations for the median voter: the dashed line to the left imputes non-filers to the tax return data while the dashed line to the right accounts for differential turnout by income. The solid line in the middle represents both the original median in the data as well as the one accounting for both modifications simultaneously.

Source: Figure E.4 in Bierbrauer et al. (2021) based on NBER TAXSIM and IRS-SOI PUF.

second order conditions for incentive compatibility are then typically checked numerically after the optimum is found. For a nice example, see Kapička (2013).

These issues are present even in models of optimal taxation when politics are not modeled.

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Appendix (not for publication)

A Proof of Corollary 1

Notice that in the case $\frac{d^2g(w)}{dw^2} - \frac{d^2g'(w)}{dw^2} = 0$ for all $w \in [\underline{w}, \overline{w}]$, the tax systems differ by a linear function, so the standard single crossing applies. We do not discuss this case further.

First we check condition (P). Suppose that $g, g' \in G''$, $\int_{\{w \in [\underline{w}, \overline{w}] | g'(w) < g(w)\}} dF(w) > \frac{1}{2}$ and $g'(w^*) \geq g(w^*)$. We must show $\int_{\underline{w}}^{\overline{w}} g'(w)dF(w) < \int_{\underline{w}}^{\overline{w}} g(w)dF(w)$. If g, g' single cross, then we have a contradiction to $\int_{\{w \in [\underline{w}, \overline{w}] | g'(w) < g(w)\}} dF(w) > \frac{1}{2}$ and $g'(w^*) \geq g(w^*)$. So they cross at least twice. They cannot cross more than twice because they are second ordered. At the left crossing point, which we call w_L, $g'(w_L) = g(w_L)$ and $\frac{dg'(w_L)}{dw} > \frac{dg(w_L)}{dw}$. At the right crossing point, which we call w_R, $g'(w_R) = g(w_R)$ and $\frac{dg'(w_R)}{dw} < \frac{dg(w_R)}{dw}$. So $\frac{d^2g(w)}{dw^2} \geq \frac{d^2g'(w)}{dw^2}$ for all $w \in [\underline{w}, \overline{w}]$. Now for all $w < w_L$, $\frac{dg'(w)}{dw} > \frac{dg(w)}{dw}$, whereas for all $w > w_R$, $\frac{dg'(w)}{dw} < \frac{dg(w)}{dw}$.

$$
\begin{aligned}
& \int_{w_R}^{\overline{w}} \left[g(w) - g'(w)\right] dF(w) \\
= & \int_{w_R}^{\overline{w}} \int_{w_R}^{w} \left[\frac{dg(w')}{dw} - \frac{dg'(w')}{dw}\right] dw' dF(w) \\
= & \int_{w_R}^{\overline{w}} \int_{w_R}^{w} \left(\int_{w_R}^{w'} \left[\frac{d^2 g(w'')}{dw^2} - \frac{d^2 g'(w'')}{dw^2}\right] dw'' + \frac{dg(w_R)}{dw} - \frac{dg'(w_R)}{dw} \right) dw' dF(w) \\
\geq & \int_{w_R}^{\overline{w}} \int_{w_R}^{w} \int_{w_R}^{w'} \frac{k}{f(w)} dw'' dw' dF(w) \\
= & \int_{w_R}^{\overline{w}} \int_{w_R}^{w} k \cdot \left(w' - w_R\right) dw' dw \\
= & \; k \cdot \int_{w_R}^{\overline{w}} \int_{w_R}^{w} \left(w' - w_R\right) dw' dw \\
= & \; k \cdot \int_{w_R}^{\overline{w}} \left(\frac{w^2}{2} - \frac{w_R^2}{2} - w w_R + w_R^2 \right) dw \\
= & \; k \cdot \int_{w_R}^{\overline{w}} \left(\frac{w^2}{2} + \frac{w_R^2}{2} - w w_R \right) dw \\
= & \; \frac{k}{2} \cdot \int_{w_R}^{\overline{w}} (w - w_R)^2 \, dw \\
= & \; \frac{k}{6} \cdot (\overline{w} - w_R)^3
\end{aligned}
$$

A similar calculation for the left side yields:

$$
\int_{\underline{w}}^{w_L} \left[g(w) - g'(w)\right] dF(w) \geq \frac{k}{6} \cdot (w_L - \underline{w})^3
$$

$$
\begin{aligned}
& \int_{w_L}^{w_R} \left[g(w) - g'(w)\right] dF(w) \\
= & \int_{w_L}^{w_R} \int_{w_L}^{w} \left[\frac{dg(w')}{dw} - \frac{dg'(w')}{dw}\right] dw' dF(w)
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{w_L}^{w_R} \int_{w_L}^{w} \left(\int_{w_L}^{w'} \left[\frac{d^2 g(w'')}{dw^2} - \frac{d^2 g'(w'')}{dw^2}\right] dw'' + \frac{dg(w_L)}{dw} - \frac{dg'(w_L)}{dw} \right) dw' dF(w) \\
\geq & \int_{w_L}^{w_R} \int_{w_L}^{w} \int_{w_L}^{w'} k dw'' dw' dw + \left(\frac{dg(w_L)}{dw} - \frac{dg'(w_L)}{dw} \right) \int_{w_L}^{w_R} \int_{w_L}^{w} dw' dF(w) \\
\geq & \int_{w_L}^{w_R} \int_{w_L}^{w} \int_{w_L}^{w'} k dw'' dw' dw - \theta \int_{w_L}^{w_R} \int_{w_L}^{w} dw' dw
\end{aligned}
$$

$$
\begin{aligned}
&= k \cdot \int_{w_L}^{w_R} \int_{w_L}^{w} \left(w' - w_L\right) dw' dw - \theta \int_{w_L}^{w_R} \int_{w_L}^{w} dw' dw \\
&= k \cdot \int_{w_L}^{w_R} \left(\frac{w^2}{2} - \frac{w_L^2}{2} - w w_L + w_L^2 \right) dw - \theta \int_{w_L}^{w_R} \left(w - w_L\right) dw \\
&= \frac{k}{2} \cdot \int_{w_L}^{w_R} \left(w - w_L\right)^2 dw - \theta \int_{w_L}^{w_R} \left(w - w_L\right) dw \\
&= \frac{k}{6} \cdot \left(w_R - w_L\right)^3 - \frac{\theta}{2} \left(w_R - w_L\right)^2
\end{aligned}
$$

Summing these three terms,

$$
\begin{aligned}
&\int_{\underline{w}}^{\overline{w}} \left[g(w) - g'(w)\right] dF(w) \qquad (1) \\
&\geq \frac{k}{6} \cdot \left(\overline{w} - w_R\right)^3 + \frac{k}{6} \cdot \left(w_L - \underline{w}\right)^3 + \frac{k}{6} \cdot \left(w_R - w_L\right)^3 - \frac{\theta}{2} \left(w_R - w_L\right)^2
\end{aligned}
$$

The last two terms are minimized when $w_R - w_L = \frac{2\theta}{k}$. So

$$
\begin{aligned}
\int_{\underline{w}}^{\overline{w}} \left[g(w) - g'(w)\right] dF(w) &\geq \frac{k}{6} \cdot \left(\overline{w} - w_R\right)^3 + \frac{k}{6} \cdot \left(w_L - \underline{w}\right)^3 + \frac{k}{6} \cdot \left(\frac{2\theta}{k}\right)^3 - \frac{\theta}{2} \left(\frac{2\theta}{k}\right)^2 \\
&= \frac{k}{6} \cdot \left(\overline{w} - w_R\right)^3 + \frac{k}{6} \cdot \left(w_L - \underline{w}\right)^3 + \frac{4\theta^3}{3k^2} - \frac{2\theta^3}{k^2} \\
&= \frac{k}{6} \cdot \left(\overline{w} - w_R\right)^3 + \frac{k}{6} \cdot \left(w_L - \underline{w}\right)^3 - \frac{2\theta^3}{3k^2}
\end{aligned}
$$

Given $\underline{w}$ and $\overline{w}$, the worst case for the first two terms, namely this sum of cubics, is: $\overline{w} - w_R = w_L - \underline{w}$. So

$$
\int_{\underline{w}}^{\overline{w}} \left[g(w) - g'(w)\right] dF(w) \geq \frac{k}{3} \cdot \left(w_L - \underline{w}\right)^3 - \frac{2\theta^3}{3k^2}
$$

We next show that the right hand side is non-negative. The following are equivalent:

$$
\begin{aligned}
\frac{k}{3} \left(\frac{\overline{w} - \underline{w} - \frac{2\theta}{k}}{2} \right)^3 &\geq \frac{2\theta^3}{3k^2} \\
\left(\frac{\overline{w} - \underline{w} - \frac{2\theta}{k}}{2} \right)^3 &\geq \left(\frac{\sqrt[3]{2}\theta}{k} \right)^3 \\
\frac{\overline{w} - \underline{w} - \frac{2\theta}{k}}{2} &\geq \frac{\sqrt[3]{2}\theta}{k} \\
\overline{w} - \underline{w} &\geq \frac{2\sqrt[3]{2}\theta}{k} + \frac{2\theta}{k} = \frac{\left(2 + 2\sqrt[3]{2}\right)\theta}{k}
\end{aligned}
$$

Condition (M) is satisfied because we can shift tax schedules by a constant:

$$
g \in G'' \Longrightarrow g' = g - \lambda \in G'' \text{ for } \lambda \in \mathbb{R}_+
$$

B Proof of Corollary 2

We will verify conditions (P') and (M') for $\underline{w} = 0$ and ϕ sufficiently large. The idea behind the proof is that the set of tax systems T'' does not change with population ϕ, but revenue each tax system does. Some of the tax systems might even be single crossing for lower ϕ, but double crossing for higher ϕ.

Notice that in the case $\frac{d^2\tau(y)}{dy^2} - \frac{d^2\tau'(y)}{dy^2} = 0$ for all $y \in [0, \overline{w}]$, the tax systems differ by a linear function, so standard single crossing applies. We do not discuss this case further.

First we check condition (P'). Suppose that $\tau, \tau' \in T''$,

$$\int_{\left\{w \in [0,\overline{w}] | u\left(y(w) - \tau(y(w)), \frac{y(w)}{w}\right) < u\left(y'(w) - \tau'(y'(w)), \frac{y'(w)}{w}\right)\right\}} dF_\phi(w) > \frac{\phi}{2} \tag{2}$$

and

$$u\left(y(w^*) - \tau(y(w^*)), \frac{y(w^*)}{w^*}\right) \geq u\left(y'(w^*) - \tau'(y'(w^*)), \frac{y'(w^*)}{w^*}\right)$$

We must show $\int_0^{\phi\overline{w}} \tau'(y'(w))dF_\phi(w) < \int_0^{\phi\overline{w}} \tau(y(w))dF_\phi(w)$. If τ, τ' single cross, then we have a contradiction to (2). So they cross at least twice. They cannot cross more than twice because they are second ordered. At the left crossing point, which we call y_L, $\frac{d\tau'(y)}{dy} > \frac{d\tau(y)}{dy}$. At the right crossing point, which we call y_R, $\frac{d\tau'(y_R)}{dy} < \frac{d\tau(y_R)}{dy}$. So $\frac{d^2\tau(y)}{dy^2} \geq \frac{d^2\tau'(y)}{dy^2}$ for all $y \in [0, \overline{w}]$. Now for all $y < y_L$, $\frac{d\tau'(y)}{dy} > \frac{d\tau(y)}{dy}$, whereas for all $y > y_R$, $\frac{d\tau'(y)}{dy} < \frac{d\tau(y)}{dy}$.

Next we compute the first order necessary condition for optimization of log-linear utility subject to the tax system. It determines $y(w)$ and $y'(w)$:

$$\max_{y \in [0,\overline{w}]} y - \tau(y) + \ln\left(1 - \frac{y}{w}\right)$$

$$1 - \frac{d\tau(y)}{dy} - \frac{1}{\left(1 - \frac{y}{w}\right)} \cdot \frac{1}{w} = 0$$

$$1 - \frac{d\tau(y)}{dy} - \frac{1}{(w-y)} = 0$$

So

$$w = \frac{1}{1 - \frac{d\tau(y)}{dy}} + y \tag{3}$$

From (3),

$$\frac{dw}{dy} = \frac{\frac{d^2\tau(y)}{dy^2}}{\left[1 - \frac{d\tau(y)}{dy}\right]^2} + 1 \tag{4}$$

$$\geq \frac{-k}{\eta} + 1 \equiv \psi > 0$$

$$\frac{dw}{dy} = \frac{\frac{d^2\tau(y)}{dy^2}}{\left[1 - \frac{d\tau(y)}{dy}\right]^2} + 1 \tag{5}$$
$$\leq \frac{k}{\eta} + 1$$

So

$$\frac{dy}{dw} \geq \frac{\eta}{k+\eta} \tag{6}$$

Notice that for all $y > y_R$, $\frac{d\tau'(y)}{dy} < \frac{d\tau(y)}{dy}$. Hence, using (3), for given w, $y'(w) \geq y(w)$ provided that $y(w) \geq y_R$. Now that we have a monotonic relationship between w and either y or y', define w_R such that $y(w_R) = y_R$ and $y'_R = y'(w_R) \geq y(w_R)$. Define w_L such that $y(w_L) = y_L$ and $y'_L = y'(w_L) \leq y(w_L)$.

Take the quadratic

$$\begin{aligned} Q_R(y) &\equiv \frac{k}{2}y_R'^2 - kyy'_R + \frac{k}{2}y^2 \\ &= \frac{k}{2}\left(y - y'_R\right)^2 \end{aligned}$$

We claim that $\tau(y) - \tau'(y) \geq Q_R(y)$ for all $y \in [y'_R, \phi\overline{w}]$. Now

$$\begin{aligned} Q_R(y'_R) &= 0 \leq \int_{y_R}^{y'_R} \left[\frac{d\tau(y)}{dy} - \frac{d\tau'(y)}{dy}\right] dy \\ &= \tau(y'_R) - \tau'(y'_R) \end{aligned}$$

.

$$\begin{aligned} \frac{dQ_R(y'_R)}{dy} &= 0 \leq \frac{d\tau(y'_R)}{dy} - \frac{d\tau'(y'_R)}{dy} \\ \frac{d^2Q_R(y)}{dy^2} &= k \leq \frac{d^2\tau(y)}{dy^2} - \frac{d^2\tau'(y)}{dy^2} \end{aligned}$$

The claim follows. To simplify notation, define $\eta' = \sqrt{\eta} + 1$. Now $\tau(y'(w)) - \tau(y(w)) \leq \eta' \cdot (y'(w) - y(w))$ so

$$\tau(y(w)) - \tau(y'(w)) \geq \eta' \cdot \big(y(w) - y'(w)\big)$$

Next,

$$\begin{aligned}
& \int_{w_R}^{\phi\overline{w}} \left[\tau(y(w)) - \tau'(y'(w))\right] dF_\phi(w) \\
= & \int_{w_R}^{\phi\overline{w}} \left(\left[\tau(y(w)) - \tau(y'(w))\right] + \left[\tau(y'(w)) - \tau'(y'(w))\right]\right) dF_\phi(w) \\
= & \int_{w_R}^{\phi\overline{w}} \left[\tau(y(w)) - \tau(y'(w))\right] dF_\phi(w) + \int_{w_R}^{\phi\overline{w}} \left[\tau(y'(w)) - \tau'(y'(w))\right] dF_\phi(w) \\
\geq & \ \eta' \cdot \int_{w_R}^{\phi\overline{w}} \left(y(w) - y'(w)\right) dF_\phi(w) + \int_{w_R}^{\phi\overline{w}} Q_R(y'(w)) dF_\phi(w) \\
\geq & \ \eta' \cdot \sup_{w\in[0,\overline{w}]} f(w) \cdot \int_{w_R}^{\phi\overline{w}} \left(y_R - y'(\phi\overline{w})\right) dw + \inf_{w\in[0,\overline{w}]} f(w) \cdot \int_{w_R}^{\phi\overline{w}} Q_R(y'(w)) dw \\
= & \ \eta' \cdot \sup_{w\in[0,\overline{w}]} f(w) \cdot \int_{y'(w_R)}^{y'(\phi\overline{w})} \left(y_R - y'(\phi\overline{w})\right) \frac{dw}{dy'} dy' + \inf_{w\in[0,\overline{w}]} f(w) \cdot \int_{y'(w_R)}^{y'(\phi\overline{w})} Q_R(y') \frac{dw}{dy'} dy'
\end{aligned}$$

From (5) and (4),

$$\begin{aligned}
\geq & \ \eta' \cdot \left(\frac{k}{\eta} + 1\right) \cdot \sup_{w\in[0,\overline{w}]} f(w) \cdot \int_{y'(w_R)}^{y'(\phi\overline{w})} \left(y_R - y'(\phi\overline{w})\right) dy' + \psi \inf_{w\in[0,\overline{w}]} f(w) \cdot \int_{y'(w_R)}^{y'(\phi\overline{w})} Q_R(y') dy' \\
= & \ \eta' \cdot \left(\frac{k}{\eta} + 1\right) \cdot \sup_{w\in[0,\overline{w}]} f(w) \cdot \left(y_R - y'(\phi\overline{w})\right) \cdot \left(y'(\phi\overline{w}) - y'_R\right) + \psi \inf_{w\in[\underline{w},\overline{w}]} f(w) \cdot \frac{k}{2} \int_{y'(w_R)}^{y'(\phi\overline{w})} \left(y' - y'_R\right)^2 dy' \\
= & \ -\eta' \cdot \left(\frac{k}{\eta} + 1\right) \cdot \left(y'(\phi\overline{w}) - y'_R\right)^2 + \psi \inf_{w\in[\underline{w},\overline{w}]} f(w) \cdot \frac{k}{6} \left(y'(\phi\overline{w}) - y'_R\right)^3
\end{aligned}$$

Define w_L to be the w satisfying $y(w_L) = y_L$. A similar calculation for the left side yields:[27]

$$\begin{aligned}
& \int_0^{w_L} \left[\tau(y(w)) - \tau'(y'(w))\right] dF(w) \\
\geq & \ -\eta' \cdot \left(\frac{k}{\eta} + 1\right) \left(y'_L - y'(0)\right)^2 + \psi \inf_{w\in[\underline{w},\overline{w}]} f(w) \cdot \frac{k}{6} \left(y'_L - y'(0)\right)^3
\end{aligned}$$

Now we are interested in

$$\left[\tau' - \tau\right]^{-1}(0) = \{y_L, y_R\}$$

Take the quadratic

$$Q(y) \equiv -\left(\theta y_L + \frac{k}{2} y_L^2\right) + \left(\theta + k y_L\right) y - \frac{k}{2} y^2$$

We claim that $\tau'(y) - \tau(y) \leq Q(y)$ for all $y \in [y_L, y_R]$. Now $Q(y_L) = \tau'(y_L) - \tau(y_L) = 0$.

$$\begin{aligned}
\frac{dQ(y_L)}{dy} &= \theta \geq \frac{d\tau'(y_L)}{dy} - \frac{d\tau(y_L)}{dy} \\
\frac{d^2Q(y)}{dy^2} &= -k \geq \frac{d^2\tau'(y)}{dy^2} - \frac{d^2\tau(y)}{dy^2}
\end{aligned}$$

[27] Notice that in fact $y'(0) = 0$. We could use these arguments for $\phi\underline{w} > 0$, but then the lower bound on ability would rise with ϕ. This is perhaps unrealistic. If we used a non-proportional population rise, we could accommodate $\underline{w} > 0$.

The claim follows. Therefore, y_R is less than or equal to the second root (other than y_L) of the equation $Q(y) = 0$. Using the quadratic formula, the solutions to $Q(y) = 0$ are: $\left\{y_L, \frac{1}{k}\left(2\theta + ky_L\right)\right\}$

Neglecting the solution $y = y_L$,

$$y_R \leq \frac{2\theta}{k} + y_L \tag{7}$$

Finally, we calculate

$$\begin{aligned}
& \int_{w_L}^{w_R} \left[\tau'(y'(w)) - \tau(y(w))\right] dF(w) \\
= & \int_{w_L}^{w_R} \left(\left[\tau'(y'(w)) - \tau'(y(w))\right] + \left[\tau'(y(w)) - \tau(y(w))\right]\right) dF(w) \\
= & \int_{w_L}^{w_R} \left[\tau'(y'(w)) - \tau'(y(w))\right] dF(w) + \int_{w_L}^{w_R} \left[\tau'(y(w)) - \tau(y(w))\right] dF(w) \\
\leq & \int_{w_L}^{w_R} \left[\tau'(y'(w)) - \tau'(y(w))\right] dF(w) + \int_{w_L}^{w_R} Q(y(w)) dF(w)
\end{aligned}$$

Now $\frac{\tau'(y'(w)) - \tau'(y(w))}{y'(w) - y(w)} \leq \eta'$, so $\tau'(y'(w)) - \tau'(y(w)) \leq \eta' \cdot \max\{0, y'(w) - y(w)\}$.

$$\begin{aligned}
& \leq \eta' \cdot \int_{w_L}^{w_R} \max\left\{0, y'(w) - y(w)\right\} dF(w) + \int_{w_L}^{w_R} Q(y(w)) dF(w) \\
& = \eta' \cdot \int_{w_L}^{w_R} \max\left\{0, y'(w) - y(w)\right\} dF(w) \\
& + \int_{w_L}^{w_R} \left[-\left(\theta y_L + k y_L - \frac{k}{2} y_L^2\right) + \left(\theta + k y_L\right) y(w) - \frac{k}{2} y(w)^2\right] dF(w) \\
& = \eta' \cdot \int_{w_L}^{w_R} \max\left\{y(w), y'(w)\right\} dF(w) + \int_{w_L}^{w_R} \left[\frac{k}{2} y_L^2 + k y_L y(w)\right] dF(w) \\
& - \int_{w_L}^{w_R} \left(\theta y_L + k y_L + \frac{k}{2} y(w)^2\right) dF(w)
\end{aligned}$$

$$\begin{aligned}
& \leq \sup_{w \in [\underline{w}, \overline{w}]} f(w) \cdot \eta' \cdot \int_{w_L}^{w_R} \max\left\{y(w), y'(w)\right\} dw + \sup_{w \in [\underline{w}, \overline{w}]} f(w) \cdot \int_{w_L}^{w_R} \left[\frac{k}{2} y_L^2 + k y_L y(w)\right] dw \\
& - \inf_{w \in [\underline{w}, \overline{w}]} f(w) \int_{w_L}^{w_R} \left(\theta y_L + k y_L + \frac{k}{2} y(w)^2\right) dw \\
& = \sup_{w \in [\underline{w}, \overline{w}]} f(w) \cdot \eta' \cdot \int_{y_L}^{y_R} \max\left\{y, y'(w(y))\right\} \frac{dw}{dy} dy + \sup_{w \in [\underline{w}, \overline{w}]} f(w) \cdot \int_{y_L}^{y_R} \left[\frac{k}{2} y_L^2 + k y_L y\right] \cdot \frac{dw}{dy} dy \\
& - \inf_{w \in [\underline{w}, \overline{w}]} f(w) \int_{y_L}^{y_R} \left(\theta y_L + k y_L + \frac{k}{2} y^2\right) \cdot \frac{dw}{dy} dy
\end{aligned}$$

$$\le \sup_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left(\frac{k}{\eta}+1\right)\cdot\int_{y_L}^{y_R}\left[\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2+ky_Ly\right]dy$$
$$-\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\int_{y_L}^{y_R}\left(\theta y_L+ky_L+\frac{k}{2}y^2\right)dy$$
$$=\sup_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left(\frac{k}{\eta}+1\right)\cdot\left[\left(\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2\right)\cdot(y_R-y_L)+ky_L\cdot\left(\frac{y_R^2}{2}-\frac{y_L^2}{2}\right)\right]$$
$$-\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left[(\theta+k)\,y_L\cdot(y_R-y_L)+\frac{k}{6}\left(y_R^3-y_L^3\right)\right]$$

$$=\sup_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left(\frac{k}{\eta}+1\right)\cdot(y_R-y_L)\cdot\left[\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2+\frac{ky_L}{2}\cdot(y_R+y_L)\right]$$
$$-\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left[(\theta+k)\,y_L\cdot(y_R-y_L)+\frac{k}{6}\left(y_R^3-y_L^3\right)\right]$$

Summing these three terms,

$$\int_{\underline{w}}^{\phi\overline{w}}\left[\tau(y(w))-\tau'(y'(w))\right]dF(w)\ge$$
$$-\eta'\left(\frac{k}{\eta}+1\right)\cdot\left(y'(\phi\overline{w})-y_R'\right)^2+\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\frac{k}{6}\left(y'(\phi\overline{w})-y_R'\right)^3$$
$$-\eta'\left(\frac{k}{\eta}+1\right)\cdot\left(y_L'-y'(\underline{w})\right)^2+\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\frac{k}{6}\left(y_L'-y'(\underline{w})\right)^3$$
$$-\sup_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left(\frac{k}{\eta}+1\right)\cdot(y_R-y_L)\cdot\left[\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2+\frac{ky_L}{2}\cdot(y_R+y_L)\right]$$
$$+\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left[(\theta+k)\,y_L\cdot(y_R-y_L)+\frac{k}{6}\left(y_R^3-y_L^3\right)\right]$$

Given $y'(\underline{w})$ and $y'(\overline{w})$, the worst case for the first 4 terms (symmetric in left and right incomes) on the right hand side is: $y'(\phi\overline{w})-y'(w_R)=y'(w_L)-y'(\underline{w})$. Using (7),

$$\int_{\underline{w}}^{\overline{w}}\left[\tau(y(w))-\tau'(y'(w))\right]dF(w)\ge$$
$$-2\eta'\left(\frac{k}{\eta}+1\right)\cdot\left(y'(\phi\overline{w})-y_R'\right)^2+\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\frac{k}{3}\left(y'(\phi\overline{w})-y_R'\right)^3$$
$$-\sup_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left(\frac{k}{\eta}+1\right)\cdot\frac{2\theta}{k}\cdot\left[\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2+\frac{ky_L}{2}\cdot(y_R+y_L)\right]$$
$$+\psi\inf_{w\in[\underline{w},\overline{w}]} f(w)\cdot\left[(\theta+k)\,y_L\cdot(y_R-y_L)+\frac{k}{6}\left(y_R^3-y_L^3\right)\right]$$
$$\ge -2\eta'\left(\frac{k}{\eta}+1\right)\cdot\left(y'(\phi\overline{w})-y_R'\right)^2+\psi\cdot\frac{k'}{3}\left(y'(\phi\overline{w})-y_R'\right)^3$$
$$-\left(\frac{k}{\eta}+1\right)\cdot\frac{2\theta'}{k}\cdot\left[\eta'\cdot\phi\overline{w}+\frac{k}{2}y_L^2+\frac{ky_L}{2}\cdot(y_R+y_L)\right]$$

Now $y'(\phi\overline{w}) - y'_R \to \infty$ uniformly in τ', since $y'(\phi\overline{w}) - y'_R \geq \left(\phi\overline{w} - \frac{2\theta}{k}\right)/2$ in the worst case for the inequality in the difference in revenues for the two taxes τ and τ'. The last expression in the equation above is increasing in ϕ for ϕ large, due to the cubic term. By (6), there is ϕ large enough so that the last expression is non-negative uniformly across $\tau, \tau' \in T''$.

Condition (M') is satisfied for $\phi \geq 1$ because we can shift tax schedules by a constant:

$$\tau \in T'' \text{ and } \int_0^{\overline{w}} \tau(y(w))dF(w) > R \Longrightarrow \exists\lambda > 0 \text{ with } \tau'(y'(w)) = \tau(y(w)) - \lambda \in T''$$

and $\int_0^{\overline{w}} \tau(y(w))dF(w) > R$ is equivalent to $\int_0^{\phi\overline{w}} \tau(y(w))dF_\phi(w) > \phi R$.