In Defence of the Endogenous Growth Theory: "Conditional" and "Unconditional" Convergence in Two-Country AK Models

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Abstract: Recent studies on cross-country (per capita) income inequality have found evidence for 'unconditional' convergence, and have interpreted this finding as a data rejection for AK growth models. This paper shows that a two-country version of the AK model with learning-by-doing externalities, extended to include international knowledge transfer, is: (*i*) consistent with the finding of 'unconditional' convergence that characterizes the world economy since mid 1980s; (*ii*) more effective than Neoclassical growth models to predict all types of convergence/divergence patterns that the empirical growth economists have documented in almost forty years of literature.

JEL classification: O11, O41, O47

Keywords: 'Unconditional' Convergence, Two-Country AK Model, Cross-Country Knowledge Transmission, Development Policies

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1 Introduction

One of the most remarkable result in the Empirical Growth Literature of the 90s is that poor countries tends to lag behind rich countries (Barro, 1991; Pritchett, 1997). Recently, a new wave of empirical studies on cross-country convergence have convincingly shown that poor countries' per capita income tends, on average, to 'unconditionally' (or 'absolute') converge towards that of rich countries since the mid 1980s. Noticeable contributions in this new stream of convergence research are the papers by Roy et al. (2016), Patel et al. (2021) and Kremer et al. (2021), who (independently) find evidence for: (*i*) emerging-market economies growing faster than advanced-frontier economies (henceforth, 'Wilde'-convergence[1]); (*ii*) poor countries catching up with rich, but at the very low average annul rate of 0.7 percent since mid-1990s; (*iii*) middle-income countries growing even faster than low-income countries.

The first two results are particularly important for the empirical growth literature because it puts into question the so-called 'iron law' of convergence, according to which countries tends to 'conditionally' converge quickly towards their own steady-state levels of income per capita at a rate of 2 percent per year. The latter, instead, poses a serious challenge on all those endogenous growth models focusing on the existence of a "middle-income trap" for developing countries, since it seems to rather predict the existence of a "middle-income trampoline".

In an attempt of providing an explanation for the emergence of 'unconditional' convergence in cross-country data, Kremer et al. (2021) put forward the idea that 'absolute' convergence might have converged towards 'conditional' convergence ('convergence to convergence' hypothesis), meaning that the phase of 'absolute' convergence characterizing the global dynamics of per capita income inequality of the last forty years can be explained by the tendency of developing countries of adopting advanced countries' institutions, policy and cultural features, as a result of the spreading, at worldwide level, of information and communication technologies. In Kremer et al.'s words:

> *"Absolute convergence did not hold initially, but, as human capital, policies, and institutions, have improved in poorer countries, the difference in institutions across countries has shrunk, and their explanatory power with respect to growth and convergence has declined. As a result, the world has converged to absolute convergence because absolute convergence has converged to conditional convergence"* (Kremer et al., 2022, page 5).

As reportedly argued by Kremer et al. (2021), this new vintage of results on cross-

[1] To best of my knowledge, Roy et al. (2016) is the first paper in this new vintage of convergence studies documenting the existence of 'absolute' convergence in the globalization era. However, when investigating about the existence of what they called Wild(W)-convergence - i.e. the tendency of emerging-market economies to grow faster than frontier economies -, these authors find that convergence occurs at an even higher speed through a process of catching-up they dubbed: 'convergence with vengeance'.

country 'absolute' convergence poses a further challenge to Endogenous Growth Theory as it neglects the capability of the AK frameworks of endogenous growth to be consistent with data, while promoting Neoclassical growth models as fully consistent with data. Again, in Kremer et al.'s words:

"[We] argue that convergence changed around 1990, and since is consistent with models of neoclassical growth and inconsistent with a class of endogenous growth theory models which predict divergence, such as AK models (Romer 1986) or some poverty trap models" (Kremer et al., 2022, page 6]

In this paper, we challenge Kremer et al.'s statement that AK models are inconsistent with data, by presenting a two-country version the Romer's (1986) model of endogenous growth extended to include cross-country learning-by-doing externalities. Our objectives are, on the one hand, to demonstrate that extended AK models are effective of replicating all the converging patterns thus far put forward by the empirical growth literature. On the other, to show to what extent growth policies can serve it well to get emerging-market economies to catch-up with the advanced ones.

We begin by showing that when countries have access to the same basin of technical knowledge, AK growth models can predict cross-country 'conditional' convergence in a way similar to that described by Barro and Sala-i-Martin (1991, 1992) and Mankiew et al. (1992); i.e., a long-run dynamic pattern where each country involved in the process of convergence tends to approach the same equilibrium growth rate, but not the same equilibrium level of per capita income. Then, we will show that if poor countries are allowed to share the same macroeconomic fundamentals of rich (e.g. saving rates, depreciation, preferences, etc.), then AK growth models can predict both the 'Wilde'-convergence type documented by Roy et al. (2016) and the standard 'absolute' (Solow) convergence scheme indicated by Patel et al. (2021) and Kremer et al. (2021).

The reason why we believe that AK-growth model works better than Neoclassical growth models in fitting data is twofold. The first motivation is based on the fact that AK models are able to explain technical progress, while Neoclassical models notoriously fail to provide an appropriate theory to explain both the creation and dissemination of technical knowledge. The second motivation, instead, relies on the very low convergence speed estimated by Patel et al. (2021) and Kremer et al. (2021), which makes Neoclassical models of exogenous growth not suitable to fit data.

As is known, in traditional Neoclassical growth models (e.g., Solow, 1956; Cass 1965; Koopmans, 1965) the assumptions that aggregate production functions display decreasing returns to capital accumulation and that the rate of technical change grows in all countries at the same exogenous rate are pivotal to determine both 'absolute' and 'conditional' convergence.[2] In contrast, in AK-like endogenous growth models à la Romer (1986), Rebelo

[2]As shown by Acemoglu and Molinas (2022), Barro's (1991) 'conditional' regressions fails to allow for

(1991) and Greiner and Semmler (1996), technical progress is country-specific and endogenously determined as a by-product of gross investment. As a result, economies that do not share the same knowledge base and that do not accumulate physical capital at the same rates tend to grow at different rates in the equilibrium ('unconditional' divergence).

In this paper, we will show that opening up the AK model to allow for endogenous international knowledge transmission can cause this framework to predict the same 'conditional' and 'unconditional' convergence scheme predicted by Neoclassical growth models, and hence to make it consistent with data.

The paper is organized as follows. Section 2 sets up the two-country AK model with cross-country knowledge externalities. Section 3 characterizes the dynamic equilibrium and solve the model for the Balanced-Growth Path (BGP) equilibrium. Section 4 numerically investigates about the dynamic properties of the BGP equilibrium and shows under which conditions the two-country AK model can generate 'conditional' and 'unconditional' convergence. Section 5 provides three extensions of the model to investigates how convergence in growth policy can affect the convergence process of poor economies. Finally, Section 6 concludes.

2 The analytical framework

We consider an asymmetric world economy made up of two countries: a technology leader (hereinafter "North"), denoted with the subscript "n", and a technology follower (hereinafter "South"), denoted with the subscript "s". In each country, households supply labor inelastically and accumulate capital assets, and firms carry out production by assembling physical capital and labor services.

To accommodate endogenous growth, throughout the paper we will assume that productivity increases as the result of learning-by-doing externalities, which cause the aggregate growth to raise through a process of Knowledge spillovers (Arrow, 1962; Romer, 1986; Rebelo, 1991; Greiner and Semmler; 1996). However, in order to differentiate between countries, we assume the existence of two types of knowledge spillovers: (*i*) localized knowledge spillovers, through which increases in each country's stock of knowledge results into more economic growth; (*ii*) cross-country knowledge spillovers, through which advances in labor productivity in North due to learning-by-doing are transmitted to South.

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heterogeneity across countries, and the same critic holds for other subsequent works in the Empirical Growth Literature, such Mankiew et al. (1992). Indeed, when testing for 'conditional' convergence, Mankiew et al. (1992) does not allow for heterogeneity across technical progress growth rates and assume that the sum of the rates of depreciation and technical progress, $n + g$ in their notations, is fixed to 0.05 to match US data. Similarly, in Barro and Sala-i-Martin (1992),each US State is assumed to share the same rate of technological progress (x in their notation) and justified by the fact that it is reasonable to assume that regions within a country share the same rate of technical progress.

The model is set in continuous time. For simplicity, we abstract from money and other nominal assets, and focus on only real quantities as in Bianconi (1995) and Bianconi and Turnovsky (1997).

2.1 Preferences and consumption

Each country $i = \{n, s\}$ is populated by an infinitely-lived representative household, each of which consists of a continuum L_i of identical individuals providing labor services in exchange of a wage. To simplify the model, we assume that the size of each household is fixed over time (i.e. no population growth) and that each individual in the world economy share the same rate of time preference, $\rho > 0$.[3]

The objective of the household i is to maximize the discounted flow of lifetime utility

$$\mathcal{U}_i = \int_0^\infty e^{-\rho t} \log c_i \mathrm{d}t, \tag{1}$$

subject to the flow budget constraint

$$\dot{k}_i = r_i k_i + w_i - c_i, \tag{2}$$

where k_i is the household i's capital stock, r_i the rental rate of capital, w_i the wage rate and c_i the level of the individual consumption expenditure of the household.

Using Pontryagin's Maximum Principle to solve this optimal control problems yields the following Euler and transversality conditions

$$\dot{c}_i = (r_i - \rho)\, c_i \tag{3}$$

$$\lim_{t\to\infty} e^{-\int_0^t r_i(v)\mathrm{d}v} k_i\,(t) = 0, \tag{4}$$

for all $i = n, s$.

2.2 Technologies and production

The production sector of each country i comprises a unit continuum of identical firms, each of which produces a unique homogeneous commodity. The production of the representative firm j of country i is given by

$$Y_{j,i} = (\mathcal{A}_i N_{j,i})^{1-\alpha} K_{j,i}^{\alpha}, \;\; \alpha \in (0,1)\,, \tag{5}$$

where α is the output elasticity of capital, supposed to be the same worldwide, $N_{j,i}$ and $K_{j,i}$ are, respectively, the firm specific levels of labor employment and physical capital, and A_i is a country-specific labor-augmenting technology parameter, whose analytical properties will be specified later on in Section 2.3.

[3]This assumption is made for simplicity. Allowing ρ to change across country, while changing quantities, does not modifies the main conclusions of the paper.

Perfect competition implies that the rates of return on capital and wages are determined by the usual marginal product conditions

$$r_i = \alpha \mathcal{A}_i^{1-\alpha} k_{j,i}^{\alpha-1} - \delta \tag{6}$$

$$w_i = (1-\alpha)\, \mathcal{A}_i^{1-\alpha} k_{j,i}^{\alpha}, \tag{7}$$

where $k_{j,i} := L_{j,i}/N_{j,i}$ is the capital-to-labor ratio of the firm j and $\delta \in (0,1)$ is the depreciation rate of capital, that we assume identical for both countries.

In the symmetric equilibrium, each firm residing in the same country finds it optimal to employ the same capital-to-labor ratio, k_i. Consequently, aggregating (5) over production units, it follows that the aggregate production function of the overall economy i is given by

$$Y_i = (\mathcal{A}_i L_i)^{1-\alpha} K_i^{\alpha}, \tag{8}$$

where $L_i = \int_0^1 N_{j,i} \mathrm{d}j$ and $K_i = \int_0^1 K_{j,i} \mathrm{d}j$ are aggregate employment and capital respectively.

2.3 Learning-by-investing and cross-country knowledge externality

As in Romer (1986), each country's knowledge stock, A_i, is assumed to reflect the positive spillover that private investments in physical capital have on the aggregate economy. However, to make the model include cross-country knowledge spillovers, in this paper we postulate that technical progress in South is also dependant upon the gross investment of North. In particular, throughout the paper we assume the following specifications for the technology parameters

$$\mathcal{A}_n = A_n^{1/(1-\alpha)} k_n, \quad \mathcal{A}_s = \begin{cases} A_s^{1/(1-\alpha)} k_s \kappa_n^{\psi} & \text{if } \kappa_n > 1 \\ A_s^{1/(1-\alpha)} k_s & \text{if } \kappa_n \leq 1 \end{cases}, \tag{9}$$

where $A_i > 0$ is given parameter capturing the effectiveness with which each country is able to generate knowledge improvements from gross investment (hereinafter the 'baseline' knowledge parameter), $\kappa_n := k_n/k_s$ is the relative capital stock of North, which we use as a metric for measuring the knowledge gap between North and South, and $\psi \in [0,1]$ is a cross-country externality parameter measuring the sensitivity with which knowledge capital in South reacts to gross investment in North.[4]

[4] More generally, the term $k_s \kappa_n^{\psi}$ appearing on the right-hand side of the productivity index of South, $\mathcal{A}_s$, can be thought of as a Weighted Generalized Mean, or Weighted Power Mean or *Hölder* Mean (after Otto Hölder), of all the productivity levels of all countries involved in the process of knowledge diffusion. *Hölder* Means are a family of mean generating functions that can include all the most important means as special cases. In the simpler case of only two countries, n and s, the Weighted *Hölder* mean with exponent $p \geq 0$ and weights $\psi_s > 0$ and $\psi_n > 0$ is: $\mu_p(k_n, k_s) = (\psi_n k_n^p + \psi_s k_s^p)^{1/p} / (\psi_n + \psi_s)$. It can be shown

In (9), A_i can be interpreted as a catch-all parameter capturing several country-specific features such as, for instance, the quality of institutions and human capital, the level of development of the financial system, the country's ability of develop/absorb new technology, etc. Everything being equal, the higher A_i, the higher the ability of the country to turn gross investment into additional knowledge.

Notice that the bigger κ_n, the higher the level of productivity of South that depends on imported technical knowledge. Indeed, when $\kappa_n > 1$, improvements in technical progress in North tends to enlarge the knowledge base of South. Since we are interested in studying convergence in a dichotomic world where North is the technology leader and South is the follower, in the rest of the paper we will restrict our attention to only those long-run equilibria where $A_n > A_s$ and $\kappa_n > 1$ hold simultaneously.

3 The perfect-foresight dynamic equilibrium

3.1 The dynamic system

In this section, we determine the dynamic system of our two-country AK model with cross-country knowledge externality. As in the symmetric equilibrium $k_{j,i} = k_i$ for all j and i, we can use (9) to substitute for $\mathcal{A}_i$ in (6) and (7) to obtain the following expressions for the rental and wage rates

$$r_n = \alpha A_n - \delta, \quad r_s = \alpha A_s \kappa_n^{\psi(1-\alpha)} - \delta \tag{10}$$

$$w_n = (1-\alpha) A_n k_n, \quad w_s = (1-\alpha) A_s \kappa_n^{\psi(1-\alpha)} k_s. \tag{11}$$

Thus, plugging (10) and (11) into (2) and (3) yields the following 4×4 system of differential equations

$$\frac{\dot{k}_n}{k_n} = A_n - \frac{c_n}{k_n} - \delta \tag{12}$$

$$\frac{\dot{k}_s}{k_s} = A_s \kappa_n^{\psi(1-\alpha)} - \frac{c_s}{k_s} - \delta. \tag{13}$$

$$\frac{\dot{c}_n}{c_n} = \alpha A_n - (\rho + \delta) \tag{14}$$

$$\frac{\dot{c}_s}{c_s} = \alpha A_s \kappa_n^{\psi(1-\alpha)} - (\rho + \delta) \tag{15}$$

that when $p = 0$, the Weighted *Hölder* mean boils down to the weighted geometric mean: $\mu_0(k_n, k_s) = k_n^{\psi_n/(\psi_n+\psi_s)} k_s^{\psi_s/(\psi_n+\psi_s)}$. The term the term $k_s \kappa_n^{\psi}$ can then be seen as a special case of $\mu_p(k_n, k_s)$ with $\psi_n + \psi_s = 1$.

Definition 1 *A dynamic equilibrium for the two-country AK model with international knowledge transmission can be defined as a set of infinite sequences for the allocations* $\{c_n, c_s, k_n, k_s\}_{t\in[0,\infty)}$ *that:* (*i*) *satisfies equations (12)-(15);* (*ii*) *fulfills the inequality constraints* $c_s \geq 0$*,* $c_n \geq 0$*,* $k_s \geq 0$*,* $k_n \geq 0$*;* (*iii*) *satisfies the transversality condition (4).*

From (13) and (15), we have that cross-country gap in knowledge capital matters for the determination of the equilibrium paths of c_s and k_s. As a result, to solve the model for the long-run equilibrium, it is convenient to reduce the dynamic system of one dimension and focus on the following re-scaled variables: the consumption-to-capital ratio of North, $x_n := c_n/k_n$, the consumption-to-capital ratio of South, $x_s := c_s/k_s$, and the cross-country knowledge capital gap, $\kappa_n = k_n/k_s$.

To this end, we combine (12)-(15) to derive the following 3×3 system of differential equations in κ_n, x_n and x_s:

$$\dot{\kappa}_n = \left[x_s - x_n + A_n - A_s \kappa_n^{\psi(1-\alpha)}\right] \kappa_n \tag{16}$$

$$\dot{x}_n = \left[x_n - \rho - (1-\alpha) A_n\right] x_n \tag{17}$$

$$\dot{x}_s = \left[x_s - \rho - (1-\alpha) A_s \kappa_n^{\psi(1-\alpha)}\right] x_s. \tag{18}$$

For any given $\kappa_n(0) > 1$, dynamic system (16)-(18) and the transversality condition (4) completely characterize the transitional dynamics of our two-country AK model with cross-country learning-by-doing externality.

3.2 The BGP equilibrium

The BGP can be defined as a set of infinite sequences for the allocations $\{c_n, c_s, k_n, k_s\}_{t\in[0,\infty)}$ satisfying Definition 1 such that: (*i*) the re-scaled variables x_n, x_s and κ_n are constant over time; (*ii*) individual consumption expenditures, c_n and c_s, and per worker capital stocks, k_n and k_s all grow at the same constant rate g.

The BGP equilibrium can be obtained from system (16)-(18) by setting $\dot{\kappa}_n = \dot{x}_n = \dot{x}_s = 0$ and then solving the resulting 3×3 static system for x_n, x_s and κ_n. This gives the following

Proposition 1 *Suppose* $\kappa_n(0) > 1$ *and* $A_n > A_s$ *hold. Then, the model predicts a unique BGP equilibrium where:* (*i*) *the consumption-to-capital ratios,* x_n^* *and* x_s^**, are the same for both countries and equal to*

$$x_n^* = x_s^* = \rho + (1-\alpha) A_n; \tag{19}$$

(*ii*) *the knowledge capital gap* κ_n^* *is strictly larger than one and equal to*

$$\kappa_n^* = (A_n/A_s)^{1/[\psi(1-\alpha)]}; \tag{20}$$

(*iii*) *the unique growth rate of the world economy is given by (14) and equates*

$$g^* = \alpha A_n - \rho; \tag{21}$$

(*iv*) *the BGP equilibrium is saddle-path stable.*

Proof. See Appendix A ■

From Proposition 1, we can extract the following key results of the model. First, from item (*iii*) of the proposition, we can conclude that what really matters for sustained long-run economic growth to emerge in the long run is the level of baseline knowledge of North, A_n. Second, from item (*ii*) of the proposition, we can conclude that the persistence over time of a cross-country knowledge gap is due to the existence of initial differences in baseline knowledge, $A_n/A_s \neq 1$. Finally, if we pick the relative per capita income of North as our chosen measure of income inequality, we can easily establish that the rise in income inequality across country can be explained by the presence of differences in the stocks of baseline knowledge according to

$$\frac{y_n^*}{y_s^*} = \left(\frac{A_n}{A_s}\right)^{1/[\psi(1-\alpha)]} = \kappa_n^*, \tag{22}$$

where $y_i = Y_i/L_i$ is the level of per capita income of country i.[5]

Since $\kappa_n^* > 1$, what (22) ultimately states is that initial cross-country differences in technological knowledge can prevent South from catching-up North in the long run. As a result, in the next section we will study the transitional properties of the model and assess to what extent a massive baseline knowledge transfer from North to South can serve it well to generate 'unconditional' convergence across countries.

4 Transitional dynamics and convergence

In this section, we will assess the adjustment dynamic properties of the two-country AK model with cross-country learning-by-doing externalities. In contrast with Kremer et al. (2021), we will show that our two-country variant is able to replicate all of the convergence patterns thus far documented by the empirical growth literature; i.e., 'unconditional' divergence, 'conditional' convergence and 'unconditional' convergence.

Consider first the special case of $\psi = 0$, i.e., the standard case of no cross-country knowledge transmission. When $\psi \to 0$, from (22) we obtain the standard AK result of 'unconditional' divergence ($y_n^*/y_s^* \to \infty$), where the Northern economy grows faster than the

[5]To obtain (22), we manipulate equation (8) to obtain: $y_i = A_i^{1-\alpha} k_i^{\alpha}$. Then, using (9) to substitute for A_i in the previous expression yields: $y_n = a_n k_n$ and $y_s = a_s \kappa_n^{\psi(1-\alpha)} k_s$. Finally, dividing y_n by y_s and then using (20) to get rid of κ_n, we obtain the expression in (22).

Southern economy.[6] However, as proved by Barro and Sala-i-Martin (1992) and Mankiew et al. (1992), such a theoretical prediction is misleading because it is not consistent with data.

Consider now the more general case of $\psi > 0$, where the size of cross-country knowledge transmission is endogenously determined by the model. Under this alternative scenario, if the world economy is initially characterized by $\kappa_n(0) > 1$, then per capita income inequality y_n^*/y_s^* stabilizes at $(A_n/A_s)^{1/[\psi(1-\alpha)]} > 1$ in the long run (see (22)), and the prediction of the model is convergence in growth rates, but not in the levels of per capita income.

As we know, this type of convergence is what economists use to call 'conditional' convergence; i.e. convergence in per capita income conditional on some structural parameter being held constant (see, among other, Barro and Sala-i-Martin,1991, 1992; Mankiew et al., 1992). As in this model all the main cross-country differences in macroeconomic fundamentals are captured by the size of the parameter A_i, for 'absolute' convergence to emerge what it is needed is to allow A_s to converge to A_n, so as to cause the initial gap in baseline knowledge to drop to zero. If this happens, then the model can predict the 'convergence to convergence' scenario indicated by Kremer et al. (2021). However, for this to occur, it is indispensable for the model to show the same transitional dynamics pattern shown by Neoclassical growth models (Solow, 1956; Cass, 1965; Koopmans, 1965).

Fortunately, whereas in standard AK models the adjustment dynamics to the BGP equilibrium occurs instantaneously (no transitional dynamics), in this model the introduction of cross-country technology transmission makes the BGP equilibrium characterizing the world economy to be asymptotically stable. Thus, if A_s increased as a result of shock, the adjustment dynamics generated by the model would become similar to that generated by Neoclassical growth models. Yet, a key question arising from our analysis is how quickly South tends to catch up with North.

To answer such a question, consider the following calibration of the exogenous parameters[7]

$$\langle \alpha = 0.33,\ \rho = 0.03,\ A_n = 0.303,\ A_s = 0.2695,\ \psi = 0.0589,\ \delta = 0.05 \rangle ,$$

[6]The reason why divergence appears when $\psi = 0$, is non difficult to grasp and can be easily explained by recalling that, without cross-country externalities, our two-country AK model reduces to describe the case of a world economy characterized by two independent economies, each of which grows over time according to her own growth rate equal to $g_i^* = \alpha A_i - \rho$.

[7]The subjective discount rate ρ is set to 0.03 to match the empirical evidence of Gollin (2002). Following the bulk of literature, we set the output elasticity of capital α at 0.33 so as to get a capital share of 1/3, and the depreciation rate δ at 0.05 so as to get annual depreciation rate close to that of US economy. The level of the baseline knowledge stock of North, A_n, is set to 0.303 so as to get a rate of growth of around 0.02 (2%) in the BGP, while that of South, A_s, is set to 0.2695 so as to get ana equilibrium value for the inequality index, y_n/y_s, of about 19.5. Finally, the cross-country externality parameter, ψ, is set to 0.0589 so as to get an half-life close to that documented by Patel et al. (2021).

and assume that, at $t = 0$, the baseline knowledge stock of South enlarges as much as to reach that of North: $A_s \to A_n = 0.303$.[8] Such a massive transfer in baseline knowledge could be explained in many ways. For instance, it can be explained by the opening up of the Southern economy to offshoring and FDI; by the increase in the worldwide diffusion of information that incentivize the use of the newest technologies and the adoption of better institutions; by the introduction, at a global level, of more growth-friendly policy measures close to those used by advanced country.

	λ_1	λ_2	λ_3
Eigenvalue	-0.0041	0.2251	0.2330

Table 1: Eigenvalues of the Jacobian matrix of the linearized system.

Under the above parametrization, the BGP quantities generated by the model are $y_n^*/y_s^* = \kappa_n^* = 19.47$ and $x_n^* = x_s^* = 0.233$, while the long-run equilibrium growth rate of both economies is equal to $g^* = 0.019$.[9] Moreover, log-linearizing (17)-(16) around the BGP equilibrium, our simulation reveals the existence of one stable (negative) and two unstable (positive) eigenvalues (see Table 1), implying that there exists a unique stable transitional path for the whole world economy.

Since the modulus of the negative eigenvalue is very close to zero, the implied half-life is very large and adds to 169.7 years. This result lines up with Patel et al.'s (2021) and Kremer et al.'s (2022) finding that emerging-market countries tend to catch-up with advanced countries only slowly over time. Moreover, using the results from Table 1 to derive the adjustment path of the cross-country income inequality, we have that the transitional dynamics followed by the relative per capita income of North y_n/y_s are that portrayed in Figure 1.

Based on Figure 1, any increase in A_s leads to an initial drop in relative per capital income of North from 19.47 to 17.32, and to a smooth adjustment towards to new long-run equilibrium value of 1 afterwards. As far as the growth rates are concerned, since the size of the baseline knowledge of North, A_n, remains unchanged, the after-shock BGP growth rate remains stuck at $g^* = 0.019$, implying that onto the transitional path the Southern economy always grows faster than the Northern economy (See Figure 2).

[8]For simplicity, throughout the section we will suppose that the transmission of knowledge from North to South is such that to make A_s exhibits a quantic jump from 0.2695 to 0.303.

[9]The result $x_n^* = x_s^*$ is due to the presence of a unique rate of time preference ρ for the entire world economy. It can be demonstrated that this result is not preserved if one allows for $\rho_n \neq \rho_s$, while all the other results concerning the existence of unique long-run growth rate for the entire world economy and the stability of the long-run equilibrium mentioned in this section remains unchanged.

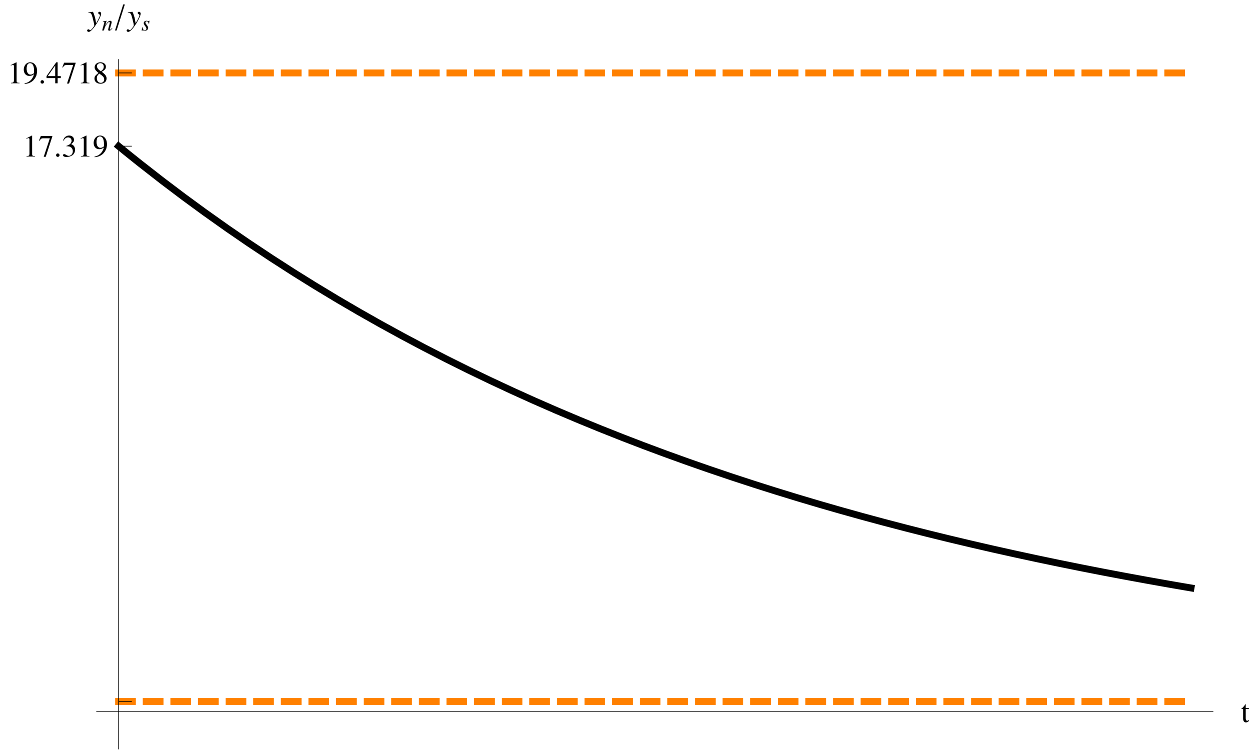

Figure 1: Transitional dynamics of the income ratio y_n/y_s due to a permanent increase in a_s from 0.1978 to 0.3.

Summing-up, what emerges from our analysis is that a two-country AK model extended to allow for international transmission of technological knowledge is not in contradiction with Patel et al.'s (2021) and Kremer et al.'s (2022) finding of 'unconditional' convergence in per capita income. Moreover, since along the adjustment path the non-frontier country (South) always grows faster than the frontier country (North), our model is also able to replicate the 'Wilde'-convergence scheme documented by Roy et al. (2016).

5 Converging to convergence

While focusing on the two catching-all parameters A_n and A_s was it useful to explain how 'unconditional' convergence might converge to 'conditional' convergence in as the simplest way as possible, this approach does not allow us to understand to what extent targeted growth policies can help non-frontier countries to bridge their per capita income gaps. For this reasons, in this section we will provide two extensions of the baseline model to show how the transformation from 'conditional' into 'absolute' convergence can be the result of a self-induced policy decision of South.

To do this, we present two extensions of the baseline model to include: (i) country-specific investment policy; (ii) productive government spending. Here our goal is to show to what extent appropriate growth policies can contribute to turn 'conditional' into 'unconditional'

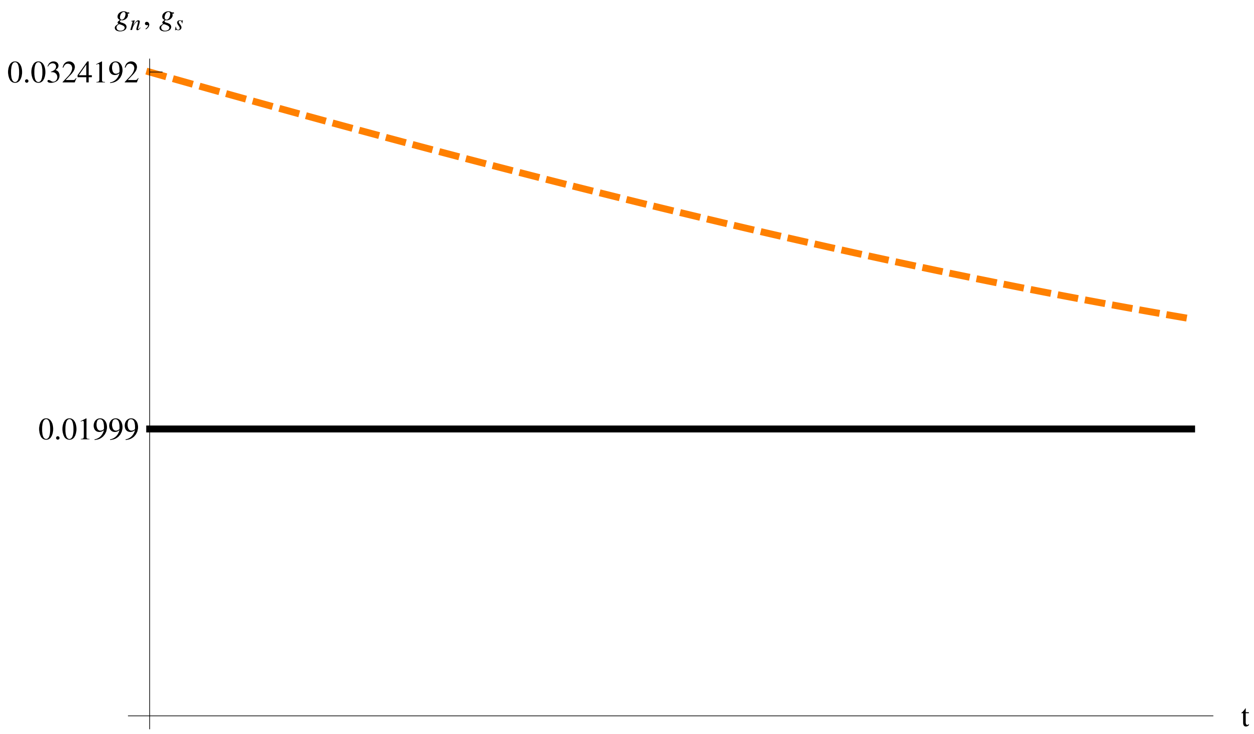

Figure 2: Transitional dynamics in growth rates. The 'dotted' line indicates the growth rate of South and the 'thick' line the growth rate of North

convergence, and possibly. As we will show, all of the theoretical results provided by this section are consistent with the 'unconditional' convergence finding of Patel et al. (2021) and Kremer et al. (2021), and the 'Wilde'-convergence documented by Roy et al. (2016).[10]

5.1 Convergence in investment policy

Consider the same two-country analytical framework of Section 2 and suppose that, at a certain point of time, each country's government decides to subsidize firms' gross investment at a rate $\sigma_i \in [0, 1)$. The introduction of the subsidy package modifies the profit function of the representative firm of country i, which becomes

$$Y_{j,i} - w_i N_{j,i} - (1 - \sigma_i)(r_i + \delta) K_{j,i}.$$

From the previous equation, it follows that differentiation with respect to $N_{j,i}$ leads to the same first-order condition with respect to labor (7), while differentiation with respect to $K_{j,i}$ gives a different first-order condition with respect to capital equal to

$$r_i = \frac{\alpha A_i^{1-\alpha} k_{j,i}^{\alpha-1}}{1 - \sigma_i} - \delta. \tag{23}$$

[10]Throughout the section, we will keep assuming that $\kappa(0) > 1$ holds initially, so that international knowledge spillovers are key for the economic growth of South. The calculations leading to the results collected in this section are available in a separate appendix upon request.

To finance the subsidy, each country's government taxes labor income at a rate $\tau_i^w \in [0,1)$. This implies a new flow budget constraint for representative households given by

$$\dot{k}_i = r_i k_i + (1-\tau_i^w)\, w_i - c_i \tag{24}$$

and a zero-deficit budget constraint for governments given by

$$\tau_i^w w_i L_i = \sigma_i\,(r_i+\delta)\, K_i. \tag{25}$$

Because labor is supplied inelastically in both countries, taxing labor income does not modify the optimal consumption decisions of households, which are still given by (3) and (4). However, substituting from (9) in (23), and the using the resulting expressions to get rid of r_i in (3), we have that the growth rates of the individual consumption expenditures, c_n and c_s, are affected by the subsidy according to

$$\frac{\dot{c}_n}{c_n} = \frac{\alpha A_n}{1-\sigma_n} - (\rho+\delta) \tag{26}$$

$$\frac{\dot{c}_s}{c_s} = \frac{\alpha A_s \kappa_n^{\psi(1-\alpha)}}{1-\sigma_s} - (\rho+\delta)\,. \tag{27}$$

Finally, substituting from (11) and (23) into the flow budget constraints of households (2), and then using (9) and (25) to substitute for $\mathcal{A}_i$ and τ_i^w, it is easy to verify that the aggregate resource constraints of, respectively, North and South are not affected by the subsidy and are still given by (12) and (13).

Equations (26) and (27), along with (12) and (13), form a system of four dynamic equations in four unknowns: c_n, c_s, k_n and k_s.

Definition 2 *A dynamic equilibrium for the two-country AK model with investment subsidy and technology transmission can be defined a set of infinite sequences for the allocations* $\{c_n,\ c_s,\ k_n,\ k_s\}_{t\in[0,\infty)}$ *and a set of infinite sequences for the fiscal packages* $\{\tau_n,\ \sigma_n\}_{t\in[0,\infty)}$ *and* $\{\tau_s,\ \sigma_s\}_{t\in[0,\infty)}$ *that: (i) satisfy equations (12), (13) (26) and (27); (ii) fulfill the inequality constraints* $c_s \geq 0,\ c_n \geq 0,\ k_s \geq 0,\ k_n \geq 0$*; (iii) satisfy the balance-budget rule (25); (iv) satisfy the transversality condition (4).*

Re-scaling variables in terms of capital per worker k_i, we can reduce of one dimension the dynamic system to obtain

$$\frac{\dot{\kappa}_n}{\kappa_n} = x_s - x_n + A_n - A_s \kappa_n^{\psi(1-\alpha)}$$

$$\frac{\dot{x}_n}{x_n} = x_n - \rho - \left(1 - \frac{\alpha}{1-\sigma_n}\right) A_n$$

$$\frac{\dot{x}_s}{x_s} = x_s - \rho - \left(1 - \frac{\alpha}{1-\sigma_s}\right) A_s \kappa_n^{\psi(1-\alpha)}.$$

Solving the above system for the BGP equilibrium, and then Taylor-expanding the system about the stationary quantities x_n^*, x_s^* and κ_n^*, it can be shown that the BGP equilibrium produced by the model is asymptotically stable, and thus that the rest point $\langle x_n^*, x_s^*, \kappa_n^* \rangle$ represents a saddle-path equilibrium for the dynamic system. Moreover, along the BGP equilibrium, cross-country income inequality is stable over time and given by

$$\frac{y_n^*}{y_s^*} = \kappa_n^* = \left[\frac{A_n / (1 - \sigma_n)}{A_s / (1 - \sigma_s)} \right]^{1/[\psi(1-\alpha)]}, \tag{28}$$

which is clearly dependent upon the subsidy policy of governments and baseline knowledge stocks.

To grasp how converging in investment policy can affect per capita income inequality worldwide, assume that $A_n = A_s = \bar{A}$ and that the Northern government provides higher fiscal incentives to capital accumulation than South, such that $\sigma_n > \sigma_s > 0$ holds. From (28), it follows that

$$\frac{y_n^*}{y_s^*} = \left(\frac{1 - \sigma_s}{1 - \sigma_n} \right)^{1/[\psi(1-\alpha)]} > 1, \tag{29}$$

and then that for 'unconditional' convergence to occur, it suffices that the Southern government adopts the same fiscal package of North, (τ_n^w, σ_n). Indeed, when South manages to get her policy package to replicate that of North, $(\tau_s^w, \sigma_s) \to (\tau_n^w, \sigma_n)$, the level of per capita income of South fully catches up with that of North, and the convergence dynamics generated by the model is similar to that portrayed in Figure 2 of Section 4.

5.2 Convergence in productive public expenditure

Results similar to those described in Section 5.1 can be obtained if we focus on productive government spending as possible source for production externality as in Barro (1990). To show this, we modify the technology parameters A_n and A_s of Section 2 to allow for an additional source of cross-country productivity externality coming from productive public expenditures. More specifically, throughout this section we will focus on the following new specifications

$$\mathcal{A}_n = a_n \left(\frac{G_n}{L_n} \right)^{1-\xi} k_n^{\xi}, \quad \mathcal{A}_s = a_s \left(\frac{G_s}{L_s} \right)^{1-\xi} \left(k_s \kappa_n^{\psi} \right)^{\xi}, \tag{30}$$

where G_i is the level of the government expenditure of country i at time t, $\xi \in [0, 1]$ is a weight parameter that we suppose identical for all countries, and a_n is a new catch-all parameter that captures all other variables that may affect the size of the baseline knowledge of country i, but productive government expenditure.

We keep assuming that governments finance expenses by taxing labor income at the rate τ_i^w, so as, at each t, the flow budget constraint of each representative household is still given by (24), while the balanced-budget rule of the each government changes to equate

$$\tau_i^w w_i L_i = G_i. \tag{31}$$

To keep working with a simple model, we finally assume that governments follow the simple spending rule of tieing the level of G_i to that of Y_i, so that the ratio $G_i/Y_i := \zeta_i \geq 0$ is constant over time. Increases in ζ_i indicate increases in productive public spending relative to GDP, and thus scenarios where governments become more involved in the determination of the long-run growth rate of the economy.

Under these modifications, the model leads to the new pair of equilibrium rental rates

$$r_n = \alpha \left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - \delta, \;\; r_s = \alpha \left(a_s \zeta_s^{1-\xi} \kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - \delta, \tag{32}$$

and to the new pair of equilibrium wage rates

$$w_n = (1-\alpha) \left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} k_n, \;\; w_s = (1-\alpha) \left(a_s \zeta_s^{1-\xi} \kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} k_s. \tag{33}$$

Using (32) to substitute for r_i in (3), we have that the two laws of motion of per capita consumption expenditures are now given by

$$\frac{\dot{c}_n}{c_n} = \alpha \left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - (\rho+\delta) \tag{34}$$

$$\frac{\dot{c}_s}{c_s} = \alpha \left(a_s \zeta_s^{1-\xi} \kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - (\rho+\delta). \tag{35}$$

Next, using (32) and (33) to replace r_i and w_i in (24), and then using (31) to get rid of the tax rate τ_i^w, it follows that the two aggregate resource constraints of North and South now differ from (12) and (13) and can be written as

$$\frac{\dot{k}_n}{k_n} = (1-\zeta_n) \left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - \frac{c_n}{k_n} - \delta \tag{36}$$

$$\frac{\dot{k}_s}{k_s} = (1-\zeta_s) \left(a_s \zeta_s^{1-\xi} \kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - \frac{c_s}{k_s} - \delta. \tag{37}$$

For any given value of ζ_n and ζ_s, the dynamic system (34)-(37) governs the dynamics of the four main endogenous variables of the model: c_n, c_s, k_n and k_s. Since our focus is on only those BGP equilibria characterized by cross-country inequality in per capita income, in the rest of the section we will assume that $\zeta_n > \zeta_s > 0$ holds initially.

Definition 3 *A dynamic equilibrium for the two-country AK model with productive government spending and international knowledge transmission can be defined a set of infinite sequences for allocations* $\{c_n, c_s, k_n, k_s\}_{t\in[0,\infty)}$ *and a set of infinite sequence of fiscal packages* $\{\tau_n, \zeta_n\}_{t\in[0,\infty)}$ *and* $\{\tau_s, \zeta_s\}_{t\in[0,\infty)}$ *that: (i) satisfy equations (34)-(37); (ii) do not violate the balanced-budget rule of governments (31); (iii) fulfill the inequality constraints* $c_s \geq 0$, $c_n \geq 0$, $k_s \geq 0$, $k_n \geq 0$*; (iv) satisfy the transversality condition (4).*

As usual, rewriting all endogenous variables in terms of the capital input, the dynamics of the model can be studied through the following 3×3 re-scaled dynamic system

$$\frac{\dot{\kappa}_n}{\kappa_n} = x_s - x_n + (1-\zeta_n)\left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]} - (1-\zeta_s)\left(a_s \zeta_s^{1-\xi}\kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]}.$$

$$\frac{\dot{x}_n}{x_n} = x_n - \rho - (1-\alpha-\zeta_n)\left(a_n \zeta_n^{1-\xi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]}$$

$$\frac{\dot{x}_s}{x_s} = x_s - \rho - (1-\alpha-\zeta_s)\left(a_s \zeta_s^{1-\xi}\kappa_n^{\xi\psi}\right)^{(1-\alpha)/[\alpha+(1-\alpha)\xi]}$$

Solving for the BGP equilibrium and then linearizing the reduced system around the triple x_n^*, x_s^* and κ_n^*, it can be shown that the BGP equilibrium is asymptotically stable, and that the BGP level of income inequality characterizing the world economy is determined by

$$\frac{y_n^*}{y_s^*} = \kappa_n^* = \left(\frac{a_n}{a_s}\right)^{1/\xi\psi}\left(\frac{\zeta_n}{\zeta_s}\right)^{(1-\xi)/\xi\psi}. \tag{38}$$

If we set $a_n = a_S = \bar{a}$, then equation (29) boils down to the simple expression: $y_n^*/y_s^* = (\zeta_n/\zeta_s)^{(1-\xi)/\xi\psi} > 1$. Therefore, when the world economy is characterized by different fiscal packages, (τ_i^w, ζ_i), the model can predict convergence in growth rates, but not in levels ('conditional' convergence). However, if the Southern government managed to rise ζ_s as much as to make its level match that of North ($\zeta_s \to \zeta_n$), then the world economy would jump onto a converging trajectory where South temporarily grow faster than North, and eventually catch up with it in the long run ('unconditional' convergence).

6 Conclusions

In this paper, we have showed that a two-country version of Romer's (1986) AK model with endogenous cross-country knowledge diffusion is consistent with the cross-country 'unconditional' evidence recently found by Patel et a. (2021) and Kremer et al. (2021). Furthermore, the paper also showed that our two-country variant of the AK growth scheme can also successfully predict other types of conditional patterns, including 'conditional' convergence (see Barro and Sala-i-Martin, 1992) and 'Wilde'-convergence (see Roy et al., 2016).

In the second part of the paper, we used the model to theoretically test whether the 'convergence to convergence' hypothesis formulated by Kremer et al. (2021) to explain the cross-country 'unconditional' convergence they found in data can be reconciled with the endogenous growth theory. Whereas poor countries are allowed to adopt the same growth policies of advanced countries (namely, investment subsidy and productive government expenditure), we find that our two-country model can predict the passage from a scenario of 'conditional' convergence towards that of 'unconditional' convergence, but at a very low convergence speed.

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Appendix

A. Proof of Proposition 1

Results in items (i) and (ii) come straightforwardly by solving the dynamic system (17)-(16) for the BGP equilibrium. The result in item (iii) can be obtained by observing that in the BGP all endogenous variables grow at the same constant rate g^*, and then by applying this result to (14). Finally, to demonstrate the saddle-path stability of the BGP equilibrium - Item (iv) of the proposition -, it suffices to show that the 3×3 Jacobian matrix of the linearized system shows one eigenvalue with negative real part and two eigenvalues with positive real part. However, this means showing that the trace of the Jacobian is positive and the determinant negative.

Taylor expanding system (17)-(16) about the rest point $(x_n^*, x_s^*, \kappa_n^*)$ yields

$$\begin{bmatrix} \dot{\kappa}_n \\ \dot{x}_s \\ \dot{x}_s \end{bmatrix} = J^* \begin{bmatrix} \kappa_n - \kappa_n^* \\ x_n - x_n^* \\ x_s - x_s^* \end{bmatrix},$$

where

$$J^* := \begin{bmatrix} -(1-\alpha)\psi & -\left(\frac{A_n}{A_s}\right)^{1/[\psi(1-\alpha)]} & \left(\frac{A_n}{A_s}\right)^{1/[\psi(1-\alpha)]} \\ 0 & \rho + (1-\alpha)A_n & 0 \\ -(1-\alpha)^2 \psi A_s \left(\frac{A_n}{A_s}\right)^{1-1/[\psi(1-\alpha)]} [\rho + (1-\alpha)A_n] & 0 & \rho + (1-\alpha)A_n \end{bmatrix}$$

is the Jacobian matrix. Straightforward computations give

$$\mathrm{Tr}J^* = 2\rho + (1-\alpha)(2-\psi)A_n > 0$$

$$\mathrm{Det}J^* = -(1-\alpha)\alpha\psi A_n [\rho + (1-\alpha)A_n]^2 < 0,$$

and this concludes the proof of the proposition.