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Some important forms of contests have participation costs and 'default allocations' where the contest prize is still awarded even when no-one actively competes. We solve a general, all-pay contest model that allows for flexible forms of these features under arbitrary asymmetry. We then use our framework to better connect the literatures on contests and sales price competition, and use this connection to solve some long-standing problems. Finally, we analyze how participation costs and default allocations can be used as novel, practical tools in contest design. Throughout, the *combined* presence of participation costs and default allocations often reverse otherwise familiar intuitions.

Keywords: All-Pay Contests; Participation Costs; Default Allocations; Clearinghouse; Sales; Contest Design

JEL Codes: C72; L13; D43

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1 Introduction

The burgeoning literature on contests analyzes situations where players compete with sunk resource investments in order to win some form of prize. This successful literature has considered many applications including R+D, rent-seeking, political campaigns, rewards in organizations, litigation, contract tendering, and conflict.[1]

In practice, in addition to the associated investment costs, players often face specific costs of participating in a contest such as entry fees, set-up costs, foregone outside options or minimum required outlays. In these cases, as we later show, the outcome of a contest can depend upon what we term as the 'default allocation' - what happens to the prize in the event that no player actively competes. As later reviewed, the existing literature has neglected this issue - there are relatively few models of contests with participation costs, and they implicitly assume that the prize is withheld when all players refrain from competing. This prevents any analysis of situations where the prize must always be allocated, or where the contest organizer cannot commit to withholding it. Important examples include tendering processes where a contract is renewed with an incumbent unless a bid is received from an entrant, policy decisions where an outcome will remain unless it is contested by a lobbyist, disputes where some default legal outcome applies unless a party starts litigation, or market settings where consumers trade with their local firm unless a rival firm advertises to them.

To help remedy these issues, our paper makes three main contributions. First, it provides a general framework that can explicitly characterize all potential equilibria in a full information, single prize all-pay contest while allowing for general forms of participation costs and default allocations, under arbitrary asymmetry. Despite its complexity, we offer a tractable characterization that rests on only two measures, which we refer to as 'reach' and 'strength'.

Second, the paper uses this framework to formally connect some recent developments in all-pay contests (e.g. Siegel 2009, 2010, 2014) to the broad family of 'clearinghouse' models that are commonly used within industrial economics and marketing to study issues such as sales price competition and price comparison platforms (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006).[2] Our framework provides a fuller bridge between these two literatures to enable them to trade theoretical and empirical insights in ways that can enhance future research on both sides. As examples, we show how this connection can help resolve two long-standing problems - by using tools from

[1]For recent reviews, see Fu and Wu (2019), Corchón and Serena (2018), Dechenaux et al. (2015), and Konrad (2009).

[2]Clearinghouse model are also used as a foundation to study consumer search, obfuscation, choice complexity and even some macroeconomic topics including nominal rigidities, output fluctuations and monetary policy. For reviews and recent examples, see Guimaraes and Sheedy (2011), Moraga-González and Wildenbeest (2012), Armstrong (2015), Spiegler (2015), Kaplan and Menzio (2016), Burdett and Menzio (2017), Bergemann et al. (2021), Armstrong and Vickers (2022) and Ronayne and Taylor (2022).

contest theory, we solve clearinghouse models under arbitrary asymmetry for the first time, and by using methods from the clearinghouse literature, we derive equilibrium uniqueness in n-player symmetric all-pay contests. After completing the latter, we also show how the combined presence of participation costs and default allocations can reverse the direction of standard comparative statics related to the 'competitiveness' of a contest (e.g. Hillman and Samet 1987, Fang et al. 2020).

Third, to further demonstrate the power and tractability of our framework, we analyze how participation costs and default allocations can be used as practical tools in contest design. Such tools have remained under-explored within the literature (as reviewed by Fu and Wu 2019 and Chowdhury et al. 2019) and so our results offer some novel and striking insights. For instance, contrary to the usual motivation for handicapping stronger players to 'level the playing field' (e.g. Baye et al. 1993, Szech 2015, Franke et al. 2018), we show how *asymmetric* participation costs or default allocation probabilities can optimally stimulate competition even in an otherwise symmetric setting. Throughout, the *combined* presence of participation costs and default allocations is key. Together, they can often reverse otherwise familiar intuitions.

In more detail, Sections 2 and 3 present our main framework - a fully asymmetric, two-player, single-prize, full-information all-pay contest with general forms of participation costs and default allocations. Each player simultaneously selects a score or 'offer', u_i. A player is termed as 'active' if they submit an offer, $u_i \geq 0$, and incur the required participation costs. Otherwise, a player is termed as 'passive' and their offer is denoted by $u_i = \phi$. As standard, the prize is awarded to the player with the highest active offer. However, the prize may also be awarded in the event where both players are passive. In this case, player i wins with a 'default allocation probability', $x_i > 0$, where $x_1+x_2 \in [0, 1]$. For any given offer u_i, player i earns $W_i(u_i)$ if she wins and $L_i(u_i)$ if she loses.

Under arbitrary asymmetry, the paper derives an equilibrium that is unique (apart from a few knife-edge parameter cases). Although the proof is long, the resulting equilibrium is tractable and neatly depends on only two measures, 'strength' and 'reach'. Broadly speaking, a player's reach determines their willingness to be active when their rival is also active, whereas a player's strength determines their willingness to be active when their rival is passive. In the previous literature, these measures would have been equivalent to each other and consistent with Siegel's (2009) definition of reach. However, in our context with *both* participation costs and default allocations, the two measures differ and prove sufficient for determining the form of equilibrium in what would otherwise be a complex problem. For instance, in equilibrium, we find that i) neither player actively competes if they both have low strength, ii) only one player actively competes if one player has high strength while the other has low reach, or iii) both players actively compete with positive probability (in a variety of possible forms) if one player has high strength and the other has sufficient reach.

After showing how the model can be easily expanded to also allow for 'indirect' participation costs in the form of minimum required outlays, Section 4 focuses on formally connecting the framework to the broad family of 'clearinghouse' sales models. As we demonstrate, this connection is particularly valuable as there are significant potential gains from trade between the two literatures. To get an initial understanding of the link, recall the seminal model of sales by Varian (1980) where a number of symmetric firms compete by simultaneously choosing their price. Each firm has a symmetric share of 'non-shopper' consumers who never consider buying from any other firm, while the remaining 'shopper' consumers buy from the firm with the lowest price. The resulting equilibrium involves mixed-strategy pricing or 'sales'. Baye et al. (1996) and Baye et al. (2012) show how a simple version of Varian's model is equivalent to a form of symmetric all-pay contest. Intuitively, i) each firm's price implies an associated surplus offer to consumers, ii) the highest offer wins the 'prize' of the shoppers' custom, iii) each firm's offer involves a sunk (opportunity) cost in the form of reduced revenues from its non-shoppers, and iv) a firm's value of winning is dependent upon the level of its surplus offer. Our framework goes beyond this initial link to connect to the wider family of clearinghouse sales models that encompass Varian (1980) (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) by allowing for positive advertising costs (via participation costs) and the possibility of winning the shoppers' custom even when no firm actively competes (via default allocation probabilities).

This connection offers substantial benefits by enabling the two literatures to share insights and methods. For instance, after discussing some empirical benefits, we demonstrate how this link can resolve two long-standing theoretical problems. First, we show how contest theory methods can be used to solve clearinghouse models for any arbitrary level of asymmetry for the first time. This should open up many new theoretical and empirical research avenues within the broad clearinghouse literature. Second, in Section 5, we demonstrate how insights from the clearinghouse literature can derive equilibrium uniqueness in n-player symmetric all-pay contests which are otherwise well-known to suffer from equilibrium multiplicity (Baye et al. 1996). Within the unique equilibrium, we then also show how the combined presence of participation costs and default allocations can provide new insights in regard to changes in the 'competitiveness' of a contest (e.g. Hillman and Samet 1987, Fang et al. 2020).[3]

Finally, Section 6 returns to the full two-player setting to analyze how participation costs and default allocations can be used as novel, practical tools in contest design. In particular, we study how a contest organizer would select the default allocation probabilities, $\{x_1, x_2\}$, and (additive) participation costs, $\{A_1, A_2\}$. Throughout, we assume

[3]Fang et al. (2020) show the effects of a range of competitiveness measures in all-pay contests under a different setting with multiple prizes and convex effort costs. In our context, we study the effect of some parallel measures, including the number of players and the division of a contest into identical sub-contests.

the organizer has an 'offer-based objective' involving any combination of expected offers, expected total offers, or expected winning offers. Despite the existing literature suggesting that greater player heterogeneity typically lowers competition (e.g. Baye et al. 1993, Szech 2015, Franke et al. 2018), our results demonstrate that an offer-orientated organizer will often find it optimal to use *asymmetric* default allocation probabilities or participation costs even when the players are otherwise symmetric. Specifically, the organizer will strictly prefer to make one player i relatively stronger by giving them a relatively lower default allocation probability, $x_i < x_j$, or participation cost, $A_i = 0 < A_j$. Intuitively, this can stimulate competition by encouraging the now stronger player i to participate more and compete harder as she understands that she now has a higher chance of winning with an active offer.[4]

Related Literature: Our focus is on full-information all-pay contests with single prizes and potentially asymmetric players. Aside from the seminal contributions by Hillman and Samet (1987), Hillman and Riley (1989) and Baye et al. (1996), the recent works by Siegel (2009, 2010, 2014) are most relevant. Siegel analyzes a broad category of 'all-pay contests' with general payoff functions and arbitrary asymmetry, but without participation costs or default allocation probabilities. In particular, Siegel (2009) develops a tractable approach involving the concept of 'reach' to derive players' equilibrium payoffs in n-player all-pay contests. Building on this, Siegel (2010) then characterizes the equilibrium in a slightly simplified version, while Siegel (2014) explores the effects of 'headstarts' where players' payoffs from losing or winning need not be strictly decreasing in their bid. In contrast, we bring some elements of these papers together within our context while allowing for general forms of costly participation and default allocations. We explicitly characterize all potential equilibria and show how they depend on each player's reach, and a new measure, which we term as 'strength'.

The role of participation costs within all-pay contests has not received a lot of attention. However, maybe confusingly, standard models without participation costs are sometimes framed in terms of participation. For instance, following Hillman and Riley (1989), asymmetric models often have one player selecting a zero bid with positive probability in a way that is interpreted as either non-participation or participation with a zero bid. Our model with participation costs has no such ambiguity because it distinguishes between non-participation, $u_i = \phi$, and zero active offers, $u_i = 0$. This is consistent with some of our applications including the clearinghouse setting where firms can reasonably advertise an offer of zero consumer surplus in a way that is qualitatively distinct from not advertising ((e.g. by advertising the monopoly price under unit demand). Aside from

[4]The optimality of asymmetric contest designs in symmetric situations has also been documented in a few other papers but our setting and contest design tools are distinct. For instance, Drugov and Ryvkin (2017) and Barbieri and Serena (2022) show how a biased contest success function can be optimal for a general family of pure-strategy contests or dynamic contest settings respectively, while Pérez-Castrillo and Wettstein (2016) show how asymmetric prizes can be optimal under private information.

our paper, a small existing literature also allows players' participation to be endogenous by introducing participation costs (e.g. Fu et al. 2015), or indirect participation costs, such as reserve prices or minimum outlays (e.g. Hillman and Samet 1987, Bertoletti 2016, Chowdhury 2017). This literature considers symmetric participation costs and assumes that i) players who do not participate cannot win, and ii) the prize is withheld if no player participates. In contrast, we introduce a general form of participation costs and allow the prize to be awarded even when no player participates. In addition, when analyzing contest design, we demonstrate how asymmetric, positive participation costs can arise endogenously.[5][6]

To our knowledge, we are the first to study default allocations. However, our modeling approach has some connection to the research on ties in all-pay contests. There, active winning bids are tied with positive probability in equilibrium because i) players' efforts are capped (e.g. Che and Gale 1998 and Szech 2015), ii) the strategy space is discrete (e.g. Cohen and Sela 2007), or iii) outright winners must win by a sufficient margin (e.g. Gelder et al. 2019). In such cases, a specified tie-break rule is used to allocate the prize, and with the exception of Szech (2015), only symmetric tie-break rules are considered. While these forms of ties in active offers never occur in our model, we specify our (potentially asymmetric) default allocation probabilities in a related fashion for instances where no player actively participates.

The clearinghouse sales framework (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006) encompasses a large range of sales models, including Varian (1980) as a special case. However, it has only been able to allow for a limited form of asymmetries in simplified settings (e.g. Narasimhan 1988, Baye et al. 1992, Wildenbeest 2011 and Arnold et al. 2011). To improve this, Shelegia and Wilson (2021) derive an equilibrium using a specific 'equilibrium' tie-break rule and offer new insights into sales competition. In contrast, the current paper uses contest theory to offer a general clearinghouse characterization for *any* default allocation probabilities under arbitrary asymmetry. This substantially expands upon the initial link between all-pay contests and Varian (1980), pioneered by Baye et al. (1996) and Baye et al. (2012) by allowing for asymmetries, positive advertising costs and the possibility of 'shopper' consumers buying in the market even when no firm advertises. Beyond our paper, Montez and Schutz (2021) explore another connection between all-pay contests and pricing in a very different setting where firms simultaneously source unobservable inventories before setting prices. Their paper focuses on inventory behavior and associated public policy, but as a side result, they show how

[5] As we later show, participation costs create a discontinuity in the players' payoffs. Duvocelle and Mourmans (2021) study some wider forms of payoff discontinuities and show how Siegel's equilibrium payoff results can still apply.

[6] In a different setting with private information, a literature on all-pay auctions considers (symmetric) participation costs in the form of entry fees that can be used to supplement the prize fund (e.g. Hammond et al. 2019 and Liu and Lu 2019). We exclude this possibility in order to focus solely on the role of costly participation per se, with no connection to the prize fund.

their equilibrium can tend to a version of the asymmetric clearinghouse equilibrium as inventory costs become fully recoverable. However, contrary to the original clearinghouse literature and our framework, they assume that 'shopper' consumers do not buy in the market if neither firm advertises (implying that default allocation probabilities are zero). Our results highlight the importance of this assumption and derive the equilibrium for all default allocation probabilities in order to fully enable the literatures on all-pay contests and clearinghouse sales price competition to exchange theoretical and empirical insights in future research.

2 Model

2.1 Assumptions

Two risk-neutral players, $i = \{1, 2\}$, consider participating in a contest to win a single prize. Each player i must simultaneously choose a bid or 'offer', u_i, where $u_i \in \{\phi\} \bigcup [0, \infty]$. If player i selects $u_i = \phi$, she refrains from making an explicit offer and is termed as 'passive'. On the other hand, player i is classed as 'active' if she submits an explicit offer $u_i \in [0, \infty]$. Given the players' chosen strategies, $S = \{u_1, u_2\}$, player i's probability of winning is then given by the following contest success function, $\Psi_i(\cdot)$:

$$\Psi_i(\cdot) = \begin{cases} 1 & \text{if } u_i \geq 0 \text{ and } u_j \in \{\phi\} \bigcup [0, u_i) \\ y_i & \text{if } u_i = u_j \text{ with } u_i \geq 0 \text{ and } u_j \geq 0 \\ x_i & \text{if } u_i = u_j = \phi \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Intuitively, player i wins outright if she submits an active offer, $u_i \geq 0$, and player j either submits a lower active offer or only participates passively. If both players submit the same active offer, then player i wins with a tie-break probability, y_i. As the exact level of y_i will prove irrelevant, we allow any $y_i \in [0, 1]$ such that $y_1 + y_2 = 1$. Finally, and most importantly, we permit the possibility that the prize is still awarded even when both players are passive. In this case, player i wins with a 'default allocation probability', $x_i \geq 0$, where $x_1 + x_2 \equiv X \in [0, 1]$. The set of default allocation probabilities, $\{x_1, x_2\}$, will turn out to be an important primitive of the game.

For a given set of strategies, S, and contest success function, $\Psi_i(\cdot)$, player i's expected payoff can then be described as follows

$$E(\Pi_i(S; \Psi_i(\cdot))) = \Psi_i(\cdot)W_i(u_i) + [1 - \Psi_i(\cdot)]L_i(u_i) \quad (2)$$

where $W_i(u_i)$ and $L_i(u_i)$ describe player i's (net) payoffs from winning and losing respectively for any given offer, including passive participation where $W_i(\phi) \equiv W_i^\phi$ and $L_i(\phi) \equiv L_i^\phi$. We then make the following additional assumptions about the payoff functions for each player i:

A1) $W_i(u_i) > L_i(u_i)$ for any given $u_i \in \{\phi\} \bigcup [0, \infty]$.

A2) For any $u_i \in [0, \infty]$, both $W_i(u_i)$ and $L_i(u_i)$ have the same unique finite maximizer, $u_i^m \in [0, \infty)$, and are strictly decreasing in $u_i > u_i^m$.

A3) $c(u_i^m) \equiv L_i^\phi - L_i(u_i^m) > 0$ and $W_i^\phi > W_i(u_i^m)$.

A1 simply assumes that the payoffs from winning are always larger than those from losing for any given offer (including passive participation, $W_i^\phi > L_i^\phi$).

A2 assumes that player i's payoffs from winning and losing both have a unique finite maximizer and are strictly decreasing in the player's offer thereafter. Moreover, although not always required, A2 also assumes that $W_i(u_i)$ and $L_i(u_i)$ have the *same* such maximizer, u_i^m. To allow for a form of non-monotonicity in player i's payoffs (or headstarts in the sense of Siegel 2009), this maximizer can be non-zero, $u_i^m \in [0, \infty)$. The maximizer can also differ across players $u_i^m \neq u_j^m$, to reflect potentially different technologies, preferences, or prior investments.[7]

A3 assumes the existence of participation costs such that $L_i^\phi > L_i(u_i^m)$ and $W_i^\phi > W_i(u_i^m)$. To understand this, and as later formalized, note that player i will never be willing to select an active offer lower than u_i^m. A3 then assumes that losing (or winning) under passive participation to gain L_i^ϕ (or W_i^ϕ) is always strictly preferred to losing (winning) under active participation to gain, at most, $L_i(u_i^m)$ (or $W_i(u_i^m)$). Without A3, the distinction between passive and active participation becomes blurred and the default allocation probabilities become ill-defined.

Finally, to help exposition, we focus on deriving the equilibria under Condition X. This ensures that both players have a strictly positive default allocation probability, as consistent with the organizer being unable to perfectly commit to withholding the prize from either player in the event that both players are passive.

$$x_i \in (0, 1) \text{ for } i = \{1, 2\} \qquad \text{(Condition X)}$$

Under A1-A3 and Condition X, we now characterize the Nash equilibria of the game for any permitted set of default allocation probabilities, $\{x_1, x_2\}$, and payoff functions, $W_i(u_i)$ and $L(u_i)$ for $i = \{1, 2\}$. To allow for mixed strategies, we define i) $(1-\alpha_i) \in [0, 1]$ as player i's probability of passive participation (with $u_i = \phi$), ii) $\alpha_i \in [0, 1]$ as player

[7] A2 is consistent with past research and many practical applications. We believe that all previous contest papers (implicitly) assume that the related functions are both maximized at zero or some positive constant, e.g. Siegel (2009). Within the clearinghouse literature, $W_i(u_i)/L_i(u_i)$ is effectively a constant ratio related to the proportions of different types of consumers, and so both functions also have a common maximizer as later detailed in Section 4.3.

i's probability of active participation on some support $u_i \in [\underline{u}_i, \bar{u}_i]$ where $0 \le \underline{u}_i \le \bar{u}_i$, and iii) $F_i(u)$ as player i's *overall* (unconditional) offer distribution on $u_i \in \{\phi\} \bigcup [0, \infty]$.[8] Lastly, it will be useful to denote $u^m = max\{u_1^m, u_2^m\} \ge 0$.

2.2 Definitions

Despite the potential complexity of our framework, we will show how the equilibria will critically depend on only two measures for each player, 'reach' and 'strength'. These two measures will drive players' participation decisions. Broadly speaking, a player's reach determines their willingness to be active when their rival is also active, whereas a player's strength determines their willingness to be active when their rival is passive. We now formally define these measures in turn.

Definition 1. For a given contest, the reach of player i, r_i, is the unique value of $u_i \ge u_i^m$ that solves

$$W_i(u_i) = L_i^\phi \tag{3}$$

if such a solution exists, and $r_i = -\infty$ otherwise. When $W_i(u_i^m) \ge L_i^\phi$, a unique solution always exists with $r_i \ge u_i^m$. When $W_i(u_i^m) < L_i^\phi$, no solution exists.[9]

Intuitively, player i will never find it optimal to provide an active offer above her 'reach', r_i, because it is defined as the active offer $u_i \ge u_i^m$ at which player i's payoff from winning for sure, $W_i(u_i)$, equals her payoff from losing for sure under passive participation, L_i^ϕ. Further, when player $j \ne i$ is active, player i can never win under passive participation and can therefore only guarantee L_i^ϕ from being passive. Hence, when player j is active, player i will prefer to submit an active offer $u_i \ge u_i^m$ only if $W_i(u_i^m) \ge L_i^\phi$ or equivalently, only if $r_i \ge u_i^m$.

Definition 2. For a given contest, the strength of player i, s_i, is the unique value of $u_i \ge u_i^m$ that solves

$$W_i(u_i) = \Omega_i \equiv L_i^\phi + x_i(W_i^\phi - L_i^\phi)\frac{c_i(u_i^m)}{b_i(u_i^m)} \tag{4}$$

if such a solution exists, and $s_i = -\infty$ otherwise, where $c_i(u_i^m) \equiv L_i^\phi - L_i(u_i^m) > 0$ and $b_i(u_i^m) \equiv W_i(u_i^m) - L_i(u_i^m) - x_i(W_i^\phi - L_i^\phi)$. When $W_i(u_i^m) \ge L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ (or

[8] To facilitate the use of $F_i(u)$, we abuse notation slightly and treat ϕ as if it were a number less than 0. Player i then sets $u_i = \phi$ with probability mass $(1 - \alpha_i) = F_i(\phi)$, and submits an active offer on $u_i \in [\underline{u}_i, \bar{u}_i]$ with aggregate probability $\alpha_i = 1 - F_i(\phi)$ where $F_i(u) = 0$ for $u < \phi$, $F_i(u) = 1$ for $u \ge \bar{u}_i$, and $F_i'(u) \ge 0$ for all u.

[9] A solution exists and is unique iff $W_i(u_i^m) \ge L_i^\phi$ because, for $u_i \ge u_i^m$, the LHS of (3) is i) at most $W_i(u_i^m)$ and ii) strictly decreasing for $u_i \ge u_i^m$, while iii) L_i^ϕ is a constant unbounded above.

equivalently when $W_i(u_i^m) \geq \Omega_i$ or $b_i(u_i^m) \geq c_i(u_i^m)$), a unique solution always exists with $s_i \geq u_i^m$. When $W_i(u_i^m) < L_i^\phi + x_i(W_i^\phi - L_i^\phi)$, no solution exists.[10]

While the definition of strength is more involved, it provides clear implications for player i's participation decision when player $j \neq i$ is passive. If player j is passive, player i's expected payoff from being passive equals $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, player i will prefer to submit an active offer $u_i \geq u_i^m$ only if $W_i(u_i^m) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ or equivalently, only if $s_i \geq u_i^m$. In more detail, when $s_i \geq u_i^m$, player i's strength is the level of active offer, $u_i \geq u_i^m$, at which her payoff from winning for sure, $W_i(u_i)$, is equal to an expression that we denote by Ω_i. Where relevant, Ω_i, can be understood as player i's expected payoff at the point where she is indifferent between being passive and submitting an active offer of u_i^m.[11]

Note the following important remarks about reach and strength. First, our use of the term 'reach' broadly parallels the existing literature, e.g. Siegel (2009). However, in the previous literature, a player's reach would always be equal to our measure of strength. To see this, note from (3) and (4) that $r_i = s_i$ if player i has i) zero participation costs, $c_i(u_i^m) \equiv L_i^\phi - L_i(u_i^m) = 0$, and/or ii) a zero default allocation probability, $x_i = 0$. This highlights the important interaction between participation costs, $c_i(u_i^m) > 0$, and default allocation probabilities, $x_i > 0$, that is the focus of the current paper; when *combined*, they imply that each player's reach is strictly larger than their strength, $r_i > s_i$ for $i = \{1, 2\}$. Hence, each player is strictly more willing to be active when their rival is active relative to when their rival is passive. Second, whereas some existing papers refer to the player with the higher reach as the 'stronger' player, we will only employ this language under Definition 3. Indeed, under our definition, it is possible that the 'stronger' player i, with $s_i \geq s_j$, can have a lower reach, $r_i < r_j$.

Definition 3. Without loss of generality, Player 1 is assigned to be the 'stronger' player (and Player 2 as the 'weaker' player) if i) $s_1 > s_2$, or ii) $s_1 = s_2$ and $u_1^m \geq u_2^m$.

Finally, to help exposition, we will sometimes focus on equilibria in 'generic' contests as defined by Definition 4. However, Appendix B later characterizes the full set of equilibria for all generic and non-generic contests, and shows that the equilibria in non-generic contests can involve some less interesting equilibrium multiplicities.

Definition 4. A 'generic' contest does not involve the following knife-edge cases: $r_2 = u_2^m$ or $s_i = u_i^m$ for any $i = 1, 2$.

[10]A solution exists and is unique iff $W_i(u_i^m) \geq \Omega_i$ because, for $u_i \geq u_i^m$, the LHS of (4) is i) at most $W_i(u_i^m)$ and ii) strictly decreasing for $u_i \geq u_i^m$, while iii) Ω_i is a constant unbounded above.

[11]More precisely, note player i's expected payoff from being passive equals $L_i^\phi + x_i(1-\alpha_j)(W_i^\phi - L_i^\phi)$. If player j never selects any active offers below u_i^m, such that $(1-\alpha_j) = F_j(u_i^m)$, and there are no ties at u_i^m, then player i's expected payoff from submitting u_i^m is $L_i(u_i^m) + (1-\alpha_j)(W_i(u_i^m) - L_i(u_i^m))$. These two payoffs are equal when $(1-\alpha_j) = c_i(u_i^m)/b_i(u_i^m)$, where $(1-\alpha_j) \in (0, 1]$ when $s_i \geq u_i^m$. Hence, after substituting $(1-\alpha_j) = c_i(u_i^m)/b_i(u_i^m)$ back in, Ω_i represents the expected payoff when player i is indifferent between ϕ and u_i^m.

3 Equilibrium Analysis

To derive the game equilibria, Section 3.1 first considers some preliminary steps before Section 3.2 provides the main characterization. Any proofs are provided in Appendix A unless stated otherwise.

3.1 Preliminaries

Lemma 1. *Any active offer, u_i, is strictly dominated for player i if a) $u_i < u_i^m$, or b) $u_i \in (u_i^m, u_j^m)$.*

This implies that player i will only consider an active offer equal to $u_i = u_i^m$ or $u_i \geq u^m \equiv max\{u_i^m, u_j^m\}$. The proof is immediate. a) Any active offer $u_i \in [0, u_i^m)$ is strictly dominated by $u_i = u_i^m$ as it would raise player i's payoffs from winning or losing (via A2), and yet never reduce her probability of winning. b) Any $u_i \in (u_i^m, u_j^m)$ is also strictly dominated by u_i^m. To see this, note from above that player j will never select any active offer $u_j < u_j^m$. Hence, moving any mass in $u_i \in (u_i^m, u_j^m)$ to $u_i = u_i^m$ would raise player i's payoffs from winning or losing, but have no effect on her probability of winning.

Lemma 2. *Suppose only one player, player i, is active with positive probability, $\alpha_i > 0$ and $\alpha_j = 0$. Then, in equilibrium, it must be that $\underline{u}_i = \bar{u}_i = u_i^m$.*

Again, the proof is immediate. In this case, player i must set $u_i \geq u_i^m$ and player j must set $u_j = \phi$ such that player i wins with probability one. Given this, by reducing u_i to u_i^m, player i can strictly increase her payoffs via A2 and still win with certainty.

Lemma 3. *In equilibrium, player i cannot put a point mass on any active offer other than $u_i = u_i^m$. Further, if $u_1^m = u_2^m = u^m$, then at most, one player can put a point mass on u^m.*

As detailed in the proof, this just follows standard mixed-strategy results - if not, at least one player would have an incentive to redistribute their probability mass elsewhere. Now denote the size of any potential point mass at u_i^m by $\beta_i \geq 0$. As player i's probability of active participation on $u_i \geq u_i^m$ is denoted by $\alpha_i \in [0, 1]$, then it must be that $\alpha_i \geq \beta_i$.

Lemma 4. *Suppose player i selects an offer strictly above u_i^m with positive probability in equilibrium such that $\alpha_i > \beta_i \geq 0$. Then, it must be that:*
a) both players make offers above u^m with positive probability and share a common upper bound, $\bar{u} \equiv \bar{u}_1 = \bar{u}_2 > u^m$,
b) any $u \in (u^m, \bar{u}]$ is a point of increase of $F_1(u)$ and $F_2(u)$,
c) on $u \in (u^m, \bar{u}]$ for $k = 1, 2$ and $l \neq k$,

$$F_k(u) = \frac{W_l(\bar{u}) - L_l(u)}{W_l(u) - L_l(u)}. \tag{5}$$

Intuitively, if $\alpha_i > \beta_i$ in equilibrium then player i makes active offers strictly above u_i^m. If so, then the other player must be doing the same otherwise player i could optimally reduce her offers towards u_i^m. Moreover, if $u_i^m < u_j^m$, then any $u \in (u_i^m, u_j^m)$ is dominated for both players and so they must make offers strictly above u^m. As consistent with standard results (without participation costs), the two players must then continuously randomize up to a common upper bound, $\bar{u}$. By deriving the players' expected payoffs and equilibrium payoffs for a given $\bar{u}$, one can then characterize the implied distribution in active offers for both players, (5).

3.2 Characterization

Building on Lemmas 1-4, we now characterize the full equilibria. To further aid exposition, it is convenient to denote the following two expressions:

$$\theta_i(u) = 1 - \frac{W_j(u) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j} \tag{6}$$

$$\sigma_i(u) = 1 - \frac{c_j(u)}{b_j(u)} \equiv 1 - \frac{L_j^\phi - L_j(u)}{W_j(u) - L_j(u) - x_j(W_j^\phi - L_j^\phi)} \tag{7}$$

Given the usual complexities with models of this sort, it is notable that we now demonstrate that any generic contest has a unique, tractable equilibrium and that the equilibrium can be reduced to five qualitatively distinct cases that only depend upon the relative sizes of s_1 and r_2. The proof is lengthy and so it is provided separately in Appendix B.

Theorem 1. *Given Condition X, there exists a unique equilibrium for any generic contest:*

i) When $s_1 < u_1^m$ (and hence $s_2 < u_2^m$), neither player is active, $\alpha_1 = \alpha_2 = 0$.
ii) When $s_1 > u_1^m$ and $r_2 \leq u^m$, player 1 is always active at u_1^m, $\alpha_1 = \beta_1 = 1$, and player 2 is always passive, $\alpha_2 = 0$.
iii) When $r_2 > u_2^m \geq s_1 > u_1^m$, player 1 selects u_1^m with probability $\beta_1 = \alpha_1 = \theta_1(u_2^m) \in (0,1)$ and player 2 selects u_2^m with probability $\beta_2 = \alpha_2 = \sigma_2(u_1^m) \in (0,1)$.
iv) When $r_2 > s_1 > u^m$, players 1 and 2 are active with probabilities $\alpha_1 = \theta_1(\bar{u}) \in (0,1)$ and $\alpha_2 = 1 - F_2(u_1^m) = \sigma_2(u_1^m) \in (0,1)$. They both randomize on $(u^m, \bar{u}]$ with $F_i(u)$ in (5) where $\bar{u} = s_1$, $\beta_1 = F_1(u^m) - (1-\alpha_1) \geq 0$ and $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.
v) When $s_1 \geq r_2 > u^m$, players 1 and 2 are active with probabilities $\alpha_1 = 1$ and $\alpha_2 = 1 - F_2(u_1^m) \in (0,1)$. They both randomize on $(u^m, \bar{u}]$ with $F_i(u)$ in (5) where $\bar{u} = r_2$, $\beta_1 = F_1(u^m) \geq 0$ and $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.

A simple example setting is later provided in Section 4.1, but to start thinking about the intuition of Theorem 1, it is useful to initially consider the (quasi-) symmetric case where $r_1 = r_2 = r$, $s_1 = s_2 = s$ and $u_1^m = u_2^m = u^m$. Here, as $r > s$, Theorem 1 collapses to a simple form involving only case i) and iv) depending solely on $s \lesseqgtr u^m$. First, suppose $s < u^m$ such that case i) applies. From Definition 2, this implies that $L^\phi + x(W^\phi - L^\phi) > W(u^m)$. Therefore, neither player wishes to be active - given that the other player is passive, a player earns $L^\phi + x(W^\phi - L^\phi)$ and has no incentive to be active in order to earn, at most, $W(u^m)$. This equilibrium is unique because the remaining possibility where both players are always active cannot be an equilibrium as one player would always deviate due to costly participation (A3). Next suppose $s > u^m$ such that case iv) applies. From Definition 2, this implies that $W(u^m) > L^\phi + x(W^\phi - L^\phi)$ such that each player has an incentive to be active if the other is passive. However, both players cannot be active with probability one in equilibrium due to the assumption of costly participation. Hence, the unique equilibrium involves both players being active with interior probability, $\alpha \in (0, 1)$, and randomizing over active offers with $F(u)$, where $\bar{u} = s > u^m$ and $\beta = 0$.

Now consider the fuller intuition of Theorem 1 with potential player asymmetry. First, consider case i). Here, the stronger player 1 has low strength, $s_1 < u_1^m$. Using Definitions 2 and 3, this implies that both players have low strength, $s_1 = s_2 = -\infty$, such that $L_i^\phi + x_i(W_i^\phi - L_i^\phi) > W_i(u_i^m)$ for $i = \{1, 2\}$. Therefore, neither player wishes to be active when the other is passive and this equilibrium can be shown to be unique. Each player i earns $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$.

Next, examine case ii) where the stronger player 1 has a relatively higher strength, $s_1 > u_1^m$, but the weaker player 2 has a relatively low reach, $r_2 < u^m$. Using Definitions 1 and 2, this implies a) $W_1(u_1^m) > L_1^\phi + x_1(W_1^\phi - L_1^\phi)$ - if player 2 is passive, player 1 has a strict incentive to be active, and b) $W_2(u^m) < L_2^\phi$ - if player 1 is active, player 2 strictly prefers to remain passive. Hence, this ensures a pure-strategy equilibrium where only player 1 actively competes. Moreover, given the specified values of reach and strength, this equilibrium is unique. Player 1 earns $W_1(u_1^m)$ while player 2 earns L_2^ϕ.

Now temporarily defer the explanations of cases iii) and iv) and jump to case v). Here, $s_1 \geq r_2 > u^m$ such that player 1 has a very high strength and the weaker player 2 has a relatively high reach. Via Definitions 1 and 2, this implies that both players are willing to be active with $\alpha_i > \beta_i$ - player 1 is willing to be active if player 2 is passive, but if player 1 is active then player 2 is also willing to be active. Specifically, by building on Lemma 4, the unique equilibrium involves a) both players mixing over active offers with $F(u)$ up to $\bar{u} = r_2 > u^m$, b) player 2 mixing over active participation with interior probability, $\alpha_2 \in (0, 1)$, but c) player 1 remaining strong enough to always be active, $\alpha_1 = 1$. This latter feature implies that the default allocation probabilities are never implemented within this case. As such, this form of equilibrium bears a qualitative resemblance to standard asymmetric equilibria without participation costs and default allocation probabilities (e.g. Hillman and Riley 1989 or Siegel 2010). Note the fact that $\alpha_1 = 1$ also implies that player 2 can only guarantee an equilibrium payoff of L_2^ϕ, while player 1 earns a payoff equal to $W_2(\bar{u}) > L_1^\phi$.

Next, move back to case iv). Here, $u^m < s_1 < r_2$, such that player 1's strength is relatively high while player 2 has a very high reach. Like in case v), this implies that both players are willing to be active with $\alpha_i > \beta_i$. Hence, once again, in the unique equilibrium, both players mix over active offers with $F(u)$ up to $\bar{u} > u^m$ and player 2 mixes over active participation with interior probability, $\alpha_2 \in (0, 1)$. However, unlike case v), $\bar{u} = s_1$ and $\alpha_1 \in (0, 1)$ as player 1 is not strong enough to be active with probability one. Consequently, the default allocation probabilities are implemented with positive probability in this case. Each player i earns payoffs higher than L_i^ϕ but strictly lower than if the players both remained passive, $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, this case has a Prisoner's Dilemma feature that is not present in the previous literature - the players would prefer everyone to remain passive, but have an individual incentive to deviate and be active.

Finally, return to case iii). Here, player 1 has a moderate strength, $s_1 \in (u_1^m, u_2^m]$ and player 2 has a relatively high reach, $r_2 > u_2^m$. Hence, this case can only occur if $u_2^m > u_1^m$. Using Definitions 1 and 2, these conditions imply $W_1(u_1^m) > L_1^\phi + x_1(W_1^\phi - L_1^\phi) > W_1(u_2^m)$ and $W_2(u_2^m) > L_2^\phi$. Intuitively, player 1 is strong enough to be active at u_1^m when player 2 is passive, but player 2 would prefer to be active at u_2^m if player 1 chooses u_1^m. Further, player 1 is not strong enough to be active at u_2^m when player 2 is passive, but player 2 would be active there if player 1 is. As a result, this case produces an unusual form of equilibrium that appears new to the literature - both players use a binary strategy to randomize between being passive and selecting their own minimum active offer, u_i^m. In equilibrium, both players earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1 - \alpha_j)$ and so like case iv), the case also has a Prisoner's Dilemma feature where the players would prefer a commitment to no active offers.

4 Examples and Connections

This section has several aims. To begin, it further illustrates the concepts of reach and strength, showcases some features of Theorem 1 and forms a base for other later parts of the paper. Specifically, Section 4.1 first offers a simple example that allows for participation costs and default allocation probabilities within an otherwise standard all-pay contest, while Section 4.2 expands this example to show our framework can also allow for indirect participation costs in the form of a reservation offer or minimum outlay. Further, Section 4.3 then goes on to provide a different example to demonstrate how Theorem 1 can be applied to clearinghouse settings. This allows us to start illustrating the benefits from connecting the all-pay contest and clearinghouse literatures.

4.1 A Simple Example

Suppose player i values the contest's prize at V_i. If player i submits an active offer of $u_i \geq 0$ she must incur i) an effort cost equal to $k_i u_i^a$ (where $k_i > 0$ and $a > 0$ are parameters), and ii) a direct participation cost, $A_i \in (0, V_i]$. Player i's default allocation probability equals $x_i \in (0, 1)$.

Using our framework, we can then denote player i's payoff functions as $W_i(u_i) = V_i - k_i u_i^a - A_i$ and $L_i(u_i) = -k_i u_i^a - A_i$. This implies $u_1^m = u_2^m = 0$, $W_i^\phi = V_i$ and $L_i^\phi = 0$. Following Definitions 1 and 2, one can compute reach and strength in (8) and (9) below. Intuitively, player i's reach and strength are both increasing in the prize value, V_i, but decreasing in the costs of making a given active bid, k_i, A_i and a. Further, player i's strength is also decreasing her default allocation probability, x_i.

$$r_i = \begin{cases} \left(\frac{V_i - A_i}{k_i}\right)^{1/a} & \text{if } r_i \geq u_i^m = 0 \Leftrightarrow V_i \geq A_i \\ -\infty & \text{if } r_i < u_i^m = 0 \Leftrightarrow V_i < A_i \end{cases} \tag{8}$$

$$s_i = \begin{cases} \left[\frac{1}{k_i}\left(V_i - \frac{A_i}{(1-x_i)}\right)\right]^{1/a} & \text{if } s_i \geq u_i^m = 0 \Leftrightarrow V_i(1-x_i) \geq A_i \\ -\infty & \text{if } s_i < u_i^m = 0 \Leftrightarrow V_i(1-x_i) < A_i \end{cases} \tag{9}$$

For ease of exposition, further suppose $k_1 = k_2 = 1$ and $a = 1$. Then, when no lower than $u_i^m = 0$, reach and strength reduce to $r_i = (V_i - A_i)$ and $s_i = V_i - \frac{A_i}{(1-x_i)}$, where $s_1 \geq s_2$ requires $V_1 - \frac{A_1}{(1-x_1)} \geq V_2 - \frac{A_2}{(1-x_2)}$. The equilibrium cases in Theorem 1 then follow straightforwardly (albeit without case iii which cannot exist given $u_1^m = u_2^m = 0$): case i) applies with no active participation if $V_1 < \frac{A_1}{(1-x_1)}$ (such that $s_1 < u^m$), case ii) applies where only player 1 is active if $V_1 - \frac{A_1}{(1-x_1)} > 0 > V_2 - A_2$ (such that $s_1 > 0 > r_2$), case iv) applies where both players mix over active offers with $\alpha_1, \alpha_2 \in (0,1)$ if $V_2 - A_2 > V_1 - \frac{A_1}{(1-x_1)} > 0$ (such that $r_2 > s_1 > 0$), and case v) applies where both players mix over active offers with $\alpha_1 = 1$ and $\alpha_2 \in (0,1)$ if $V_1 - \frac{A_1}{(1-x_1)} \geq V_2 - A_2 > 0$ (such that $s_1 \geq r_2 > 0$).[12]

[12] Specifically, case iv) has $\bar{u} = s_1 = V_1 - \frac{A_1}{(1-x_1)}$, $\Pi_1^* = W_1(\bar{u}) = A_1 \frac{x_1}{(1-x_1)}$, $\Pi_2^* = W_2(\bar{u}) = V_2 - A_2 - V_1 + \frac{A_1}{(1-x_1)}$, $\alpha_1 = \theta_1(\bar{u}) = 1 - \left(\frac{\Pi_2^*}{x_2 V_2}\right)$, $\alpha_2 = 1 - F_2(0) = \frac{A_1}{(1-x_1)V_1}$, and $F_i(u_i) = \frac{\Pi_j^* + u_i + A_j}{V_j}$ for $i = \{1, 2\}$, and case v) has $\bar{u} = r_2 = V_2 - A_2$, $\Pi_1^* = W_1(\bar{u}) = (V_1 - V_2) + (A_2 - A_1)$, $\Pi_2^* = W_2(\bar{u}) = 0$, $\alpha_1 = 1$, $\alpha_2 = 1 - F_2(0) = \frac{V_2 - A_2}{V_1}$, $F_1(u_1) = \frac{u_1 + A_2}{V_2}$ and $F_2(u_2) = 1 - \left(\frac{V_2 - A_2}{V_1}\right) + \left(\frac{u_1}{V_1}\right)$.

4.2 An Example with Indirect Participation Costs

In contrast to direct participation costs like those in the example above, other 'indirect' forms of participation costs can derive from a minimum required outlay or reservation offer. In such settings, any valid active offer must be weakly larger than a reservation offer, $u^R \geq 0$. From A2 and Lemma 1, the reservation offer will have no effect on equilibrium if $u^R \leq u_i^m$. However, if $u^R > u_i^m$, the reservation offer can create additional indirect participation costs by prompting player i to submit an active offer at a higher level than she might have done otherwise.

This can be easily captured within our framework by slightly modifying the game. First, without loss, we can modify each player i's win and loss functions to equal zero for $u_i \in [0, u^R)$ but to remain otherwise unchanged. Denote these as $\tilde{W}_i(\cdot)$ and $\tilde{L}_i(\cdot)$, respectively. From above, we then know that player i will only consider any active offer $u_i \geq max\{u_i^m, u^R\}$. Hence, player i's new maximizer becomes $\tilde{u}_i^m \equiv max\{u_i^m, u^R\}$. Second, we can use these to calculate a modified level of strength, $\tilde{s}_i$, from (4), and apply Theorem 1 to the modified game. For instance, consider the introduction of a reservation offer, $u^R > u_i^m = 0$, into our previous example with $k_1 = k_2 = 1$ and $a = 1$. This reduces player i's willingness to compete when their rival is passive by decreasing their strength, $\tilde{s}_i = V_i - \frac{(A_i + x_i \tilde{u}_i^m)}{(1-x_i)}$ when $\tilde{s}_i \geq \tilde{u}_i^m = u^R$, and makes it less likely that the equilibrium falls into a case involving more active participation.

4.3 Connection to Clearinghouse Models

Here, we put forward a different example to lay out the connection between our framework and the clearinghouse sales models that are popular workhorses in IO and marketing (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006). We then begin to discuss some advantages of establishing this connection, before demonstrating some specific benefits more formally in Section 5 where we expand the framework to $n > 2$ symmetric players.

To start, we present a relatively general version of a duopoly clearinghouse model and then show how this can be translated into our framework. Suppose there are two firms, $i = \{1, 2\}$, that each sell a single (potentially differentiated) good. All consumers have the same product preferences and so given firm i's price, p_i, each consumer has an identical demand function for firm i's good, $D_i(p_i)$. Hence, given firm i has a constant marginal cost, $k_i \geq 0$, firm i's potential profits *per-consumer* equal $\pi_i(p_i) = (p_i - k_i)D_i(p_i)$. We assume these profits are strictly quasi-concave in p_i with a unique maximizer at firm i's monopoly price, p_i^m.

Consumers are split into two types. Each firm i has a base of 'non-shopper' consumers with mass $\lambda_i > 0$. Such consumers only consider purchasing from their associated firm. In addition, there is a group of 'shopper' consumers with mass $S > 0$. These consumers are initially allocated to the firms in respective proportions, x_1 and x_2. However, any shoppers allocated to firm i become aware of firm $j \neq i$ iff firm j advertises. Hence, if firm j does not advertise, the shoppers allocated to firm i only consider firm i but if firm j advertises, then the shoppers assigned to firm i trade with the firm offering the best deal (using any tie-breaking rule in the event of a tie). Within a one-shot game, each firm i simultaneously selects its price, p_i, and whether to advertise for a fixed cost, $A_i > 0$.

We now translate the model into our framework. First, given firm i's price, it is straightforward to calculate firm i's implied utility offer, u_i, its monopoly utility offer, u_i^m, and its per-consumer profits in terms of u_i, $\pi_i(u_i)$.[13] One can then construct firm i's payoffs from winning and losing as follows. Suppose firm i opts to be 'active' by advertising. If it has the highest offer, it wins all the shoppers to receive $W_i(u_i) = (S + \lambda_i)\pi_i(u_i) - A_i$, but otherwise, it earns $L_i(u_i) = \lambda_i \pi_i(u_i) - A_i$. Alternatively, if firm i opts to be passive by not advertising, then it will optimally offer u_i^m. If firm j advertises, then firm i will then only trade with its non-shoppers to obtain $L_i^\phi = \lambda_i \pi_i(u_i^m)$. However, if firm j is also passive, firm i will also retain its share of shoppers to receive $(x_i S + \lambda_i)\pi_i(u_i^m)$. Equivalently, in the language of the framework, when both firms are passive firm i will earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ where $W_i^\phi = (S + \lambda_i)\pi_i(u_i^m)$. Finally, one can verify that A1-A3 and Condition X apply given $\lambda_i > 0$, $S > 0$, $x_i > 0$, and $A_i > 0$. The measures of strength and reach can then be calculated, and Theorem 1 can be stated to fully derive the market equilibrium.

We now offer an initial discussion of how this connection between the literatures on all-pay contests and clearinghouse sales competition can offer substantial benefits by enabling the two literatures to trade methods and insights.

First, contest theory can help the clearinghouse literature. By importing tools from contest theory, the above example shows how our framework is able to characterize the full clearinghouse equilibrium under arbitrary asymmetry; allowing a tractable analysis of asymmetries in the firms' mass of non-shoppers, product demands, marginal costs, advertising costs, and default allocation probabilities. Previously, this has been a long-standing problem within the clearinghouse and associated literature where past research has only been able to consider some limited forms of firm asymmetries in simplified or special settings.[14] Here, through the use of contest theory, our full characterization should open up new lines of research in terms of theory, policy advice and empirical applications.

Second, in turn, with this connection established, the clearinghouse literature can also aid the contest literature. We illustrate this in two ways. a) Unlike the contest literature, the clearinghouse literature can offer a developed body of field evidence. Empirical evidence outside the lab is less common within the contest literature due to the difficulties

[13]Specifically, as the consumers have identical product preferences, all consumers value firm i's offer with the associated consumer surplus, $u_i = CS_i(p_i) = \int_{p_i}^{\infty} D_i(x)dx$, where $u_i^m = CS_i(p_i^m)$. To then calculate firm i's per-consumer profit function in terms of u_i, one can denote $p_i(u_i) = CS_i^{-1}(u_i)$ and $d_i(u_i) = D_i(p_i(u_i))$ to obtain $\pi_i(u_i) = d_i(u_i)(p_i(u_i) - k_i)$.

[14]For instance, see Narasimhan (1988), Baye et al. (1992), Wildenbeest (2011), Arnold et al. (2011), and Shelegia and Wilson (2021).

of observing behaviors, such as effort (see Dechenaux et al. 2015).[15] However, field evidence within the clearinghouse literature is better developed as firms' pricing behaviors can be more readily observed. As such, once viewed through the lens of our framework, the empirical clearinghouse literature can offer all-pay contest literature some established field results and statistical methods. These include a range of tests, and evidence, for the use of equilibrium mixed strategies, structural estimation techniques and methods for adjusting for player asymmetries.[16] b) The clearinghouse literature can also help inform the contest literature in regards to a long-standing problem of equilibrium selection when there are $n > 2$ symmetric players. We explore this in the next section.

5 Equilibrium with $n > 2$ Symmetric Players

This section now extends our framework to more than two players, $n > 2$. It is well known that a general analytical characterization of full information all-pay contests under arbitrary asymmetry for $n > 2$ is inherently difficult, even without our additional complications of participation costs and default allocations.[17] Therefore, we opt to focus on a symmetric setting instead. Here, the standard all-pay contest literature has a long-standing problem of equilibrium multiplicity; in addition to the symmetric equilibrium, there also exists a continuum of asymmetric equilibria (Baye et al. 1996).

In Section 5.1, we first demonstrate how the presence of participation costs and default allocation probabilities can resolve this issue to ensure that only the symmetric equilibrium remains. To do so, we build on the clearinghouse literature where a parallel problem exists; the symmetric clearinghouse setting with zero advertising costs (à la Varian 1980) also has an infinite number of equilibria when there are more than two firms. However, Arnold and Zhang (2014) show how the presence of positive advertising costs à la Baye and Morgan (2001) is enough to ensure equilibrium uniqueness. We import and generalize their approach into our contest setting, while also revealing the importance of their implicit assumption of positive default allocation probabilities.

Given this unique equilibrium, Section 5.2 then goes on to show how participation costs and default allocation probabilities can also have qualitatively important effects in reversing standard comparative statics. For instance, amid the recent interest in the 'competitiveness' of all-pay contests (e.g. Fang et al 2020), we show i) how an individual's expected offer, $E(u)$, can be *increasing*, rather than decreasing, in the number of players, and ii) rather than being independent to scaling, we show how dividing the contest's prize and players across $m > 1$ symmetric sub-contests always *reduces* expected offers.

[15] Some notable exceptions include Liu et al. (2014) and Boudreau et al. (2016).

[16] For example, see Wildenbeest (2011), Allen et al. (2014), Lach and Moraga-González (2017), Pennerstorfer et al. (2020), and Shelegia and Wilson (2021).

[17] For more details and the best findings in this regard, see Siegel (2009, 2010 and 2014).

5.1 Unique Equilibrium

In their seminal paper, Baye et al. (1996) show how a standard, symmetric, single-prize all-pay contest has an infinite number of equilibria when there are more than two players. This has remained a long-standing problem within the literature, providing uncertainty over players' predicted behavior. Specifically, such multiple equilibria would have the following features (translated into our set-up and notation). Due to the absence of participation costs, each player is always active with probability one, $\alpha = 1$. There then exists i) a unique symmetric equilibrium where all players mix over $[u^m, \bar{u}]$ with no mass at u^m, and ii) a continuum of asymmetric equilibria where at least two players mix over $[u^m, \bar{u}]$, while others mix over $[\underline{u}_i, \bar{u}]$ with a positive mass point at u^m, where $\underline{u}_i > u^m$ is a free individual parameter (and where the relevant player i bids u^m with probability one if $\underline{u}_i \geq \bar{u}$).

However, we now show how participation costs and default allocation probabilities can guarantee equilibrium uniqueness. To proceed under $n \geq 2$ symmetric players, we maintain assumptions A1-A3 and an n-player version of Condition X such that $x_i = (X/n) \in (0, 1)$ for all i:

Proposition 1. *Suppose there are $n \geq 2$ symmetric players and $x_i = (X/n) \in (0, 1)$ for all i. Then the unique equilibrium of any generic or non-generic contest is symmetric:*
i) When $s \leq u^m$, all players are passive, $\alpha_i = 0\ \forall i$.
ii) When $s > u^m$, all players are active with probability $\alpha_i = \alpha \in (0, 1)$ in (10). They all randomize on $[u^m, \bar{u}]$ with $F_i(u) = F(u)$ in (11) where $\bar{u} = s$ and $\beta_i = 0\ \forall i$.

$$\alpha = 1 - \left(\frac{c(u^m)}{b(u^m)} \right)^{\frac{1}{n-1}} \tag{10}$$

$$F(u) = \left(\frac{W(\bar{u}) - L(u)}{W(u) - L(u)} \right)^{\frac{1}{n-1}} \tag{11}$$

Proposition 1 shows how all asymmetric equilibria disappear and that only the symmetric equilibrium remains once both participation costs and default allocation probabilities become positive. Provided participation is not too costly, this equilibrium involves the players mixing between passive and active participation with interior probability, and randomizing over $[u^m, \bar{u}]$ without mass at u^m. This applies for both generic and non-generic contests.

Although lengthy to prove, the intuition of Proposition 1 can be understood as follows. All the potential asymmetric equilibria involve at least one player using mass at the lowest possible active offer, u^m. Such an offer at u^m is relatively uncompetitive because at least two other players always mix over $[u^m, \bar{u}]$. Hence, whenever active participation is costly, the use of such mass points at u^m becomes dominated and cannot be part of equilibrium behavior. Thus, only symmetric equilibria can remain. Then, given positive default allocation probabilities, one can use a logic akin to Theorem 1, to show that only a single symmetric equilibrium exists.

5.2 Equilibrium Features

We now briefly discuss some interesting features of the unique equilibrium. Specifically, in light of the recent interest in the 'competitiveness' of all-pay contests (e.g. Fang et al. 2020), we show how participation costs and default allocations can reverse some standard results in relation to i) an increase in the number of players, and ii) 'scaling'. To see these features most easily, we place some more structure on our symmetric $n \geq 2$ model by focusing on the example settings from Section 4 under linear effort costs, ku, with $a = 1$.

As a preliminary step, one can verify the comparative statics with respect to our key variables: participation costs, A, and the (total) default allocation probability, X. Following an increase in A or X, passive participation becomes relatively more attractive, lowering each player's strength, s. Hence, as expected, in the active equilibrium (where $s > u^m$), each player lowers their probability of being active, reduces their offers in the sense of first-order stochastic dominance, and earns higher expected payoffs.

More substantially, now consider a change in competitiveness due to an increase in the number of players, n. In a standard, symmetric all-pay contest (without participation costs or default allocations), it is well-known that expected individual offers, $E(u)$, are decreasing in the number of players, and total expected offers, $nE(u)$, are independent of n (e.g. Hillman and Samet 1987). Similarly, in the parallel clearinghouse literature with zero advertising costs à la Varian (1980), it is also well-known that expected prices, $E(p)$, are increasing in the number of firms (e.g. Morgan et al. 2006). Intuitively, a higher number of players diminishes the chance of any given player winning the contest and so discourages them from competing aggressively. In contrast, in our framework with participation costs and default allocation probabilities, Proposition 1 suggests that $E(u)$ and $nE(u)$ can both *rise* in response to more players. Hence, if participation costs vary across different settings, this result could help understand the varied empirical findings regarding how contest offers are affected by the number of players (see Dechenaux et al. 2015) or the empirical literature on how the number of firms affects market prices in clearinghouse settings (e.g. Allen et al. 2014, Lach and Moraga-González 2017).

To see this result most easily, it is mathematically convenient to use the example from Section 4.2 that considers participation costs in the form of a reservation offer, $u^R > u^m = 0$. By denoting passive offers as zero, $\phi \equiv 0$, one can then show that $E(u) = \int_{u^R}^{\bar{u}} uf(u)du$ equals $\frac{V}{n} - \left(\frac{u^R}{(n-X)}\right)T$ where $T = X + (1-X)\left(\frac{nu^R}{V(n-X)}\right)^{\frac{1}{n-1}}$. For any $n \geq 2$ and any positive X, it then follows that total expected offers, $nE(u)$, are strictly increasing in n for any $u^R > 0$, and $E(u)$ is strictly *increasing* in n if u^R is sufficiently close to the boundary for active participation, $u^R \to s$. Intuitively, once participation costs and default allocation probabilities are positive, an increase in n now generates a second, opposing effect that makes the players more aggressive. Under this effect, an increase in n prompts the players to make higher offers by reducing each player's chance of winning when passive via their default allocation probability, $x = X/n$. This effect is strongest when participation costs are relatively large. Indeed, when participation costs are close to their largest level while still permitting active offers, our result shows how this effect can dominate the standard effect on $E(u)$.

Now consider a different change in competitiveness in the form of 'scaling' (Fang et al. 2020). In our single prize context, scaling reduces to a comparison between a grand contest involving the full prize, V, and all n symmetric players, versus an alternative where the contest is divided into $m > 1$ parallel, identical sub-contests, each with a single prize of $\hat{V} = V/m$ and $\hat{n} = n/m \geq 2$ distinct players (where we assume the default allocation probabilities are adjusted in each sub-contest, holding X constant, such that $\hat{x} = X/\hat{n}$). In a standard, symmetric all-pay contest (without participation costs or default allocations), one can use well-known results (e.g. Hillman and Samet 1987) to verify that scaling does not change the level of expected offers, $E(u)$. Intuitively, each player optimally responds to the reduced prize and the reduced number of rivals by using an offer distribution with a lower variance, but an equal expected offer. However, within our framework, Proposition 1 implies that a grand contest can produce strictly higher expected offers. For instance, using the example from Section 4.2 again, a player's expected offer within a sub-contest will equal $E(u) = \frac{\hat{V}}{\hat{n}} - \left(\frac{u^R}{(\hat{n}-X)}\right)\hat{T}$ where $\hat{T} = X + (1-X)\left(\frac{\hat{n}u^R}{V(\hat{n}-X)}\right)^{\frac{1}{\hat{n}-1}}$. (Ignoring any integer issues), it then follows that $\frac{\partial E(u)}{\partial m} < 0$ for u^R sufficiently close to the boundary for active participation, $u^R \to s$. Hence, for high participation costs, each player's expected offers (and therefore total expected offers, $nE(u)$) are maximized at $m = 1$ as consistent with using a grand contest.[18]

6 Contest Design

To further demonstrate the benefits of our framework, this final section takes a different direction by returning to our full *two-player* setting in order to analyze how a contest organizer would optimally design the most novel features of our contest: i) the default allocation probabilities, and ii) the participation costs. As later detailed, these contest design tools have received little or no attention within the existing literature (e.g. Chowdhury et al. 2019 and Fu and Wu 2019).

[18]Specifically when $u^R \to V\left(\frac{1}{m} - \frac{X}{n}\right)$ so that $u^R \to \tilde{s}$, $\frac{\partial E(u)}{\partial m} \to -\frac{V}{m(n-m)} < 0$. Note this result need not depend on $X > 0$. Indeed, it can hold in Hillman and Samet's (1987) model of positive participation costs with zero default allocation probabilities - yet to our knowledge, this has not been noted within the literature.

We will assume that the organizer wishes to maximize offers. Specifically, a contest design will be referred to as *'offer-maximizing'* if it maximizes any combination of the sum of total expected offers, $E(u_1) + E(u_2)$, and the expected winning offer, $E(u_{max})$. Aside from capturing familiar contest objectives related to the associated level of effort or bids, this objective can also correspond to consumer surplus in a clearinghouse sales context. Throughout, whenever the players are mixing over the interval $(u^m, \bar{u}]$, we refer to an 'improvement' (or 'reduction') in player i's offers in the sense of first-order stochastic dominance (FOSD). (Holding constant player j's strategy), such an improvement (or reduction) ensures both an increase (or decrease) in player i's expected offer, $E(u_i)$, and the contest's expected winning offer, $E(u_{max})$.[19]

6.1 Default Allocation Probabilities

To begin, we consider how an organizer would optimally manipulate the default allocation probabilities, $\boldsymbol{x} \equiv \{x_1, x_2\}$, under the assumption that these can be credibly announced at the start of the game. We argue that the use of $\boldsymbol{x}$ offers a practical, low-cost form of contest design that has remained unstudied within the previous literature.

While the spirit of our results can be shown more generally, we focus on the following setting to present our findings most cleanly. First, aside from $\boldsymbol{x}$, we assume the players are otherwise symmetric. Hence, from Definitions 1 and 2, both players will always have the same reach, $r_1 = r_2 = r$, but will vary in strength whenever $x_i \neq x_j$. Second, as consistent with Condition X, we assume the organizer is unable to commit to an individual default allocation probability that is lower than some $\underline{x} \in (0, 0.5)$ for either player. Hence, the organizer must select $x_i \geq \underline{x}$ for $i = \{1, 2\}$ such that $x_1 + x_2 = X \in [2\underline{x}, 1]$. For all permitted $\boldsymbol{x}$, this implies that reach remains larger than strength, $r > s_i$, for $i = \{1, 2\}$. Finally, we ensure the players have some basic potential to be active. Specifically, we assume $W(u^m) > L^\phi + \underline{x}(W^\phi + L^\phi)$ such that $s_i > u^m$ when $x_i = \underline{x}$ for any player i.

As some preliminary comparative statics, consider the effects of a marginal increase in x_j while holding x_i constant (when $x_1 + x_2 < 1$). From (3)-(4), this will strictly reduce player j's strength (when $s_j > u^m$), but leave s_i and r unchanged. Hence, we know that player i will be the stronger player, with $s_i > s_j$, whenever $x_i < x_j$. We can now state the following for any generic or non-generic contest:

Proposition 2. *In any contest under our assumptions, it is always strictly offer-maximizing to set $x_i = \underline{x}$ and $x_j = 1 - \underline{x}$.*

[19]Technically, it is sufficient to define a FOSD improvement (or reduction) to occur when a) the player's new offer distribution $\hat{F}_i(u)$ is weakly less (greater) than their original offer distribution $F_i(u)$ for all active offers, $u \in [0, \infty]$, b) the player's new probability of being passive, $(1 - \hat{\alpha}_i)$, is weakly less (greater) than their original probability of being passive, $(1 - \alpha_i)$, and c) either i) $\hat{F}_i(u) < (>) F_i(u)$ for at least some $u \in [0, \infty]$, and/or ii) $(1 - \hat{\alpha}_i) < (>)(1 - \alpha_i)$.

Despite the players being otherwise symmetric, Proposition 2 indicates that an offer-maximizing organizer will never wish to select a symmetric set of default allocation probabilities under our conditions. Instead, the organizer can always do strictly better by decreasing the default allocation probability of one player and increasing the other's until $x_i = \underline{x}$ and $x_j = 1 - \underline{x}$. In this sense, it is optimal to 'favor' one of the players. Moreover, in the extreme, when $\underline{x}$ is close to zero, it is optimal to make one of the players the 'default winner' in the event that both players are inactive, with $x_j \to 1$. Thus, in the context of our example settings, i) favoring a default winner in tendering contexts may stimulate more competition for a contract, and ii) the use of more balanced default allocation probabilities in legal/policy decisions may reduce legal/lobbying expenditures.

Much of the intuition can be gained from inspecting case iv) in Theorem 1 where $u^m < s_i < r$, and where we assume i is the stronger player with $s_i \geq s_j$ (such that $x_i \leq x_j$). First, consider a decrease in player i's default allocation probability, x_i, while holding x_j constant. This reduces i's expected payoff from being passive, and thereby increases her strength and prompts her to be more aggressive. In turn, via a form of strategic complementarity, player j responds by also becoming more aggressive. Hence, expected offers increase. Second, consider an increase in player j's default allocation probability, x_j, while holding x_i constant. This enhances j's expected payoffs from being passive and so reduces her strength. Ceteris paribus, this encourages j to be less aggressive. However, player i responds by competing harder as she understands that she now has a higher chance of winning with an active offer, and this then encourages j to be more aggressive too. In equilibrium, these two conflicting incentives for player j cancel, leaving only player i to be more aggressive. The results of Proposition 2 then follow by combining the effects of decreasing x_i and increasing x_j.

The existing literature on all-pay auctions has suggested that when players become more asymmetric, then both players are likely to compete less aggressively via the 'discouragement effect' (e.g. Baye et al. 1993). More recent results on contest design build on this to show how competition can be increased by handicapping the ex ante stronger player and favoring of the ex ante weaker player in order to 'level the playing field'.[20] Somewhat similarly, we demonstrate how an organizer can induce more fierce competition by using asymmetric default allocation probabilities. However, our results differ in two important ways. First, in contrast to much of the literature, we show how an asymmetric contest design can stimulate competition even when the players are otherwise symmetric.[21] Second, and more unusually, rather than leveling the playing field, our results suggest that offer-maximizing organizers should use default allocation probabili-

[20]For instance, Szech (2015) and Franke et al. (2018) show this in relation to the use of tie-break rules or headstarts/multiplicative biases, respectively.

[21]As noted in the introduction, a small stream of literature has found a similar principle can also apply but these focus on different settings and different mechanisms (e.g. Drugov and Ryvkin 2017, Barbieri and Serena 2022, and Pérez-Castrillo and Wettstein 2016).

ties to create or enhance any difference between the two players' strengths. Indeed, the weaker (stronger) player with a relatively higher (lower) default allocation probability should optimally be made even weaker (stronger) by increasing (reducing) their value of x_i.

6.2 Participation Costs

We now move on to consider the optimal design of participation costs under the assumption that the contest organizer can manipulate the players' individual, and potentially asymmetric, costs of being active, $A_1 \geq 0$ and $A_2 \geq 0$, at the start of the game. To focus on the optimal design of participation costs per se, we assume that $\boldsymbol{A} = \{A_1, A_2\}$ comprises of players' set-up costs. This rules out the possibility that participation costs take the form of entry fees which are used to enhance the prize fund or the value of winning.[22]

While the spirit of our results can be shown more broadly, we focus on the following setting. First, apart from, $\boldsymbol{A}$, we assume the players are otherwise symmetric. Unlike the previous subsection, this implies that the players will vary in both reach and strength whenever $A_i \neq A_j$. Second, for each player i and for all active offers, we re-define the payoff functions as follows: $L(u) \equiv l(u) - A_i$ and $W(u) \equiv w(u) - A_i$ where $w(u)$ and $l(u)$ satisfy versions of A1-A3 and where $u^m \geq 0$ is their maximizer. In particular, to ensure that costly participation remains in A3 even if $A_i = 0$, we assume some other form of exogenous participation costs exists such that $L^\phi > l(u^m)$ and $W^\phi > w(u^m)$. Finally, we maintain some basic potential for the players to be active. Specifically, we let $w(u^m) > L^\phi + x(W^\phi + L^\phi)$ such that $s_i > u^m$ when $A_i = 0$ for any player i.

As some preliminary comparative statics, consider the effects of a marginal increase in A_j while holding A_i constant. From (3)-(4), this will strictly reduce player j's reach and strength (when they exceed u^m), but leave player i's reach and strength unchanged. Hence, we know that player i will have the higher strength and reach, $s_i > s_j$ and $r_i > r_j$, whenever $A_i < A_j$. We can now state the following for any generic or non-generic contest:

Proposition 3. *In any contest under our assumptions, it is always strictly offer-maximizing to set* $A_i = 0$ *and* $A_j = \bar{A} \equiv \frac{x(W^\phi - L^\phi)(L^\phi - l(u^m))}{w(u^m) - l(u^m) - x(W^\phi - L^\phi)} > 0$ *such that* $s_1 = r_2 > u^m$.

One may have predicted that the offer-maximizing design would involve zero participation costs for both players. Indeed, this is easy to show if participation costs are forced to be symmetric, $A_1 = A_2$. However, Proposition 3 demonstrates that such logic is incorrect if the contest designer can employ an asymmetric contest design. Indeed, like the

[22] In their private information setting, Hammond et al. (2019) and Liu and Lu (2019) show how positive (symmetric) participation costs can be optimal due to their effects on i) deterring low ability entrants, and ii) generating funds to increase the prize fund which encourages more effort from high ability entrants. In contrast, in a setting without these effects, we consider the optimal design of potentially, asymmetric participation costs per se.

previous subsection, despite the players being otherwise symmetric, an offer-maximizing organizer will optimally use an asymmetric design, with $A_i = 0$ and $A_j = \bar{A} > 0$. Hence, this suggests that asymmetric and positive positive participation costs may arise endogenously. Thus, within our examples, this implies that i) a contest organiser may wish to use asymmetric participation costs to stimulate higher bids in symmetric tendering contexts, and ii) discriminatory advertising costs or platform fees can be *pro-competitive* in clearinghouse contexts.

Much of the intuition can be understood within equilibrium case iv) where $u^m < s_i < r_j$ and $s_i \geq s_j$ (such that $A_i \leq A_j$). First, for reasons similar to Proposition 2, a decrease in the (weakly) stronger player's participation cost, A_i, increases i's strength and prompts her and her rival to be more aggressive. This is consistent with the instinctive logic of optimally lowering participation costs. However, an *increase* in the weaker player's participation cost, A_j, also acts to make player i *more* aggressive with no change in player j's behavior. The logic of this is also similar to that in Proposition 2; the rise in A_j prompts player i to be more competitive because she understands that she has an improved chance of winning. Hence, it is this effect that ensures $A_1 = A_2 = 0$ is not optimal. Specifically, within case iv), the designer faces incentives to decrease A_i while increasing A_j such that s_i rises and r_j falls until $s_i = r_j$. At this point, it is offer maximizing to set the lowest values of $\{A_i, A_j\}$ possible while still ensuring set $s_i = r_j$, which gives $A_i = 0$ and $A_j = \bar{A} > 0$.

Again, the standard logic of leveling the playing field does not apply under positive participation costs and default allocation probabilities. Indeed, the weaker (stronger) player with the relatively higher (lower) participation costs should optimally be made even weaker (stronger) by increasing (reducing) their participation costs.[23][24] Once more, the combined presence of *combined* presence of participation costs and default allocations is key as $\bar{A}$ would equal zero if $x = 0$.

7 Conclusion

Players often face direct costs of participating in contests. In such cases, the outcome can depend upon the 'default allocation' - how the prize is awarded if no player actively competes. The existing literature with participation costs has neglected this issue by implicitly assuming that the prize is only awarded under active participation. However,

[23] As we later show, participation costs create a discontinuity in the players' payoffs. Duvocelle and Mourmans (2021) study some wider forms of payoff discontinuities and show how Siegel's equilibrium payoff results can still apply.

[24] In a different setting with private information, a literature on all-pay auctions considers (symmetric) participation costs in the form of entry fees that can be used to supplement the prize fund (e.g. Hammond et al. 2019 and Liu and Lu 2019). We exclude this possibility in order to focus solely on the role of costly participation per se, with no connection to the prize fund.

in practice, there are many important situations where this does not apply. To help better address these issues, our paper makes three main contributions. First, it provides a general, tractable framework that can explicitly characterize all potential equilibria in all-pay contests while allowing for general forms of both participation costs and default allocations, under arbitrary asymmetry. Second, the paper uses this framework to formally connect some recent developments in all-pay contests (Siegel 2009, 2010, 2014) to the broad family of 'clearinghouse' models on sales price competition that are popular workhorses in IO and marketing (e.g. Baye and Morgan 2001, Baye et al. 2004, Baye et al. 2006). The documented link offers substantial benefits for future research by enabling the two literatures to trade theoretical and empirical insights. As examples, we use this link to resolve two long-standing problems: addressing asymmetry in clearinghouse models and equilibrium uniqueness in symmetric n-player all-pay contests. Finally, we analyze how participation costs and default allocations can be used as new, practical tools in contest design. Throughout, we show how the *combined* presence of participation costs and default allocations is key. Together, they can often reverse otherwise familiar intuitions.

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Appendix A: Main Proofs

Proof of Lemma 3. By adapting well-known results in the contest or clearinghouse literatures (e.g. Narasimhan 1988, Hillman and Riley 1989, Baye et al. 1992, Baye et al. 1996), we know that in equilibrium: i) no player will ever use a point mass at $u > u^m$, and ii) if one player has a point mass at u, then the other player will not. Hence, when combined with Lemma 1, player i can only possibly use a point mass at u_i^m or u_j^m. We now prove that player i will never put a point mass at $u_j^m \neq u_i^m$. First, suppose $u_i^m > u_j^m$. A point mass at u_j^m cannot be optimal as $u_i = u_j^m$ is dominated via Lemma 1. Second, suppose $u_i^m < u_j^m$. By reversing the previous argument together with Lemma 1, we know that player j will never select $u_j \in [u_i^m, u_j^m)$. Thus, if player i had a mass point at u_j^m, then she would optimally deviate by moving the mass from u_j^m to u_i^m in order to increase her payoffs from winning (or losing) without affecting her probability of winning. □

Proof of Lemma 4. Suppose $\alpha_i > \beta_i \geq 0$. a) From Lemma 1, we know no player will set an active offer in the interval $(\min(u_i^m, u_j^m), u^m)$. Hence, given Lemmas 2 and 3, player i must make offers above u^m with positive probability. For this to be optimal, it must be that player j also makes offers above u^m with positive probability, otherwise i would deviate. Hence, $\bar{u}_1, \bar{u}_2 > u^m$. By adapting standard well-known results (e.g. Narasimhan 1988, Hillman and Riley 1989, Baye et al. 1992, Baye et al. 1996), one can then demonstrate $\bar{u}_1 = \bar{u}_2$ as well as b). For c), we know that any player $l = \{1, 2\}$ has an expected payoff from any $u \in (u^m, \bar{u}]$ equal to $L_l(u) + F_k(u)[W_l(u) - L_l(u)]$ given $k \neq l$. For player l to mix over $u \in (u^m, \bar{u}]$, she must earn the same equilibrium payoffs, Π_l^*, over this interval. At $u_l = \bar{u}$, she can guarantee to win (as there are no mass points at $\bar{u}$). Hence, it must be that $\Pi_l^* \equiv W_l(\bar{u})$ which implies the unique active offer distribution $F_k(u)$ in (5). □

The proof of Theorem 1 is contained separately in Appendix B.

Proof of Proposition 1. The proof proceeds in a series of steps that build on those in Arnold and Zhang (2014):

STEP 1: To begin, it is trivial to reproduce versions of Lemmas 1-3 for the case of $n > 2$ symmetric players. Further, by following standard results in the literature, it is also straightforward to note a few additional results (without full proof) that apply when at least two players are active with positive probability. i) At least one player must have a lower bound of their support, $\underline{u}_i$, equal to u^m. To show this, suppose one or more players share the lowest lower bound, $min\{\underline{u}_j\} > u^m$. Then at least one such player would optimally deviate by relocating probability mass from just above the lower bound to u^m because this would strictly increase their payoffs via A2 and yet leave their probability of winning (nearly) unchanged. ii) There can be no interval of active offers, $u \in (u', u'')$ with $u' < u''$, that is only in the support of one player. If so, that player's expected payoffs would be decreasing across the interval and so they would optimally reallocate the probability mass in the interval to the lower end of the interval. iii) These two results then imply that if at least two players are active with positive probability, then the lower bound for at least two players, $\underline{u}_i$ and $\underline{u}_j$, must equal u^m.

STEP 2: In equilibrium, no player i can have $\alpha_i = 1$. We prove this by contradiction across three exhaustive cases. First, suppose there are at least two players, i and j, with $\alpha_i = \alpha_j = 1$. In this case, it must be that $\underline{u}_i = \underline{u}_j = u^m$. If not, with $\underline{u}_i < \underline{u}_j$, then player i would always wish to deviate. Further, from Lemma 3, we also know there can be no ties in active offers within any equilibrium. Hence, players i and j must lose whenever they select u^m and so earn an equilibrium payoff of $L(u^m)$. They would then wish to deviate to being passive to earn L^ϕ via A3. Hence, at most, only one player can have $\alpha_i = 1$. Second, suppose $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$. For this to be an equilibrium, i cannot wish to deviate to being passive, and any player j cannot wish to deviate to $u^m+\varepsilon$ for sufficiently small ε. This requires $W(u^m) \geq L^\phi + x(W^\phi - L^\phi)$ and $L^\phi \geq W(u^m + \varepsilon)$ respectively, which provides a contradiction for small enough ε given $x(W^\phi - L^\phi) > 0$. Finally, suppose only one player i has $\alpha_i = 1$, and at least one player j has $\alpha_j \in (0, 1)$. From Step 1, we know at least two players k and l must have $\underline{u}_k = \underline{u}_l = u^m$. This cannot be an equilibrium if player i is neither k or l, as then players k and l will definitely lose at u^m and so would prefer to deviate. Hence, suppose player i equals k. If so, then i has to put a point mass on u^m or else l would lose for sure at u^m and so would wish to deviate. Therefore, given $\alpha_i = 1$ and $\alpha_h < 1$ $\forall h \neq i$, all players other than i must have an equilibrium payoff $\Pi^*_h = L^\phi$, while player i earns $\Pi^*_i = L^\phi + x(W^\phi - L^\phi)\Pi_{h\neq i}(1-\alpha_h) > L^\phi$. From $\Pi^*_h = L^\phi$ and $\alpha_j \in (0, 1)$, it must be that $\bar{u}_j = r$ otherwise j would deviate to above $\bar{u}$ if $\bar{u} < r$. However, from $\Pi^*_i > L^\phi$, it must be that $\bar{u}_i < \bar{u}_j = r$. Yet this leads to a contradiction: at $u_i = \bar{u}_i$, i's payoff is lower than Π^*_i. At $\bar{u}_i$, player j who randomizes just above $\bar{u}_i$ has an expected payoff equal to $\Pi^*_j = L(\bar{u}_i) + (W(\bar{u}_i) - L(\bar{u}_i))\Pi_{h\neq j}F_h(\bar{u}_i)$ and this must equal $\Pi^*_j = L^\phi$. However, at $\bar{u}_i$, player i earns $L(\bar{u}_i) + (W(\bar{u}_i) - L(\bar{u}_i))\Pi_{h\neq i}F_h(\bar{u}_i)$

and this must be less than L^ϕ because $F_j(\bar{u}_i) < 1$ and $F_i(\bar{u}_i) = 1$.

STEP 3: In equilibrium, any player who actively participates with positive probability has the same upper bound, $\bar{u}$. Suppose not. Specifically, suppose there are two active players, i and j, with $\bar{u}_i \equiv \bar{u} > \bar{u}_j > u^m$. From Step 1, we know it must be that $F_i(\bar{u}) = 1$ and $F_i(\bar{u} - \varepsilon) < 1$ for any $\varepsilon > 0$, such that $F_i(\bar{u}_j) \in (0, 1)$. As there can be no ties, we also know that $\Pi_i^* = W(\bar{u})$. For this to be equilibrium, we require i) player i's expected payoff at $\bar{u}$ to be weakly larger than her expected payoff at $\bar{u}_j$: $W(\bar{u}) \geq L(\bar{u}_j) + (W(\bar{u}_j) - L(\bar{u}_j))\Pi_{h\neq i}F_h(\bar{u}_j)$, and ii) player j's expected payoff at $\bar{u}_j$ to be weakly larger than her expected payoff at $\bar{u}$: $L(\bar{u}_j) + (W(\bar{u}_j) - L(\bar{u}_j))\Pi_{h\neq j}F_h(\bar{u}_j) \geq W(\bar{u})$. However, this leads to a contradiction because both inequalities cannot hold simultaneously as $\Pi_{h\neq i}F_h(\bar{u}_j) > \Pi_{h\neq j}F_h(\bar{u}_j)$ given $F_i(\bar{u}_j) \in (0, 1)$ and $F_j(\bar{u}_j) = 1$. Therefore, all active players must set a common upper bound, $\bar{u}$, and so achieve an equilibrium payoff, $W(\bar{u})$.

STEP 4: In equilibrium, all players must have $\alpha_i = \alpha \in [0, 1)$. From above, we know that each player must be passive with positive probability as no player can be active with probability one. Thus, any player i must earn equilibrium payoffs of $\Pi_i^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h\neq i}(1-\alpha_h)$. If any player i is active with positive probability, $\alpha_i \in (0, 1)$, then we further know that $\Pi_i^* = W(\bar{u})$ such that $W(\bar{u}) = L^\phi + x(W^\phi - L^\phi)\Pi_{h\neq i}(1 - \alpha_h)$. Clearly, it then follows that any player i with $\alpha_i \in (0, 1)$ must share the same value of $\alpha_i = \alpha \in (0, 1)$. Further it cannot be that one or more players have $\alpha_i = \alpha \in (0, 1)$ while one or more players have $\alpha_k = 0$ as player k would then earn strictly lower expected payoffs than an active player i and so wish to deviate to $\bar{u}$, a contradiction; $\Pi_k^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h\neq k}(1 - \alpha_h) < \Pi_i^* = L^\phi + x(W^\phi - L^\phi)\Pi_{h\neq i}(1 - \alpha_h) = W(\bar{u})$.

STEP 5: If $s \leq u^m$, then the equilibrium is unique and symmetric with $\alpha = 0$. From Step 4, we know $\alpha_i = \alpha \in [0, 1)$ for all i. Given $s \leq u^m$, we know $L^\phi + x(W^\phi - L^\phi) \geq W(u^m)$. In this case, there is always an equilibrium at $\alpha = 0$ as no player has a strict incentive to deviate to $\alpha_i > 0$ as they earn $L^\phi + x(W^\phi - L^\phi)$ by being passive and $W(u^m)$ at most from being active. Moreover, there is never an equilibrium with $\alpha \in (0, 1)$ as this would require any individual player to be indifferent between being passive and being active at u^m, such that $L^\phi + x(W^\phi - L^\phi)(1-\alpha)^{n-1} = L(u^m) + (W(u^m) - L(u^m))(1-\alpha)^{n-1}$. However, given $s \leq u^m$, this can never hold for $\alpha \in (0, 1)$. Hence, the only possible equilibrium involves $\alpha = 0$.

STEP 6: If $s > u^m$, then the equilibrium is unique and symmetric with $\alpha \in (0, 1)$ and $F_i(u) = F(u)$ for all i and for all $u \in [u^m, \bar{u}]$. If $s > u^m$, then $L^\phi + x(W^\phi - L^\phi) < W(u^m)$. Hence, there can be no equilibrium with $\alpha = 0$ as a player would wish to deviate to u^m instead. Therefore, it must be that $\alpha \in (0, 1)$. Next, we show that for any two players, i and j, it cannot be that $F_i(u') > F_j(u')$ for some offer $u' > u^m$. First, suppose that both players select u' with positive probability. $F_i(u') > F_j(u')$ then implies that player j has a higher probability of winning at u' and so the two players cannot have the same equilibrium payoffs, contrary to an earlier result, $\Pi_i^* = W(\bar{u})\ \forall i$. Second, suppose u' is only selected with positive probability by player i and not j. From Step 3, we know $\bar{u}_i = \bar{u}_j = \bar{u} > u^m$, and so there would have to be some $\hat{u} \in (u', \bar{u})$ in the support of both players, with $F_i(\hat{u}) > F_j(\hat{u})$ (unless player j has a mass point at $\hat{u}$ but this is ruled out as we know mass points can only arise at u^m for one player, via a n-player version of Lemma 3). Hence, like above, this leads to a contradiction as the two players cannot have the same equilibrium payoffs. Third, suppose u' is only selected with positive probability by player j and not i. At u', we know player j must earn $L(u') + (W(u') - L(u'))(1 - F_i(u'))\Pi_{k\neq i,j}(1 - F_k(u')) = \Pi_j^*$, while player i would earn $\Pi_i(u') = L(u') + (W(u') - L(u'))(1 - F_j(u'))\Pi_{k\neq i,j}(1 - F_k(u'))$. Given $F_i(u') > F_j(u')$, this leads to $\Pi_i(u') > \Pi_j^* = W(\bar{u})$, which again gives a contradiction. Fourth, suppose u' is not selected by either player with positive probability. In this case, consider the highest offer below u' which is selected by at least one player with positive probability, u''. Such an offer $u'' > u^m$ has to exist, otherwise $\alpha = 0$. As there are no point masses above u^m, we must have $F_i(u'') = F_i(u')$ and $F_j(u') = F_j(u'')$. Hence, if $F_j(u') > F_j(u')$ then $F_j(u'') > F_j(u'')$ and so one can then apply the previous deductions again to show a contradiction with u'' instead of u'.

Lastly, for any two players, i and j, it also cannot be that $F_i(u^m) > F_j(u^m)$. From above, we know for all i: $\alpha_i = \alpha$ and $F_i(u) = F(u)$ for $u > u^m$. Hence, it must also be that $F_i(u^m) = F(u^m)\ \forall i$. Therefore, when $s > u^m$, the equilibrium is unique and symmetric.

STEP 7: Finally, when $s > u^m$, we derive the equilibrium values of α and $F(u)$ in (10) and (11), together with $\bar{u} = s$ and $\beta = 0$. First, given $F_i(u^m) = F(u^m)$, all the players could, in principle, use an identical mass point at u^m. However, this is ruled out by a n-player version of Lemma 3 which says that only one player at most can use such a mass point. Hence, $\beta = 0$. Second, given $\alpha \in (0, 1)$, each player must be indifferent between i) being passive and selecting u^m, such that $\Pi_i^* = L^\phi + x(W^\phi - L^\phi)(1 - \alpha)^{n-1} = L(u^m) + (W(u^m) - L(u^m))(1 - \alpha)^{n-1}$, and ii) selecting any $u \in (u^m, \bar{u}]$, such that $\Pi_i^* = W(\bar{u}) = L(u) + F(u)^{n-1}(W(u) - L(u))$. By rearranging, these provide (10) and (11). Finally by setting $\Pi_i^* = W(\bar{u}) = L^\phi + x(W^\phi - L^\phi)(1 - \alpha)^{n-1}$ and inserting (10) for the value of α, one can show that $\bar{u} = s$. One can then verify that $1 - \alpha = F(u^m)$ such that $\beta = 0$ as required. □

Contest Design Proofs

For ease of exposition, the proofs for Propositions 2 and 3 make references to the cases of Theorems 1 and 2 in abbreviated form, e.g. T1i refers to case i of Theorem 1, and T2a refers to case a of Theorem 2.

Proof of Proposition 2. We proceed through a number of steps. Given our assumptions, it is useful to firstly summarize which equilibrium cases are relevant (across both generic and non-generic contests in Theorems 1 and 2). Specifically, from the text, we know $u_1^m = u_2^m = u^m$ and $r = r_1 = r_2$, and for any player i, we also know $s_i > u^m$ when $x_i = \underline{x}$ and $r > s_i$ for any permitted x_i. Therefore, the following cases can never apply: T1ii, T1iii, T1v, T2b, T2c and T2d. This leaves T1i) where $\alpha_1 = \alpha_2 = 0$ (if $s_1 < u^m < r$), T1iv) where $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1, 2\}$ (if $r > s_1 > u^m$), and T2a) where $\alpha_2 = 0$ and $\alpha_1 = \beta_1 \in [0, 1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$ (if $r > s_1 = u^m$).

From these, it is immediate that any $\boldsymbol{x}$ consistent with T1i can never be offer maximizing as both players would always be passive. Further, any $\boldsymbol{x}$ consistent with T2a can never be offer maximizing either because it would be dominated by some $\boldsymbol{x}$ consistent with T1iv. Intuitively, in T1iv, i) player 2 is active with positive, rather than zero, probability, and ii) player 1 is active with a higher probability (as $\theta_1(u^m) < \theta_1(\bar{u})$ for $\bar{u} > u^m$) and has an average active offer that is strictly higher than u^m. Hence, the offer-maximizing level of $\boldsymbol{x}$ must lie within the remaining case, T1iv. To understand more, the following lemma details the comparative statics within T1iv.

Lemma 5. *Let $r > s_1 > u^m$ such that T1iv applies. Then a) a marginal increase in x_1 reduces the offers of both players in the sense of FOSD, and b) a marginal increase in x_2 improves player 1's offers in the sense of FOSD but leaves player 2's offers unchanged.*

Proof. In T1iv, given $u_1^m = u_2^m = u^m$, we know $\bar{u} = s_1$, $F_1(u) = F_2(u) = \frac{W(s_1) - L(u)}{W(u) - L(u)}$, $1 - \alpha_1 = \frac{W(s_1) - L^\phi}{(W^\phi - L^\phi)x_2} \in (0, 1)$, $\beta_1 = F_1(u^m) - (1 - \alpha_1)$, $1 - \alpha_2 = F_2(u_1^m)$, $\beta_2 = 0$ and $\Pi_1^* = \Pi_2^* = W(s_1)$. a) From the text, we know $\partial s_1 / \partial x_1 < 0$. Hence, both players' offers reduce in the sense of FOSD because $F_1(u)$, $F_2(u)$, $(1 - \alpha_1)$ and $(1 - \alpha_2)$ are all strictly increasing in x_1. b) From the text, also recall $\partial s_1 / \partial x_2 = 0$. Thus, the only changes that occur involve a decrease in $1 - \alpha_1$ and an associated increase in β_1, $\frac{\partial(1-\alpha_1)}{\partial x_2} = -\frac{\partial \beta_1}{\partial x_2} < 0$. Hence, player 1's offers improve in the sense of FOSD, but player 2's offers remain unchanged. □

To complete the proof of Proposition 2, suppose $s_i \geq s_j$ (such that $x_i \leq x_j$). Then within T1iv where $u^m < s_i < r$, we know that both players offers are strictly reducing in x_i in the sense of FOSD, and an increase in x_j will improve player i's offer, while leaving player j's unchanged. Hence, it is offer-maximizing to decrease x_i until $x_i = \underline{x}$ and raise x_j until $x_j = 1 - \underline{x}$. Such a change will raise s_i and lower s_j, maintaining $s_i > s_j$. At such a point, $u^m < s_i < r$ still applies because we know $s_i > u^m$ when $x_i = \underline{x}$ and $s_i < r$ for all valid x_i.

Proof of Proposition 3. We proceed through a number of steps. Given our assumptions, it is useful to firstly summarize which equilibrium cases are relevant (across both generic and non-generic contests in Theorems 1 and 2). Specifically, from the text, we know $u_1^m = u_2^m = u^m$, and for any player i we also know $s_i > u^m$ when $A_i = 0$, and $r_i > s_i$ for any $A_i \geq 0$. Therefore, cases T1iii and T2c can never apply. However, all other cases remain possible. This leaves T1i and T2b where $\alpha_1 = \alpha_2 = 0$, T2a where $\alpha_2 = 0$ and $\alpha_1 = \beta_1 \in [0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$, T1ii and T2d where $\alpha_2 = 0$ and $\alpha_1 = \beta_1 = 1$, T1iv where $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1,2\}$, and T1v where $1 = \alpha_1 > \beta_1 \geq 0$ and $1 > \alpha_2 > \beta_2 \geq 0$.

From these, it is immediate that any $\boldsymbol{A}$ consistent with T1i and T2b can never be offer maximizing as both players would always be passive. Further, any $\boldsymbol{A}$ consistent with T1ii, T2a, or T2d (which all have $\alpha_1 = \beta_1 \in (0,1]$ and $\alpha_2 = 0$) can never be offer maximizing either because it would be dominated by some $\boldsymbol{A}$ consistent with T1v. Intuitively, in T1v (where $1 = \alpha_1 > \beta_1 \geq 0$ and $1 > \alpha_2 > \beta_2 \geq 0$), we know that i) player 2 is active with positive, rather than zero, probability, and ii) player 1 is active with a weakly higher probability and has an average active offer strictly above u^m. Hence, the offer-maximizing $\boldsymbol{A}$ must lie somewhere within the remaining cases, T1iv and T1v. To understand more, the next two lemmas detail the comparative statics within these two cases.

Lemma 6. *Let $u^m < r_2 \leq s_1$ such that T1v applies. Then a) a marginal increase in A_1 leaves both players' offers unchanged, while b) a marginal increase in A_2 reduces both players' offers in the sense of FOSD.*

Proof. In T1v, given $u_1^m = u_2^m = u^m$, we know $\bar{u} = r_2$, $F_1(u) = F_2(u) = \frac{w(r_2)-l(u)}{w(u)-l(u)}$, $\alpha_1 = 1$, $1 - \alpha_2 = F_2(u_1^m)$, $\beta_1 = F_1(u^m)$, $\beta_2 = 0$, $\Pi_1^* = w(r_2) - A_1$ and $\Pi_2^* = w(r_2) - A_2 \equiv L^\phi$. a) From the text we know $\partial r_2/\partial A_1 = 0$. Hence, a marginal increase in A_1 has no impact on the players' offers because $F_1(u)$, $F_2(u)$, $(1-\alpha_1)$, and $(1-\alpha_2)$ all remain unchanged. b) From the text, also recall $\frac{\partial r_2}{\partial A_2} = \frac{1}{w'(r_2)} < 0$. Hence, both player's offers reduce because $F_1(u)$, $F_2(u)$ and $(1-\alpha_2)$ are all strictly increasing in A_2 via r_2, while α_1 is independent of A_2. □

Lemma 7. *Let $u^m < s_1 < r_2$ such that case T1iv applies. Then a) a marginal increase in A_1 reduces both players' offers in the sense of FOSD, and b) a marginal increase in A_2 improves player 1's offers in the sense of FOSD but leaves player 2's offers unchanged.*

Proof. In T1iv, given $u_1^m = u_2^m = u^m$, we know $\bar{u} = s_1$, $F_1(u) = F_2(u) = \frac{w(s_1)-l(u)}{w(u)-l(u)}$, $1 - \alpha_1 = \frac{w(s_1)-A_2-L^\phi}{(W^\phi-L^\phi)x} \in (0,1)$, $\beta_1 = F_1(u^m) - (1-\alpha_1)$, $1 - \alpha_2 = F_2(u_1^m)$, $\beta_2 = 0$ and $\Pi_i^* = w(s_1) - A_i$ for $i = 1,2$. a) From the text we know $\partial s_1/\partial A_1 < 0$. Hence, both players' offers reduce in the sense of FOSD as $F_1(u)$, $F_2(u)$, $(1-\alpha_1)$ and $(1-\alpha_2)$ are all increasing in A_1. b) From the text, also recall $\partial s_1/\partial A_2 = 0$. Thus, in terms of offers, the only changes that occur involve a decrease in $1-\alpha_1$ and an associated increase in β_1, $\frac{\partial(1-\alpha_1)}{\partial A_2} = -\frac{\partial \beta_1}{\partial A_2} < 0$. Hence, player 1's offers improve in the sense of FOSD, but player 2's offers remain unchanged. □

To complete the proof of Proposition 3, suppose $A_i \leq A_j$ such that $s_i \geq s_j$ and $r_i \geq r_j$. First, consider case T1iv where $u^m < s_i < r_j$. Here, we know that a marginal reduction in A_i (and associated increase in s_i) will improve both players' offers, and a marginal increase in A_j (and associated reduction in r_j) will improve player i's offers, but leave j's unchanged. Hence, within this case, it is strictly offer maximizing to reduce A_i and increase A_j until the boundary point where we approach $s_i = r_j$. At this point, we enter case T1v where $u^m < r_j \leq s_i$. In this case, we know that a marginal reduction in A_j (and associated increase in r_j) can improve both player's offers. Therefore, we know that reducing A_j until the point where $r_j = s_i > u^m$ must be strictly offer-maximizing.

Implementing the point $r_j = s_i > u^m$ by manipulating $\{A_i, A_j\}$ is always possible given our assumption that $r_i > s_i > u^m$ when $A_i = 0$. Indeed, there are an infinite number of pairs of $\{A_i, A_j\}$ for which $r_j = s_i > u^m$. With use of (3) and (4), any such pair must satisfy $A_i + \frac{x(W^\phi - L^\phi)(L^\phi - (l(u^m) - A_i))}{w(u^m) - l(u^m) - x(W^\phi - L^\phi)} = A_j$ or equivalently, $A_j = \frac{A_i(w(u^m) - l(u^m)) + x(W^\phi - L^\phi)(L^\phi - l(u^m))}{w(u^m) - l(u^m) - x(W^\phi - L^\phi)}$. From Lemma 6, the offer maximizing pair must be the one with the lowest value of A_j. Hence, it is offer maximizing to set $A_i = 0$ and $A_j = \frac{x(W^\phi - L^\phi)(L^\phi - l(u^m))}{w(u^m) - l(u^m) - x(W^\phi - L^\phi)} \equiv \bar{A}$ where $\bar{A} > 0$ due to our assumptions $L^\phi > l(u^m)$ and $w(u^m) > L^\phi + x(W^\phi + L^\phi)$.

Appendix B: Proof of Theorem 1

This appendix provides the proof of Theorem 1 by deriving a more general result, Theorem 2, which characterizes the set of equilibria for *both* generic *and* non-generic contests. For convenience, it is useful to define

$$\delta_i(u) = 1 - \frac{W_i(u) - L_i(u_i^m)}{W_i(u_i^m) - L_i(u_i^m)}. \tag{12}$$

Theorem 2. *Given Condition X, the equilibrium in any generic or non-generic contest follows Theorem 1 unless any of the following knife-edge cases apply. If so, the equilibrium is potentially non-unique:*

a) When $s_1 = u_1^m$, player 2 is always passive, $\alpha_2 = 0$, but player 1 selects u_1^m with any probability $\alpha_1 = \beta_1 \in [0, 1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$.

b) When $r_2 = u_2^m$ and $s_1 < u_1^m$, then neither player is active, $\alpha_1 = \alpha_2 = 0$.

c) When $r_2 = u_2^m$, $s_1 > u_1^m$ and $u_1^m < u_2^m$, player 1 is always active with u_1^m, $\alpha_1 = \beta_1 = 1$, and player 2 selects u_2^m with any probability $\alpha_2 = \beta_2 \in [0, \min\{\sigma_1(u_1^m), \delta_1(u_2^m)\}]$.

d) When $r_2 = u_2^m$, $s_1 > u_1^m$ and $u_1^m \geq u_2^m$, player 1 is always active with u_1^m, $\alpha_1 = \beta_1 = 1$, and player is always passive, $\alpha_2 = 0$.

The proof of Theorem 2 proceeds as follows. Step 1 provides an exhaustive list of possible equilibrium forms. Step 2 defines some further features for each equilibrium form, and characterizes some necessary parameter conditions for each form to exist. Finally, Step 3 shows how these parameter conditions are enough to characterize the equilibria for the entire parameter space in a way that is consistent with Theorem 2.

Step 1: Possible Equilibrium Forms

Lemmas 1-4 offer a start in thinking about possible equilibrium forms. However, to narrow this down further, Lemma 8 shows that both players cannot be active in equilibrium with probability one.

Lemma 8. *Suppose both players are active with positive probability in equilibrium such that $\alpha_k > 0$ for $k = 1, 2$. Then it cannot be that $\alpha_1 = \alpha_2 = 1$. Instead, either i) $\alpha_k = \beta_k > 0$ for $k = 1, 2$, in which case it must be that $u_j^m > u_i^m$ and $\alpha_j = \beta_j \in (0, 1)$ for some j, or ii) $\alpha_k > \beta_k \geq 0$ for $k = 1, 2$, in which case it must be that $\alpha_2 \in (0, 1)$.*

Proof of Lemma 8. Suppose both players are active with positive probability in equilibrium such that $\alpha_k > 0$ for $k = 1, 2$. Then, we know it must be that either i) $\alpha_k = \beta_k > 0$ for $k = 1, 2$, or ii) $\alpha_k > \beta_k \geq 0$ for $k = 1, 2$ because Lemma 4 rules out the possibility that $\alpha_i > \beta_i \geq 0$ but $\alpha_j = \beta_j > 0$. First consider i). Here, two initial conditions must hold. First, form Lemma 3, it must be that $u_1^m \neq u_2^m$. Hence, without loss, let $u_j^m > u_i^m$. Then it must be that $\alpha_j = \beta_j < 1$. If instead, $\alpha_j = \beta_j = 1$, then player i would lose with certainty by selecting u_i^m, and so would deviate to $\alpha_i = \beta_i = 0$ to earn $L_i^\phi > L_i(u_i^m)$ via A3. Now consider ii). Here, it cannot be that $\alpha_2 = 1$. We prove this by contradiction under two exhaustive cases. First, suppose $\alpha_2 = 1$ with $\alpha_1 = 1$ and let $u_i^m = u^m > u_j^m$. From Lemma 4, we know both players must mix on $(u^m, \bar{u}]$ and that player i must select u_i (arbitrarily close to) u^m with positive probability, with no ties at such a point. Given $\alpha_j = 1$, i must always lose when making such an offer and so earn an equilibrium payoff (arbitrarily close to) $L_i(u_i^m = u^m)$. However, i would then strictly prefer to deviate by setting $u_i = \phi$ as $L_i^\phi - L_i(u_i^m) > 0$ via A3. Second, suppose $\alpha_2 = 1$ with $\alpha_1 \in (0, 1)$. Given $\alpha_2 = 1$, player 1 will earn L_1^ϕ when passive. From Lemma 4, we know that player 1 must must mix up to $\bar{u}$ and that player 1 will win with certainty at $\bar{u}$, earning $W_1(\bar{u})$. Hence, for player 1 to mix between ϕ and $\bar{u}$, we require $L_1^\phi = W_1(\bar{u})$. This implies $\bar{u} = r_1$. For $\alpha_2 = 1$, we need to rule out any deviations to ϕ and so we require $W_2(\bar{u}) \geq L_2^\phi + (1-\alpha_1)x_2(W_2^\phi - L_2^\phi)$. From the definition of strength, the RHS is equivalent to $W_2(s_2)$. Hence, we require $W_2(\bar{u}) \geq W_2(s_2)$ which implies $\bar{u} \leq s_2$. Therefore, when combined with $\bar{u} = r_1$ and $s_1 < r_1$, we require $s_1 < r_1 = \bar{u} \leq s_2$. This implies $s_1 < s_2$; a contradiction via Definition 3. □

Using this with the results from Lemmas 1-4, we can now list the possible equilibrium forms as follows.

Lemma 9. *Given Lemmas 1-4 and 8, the only possible equilibrium forms are:*
1. Neither player is active, $\alpha_1 = \alpha_2 = 0$.
2. Player i *is always active with* u_i^m, $\alpha_i = \beta_i = 1$, *and player* j *is always passive,* $\alpha_j = 0$.
3. Player i *randomizes between being active at* u_i^m *and being passive,* $\alpha_i = \beta_i \in (0, 1)$, *while player* j *is always passive,* $\alpha_j = 0$.
4. Each player k *randomizes between being active at* u_k^m *and being passive,* $\alpha_k = \beta_k \in (0, 1)$, *for* $k = 1, 2$, *where* $u_j^m > u_i^m$.
5. Player j *randomizes between being active at* u_j^m *and being passive,* $\alpha_j = \beta_j \in (0, 1)$, *but player* i *is always active with* u_i^m, $\alpha_i = \beta_i = 1$, *where* $u_j^m > u_i^m$.
6. Both players are active above u^m *and active with interior probability,* $1 > \alpha_i > \beta_i \geq 0$ *for* $i = 1, 2$.
7. Both players are active above u^m *where one player, player 2, is active with interior probability,* $1 > \alpha_2 > \beta_2 \geq 0$, *and one player, Player 1, is active with probability one,* $1 = \alpha_1 > \beta_1 \geq 0$.

Proof of Lemma 9. By definition, any equilibrium must have $\alpha_i \geq \beta_i \geq 0$ for $i = \{1, 2\}$. Hence, the only possible outcomes can be exhaustively listed by a) $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, b) $\alpha_i = \beta_i > 0$ and $\alpha_j = \beta_j \geq 0$, and c) $\alpha_i > \beta_i \geq 0$ for at least one player i. We now show how these possible outcomes are fully covered by equilibrium forms 1-7 in the Lemma. First, a) corresponds directly to form 1. Second, we can split b) into four sub-cases that directly correspond to forms 2, 3, 4, and 5 respectively. Note that Lemma 8 rules out $\alpha_i = \beta_i = 1$ for both players, and also ensures that $u_j^m > u_i^m$ must hold in forms 4 and 5. Finally, if c) applies then we know from Lemma 4 that $\alpha_j > \beta_j \geq 0$ must also apply, with both players being active above u^m. Hence, c) can be split into two sub-cases that correspond directly to forms 6 and 7. In form 7, note it cannot be that $\alpha_2 = 1$ due to Lemma 8. □

Step 2: Further Results on each Equilibrium Form

We now detail some further features of the equilibrium forms and define some necessary parameter conditions for the existence of each form. These results apply for both generic and non-generic contests.

Lemma 10. *Equilibrium Form 1:* $\alpha_1 = \alpha_2 = 0$ *is an equilibrium iff* $s_i \leq u_i^m$ *for* $i = \{1, 2\}$.

Proof of Lemma 10. If $\alpha_1 = \alpha_2 = 0$, then each player i expects to earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. For $\alpha_1 = \alpha_2 = 0$, we require no player i to have an incentive to deviate by submitting an active offer, even if they were to win at u_i^m with probability one. This requires $W_i(u_i^m) \leq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ for $i = \{1, 2\}$. From the definition of strength this is equivalent to $s_i \leq u_i^m$ for $i = \{1, 2\}$. □

Lemma 11. *Equilibrium Form 2: $\alpha_i = \beta_i = 1$ and $\alpha_j = 0$ is an equilibrium iff $i = 1$, $j = 2$, $s_1 \geq u_1^m$ and $r_2 \leq u^m$.*

Proof of Lemma 11. If $\alpha_i = \beta_i = 1$ and $\alpha_j = 0$ then from Lemma 2, player i earns $W_i(u_i^m)$ and player j earns L_j^ϕ. For this to be an equilibrium, it is first necessary that player i has no incentive to deviate to $u_i = \phi$ to earn $L_i^\phi + x_i(W_i^\phi - L_i^\phi)$. Hence, we require $W_i(u_i^m) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)$, which from the definition of strength, gives $s_i \geq u_i^m$. Second, it is necessary that player j has no incentive to deviate. If j deviated, she would optimally deviate to either i) just above u^m to earn slightly below $W_j(u^m)$ if $u_i^m = u^m \geq u_j^m$, or ii) u^m to earn $W_j(u^m)$ if $u_i^m < u_j^m = u^m$. Hence, as a necessary condition, we require $L_j^\phi \geq W_j(u^m)$, which by using the definition of reach requires $r_j \leq u^m$. Thus, this equilibrium requires $s_i \geq u_i^m$ and $r_j \leq u^m$. As we now prove, these two conditions cannot both hold unless $i = 1$ and $j = 2$. We proceed by contradiction. Suppose $j = 1$ such that $r_1 \leq u^m$. As $x_1 > 0$, this gives $s_1 < r_1 \leq u^m$. First suppose $u^m = u_2^m$. This then implies $s_1 < u_2^m$ which when combined with Definition 3, gives $s_2 \leq s_1 < u_2^m$ such that $s_2 \geq u_2^m$ can never apply. Finally, suppose $u^m = u_1^m$. Then $s_1 < r_1 \leq u^m$ gives $s_1 < u_1^m$ which implies $s_1 = -\infty$ from Definition 2. From Definition 3, this further implies $s_2 \leq s_1 = -\infty$ such that $s_2 \geq u_2^m \geq 0$ can also never apply. Hence, it must be that $i = 1$ and $j = 2$. $\square$

Lemma 12. *Equilibrium Form 3: $\alpha_i = \beta_i \in (0, 1)$ and $\alpha_j = 0$ is an equilibrium iff $i = 1$, $j = 2$, $s_1 = u_1^m$ and $\alpha_1 = \beta_1 \in (0, 1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$.*

Proof of Lemma 12. Suppose $\alpha_i = \beta_i \in (0, 1)$ and $\alpha_j = 0$. First, in order for player i to be willing to mix between u_i^m and ϕ, we require $W_i(u_i^m) = L_i^\phi + x_i(W_i^\phi - L_i^\phi)$ given $\alpha_j = 0$. This implies $s_i = u_i^m$ from the definition of strength. Further, as player j is passive, she must earn $\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1 - \alpha_i)$. To be an equilibrium, we require neither player to have an incentive to deviate. For i, this is trivial because she has no other profitable deviations. For j, we proceed to consider two exhaustive situations: $u_j^m \geq u_i^m$ and $u_j^m < u_i^m$.

Begin with the situation with $u_j^m \geq u_i^m$. Here, player j could deviate from ϕ to u_j^m (or just above u_j^m if $u_i^m = u_j^m$). To rule this out, we need $\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1 - \alpha_i) \geq W_j(u_j^m)$ which is equivalent to $\alpha_i \leq \theta_i(u_j^m)$. For later, it is useful to note that this condition binds, that is $\theta_i(u_j^m) < 1$, when $W_j(u_j^m) > L_j^\phi \leftrightarrow u_j^m < r_j$. But more immediately, note that to allow $\alpha_i > 0$ as required, we need $\theta_i(u_j^m) > 0$ or equivalently, $W_j(u^m) < L_j^\phi + x_j(W_j^\phi - L_j^\phi)$. Via the definition of strength, this implies $s_j < u_j^m$, which in turn implies $s_j = -\infty$. Hence, as $s_i = u_i^m \geq 0$, it must be that $i = 1$ and $j = 2$ from Definition 3. Given these player identities, we require $W_2(u^m) < L_2^\phi + x_2(W_2^\phi - L_2^\phi)$ which implies from (7) that $\sigma_1(u_2^m) < 0$.

Now consider the other situation with $u_j^m < u_i^m$. Here, we need to consider two possible deviations by player j to a) u_j^m, or b) just above u_i^m. First consider deviation a). This will not be optimal if

$$\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) \geq L_j(u_j^m) + (W_j(u_j^m) - L_j(u_j^m))(1-\alpha_i) \tag{13}$$

Given $L_j^\phi > L_j(u_j^m)$ from A3, this condition holds for any α_i if $W_j(u_j^m) \leq L_j(u_j^m) + x_j(W_j^\phi - L_j^\phi)$. The condition also continues to hold for higher $W_j(u_j^m)$ if $W_j(u_j^m) \leq L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ (such that $s_j \leq u_j^m$) because there, even at $\alpha_i = 0$, (13) holds. For $W_j(u_j^m) > L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ (such that $s_j > u_j^m$), then one can rearrange (13) to require $\alpha_i \geq \sigma_i(u_j^m) = 1 - \frac{L_j^\phi - L_j(u_j^m)}{W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi)}$. Now consider deviation b) to just above $u_i^m > u_j^m$. To rule this out we require $\Pi_j^* = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) \geq W_j(u_i^m)$ or equivalently, $\alpha_i \leq \theta_i(u_i^m) = 1 - \frac{W_j(u_i^m) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j}$.

We now explore deviations a) and b) in three exhaustive parameter regions and show that in each case, it must be that $\alpha_i = \beta_i \in (0,1)$ must lie within the interval, $[\sigma_i(u_j^m), \theta_i(u^m)]$, as required. Finally, we then prove that in each case, it must be that $i = 1$ and $j = 2$.

First, suppose $s_j \leq u_j^m$. Here, deviation a) is never profitable using earlier results, but to rule out deviation b) we need $\alpha_i \leq \theta_i(u_i^m)$. To then allow $\alpha_i > 0$ as required, we need $\theta_i(u_i^m) > 0$ or equivalently, $W_j(u_i^m) < L_j^\phi + x_j(W_j^\phi - L_j^\phi)$. Given $u_i^m > u_j^m$, this always holds because $W_j(u_i^m) < W_j(u_j^m) \leq L_j^\phi + x_j(W_j^\phi - L_j^\phi)$ where the last part follows from $s_j \leq u_j^m$. So, if $s_j \leq u_j^m$ then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i \leq \theta_i(u_i^m)$.

Second, suppose $s_j > u_j^m$ and $r_j \leq u_i^m$. Deviation b): the latter condition on r_j implies $\theta_i(u_i^m) \geq 1$ such that any $\alpha_i = \beta_i \in (0,1)$ will automatically satisfy $\alpha_i \leq \theta_i(u_i^m)$. However, to rule out deviation a), given $s_j > u_j^m$, we know from earlier results that we require $\alpha_i \geq \sigma_i(u_j^m)$. Hence, to allow $\alpha_i = \beta_i \in (0,1)$ we need $\sigma_i(u_j^m) < 1$. This is assured by $s_j > u_j^m$ and A3: $L_j^\phi - L_j(u_j^m) > 0$ and (which in turn gives $W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi) > 0$). So if $s_j > u_j^m$ and $r_j \leq u_i^m$ then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i \geq \sigma_i(u_j^m)$.

Third, suppose $s_j > u_j^m$ and $r_j > u_i^m$. From earlier results we need both $\alpha_i \geq \sigma_i(u_j^m)$ and $\alpha_i \leq \theta_i(u_i^m)$ to rule out deviations b) and a), respectively. Whilst $s_j > u_j^m$ and $r_j > u_i^m$ ensure that $\sigma_i(u_j^m) < 1$ and $\theta_i(u_i^m) > 0$ as required, we still need to ensure $\theta_i(u_i^m) \geq \sigma_i(u_j^m)$ in order for some $\alpha_i = \beta_i \in (0,1)$ to be possible. This condition can be rewritten as

$$1 - \frac{W_j(u_i^m) - L_j^\phi}{(W_j^\phi - L_j^\phi)x_j} \geq 1 - \frac{L_j^\phi - L_j(u_j^m)}{W_j(u_j^m) - L_j(u_j^m) - x_j(W_j^\phi - L_j^\phi)}$$

or $W_j(u_i^m) \leq W_j(s_j)$, which is equivalent to $s_j \leq u_i^m$. So if $s_j > u_j^m$ and $r_j > u_i^m \geq s_j$, then any $\alpha_i = \beta_i \in (0,1)$ can be an equilibrium provided $\alpha_i = \beta_i \in [\sigma_i(u_j^m), \theta_i(u_i^m)]$.

Finally, notice that in all three regions, it follows that $i = 1$ and $j = 2$ from Definition 3. In the first region, we require $s_j \le u_j^m < u_i^m = s_i$ and so $i = 1$. In the second region, we require $s_j > u_j^m$ and $r_j \le u_i^m = s_i$, and so $s_i > s_j$ follows from $s_j < r_j$ for $s_j \ge u_j^m$. In the third region, we require $s_j > u_j^m$ and $r_j > s_i = u_i^m \ge s_j$, and so even if $s_i = s_j$ the fact that $u_i^m > u_j^m$ implies $i = 1$ from Definition 3.

To summarize, when $u_i^m > u_j^m$, an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ can arise iff $i = 1$, $j = 2$ and either i) $s_2 \le u_2^m < u_1^m = s_1$ and $\alpha_1 \le \theta_1(u_1^m)$ (where $\sigma_1(u_2^m) \le 0$); ii) $s_2 > u_2^m$, $r_2 \le u_1^m = s_1$, and $\alpha_1 \ge \sigma_1(u_2^m)$ (where $\theta_1(u_1^m) \ge 1$), or iii) $s_2 > u_2^m$, $r_2 > u_1^m = s_1$, and $\alpha_1 \in [\sigma_1(u_2^m), \theta_1(u_1^m)]$ (where $0 < \sigma_1(u_2^m) \le \theta_1(u_1^m) < 1$). Therefore, in all three cases $i = 1$, $j = 2$, $\alpha_1 \in (0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$. Furthermore, when $u_j^m = u^m \ge u_i^m$, we also found an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ can arise iff $i = 1$, $j = 2$, $s_1 = u_1^m \le u_2^m$, $s_2 < u_2^m$, $\alpha_1 \le \theta_1(u^m)$, and $\sigma_1(u_2^m) \le 0$. Thus, overall, an equilibrium with $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = 0$ arises iff $i = 1$, $j = 2$, $s_1 = u_1^m$ and $\alpha_1 = \beta_1 \in (0,1) \cap [\sigma_1(u_2^m), \theta_1(u^m)]$. □

Lemma 13. *Equilibrium Form 4:* $\alpha_i = \beta_i \in (0,1)$, $\alpha_j = \beta_j \in (0,1)$ *with* $u_j^m > u_i^m$ *is an equilibrium iff* $i = 1$, $j = 2$, $u_1^m < s_1 \le u_2^m < r_2$, $\alpha_1 = \beta_1 = \theta_1(u_2^m)$, *and* $\alpha_2 = \beta_2 = \sigma_2(u_1^m)$.

Proof of Lemma 13. Suppose $\alpha_i = \beta_i \in (0,1)$ and $\alpha_j = \beta_j \in (0,1)$ with $u_j^m > u_i^m$. Given this, for player j to mix over ϕ and u_j^m, we require $L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i) = W_j(u_j^m)$. This implies that player i must have $\alpha_i = \beta_i = \theta_i(u_j^m)$. For $\theta_i(u_j^m) \in (0,1)$ as required, one then needs $s_j < u_j^m < r_j$. Given $\alpha_i = \beta_i = \theta_i(u_j^m)$ and $u^m = u_j^m$, there are no possible profitable deviations for player j. Now for player i to mix with $\alpha_i = \beta_i \in (0,1)$ requires $L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1-\alpha_j) = L_i(u_i^m) + (1-\alpha_j)(W_i(u_i^m) - L_i(u_i^m))$. This implies player j must have $\alpha_j = \sigma_j(u_i^m)$ and an associated equilibrium payoff equal to $W_i(s_i)$. For $\sigma_j(u_i^m) \in (0,1)$, we require $s_i > u_i^m$. Further, to ensure player i does not wish to deviate to just above u_j^m we also require $W_i(s_i) \ge W_i(u_j^m)$ or $s_i \le u_j^m$. Thus, we need $u_i^m < s_i \le u_j^m$ together with $s_j < u_j^m < r_j$. From Definition 2, note that $s_j < u_j^m$ implies $s_j = -\infty$. Hence, it must be that $i = 1$ and $j = 2$ via Definition 3 as $s_j = -\infty < 0 \le u_i^m < s_i$. □

Lemma 14. *Equilibrium Form 5:* $\alpha_i = \beta_i = 1$, $\beta_j = \alpha_j \in (0,1)$ *with* $u_j^m > u_i^m$ *is an equilibrium iff* $i = 1$, $j = 2$, $r_2 = u_2^m > u_1^m$, $s_1 > u_1^m$, *and* $\alpha_2 = \beta_2 \in (0,1) \cap (0, \min\{\delta_2(u_2^m), \sigma_2(u_1^m)\}]$.

Proof of Lemma 14. Given $\alpha_i = \beta_i = 1$, player j can only earn L_j^ϕ when passive. However, given $u_j^m > u_i^m$, player j will earn $W_j(u_j^m)$ when active. Hence, for player j to mix with $\beta_j = \alpha_j \in (0,1)$, we require $L_j^\phi = W_j(u_j^m)$ such that $u_j^m = r_j$. As $x_j > 0$, this implies $u_j^m = r_j > s_j$ and so it must be that $s_j = -\infty$. We also require player i to have no incentive to deviate from u_i^m to i) ϕ or ii) just above u_j^m. To rule out i), we require $L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j) \geq L_i^\phi + x_i(W_i^\phi - L_i^\phi)(1-\alpha_j)$ or $\alpha_j = \beta_j \leq \sigma_j(u_i^m)$. To rule out ii), we require $L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j) \geq W_i(u_j^m)$ or $\alpha_j = \beta_j \leq 1 - \frac{W_i(u_j^m)-L_i(u_i^m)}{W_i(u_i^m)-L_i(u_i^m)} = \delta_j(u_j^m)$. Thus, we require $\alpha_j = \beta_j \leq \min\{\delta_j(u_j^m), \sigma_j(u_i^m)\}$. Hence, to allow for $\alpha_j = \beta_j > 0$, we require $\min\{\delta_j(u_j^m), \sigma_j(u_i^m)\} > 0$. Given $u_j^m > u_i^m$, this is satisfied if $s_i > u_i^m$. When combined with $s_j < u_j^m = r_j$ such that $s_j = -\infty$, it must be that $i = 1$ and $j = 2$ via Definition 3 as $s_j = -\infty < 0 \leq u_i^m < s_i$. □

Lemma 15. *Equilibrium Form 6:* $1 > \alpha_i > \beta_i \geq 0$ *for* $i = \{1, 2\}$ *is an equilibrium iff* $r_2 > s_1 > u^m$, $\bar{u} = s_1$, $\alpha_1 = \theta_1(\bar{u})$, $\alpha_2 = \sigma_2(u_1^m)$, $\beta_1 = F_1(u^m) - (1-\alpha_1) \geq 0$, *and* $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.

Proof of Lemma 15. Suppose $1 > \alpha_i > \beta_i \geq 0$ for $i = \{1, 2\}$. From Lemma 4, each player k must mix over $u_k \in \{\phi\} \bigcup (u^m, \bar{u}]$. For this to be part of equilibrium, each player k must earn their equilibrium payoff, Π_k^*, from any such u_k. Thus, to be indifferent between ϕ and $\bar{u}$ specifically, requires $\Pi_k^* = L_k^\phi + x_k(1-\alpha_l)(W_k^\phi - L_k^\phi) = W_k(\bar{u})$ such that $\alpha_l = 1 - \frac{W_k(\bar{u})-L_k^\phi}{x_k(W_k^\phi - L_k^\phi)} \equiv \theta_l(\bar{u})$ for any k, $l \neq k \in \{1, 2\}$.

Without loss let $u_i^m \leq u_j^m$. Initially consider a first possibility where $\beta_i > 0$. Then, player i must earn its equilibrium payoff, Π_i^*, from setting u_i^m. Hence, $\Pi_i^* = L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j)$. By setting this equal to the previous expression, $\Pi_i^* = L_i^\phi + x_i(1-\alpha_j)(W_i^\phi - L_i^\phi)$, one obtains an alternative expression for $\alpha_j = \sigma_j(u_i^m)$. Hence, by setting $\alpha_j = \sigma_j(u_i^m) = \theta_j(\bar{u})$, we find $\Pi_i^* = W_i(s_i)$ such that $\bar{u} = s_i$. Now consider player j. She must earn her equilibrium profit when selecting select u_j (arbitrarily close to) u^m. Thus, $\Pi_j^* = L_j(u^m) + (W_j(u^m) - L_j(u^m))(1-\alpha_i+\beta_i)$. By setting this equal to $\Pi_j^* = W_j(\bar{u})$, it gives $\beta_i = \frac{W_j(\bar{u})-L_j(u^m)}{W_j(u^m)-L_j(u^m)} - \theta_i(\bar{u}) \equiv F_i(u^m) - (1-\alpha_i)$. By rearranging the expression for β_i, we then require $s_i > s_j$ to ensure $\beta_i > 0$ as assumed. Hence it must be that $i = 1$ and $j = 2$. By definition it follows that $\beta_2 = F_2(u^m) - (1-\alpha_2)$. Then, using the definition of strength, one can show $(1-\alpha_2) = F_2(u_1^m)$ such that $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$ given $u_1^m \leq u_2^m = u^m$. Lastly, given $s_1 > s_2$, to ensure $\alpha_k \in (\beta_k, 1)$ for $k \in \{1, 2\}$, we require $r_2 > s_1 > u^m$.

Now continue to assume $u_i^m \leq u_j^m$, but consider the remaining possibility with $\beta_i = 0$. For player j to be indifferent between setting $u_j^m = u^m$ and being passive, we require $L_j(u_j^m) + (W_j(u_j^m) - L_j(u_j^m))(1-\alpha_i) = L_j^\phi + x_j(W_j^\phi - L_j^\phi)(1-\alpha_i)$. Hence, one obtains an alternative expression for $\alpha_i = \sigma_i(u_j^m)$. Then by setting $\alpha_i = \sigma_i(u_j^m) = \theta_i(\bar{u})$, we find $\Pi_j^* = W_j(s_j)$ such that $\bar{u} = s_j$. Given $\beta_i = 0$, player i should not want to deviate to u_i^m. Hence, we require $\Pi_i^* = W_i(\bar{u}) \geq L_i(u_i^m) + (W_i(u_i^m) - L_i(u_i^m))(1-\alpha_j)$. After rearranging, this gives $W_i(s_j) \leq L_i^\phi + \frac{c_i(u_i^m)}{b_i(u_i^m)}(W_i^\phi - L_i^\phi)x_i = W_i(s_i)$ or $s_i \leq s_j$. Given $\beta_j \geq 0$, player i should also earn Π_i^* by setting u_i just above u_j^m such that $\Pi_i^* = W_i(\bar{u}) =$

$L_i(u_j^m) + (W_i(u_j^m) - L_i(u_j^m))(1 - \alpha_j + \beta_j)$. This confirms that $\beta_j = F_j(u_j^m) - (1 - \alpha_j)$. This equals zero if $s_i = s_j$ but is otherwise positive. Hence, when $s_1 = s_2$, player i can either be 1 or 2, but when $s_i < s_j$ then it must be that $j = 1$ and $i = 2$ from Definition 3. Either way, this again confirms that $\bar{u} = s_1$ and $\beta_1 = F_1(u^m) - (1 - \alpha_1) \geq 0$ and $\beta_2 = F_2(u^m) - (1 - \alpha_2) = F_2(u^m) - F_2(u_1^m) = 0$. Further, again, we require the same conditions, $r_2 > s_1 > u^m$, to ensure $\alpha_k \in (0, 1)$ and $\alpha_k > \beta_k$ for $k \in \{1, 2\}$.

Finally, we also need to verify that $F_k(u)$ in (5) is well-behaved for both $k = \{1, 2\}$ with i) $F_k(\bar{u}) = 1$, and ii) $F_k'(u) > 0$ for all $u \in (u^m, \bar{u}]$. i) is satisfied automatically. For ii), note $F_k'(u)$ has the same sign as $-L_l'(u)[W_l(u) - W_l(\bar{u})] - W_l'(u)[W_l(\bar{u}) - L_l(u)]$, and that this is guaranteed to be positive for all $u \in (u^m, \bar{u}]$ if $W_l(\bar{u}) > L_l(u^m)$ for $l = \{1, 2\}$. As $L_l^\phi > L_l(u^m)$, this condition would be satisfied if $W_l(\bar{u}) \geq L_l^\phi$. Using the definition of reach, this requires $r_l \geq \bar{u}$ for $l = \{1, 2\}$ which follows given $r_2 > \bar{u} = s_1 > u^m$, and $r_1 > s_1$. □

Lemma 16. *Equilibrium Form 7:* $1 > \alpha_2 > \beta_2 \geq 0$ *and* $1 = \alpha_1 > \beta_1 \geq 0$ *is an equilibrium iff* $s_1 \geq r_2 > u^m$, $\bar{u} = r_2$, $\alpha_2 = 1 - F_2(u_1^m)$, $\beta_1 = F_1(u^m) > 0$ *and* $\beta_2 = F_2(u^m) - F_2(u_1^m) \geq 0$.

Proof of Lemma 16. Suppose $1 > \alpha_2 > \beta_2 \geq 0$ and $1 = \alpha_1 > \beta_1 \geq 0$. From Lemma 4, each player k must mix over $u_k \in \{\phi\} \bigcup (u^m, \bar{u}]$. First, it must be true that $\beta_1 > 0$ in equilibrium. If not, with $\beta_1 = 0$, then player 2 would always lose when choosing u_2 arbitrarily close to u^m and so she would prefer to deviate to ϕ instead as $L_2^\phi - L_2(u_2^m) > 0$ via A3. Second, it then follows that $\beta_2 = 0$ when $u_1^m \geq u_2^m$. To understand this, note that given $\beta_1 > 0$, β_2 must equal zero if $u_1^m = u_2^m$ from Lemma 3. Further, if $u_1^m > u_2^m$ then player 2 will always lose at u_2^m given $\alpha_1 = 1$. Therefore, player 2 would optimally set $\beta_2 = 0$ and instead, deviate to ϕ as $L_2^\phi - L_2(u_2^m) > 0$ via A3. Third, for $\alpha_2 \in (0, 1)$, player 2 must earn Π_2^* from any $u_2 \in \{\phi\} \bigcup (u^m, \bar{u}]$. Hence, she must be indifferent between setting u i) equal to ϕ, ii) just above u^m, and iii) equal to $\bar{u}$. Given $\alpha_1 = 1$, this implies $\Pi_2^* = L_2^\phi = L_2(u^m) + (W_2(u^m) - L_2(u^m))\beta_1 = W_2(\bar{u})$ such that $\bar{u} = r_2$ and $\beta_1 = \frac{W_2(\bar{u}) - L_2(u^m)}{W_2(u^m) - L_2(u^m)} \equiv F_1(u^m)$. Fourth, to ensure $\bar{u} > u^m$, we require $r_2 > u^m$. (This also ensures $F_1(u^m) > 0$ given $L_2^\phi > L_2(u^m)$ via A3.) Fifth, given $\alpha_1 > \beta_1 > 0$, player 1 must earn Π_1^* from any $u_1 \in [u^m, \bar{u}]$. Given $\beta_2 = 0$ when $u_1^m \geq u_2^m$, player 1 must earn $\Pi_1^* = L_1(u_1^m) + (1 - \alpha_2)(W_1(u_1^m) - L_1(u_1^m))$ by selecting $u_1 = u_1^m$. By setting this equal to $\Pi_1^* = W_1(\bar{u})$, one obtains $\alpha_2 = 1 - \frac{W_1(\bar{u}) - L_1(u_1^m)}{W_1(u_1^m) - L_1(u_1^m)} \equiv 1 - F_2(u_1^m)$. Given $\bar{u} > u^m$, our previous condition, $r_2 > u^m$, ensures $\alpha_2 \in (0, 1)$ as required. It then follows that player 2 has a mass point at u_2^m of size $\beta_2 = F_2(u^m) - (1 - \alpha_2) = F_2(u^m) - F_2(u_1^m)$. As consistent with our earlier claim, this is positive if $u_1^m < u_2^m$, and zero if $u_1^m \geq u_2^m$. Further, player 1 should not want to deviate to being passive, so we require $W_1(\bar{u}) \geq L_1^\phi + x_1(W_1^\phi - L_1^\phi)(1 - \alpha_2) \equiv W_1(s_1)$.

This implies $s_1 \geq r_2$. So overall, we require $s_1 \geq r_2 > u^m$. (Finally, we need to verify that $F_k(u)$ in (5) is well-behaved for both $k = \{1, 2\}$ with i) $F_k(\bar{u}) = 1$, and ii) $F'_k(u) > 0$ for all $u \in (u^m, \bar{u}]$. Using the details from the proof for Lemma 15, this requires $r_k \geq \bar{u}$ for $k = \{1, 2\}$. Here, this follows given $r_1 > s_1 \geq \bar{u} = r_2 > u^m$.) □

Step 3: Characterizing the Parameter Space

To complete the derivation, Step 3 uses the results from Step 2 to identify the possible equilibria in each region and show how the equilibrium results are consistent with Theorems 1 and 2.

First, by using Step 2, it is tedious but straightforward to show that the equilibria detailed in Lemmas 10-16 cover all valid parameter cases under our assumptions and definitions. Hence, at least one equilibrium form exists in each possible parameter constellation.

Second, we need to show that Theorems 1 and 2 cover all possible equilibria, that each equilibrium is correctly detailed within the Theorems, and that the equilibria within Theorem 1 are unique. To proceed, we work through Lemmas 10-16 in reverse order. The necessary and sufficient conditions regarding the levels of reach and strength in Lemma 16, $s_1 \geq r_2 > u^m$, are not compatible with any other Lemma from Step 2 and are fully captured by case v) in Theorem 1. Similarly, the necessary and sufficient conditions in Lemma 15 are not compatible with any other Lemmas from Step 2 and are fully covered by case iv) Theorem 1. The necessary and sufficient conditions in Lemma 14 are fully covered by case c) of Theorem 2. However, at these conditions, Lemma 11 can also apply and so case c) of Theorem 2 permits $\alpha_2 = 0$ as well as $\alpha_2 \in (0, 1)$. The necessary and sufficient conditions in Lemma 13 are not compatible with any other Lemmas from Step 2 and are fully covered by cases iii) Theorem 1. The necessary and sufficient conditions in Lemma 12 are fully covered by case a) of Theorem 2. However, at these conditions, the necessary and sufficient conditions for Lemmas 11 and 10 can also apply if $r_2 \leq u^m$ or $s_2 \leq u_2^m$, respectively. Nevertheless, these equilibrium possibilities are still consistent with Theorem 2a because $\theta_1(u^m) \geq 1$ if $r_2 \leq u^m$, and $\sigma_1(u_2^m)$ and $\theta_1(u^m)$ equal zero if $s_2 \leq u_2^m$. The necessary and sufficient conditions in Lemma 11 are not compatible with any other Lemma from Step 2 and are fully covered in Theorem 1 case ii), apart from the overlap situations that we have already covered, and apart from the situation where $s_1 > u_1^m$ and $r_2 = u_2^m \leq u_1^m$ but this is covered by case d) of Theorem 2. Finally, the necessary and sufficient conditions in Lemma 10 are not compatible with any other Lemma from Step 2 and are fully covered in Theorem 1i) apart from the overlap situations that we have already covered, and apart from the situation where $s_1 < u_1^m$ and $r_2 = u_2^m$ but this is covered by case b) of Theorem 2.