A synthesis of local and effective tax progressivity measurement

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Abstract
This paper examines theoretical properties of local and global measures of income tax progressivity. In particular, consistency property of global measures with local measures is analyzed. Using a normative approach, an index of performance in effective progression underlying a tax system, in relation to that of a 'norm', is suggested and analyzed. The norm chosen here is the welfare level associated with the post-tax distribution resulting from an inequality minimizing taxation policy which maintains pre-tax rank orders of tax payers and does not impose any additional tax burden on them, given that the pre-tax distribution is fixed as well. As the actual post-tax welfare increases, effective progression (hence performance) improves, which ensures that it is possible to elevate the level of performance sequentially, as may be desired by a policy maker, towards achieving the norm welfare.

JEL classification: D31, D63, H64

Keywords: Taxation, effective progression, inequality minimization, welfare, performance, policy

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1. Introduction

'Effective progression' of an income taxation program refers to the extent to which the underlying post-tax distribution becomes equitable. (The term 'effective progression' was introduced by Musgrave and Thin (1948).) In other words, tax progressivity is an effective way of reducing income inequality. An effectively progressive tax structure imposes lower tax burden on persons with lower incomes than on the persons earning higher incomes. Thus, under effective progression incomes are redistributed more equally after tax. Since high inequality in a society is undesirable for many reasons, from a policy perspective redistributive impact of a progressive taxation scheme is an important concern of a social planner[1].

Measures of tax progressivity suggested in the literature can be classified into two subgroups, local and global. While a local measure is concerned with the evaluation of a tax structure at individual income levels, a global or summary measure looks at the effect of the tax structure on the income distribution as a whole.

In an innovative paper, Musgrave and Thin (1948) suggested the following four local measures of progression, of which the first two measures were proposed earlier by Pigou (1928) :(a) the average rate progression (b) marginal rate progression), (c) liability progression and (d) residual progression Several authors, including Jakobsson (1976), Lambert (1985, 2001) and Pfingsten (1987) examined properties of these measures from different perspectives.

Connection between local progressivity of a tax function and inequality reduction was hinted at by Musgrave and Thin (1948). Kakwani (1977) made this formulation more precise. Jakobsson (1976) was the first to demonstrate equivalence between tax progressivity and inequality reduction. Further contributions along this line were made, among others, by Fellman, 1976). Eichhorn, Funke and Richter (1984), Moyes (1988), Pfingsten (1988) and Chakravarty and Sarkar (2022).

Section 3 of this article presents rigorous formulations of the four local measures, investigates their properties from different perspectives and reviews the literature studying their connections with inequality reduction under alternative notions of

[1] Chakravarty and Sarkar (2021) made an extensive discussion on consequences of high inequality.

inequality invariance. Recently local progressivity has been interpreted in terms of lower 'bipolarization', a notion that has been related to the 'shrinking middle class' (Carbonell-Nicolau and Llavador, 2021). Therefore Section 3 present a rigorous discussion on this relationship also.

Musgrave and Thin (1948) proposed a global measure of effective progression which indicates shifts toward equality in the distribution of income caused by the taxation scheme. It is defined as the ratio between the Gini equality indices for the post-and pre-tax distributions of income. This index pioneers the idea of relating global progressivity to inequality. The innovative contribution of Musgrave and Thin has motivated many researchers to propose alternatives and variants of their measure. Most of the progressivity metrics suggested in these articles are Gini-based measures. Section 3 of our paper reviews this literature as well.

As a background material we discuss some preliminaries in the next section. Often from policy point of view it may be desirable to improve the level of effective progression without imposing any additional tax burden on the individuals. For instance, in a recent contribution, Datta, Ray and Teh (2021) looked at the redistributive effect of the Indian tax system using the Liu (1985)-Pfahler (1987) index of effective progression for the period 2011-18. They noted that this effect has been around 0.05 over the period. They argued that this low value of effective progression, equivalently, low performance of the tax system in terms of effective progression, is mostly a consequence of low average tax rate of around 9-10%, which remained unchanged over the period considered. Therefore, in order to increase the performance of effective progression by lowering the post-tax inequality, a social planner may recommend the use of higher average tax rates as appropriate tool. If average tax rates are increased proportionally at all income levels, effective progression increases (Liu, 1985). In Section 4 of the paper we clearly establish how a measure of effective progression can be applied for this purpose. Finally, Section 5 concludes.

2. Preliminaries

For ease of exposition we subdivide this section into three subsections.

2.1 Inequality and Welfare

We denote an income distribution in an n-person society by a vector $x = (x_1, x_2, ..., x_n)$, where $x_i > 0$ is the income of person i, $i = 1,2,..., n$. Any income distribution $x = (x_1, x_2, ..., x_n)$ is assumed to be illfare ranked, that is, $x_1 \le x_2 \le ..... \le x_n$. Let D^n_{++}, the strictly positive part of the n dimensional Euclidean space R^n, stand for the set of all illfare ranked income distributions in an n person society, where all incomes are positive.

Anonymity of an inequality standard means that inequality in $x = (x_1, x_2, ..., x_n)$ is same as the inequality in $\left(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}\right)$, where π is a permutation of $(1,2,..,n)$. Under this postulate we need information only on incomes for inequality measurement. This enables us to define an inequality standard I directly on D^n_{++}. An inequality index I is a non-constant continuous function defined on D^n_{++}, taking values in R_+, the non-negative part of the real line R. Formally, $I : D^n_{++} \to R_+$. Since inequality is meaningless for $n = 1$, in the remainder of the paper we will deal with a fixed population of size $n > 1$. For any $x \in D^n_{++}$, let $\mu(x)$ stand for the mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ of x.

For any $x \in D^n_{++}$, $y \in D^n_{++}$ we say that y is deduced from x by a progressive transfer, a Robin Hood operation, if for some pair (i, j), with $x_i < x_j$, $y_i = x_i + c$, $y_j = x_j - c$, $c > 0$ and $y_k = x_k$ for all $k \ne i, j$. That is, the distribution y is derived from the distribution x by transferring the positive quantity of income c from person j to person i who has a lower income. An inequality standard is said to satisfy the Pigou-Dalton transfer principle (transfer principle, for short) or the Robin Hood principle if $I(y) < I(x)$.

Under anonymity only those transfers are allowed that maintain rank orders of the individuals. An anonymous inequality index satisfying the transfer principle is strictly S-convex (Dasgupta, Sen and Starrett, 1973)[2]. In addition to the two basic postulates we also assume a normalization condition, that is, for $x \in D^n_{++}$, $I(x) = 0$ if and only if x is perfectly equal.

The Lorenz curve of any $x \in D^n_{++}$ is the graph of the cumulative income proportion $L\left(x, \frac{j}{n}\right) = \frac{\sum_{i=1}^{j} x_i}{n\mu(x)}$ enjoyed by the bottom $\frac{j}{n}$ proportion of the population against $\frac{j}{n}$, where $,1 \le j \le n$ .Likewise, the Bonferroni curve of $x \in D^n_{++}$ is obtained by graphing the ratios $B\left(x, \frac{j}{n}\right) = \frac{\frac{\sum_{i=1}^{j} x_i}{n\mu(x)}}{\frac{j}{n}}$ against $\frac{j}{n}$, where $,1 \le j \le n$. For $x, y \in D^n_{++}$, y is said to be Lorenz(respectively Bonferroni) superior to x if $L\left(y, \frac{j}{n}\right) \ge L\left(x, \frac{j}{n}\right)$(respectively $B\left(y, \frac{j}{n}\right) \ge B\left(x, \frac{j}{n}\right)$) for all $,1 \le j \le n$, with $>$ for at least one $j < n$.

An inequality metric is of relative or absolute category according as it is homogenous or translatable of degree zero. Analytically, $I : D^n_{++} \to R_+$. is relative if

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2 Technically, a function $F : D^n_{++} \to R$ is said to be S-convex if for all $x \in D^n_{++}$ and for all bistochastic matrices Q of order n , $F(xQ) \le F(x)$, where a bistochastic matrix of order n is an $n \times n$ non-negative matrix each of whose rows and columns sums to unity. Strict S-convexity of F requires that the weak inequality should be replaced by a strict inequality whenever xQ is not a permutation of x. A function $F : D^n_{++} \to R$ is strictly S-concave if $-F$ is strictly S-convex (see Marshall, Olkin and Arnold, 2011).

for all $x \in D^n_{++}$ and all positive scalars c, $I(cx) = I(x)$, and I is absolute if $I(x + \alpha 1^n) = I(x)$, where α is a scalar such that $x + \alpha 1^n \in D^n_{++}$ and 1^n is the *n*-coordinated vector of ones. A relative inequality index is called a compromise index if on multiplication with the mean it becomes an absoluter index.

Given any two income distributions $x, y \in D^n_{++}$, for any strictly S-convex relative inequality standard I, the inequality $I(y) < I(x)$ is necessary and sufficient for y to be Lorenz dominant over x. Unanimous ranking of income distributions by absolute inequality indices can be developed using the absolute Lorenz curve. The absolute Lorenz curve $AL\left(x, \frac{j}{n}\right)$ of a distribution $x \in D^n_{++}$ is the plot of $\frac{\sum_{i=1}^{j}\left(x_i - \mu(x)\right)}{n}$, the population size normalized cumulative income deviations of the first j persons from the mean at the population fraction $\frac{j}{n}$ against $\frac{j}{n}$, where $1 \le j \le n$. For $x, y \in D^n_{++}$, y is said to be absolute Lorenz superior to x if $AL\left(y, \frac{j}{n}\right) \ge AL\left(x, \frac{j}{n}\right)$ for all $1 \le j \le n$, with $>$ for at least one j. Given $x, y \in D^n_{++}$, for any strictly S-convex absolute inequality metric A, $A(y) < A(x)$ holds if only if y is absolute Lorenz dominant over x (Moyes, 1987).

We now formally define the two compromise indices to be used for our analysis. For $x \in D^n_{++}$, the Gini inequality metric I_G is defined by averaging the absolute values of pairwise income differences:

$$I_G(x) = \frac{1}{2n^2\mu(x)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| x_i - x_j \right| . \tag{1}$$

Since incomes in D^n_{++} are illfare ranked, we can rewrite $I_G(x)$ as

$$I_G(x)=1-\frac{1}{n^2\mu(x)}\sum_{i=1}^{n}[2(n-i)+1]x_i . \tag{2}$$

The second compromise inequality metric we consider is the recently revived Bonferroni index I_B which, for any $x \in D^n_{++}$, is defined as

$$I_B(x)=1-\frac{1}{n\mu(x)}\sum_{i=1}^{n}\mu_i(x), \tag{3}$$

where $\mu_i(x)=\frac{1}{i}\sum_{j=1}^{i}x_j$ is the i^{th} partial mean of x, $i=1,2,......,n$.

Since these two indices are normalized they are bounded from below by zero. An attractive feature of the two metrics is their satisfaction of source decomposability; if the individuals earn incomes from different sources and their rank orders across sources are the same, then overall inequality is a mean-weighted sum of source-wise inequality quantities, where the non-negative weights add up to one (Weymark, 1981 and Chakravarty and Sarkar, 2021a). Each of them possesses a nice geometric interpretation. While the Gini index can be expressed as twice the area enclosed between the Lorenz curve and the line of equality, the Bonferroni index is the area between the Bonferroni curve and the horizontal line at 1. A rank preserving Robin Hood operation reduces the Bonferroni by a larger amount the poorer the donor is, provided that the number of persons between the donor and the recipient is fixed. But the Gini is equally sensitive to such an operation at all income positions. Of the two indices while the Gini is suitable for cross population comparison of inequality since it is population replication invariant, the Bonferroni is not so.

In view of the compromise characteristic of I_G and I_B we can define their respective absolute counterparts A_G and A_B as follows:

$$A_G(x)=\mu(x)-\frac{1}{n^2}\sum_{i=1}^{n}[2(n-i)+1]x_i , \tag{4}$$

and

$$A_B(x) = \mu(x) - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i} x_j . \tag{5}$$

In order to provide welfare theoretic interpretations of these two indices, we assume the Atkinson (1970)-Kolm (1969)-Sen (1973) (AKS) framework. An AKS social welfare function (SWF) $W : D_{++}^n \to D_{++}^1$ is a continuous and surjective (or, onto) function, that is, every element in the codomain of W is mapped to by an element of the domain D_{++}^n. For any $x \in D_{++}^n$ the AKS SWFs W_G and W_B that correspond respectively to I_G and I_B respectively (and also respectively to A_G and A_B), in a monotonically decreasing way, are given by

$$\begin{aligned} W_G(x) &= \mu(x)\left(1 - I_G(x)\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\left((2n-i)+1\right)x_i \\ &= \mu(x) - A_G(x). \end{aligned} \tag{6}$$

$$\begin{aligned} W_B(x) &= \mu(x)\left(1 - I_B(x)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i} x_j \\ &= \mu(x) - A_B(x). \end{aligned} \tag{7}$$

(See Blackorby and Donaldson (1978), Donaldson and Weymark (1980) and Chakravarty and Sarkar (2021a)).

These two regular (that is, continuous and strictly S-concave) social evaluation functions attain their upper bound $\mu(x)$ when all the incomes are equal. They also satisfy the strong Pareto principle (increasingness in individual incomes) and distributional homogeneity. An SWF $W : D_{++}^n \to D_{++}^1$ is called distributionally homogenous if for all $x \in D_{++}^{n}$, $W(cx+\alpha 1^{n}) = cW(x)+\alpha 1^{n}$, where $c > 0$ is a scalar and α is a scalar such that $cx+\alpha 1^{n} \in D_{++}^{n}$ and 1^{n} is the *n*-coordinated vector of ones[3].

2.2 Bipolarization

It is also necessary to present some preliminaries on bipolarization, since progressivity has been shown to be related to bipolarization reduction (Carbonell-Nicolau and Llavador, 2021). Bipolarization is represented in terms of distances of the two groups lying on the two sides of the median from the median itself. With a shrinkage of the middle class, defined in terms of concentration of population in some range around the median, these distances increase, which in turn lead to an increase in bipolarization. In other words, more people are now concentrated in the income groups below and above the median. This is likely to generate social conflict. Thus, a high bipolarization is undesirable in a society. In contrast, a large and rich middle class contributes to the society's well-being in many ways, including supporting with a higher amount of tax revenue and providing highly educated and technical professionals.

Intrinsic to the concept of bipolarization are increased spread and increased bipolarity. According to increased spread a reduction (respectively an increment) in any income below (respectively above) the median increases bipolarization. Since such a change makes the subgroups below and above the median more alienated, polarization should go up[4]. Increased bipolarity demands that a progressive transfer of income

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[3] Distributional homogeneity becomes linear homogeneity and unit translatability if $\alpha = 0$ and $c = 1$ respectively. However, not all linear homogenous and unit translatable SWFs are distributionally homogenous. For instance, the Atkinson (1970) linear homogenous and the Kolm (1976) unit translatable SWFs are not distributionally homogenous. Chakravarty and Dutta (1987) employed distributional homogeneity to axiomatize economic distance between two distributions that represents the well-being of one population relative to that of the other. Bossert (1990) used this property as a postulate to characterize the 'single-series Ginis'.

[4] The increased spread postulate parallels Cowell's (1985) 'Principle of Monotonicity in Distance', which stipulates that if for two income distributions one varies from the other with respect to only one person's income, then the one that represents higher distance from equality with respect to this person's income should indicate higher level of inequality than the other. (See also Cowell and Flachaire (2017).)

between any two persons on the either side of the median should increase bipolarization. This is because under such a change the individuals on the same side of the median feel more identified. Thus, bipolarization contains an inequity-like feature, increased spread, and equity-like feature, increased bipolarity[5].

For any $x \in D^n_{++}$, let $m(x)$ stand for the median of x. If n is odd, $\frac{(n+1)^{th}}{2}$ observation in x is the median. But if n is even, it is customary to take the mean of the $\frac{n^{th}}{2}$ and $\frac{(n+1)^{th}}{2}$ observations in x as the median. For illustrative purpose consider the distributions $x = (4,9,10,12,15)$ and $y = (2,2,3,7,9,11)$, then $m(x) = 10$ and $m(y) = 5$.

Symmetric mean of order $0 < r < 1$ of the absolute values of the deviations of individual incomes from the median can be employed to construct a relative bipolarization index (Chakravarty, 2015). Formally for any $x \in D^n_{++}$ this index, which we denote by H_θ, is defined as

$$H_\theta(x) = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\left|m(x) - x_i\right|^\theta\right)^{\frac{1}{\theta}}}{m(x)}. \tag{8}$$

The restriction $0 < \theta < 1$ ensures that the continuous function H_θ satisfies the postulates increased spread and increased bipolarity. For a given $x \in D^n_{++}$, an increase in the value

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Chakravarty and Sarkar (2022a) analytically examined implications of this postulate on several well-known inequality measures.

[5] This notion of polarization differs from the concept of 'multipolar' polarization, introduced by Esteban and Ray (1994). (For a discussion, see Duclos and Taptué (2015).) Amiel, Cowell and Ramos (2010) employed a questionnaire method to study people's attitude towards different postulates of polarization.

of θ increases H_{θ}. Note that H_{θ} is a compromise polarization metric; when multiplied by the median it becomes an absolute bipolarization index.

In order to relate progressivity to bipolarization, it becomes necessary to define the relative bipolarization curve. The first segment of the relative bipolarization curve $RB\left(x,\frac{j}{n}\right)$ of the distribution $x \in D^{n}_{++}$ is the graph of $\sum_{j \leq i < \bar{n}} \frac{(m(x)-x_i)}{nm(x)}$, the cumulative income shortfalls, normalized by the factor $nm(x)$, of the first j individuals with incomes below the median from the median itself , against the population proportion $\frac{j}{n}$, where $1 \leq j < \bar{n}$, $\bar{n}=\frac{n+1}{2}$. The remaining segment of the curve is obtained by plotting the cumulative normalized income excesses of j individuals with incomes above the median $\sum_{\bar{n} \leq i \leq j} \frac{(x_i - m(x))}{nm(x)}$ over the median itself, against the population proportion $\frac{j}{n}$, where $\bar{n} \leq j \leq n$. For $x, y \in D^{n}_{++}$ we say that y depolarization dominates x, equivalently x polarization dominates y, if $RB\left(y,\frac{j}{n}\right) \leq RB\left(x,\frac{j}{n}\right)$ for all $1 \leq j \leq n$, with < for at least one j (Chakravarty, 2015)[6]. For arbitrary $x, y \in D^{n}_{++}$, y depolarization dominates x if and only if $H(y) < H(x)$, where H is any relative bipolarization index that satisfies anonymity, increased spread and increased bipolarity (Chakravarty, 2015).

2.3 Taxation

[6] If the two distributions have the same median, then it coincides with the Foster-Wolfson (2010) polarization ordering-or its equivalent formulation proposed by Wang and Tsui (2000)-and the ordering $\geq_0$ suggested in Theorem 1 of Bossert and Schworm (2008).

A taxation method f is a function $f: D^1_{++} \to R_+$ that associates a pre-tax income $u \in D^1_{++}$ to a tax $f(u) \in R_+$. That is, $f(u)$ tax liability levied on the income level u and $u - f(u) \in D^1_{++}$ is the corresponding post-tax income. Note that we do not restrict ourselves to positive taxes. Thus, a positive income earner may be a zero tax payer. Throughout this paper for any given (pre-tax) income distribution $x \in D^n_{++}$, we will write $(T_1, T_2, \ldots, T_n) \in R^+_n$ for the associated tax profile, where $T_i = f(x_i)$ is the tax paid by individual i, $1 \le i \le n$, and R^n_+ is the non-negative part of the $n-$ dimensional Euclidean space with the origin deleted. For the pre-tax income distribution x, the associated post-tax income distribution $(x - T)$ will be denoted by $y \in D^n_{++}$, where $y_i = (x_i - T_i)$ $1 \le i \le n$. We will assume throughout the paper that f is continuously differentiable.

We further assume that taxes are incentive preserving (Fei, 1981). Under this condition individuals' ranks in the pre-tax distribution are maintained in the post-tax situation. It acts as a stimulant for individual earners to earn more. Since a Robin Hood operation does not alter the rank orders of affected persons, all inequality indices that agree with the Lorenz ordering (Lorenz-consistent, for short) satisfy this condition. In the literature incentive preservation is also referred to as horizontal equity (Blackorby and Donaldson, 1984). Analytically, incentive preservation requires that for any $x_i < x_j$, we must have $y_i \le y_j$, where $i \ne j = 1, 2, \ldots, n$. For any income distribution $x \in D^n_{++}$ and the associated tax profile T, we write Γ for the total tax $\sum_{i=1}^{n} T_i$, $\overline{T} = \frac{\sum_{i=1}^{n} T_i}{n}$ for the average tax and $t=\left(\frac{T_1}{x_1},\frac{T_2}{x_2},.......,\frac{T_n}{x_n}\right)$ for the vector of tax rates. We denote the total tax as a proportion of total income by $\bar{t}=\frac{\sum_{i=1}^{n}T_i}{\sum_{i=1}^{n}x_i}$.

As we will see in the next section, some of the tax progressivity results developed in the literature can be related to the inequality minimizing taxation (IMT) principle suggested by Chakravarty and Sarkar (2022). We, therefore, conclude this background section with a discussion on the IMT principle.

Given $x\in D_{++}^{n}$, we denote the partial means of the right tails of x by $m_i=\frac{1}{n-i+1}\sum_{j=i}^{n}x_j$, $i=1,2,......,n$. Assuming that a total Γ of tax proceeds is to be raised from x, define the positive integer λ as follows:

$$\lambda=\min\{i\in\{1,2,....,n\}:(n-i+1)(x_i-m_i)\le\Gamma\}. \tag{9}$$

Next, define $y\in D_{++}^{n}$ to be the distribution $F(x,\Gamma)$ as follows:

$$y_i=\begin{cases}x_i \ \ for\ 1\le i\le(\lambda-1),\\ x_\lambda-\dfrac{\Gamma}{n-\lambda+1}\ for\ \lambda\le i\le n.\end{cases} \tag{10}$$

Evidently, if the tax volume Γ equals 0, then $F(x,\Gamma)=x$.

From now on, given $x\in D_{++}^{n}$ we denote the tax vector and the post-tax income profile associated with an IMT program raising an aggregate tax $\Gamma>0$, by $T^{IM}=\left(T_1^{IM},T_2^{IM},....,T_n^{IM}\right)$ and $y^{IM}=\left(y_1^{IM},y_2^{IM},....,y_n^{IM}\right)$ respectively. By t^{IM} we mean the vector $\left(\frac{T_1^{IM}}{x_1},\frac{T_2^{IM}}{x_2},....,\frac{T_n^{IM}}{x_n}\right)$, the vector of tax rates under an IMT scheme.

We may illustrate the idea using an example.

Example 1: Let the pre-tax income distribution be $x=(4,8,10,16,19,25)$. The partial means of the right tails of x are 25, 22, 20, 17.5, 15.6 and 13. 67. Assume that the total tax amount is 30. Then $\lambda=4$.The implied post-tax distribution becomes $y^{IM}=\left(y_1^{IM},y_2^{IM},y_3^{IM},y_4^{IM},y_5^{IM},y_6^{IM}\right)=(4,8,10,10,10,10)$.

Proposition 3 along with Corollary 2 of Chakravarty and Sarkar (2022) enables us to state the following:

Theorem 1(Chakravarty and Sarkar, 2022): Suppose the total tax $\Gamma>0$ is to be raised from a society with income distribution $x\in D_{++}^n$. Define $F(x,\Gamma)$ using (10). Then the underlying schedule $x-F(x,\Gamma)$ of taxes is an IMT program for any continuous, normalized inequality index that satisfies anonymity and the Pigou-Dalton transfer principle.

An innovative feature of this result is that it does not rely on any notion of inequality invariance. Thus, it holds for all indices that satisfies any notion of invariance, say relative or absolute or the Bossert-Pfingsten (1990) intermediate invariance and their variants. (See Chakravarty (2015) for a discussion on different concepts of inequality invariance.)

3. An Overview of Local and Effective Tax Progressivity Measures

To make the presentation articulate, we partition this section into two subsections.

3.1 Local Measures of Tax progressivity

The four measures of progressivity we wish to analyze here are based on point-wise concept of progression. Now, given that $f(u)$ is the tax liability levied on the income level u, the average tax rate is $a(u)=\frac{f(u)}{u}$, the tax liability as a proportion of income, and the marginal tax rate is $m(u)=\frac{df(u)}{du}$. The four local measures can now be formally defined as (i)Average Rate Progression): The rate of change of average tax rate: $AP(u)=\frac{d(a(u))}{du}$.

A positive value of this coefficient means that the underlying taxation scheme is progressive, for a proportional taxation structure its value is 0 and for regressivity its value is negative.

(ii) Marginal Rate Progression (MP): The rate of change in marginal tax rate: $MP(u)=\frac{dm(u)}{du}$.

This measure will be 0 for proportional taxation, exceed 0 when taxation is progressive and fall short of 0 when taxation is regressive.

(iii) Liability Progression (LP): The elasticity of tax liability with respect to before-tax income: $LP(u)=\frac{df(u)}{du}.\frac{u}{f(u)}$.

If value of this indicator equals 1 taxation is proportional, if the value exceeds 1 taxation is progressive and if the value falls short of 1 taxation is regressive.

(iv)Residual Progression (RP): The elasticity of after-tax income with respect to before-tax income: $RP(u)=\frac{d(u-f(u))}{du}.\frac{u}{(u-f(u))}$.

If this measure equals 1 taxation is proportional, if it exceeds 1 taxation is regressive, and if it falls short of 1taxation is progressive.

While for the first three measures higher values indicate higher level of progressivity, for the residual income progression measure decreasing numerical values are exhibited when progression increases. Pfahler (1984) and Lambert (1985) made a systematic comparison between residual progression and liability progression, and noted that there exist tax changes which increase progressivity with respect to one of them but are progressivity-neutral with respect to the other.

Lambert (1985, 2001) categorized the following three particularities of a progressive taxation, which can be related to three of the local Musgrave-Thin measures unambiguously:

(a)Departure from proportionality (the distribution of tax burden is more unequal than the pre-tax distribution), (b)Redistributive effect (the post-tax income distribution has less inequality than the pre-tax distribution),

(c)Revenue responsiveness (total tax revenue responds to equi -proportionate growth in incomes elastically),

The average rate progression, the liability progression and the residual progression measures correspond respectively to the characteristics revenue responsiveness, departure from proportionality and redistributive effect of a tax system. Given that the tax function is twice continuously differentiable, if the marginal tax rate is increasing monotonically, then the definitions of average rate progression and the marginal rate progression measures are conceptually equivalent(Stroup, 2005).

In his highly interesting paper Jakobsson (1976) addressed the following problem: Does there exist a local measure of progression such that if it regards one taxation system as more progressive than another everywhere, then the former is also more redistributive than the latter? The following proposition shows that the only measure to meet this requirement is residual progression.

Proposition 2(Jakobsson, 1976): Consider two taxation schedules f_1 and f_2 raising the same amount of revenue from the before-tax distribution $x \in D^n_{++}$. Let $y^1, y^2 \in D^n_{++}$ be the resulting after-tax distributions. The corresponding residual progression measures are denoted by RP_1 and RP_2 respectively. Then the following statements are equivalent:

(a) $RP_1(u) < RP_2(u)$ at all income points u.

(b) y^1 Is Lorenz superior to y^2.

Since the after-tax distributions y^1 and y^2 have the same mean income, by the Dasgupta-Sen_-Starrett theorem it follows that $W\left(y^1\right) > W\left(y^2\right)$ for all strictly S-concave SWFs.

Proposition 2 shows that of two taxation schedules, one has lower level of residual progression than the other at all income points, if and only if for every unequal pre-tax income distribution the post-tax distribution for the former is regarded as more equal than that for the latter by the Lorenz criterion.

In the context of inequality minimizing taxation policy a somewhat similar result was demonstrated by Chakravarty and Sarkar (2022). Below we state the result formally.

Theorem 3(Chakravarty and Sarkar, 2022): Suppose that an incentive preserving taxation method and an inequality minimizing taxation method raise the same amount of revenue from the before-tax distribution $x \in D^n_{++}$. Denote the corresponding after-tax distributions by z and y^{IM} respectively. Then y^{IM} is Lorenz superior to z.

This theorem unambiguously establishes that no taxation scheme can be more redistributive than an inequality minimizing taxation program.

Jakobsson (1976) also demonstrated a result for liability progression that parallels Proposition 2.

Proposition 4(Jakobsson, 1976): Consider two taxation schedules f_1 and f_2 raising the same amount of revenue from before-tax distribution $x \in D^n_{++}$. Let $T^1, T^2 \in D^n_{++}$ be the associated distributions of taxes. The corresponding liability progression measures are denoted by LP_1 and LP_2 respectively. Then the following statements are equivalent:

(a) $LP_1(u) < LP_2(u)$ at all income points u.

(b) T^1 is Lorenz superior to T^2.

This result claims that between two schedules, one has lower level of liability progression than the other at all income points, if and only if for every unequal pre-tax income distribution the distribution of tax burdens resulting from the former is more equal than that for the latter. This means that income-by-income dominance of liability progression is necessary and sufficient to incorporate non-egalitarian bias into the distribution of tax burdens. Thus, while liability progression is concerned with the distribution of tax burdens, residual progression examines the impact of a tax program on the after-tax income distribution.

Liu (1985) demonstrated increasingness of three local measures under proportional increase in average tax rates. This is stated in the following proposition formally.

Proposition 5 (Liu, 1985): Suppose the average tax rates are raised proportionally at all income points. Then a taxation schedule that is everywhere point-wise progressive will become point-wise more progressive with respect to the local measures average rate progression, marginal rate progression, and residual progression; but not with respect to the liability progression measure.

This result combined with Proposition 2 establishes that under proportional increases in average rate progressions, residual progressions decrease and the tax structure becomes more redistributive.

A corollary of Proposition 1 of Jakobsson (1976) claims that the post-tax profile of incomes for a tax system is Lorenz dominant over its pre-tax counterpart if and only the tax system is progressive everywhere, where a progressive tax system is 'considered 'more progressive' than a proportional tax' (op. cit., p.161).As Eichhorn, Funke and Richter (1984) argued, this conclusion lacks formal accuracy. They have demonstrated rigorously that the post-tax income profile for a tax system is more equal than its pre-tax twin if and only if the average tax liability is increasing with income. Formally,

Theorem 6(Eichhorn, Funke and Richter, 1984): Consider the taxation scheme f raising some positive amount of revenue from the before-tax distribution $x \in D^n_{++}$. Let $y \in D^n_{++}$ be the resulting after-tax distribution. Then the following statements are equivalent:

(a)The taxation scheme f is incentive preserving and the average tax rate is increasing.

(b) y is Lorenz superior to x.

(c) $I(y) < I(x)$ for any strictly S-convex relative inequality index I.

As Carbonell-Nicolau and Llavador (2021) demonstrated the Eichhorn-Funke-Richter result can be interpreted in terms of depolarization dominance. Formally,

Theorem 7 (Carbonell-Nicolau and Llavador, 2021) Consider the taxation schedule f raising some positive amount of revenue from the before-tax distribution $x \in D^n_{++}$. Let $y \in D^n_{++}$ be the resulting after-tax distribution. Then the following statements are equivalent:

(a)The taxation scheme f is incentive preserving and the average tax rate is increasing.

(b) y is depolarization superior to x.

(c) $H(y) < H(x)$ for all relative bipolarization indices H that satisfy the postulates anonymity, increased spread and increased bipolarity.

Thus, while the Eichhorn-Funke-Richter result claims inequality reduction under increasing average tax rate, the Nicolau -Llavador result shows that increasing average tax rate improves the position of the middle class by making the income distribution more depolarized.

Theorem 6 relies on relative inequality indices. The absolute counterpart to Theorem 6 was established by Moyes (1988). In the Moyes theorem increasing average tax rate is replaced by minimal increasingness. A tax function $f : D^1_{++} \to R^1_{+}$ is said to be minimally increasing if the tax liability increases with income, that is, given $u, v \in D^1_{++}$, where $u < v$, we have $f(u) < f(v)$. The following theorem specifies necessary and sufficient conditions for absolute inequality reduction under taxation.

Theorem 8(Moyes, 1988): Consider the taxation scheme f raising some positive amount of revenue from the before-tax distribution $x \in D^n_{++}$. Let $y \in D^n_{++}$ be the resulting after-tax distribution. Then the following statements are equivalent:

(a)The taxation scheme f is incentive preserving and minimally increasing.

(b) y is absolute Lorenz superior to x.

(c) $A(y) < A(x)$ for any strictly S-convex absolute inequality index A.

Thus, absolute inequality of incomes is reduced by taxation if and only if tax liability is increasing with income[7].

We conclude this section by stating that inequality minimizing taxation is sufficient but not necessary for average rate progression and minimal progressivity.

Theorem 9(Chakravarty, Pal, Pal and Sarkar, 2022): Inequality minimizing taxation implies average rate progressivity (hence depolarization) and minimal progressivity. But the reverse implications are not true.

3.1 Effective Measures of Tax progressivity

Quite often policy makers find it useful to identify whether a taxation schedule is globally progressive, instead of being locally progressive, and to make normative statements on redistributive effects in relation to its progressivity. In this subsection we make a brief analytical scrutiny of measures of effective progression.

Following Jakobsson's (1976) contribution we may take the Lorenz dominance relation as the appropriate criterion for ranking income distributions in terms of progressivity. From Jakobsson's (1976, p.165) demonstration it follows that a tax schedule is globally progressive if and only if its residual income progression is less than unity at all income positions , but this is so if and only if there is average rate progression at all income levels. Thus, a globally progressive taxation schedule will also be point-wise progressive at all income points. It also follows that a taxation schedule can be treated as globally progressive if and only if the post-tax income distribution is Lorenz dominant over the pre-tax income distribution**.** Thus, an inequality index obeying the Lorenz criterion can be used to construct an effective measure of progressivity.

Musgrave and Thin's (1948) classical measure of progressivity is defined as

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[7] Pfingsten (1988) adopted the Bossert-Pfingsten (1990) intermediate inequality invariance which contains the relative and absolute approaches as polar cases and developed quite a general condition between inequality reduction and tax progression(see also Pfingsten , 1986).

$$P_{MT}(x,t)=\frac{1-I_G(y)}{1-I_G(x)}. \quad (11)$$

(Musgrave and Thin (1948) credited Dalton (1936) with the original insight. See also Sykes, Smith and Formby, 1985.) Given a pre-tax distribution of incomes, as the post-tax distribution becomes more equitable, effective progressivity, as measured by P_{MT}, increases. Hence the discussion presented above provides a theoretical underpinning of the Musgrave-Thin measure as an appropriate index of global progressivity.

Blackorby and Donaldson (1984) proposed general ethical indices of relative and absolute effective progression. Assuming that W is regular, increasing and linear homogenous, the Blackorby- Donaldson general relative effective progressivity metric $P^r_{BD}(x,t)$ is defined as the proportionate increase in the actual post-tax welfare over what it would be if the same amount of tax were levied proportionally on the individuals. Formally,

$$P^r_{BD}(x,t)=\frac{W(y)}{W(x(1-\bar{t}))}-1. \quad (12)$$

Since W is linear homogenous, by construction P^r_{BD} is a relative index. By linear homogeneity of W we can write the denominator of (12) as $(1-\bar{t})W(x)$. Note that $(1-\bar{t})=\frac{\mu(y)}{\mu(x)}$. Hence we can rewrite (12) as

$$P^r_{BD}(x,t)=\frac{\frac{W(y)}{\mu(y)}-\frac{W(x)}{\mu(x)}}{\frac{W(x)}{\mu(x)}}. \quad (13)$$

Given the before-tax distribution $x \in D^n_{++}$, an increase in after-tax welfare increases P^r_{BD}. By the Dasgupta-Sen-Starett theorem the new post-tax distribution becomes Lorenz dominant over the initial one. Hence by Proposition 2 the residual progressivity measure goes down at all income levels. This establishes that the

Blackorby-Donaldson measures of effective progression are consistent with this measure of local progression.

Assume that social evaluation is done with respect to the Gini SWF. Then $P_{BD}^{r}(x,t)$ in (13) comes to be

$$P_{BD}^{rG}(x,t)=\frac{I_G(x)-I_G(y)}{1-I_G(x)}. \tag{14}$$

The Musgrave-Thin index $P_{MT}(x,t)$ can be rewritten as:

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$$P_{MT}(x,t)=P_{BD}^{rG}(x,t)+1. \tag{15}$$

Thus, $P_{BD}^{r}(x,t)$ may be treated as a generalization of P_{MT} to an arbitrary regular, increasing, linear homogenous SWF. While $P_{BD}^{r}(x,t)$ is positive, zero or negative according as the tax structure is progressive, proportional or regressive; P_{MT} takes on the value one for proportionality.

In (13) if we employ the Gini SWF and replace the mean incomes by the corresponding totals, then the numerator becomes the Kiefer (1985) measure of global progression.

Liu (1985) and Pfahler (1987) independently suggested the use of the numerator of $P_{BD}^{rG}(x,t)$ as an effective progressivity standard. The Liu-Pfahler index $P_{LP}(x,t)$ can be written in terms of $P_{BD}^{rG}(x,t)$ as

$$P_{LP}(x,t)=P_{BD}^{rG}(x,t)\left(1-I_G(x)\right). \tag{16}$$

Thus, P_{LP} is ordinally equivalent to a member of the Blackorby-Donaldson relative index[8].

Kakwani (1977a) and Suits (1977) suggested indices of global progressivity that look at the extent of deviation of a tax system from proportionality. The Suits measure is based on the deviation from the line of proportionality of a Lorenz curve defined as the plot of the cumulative proportions of the total tax liability against the cumulative proportions of the total pre-tax income. (See also Hainsworth, 1964 and Kienzle, 1980.) The Kakwani index is defined as the difference between the Gini indices of the distribution of tax liability and pre-tax income distribution. Formally, the Kakwani (1977a) index can be defined as

$$P_K(x,t) = I_G(t) - I_G(x). \tag{17}$$

Given the before-tax distribution $x \in D^n_{++}$, assume that the distribution of taxes T becomes more unequal so that the richer bear higher burden of taxes. Denote the new tax distribution by T'. Evidently, the value of the Kakwani index increased because of this change in the distribution of taxes. It is also true that T is Lorenz dominant over T'. By Proposition 4 liability progression at all income positions is higher under T' than under T. This demonstrates consistency of the Kakwani index with the local measure liability progression. Thus, P_K represents departure from proportionality, not redistributive effect. These two are different features of a taxation system.

Khetan and Poddar (1976) suggested the use of

$$P_{KP}(x,t) = \frac{1 - I_G(x)}{1 - I_G(t)}, \tag{18}$$

as a summary measure of overall progressivity. Since

$$P_{KP}(x,t) = 1 + \frac{P_{KP}(x,t)}{1 - I_G(t)}, \tag{19}$$

[8] Gini index-based progressivity metrics were also suggested, among others, by Reynolds and Smolensky (1977) and Pechman and Okner (1980). They are closely related to the Musgrave-Thin measure.

the two indices are closely related. Given that the pre-tax distribution is fixed, all permutations of taxes are regarded as equally progressive by these metrics. Hence, in the context of applications it is necessary to ensure that the pre-tax incomes and taxes are ordered in the same way. (This is ensured in Kakwani, 1977a). Further, since the before-tax distribution is fixed, an equi-proportionate change in tax levels can generate significant changes in post-tax inequality. The tax-scale invariance condition satisfied by these two indices should, therefore, be avoided when using them for applied purpose. (An extensive. (A comprehensive analysis on this issue was provided by Blackorby and Donaldson (1984)).[9]

While Blackorby and Donaldson (1984) looked at global progressivity in terms of welfare deviation from a reference distribution (proportional), no other contribution addressing the issue incorporates deviation from the welfare of a reference distribution in its basic formulation. A common feature of all the existing indicators of overall progressivity is that each of them satisfies some specific notion of inequality invariance.

Ebert (1992) axiomatically characterized a global measure of progressivity as the geometric average of the residual progression local measures at different income points. Formally, the Ebert measure is given by

$$P_E(x,t) = \prod_{i=1}^{n} \left(RP(x_i)\right)^{\frac{1}{n}}. \tag{20}$$

The index P_E regards a taxation system as globally progressive if $P_E < 1$, globally proportional if $P_E = 1$ and globally regressive if $P_E > 1$. Of two taxation systems if one has lower level of residual progressions at all income points than the other, then P_E regards the former as globally more progressive than the latter. This establishes redistributive property of P_E. The measure P_E is different from other measures of

[9] For further discussions on the Kakwani and Suits indices, see, among others, Formby, Seaks and Smith(1981), Formby and Skyes(1984), Silber (1994), Kakwani and Son(2020), Mantovani , Pellegrino and Vernizzi(2020) and Enami , Larroulet and Lustig (2022). The Musgrave-Thin and the Kakwani/Suits indices may generate conflicting results (Formby, Smith and Thistle, 1990 and Silber, 1994).

global progression in two ways. First, it uses a local measure of progression to derive an overall measure. Second, its axiomatic foundation enables us to understand the properties which uniquely identify the index.

Blackorby and Donaldson (1984) also suggested a general ethical index of absolute progressivity P_{BD}^{a} by using the reference distribution where equal-yield taxes are the same across persons. Since our income domain is D_{++}^{n}, the formulation allows only those values of Γ such that none of after-tax incomes is non-positive. Thus, it is required that the average tax should be less than the lowest income. Assuming that W is regular, increasing and unit translatable, the Blackorby-Donaldson (general) absolute effective progressivity standard $P_{BD}^{a}(x,T)$ can be defined formally as

$$P_{BD}^{a}(x,T) = W(y) - W\left(x - \overline{T}1^{n}\right). \tag{21}$$

In words, P_{BD}^{a} is the excess of actual post-tax welfare over what it would be if the equal yield taxes are raised on an equal basis. If $I_{A} : D_{++}^{n} \to R_{+}$ is a general absolute inequality index defined as $I_{A}(x) = \mu(x) - W(x)$, then using unit translatability of W we can rewrite P_{BD}^{a} as

$$P_{BD}^{a}(x,T) = I_{A}(x) - I_{A}(y), \tag{22}$$

which is simply the difference between the pre-and post-tax inequality levels. If we employ the Gini evaluation function in (6), then P_{BD}^{a} becomes the absolute sister of Liu-Pfahler index. An alternative of interest arises if we use the absolute Bonferroni index in (22).

4. The Performance Indices

In order to judge the performance of effective progression, it becomes necessary to design a benchmark profile of post-tax incomes such that as the welfare(equity) of actual post-tax incomes increases its increased effective progression gets closer to that of the benchmark profile. It is quite sensible to express the performance of a social/economic indicator in comparative terms instead of looking at the issue on an 'absolute basis'. For instance, the performance of each of the three dimension-wise indicators (life expectancy at birth, literacy and per capita real GDP) used for the construction of the human development index of a country is expressed by comparing the actual value of the dimensional achievement with the maximum and minimum values it can assume. Dijkstra and Hanmer (2000) suggested a summary measure of female advantage in different dimensions of human well-being by using the ratios between female-to-male achievements in the dimensions. Patients often judge the performances of a private health care body by making a systematic comparison with the facilities available from the public health care system. Sometimes an administrative body prepares its performance report by taking the previous performance as the status quo.

Theorem 3 of Section 2 shows that the post-tax income profile corresponding an IMT structure cannot be Lorenz inferior to the post-tax income distribution resulting from an alternative taxation scheme collecting the same amount of revenue. Therefore, from egalitarian perspective it becomes quite sensible to choose the post-tax income distribution associated with an IMT (a welfare maximizing taxation) as the benchmark.

Below we suggest general ethical indices of comparative performance in effective progression by comparing the level of actual effective progression with that based on the selected norm. The effective progressivity standards we use for this purpose are the Blackorby-Donaldson ethical indices of progression since they are directly welfare-based indices. Given that equal absolute taxation may make some of the after-tax incomes negative, we propose indices of relative variety only.

For simplicity of analysis we will deal only with regular, distributionally homogenous SWFs. For illustrative purpose, we will use the Gini and Bonferroni SWFs.

Our (relative) index Q^r is defined as the ratio between the Blackorby-Donaldson relative effective progressivity index for the actual after-tax incomes and that for the after-tax profile resulting from the norm. Formally,

$$Q^r\left(x,t,t^{IM}\right)=\frac{\dfrac{W(y)}{W(x(1-\bar{t}))}-1}{\dfrac{W\left(y^{IM}\right)}{W(x(1-\bar{t}))}-1}, \tag{23}$$

which we can rewrite as

$$Q^r\left(x,t,t^{IM}\right)=\frac{W(y)-W(x(1-\bar{t}))}{W\left(y^{IM}\right)-W(x(1-\bar{t}))}. \tag{24}$$

As W is distributionally homogenous, Q^r is scale invariant. Since x, $\bar{t}$, y^{IM} are given, Q^r increases if $W(y)$ increases, that is, if the tax structure becomes more equitable. In other words, Q^r increases if, under ceteris paribus assumptions, effective progression increases. As a result residual progression measures at all income positions decrease. We refer to Q^r as an effective progression performance index since a higher value of Q^r represents a better performance in the sense of increased effective progression. It is comparative in nature because its construction relies on comparison with effective progression corresponding to an IMT structure.

The performance yardstick Q^r is negative for regressivity, zero for proportionality and positive for progressivity. It is continuous and bounded above by one, where the upper bound is achieved when taxes are apportioned in an inequality minimizing basis, that is, when after-tax welfare under the IMT structure materializes as the actual after-tax welfare.

We can rewrite Q^r in (21) in terms of a compromise relative inequality metric as

$$Q^r\left(x,t,t^{IM}\right)=\frac{I(x)-I(y)}{I(x)-I\left(y^{IM}\right)}. \tag{25}$$

Thus, while in (24) as actual after-tax welfare increases, Q^r increases; in (25) as actual after-tax inequality decreases, Q^r increases. These two statements are equivalent under the given assumptions.

When social evaluation is done with respect to the Gini and Bonferroni SWFs, the respective expressions for Q^r are given by

$$Q_G^r\left(x,t,t^{IM}\right)=\frac{\frac{1}{\mu(y)n^2}\sum_{i=1}^{n}((2n-i)+1)y_i-\frac{1}{\mu(x)n^2}\sum_{i=1}^{n}((2n-i)+1)x_i}{\frac{1}{\mu(y)n^2}\sum_{i=1}^{n}((2n-i)+1)y_i^{IM}-\frac{1}{\mu(x)n^2}\sum_{i=1}^{n}((2n-i)+1)x_i}$$

, (26)

and

$$Q_B^r\left(x,t,t^{IM}\right)=\frac{\frac{1}{\mu(y)n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}y_j-\frac{1}{\mu(x)n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}x_j}{\frac{1}{\mu(y)n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}y_j^{IM}-\frac{1}{\mu(x)n}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}x_j} . \quad (27)$$

From policy viewpoint the performance standard represents sensitivity of effective progression towards the norm. A low level of sensitivity is undesirable for the society since high magnitude of inequality may fuel unrest in the society. A higher level of sensitivity also makes the middle class of the economy better off in the sense of lower extent of bipolarization (Chakravarty, 2015 and Carbonell-Nicolau and Llavador, 2021)[10].

We may now numerically illustrate the formulae (26) and (27) by an example.

Example 2: Consider the pre-tax income distribution $x=(4,8,10,16,19,25)$ of Example 1. Consider the alternative tax vector $T=(0,1,3,5,8,13)$ collecting the same amount of revenue 30 from x as the IMT scheme. The incentive preserving tax program underlying

[10] Chakravarty and Sarkar (2021a) addressed a related problem from a different perspective. They analyzed the duality problem of arising at an income distribution whose inequality values with respect to the Gini and Bonferroni indices coincide with their respective targeted (lower) values.

T is both average progressive and minimally progressive. The resulting after tax distribution becomes $y = (4,7,7,11,11,12)$. Then given that the after-tax distribution resulting from the IMT scheme is $(4,8,10,10,10,10)$, the value of $Q_G^r\left(x,t,T^{IM}\right)$ comes to be

$$Q_G^r\left(x,t,t^{IM}\right) = \frac{0.292683 - 0.179467}{0.292683 - 0.115385} = 0.65. \qquad (28)$$

Likewise, the value of $Q_B^r\left(x,t,t^{IM}\right)$ for this example becomes

$$Q_B^r\left(x,t,t^{IM}\right) = \frac{0.367073 - 0.241987}{0.367073 - 0.184615} = 0.51 \qquad (29)$$

Since for any unequal income distribution the Gini is bounded above by the Bonferroni (Chakravarty and Sarkar, 2021a), for each of the three distributions we have considered the Bonferroni value is higher.

Given a regular, distributionally homogenous SWF, there exists a corresponding effective performance index. These indices will differ in the way how a social planner decides to aggregate individual incomes to arrive at a summary measure of welfare.

As an intermediate step a social planner's objective may be to look for a tax structure whose effective progressivity is higher than that of the existing one but less than the corresponding figure for the norm. This bears some similarity with checking a country's sequential success towards achieving the Millennium Development Goals. 'Success will require sustained action ……between now and the deadline' (Annan, 2005). As we demonstrate, continuity of the comparative performance indices ensures that it is possible to ensure the existence of such a tax structure.

We now formally demonstrate the existence of a tax profile with a higher value of performance than the existing one but still with a lower performance than the IMT structure.

Theorem 10: Given the pre-tax income distribution $x \in D_{++}^n$, aggregate tax $\Gamma > 0$ and inequality minimizing tax profile t^{IM}, let c_1 be the value of (relative) effective progressivity improvement index Q^r corresponding to the current distribution y of after-tax incomes. Given a higher level of effective progressivity improvement c_2, where $c_1 < c_2 < 1$, there must exist a tax vector $T' \in R_n^+$, whose after-tax income distribution $y' = (x - T')$ when replaces y as the argument of W in Q^r, makes the value of Q^r equal to c_2.

Proof: To demonstrate this claim we employ the Intermediate Value Theorem which states that for a real valued continuous function g defined on the closed interval $[a,b]$, where $g(a) < g(b)$ and a number v such that $g(a) < v < g(b)$, then there exists a point $z \in (a,b)$ such that $g(z) = v$ (Rudin, 1976, p.93). Since $x \in D_{++}^n$, $\Gamma > 0$, t^{IM} (hence y^{IM}) are fixed, we can rewrite Q^r as $H(W(y))$. Given that $c_1 < c_2 < 1$, by an application of the Intermediate Value Theorem for the continuous function over $[c_1, 1]$, a value of the surjective function $W(y)$, say $W(y')$, must exist such that $W(y') = c_2$. (Since $c_1 < c_2$, we have $W(y) < W(y')$. Recall that welfare is expressed as a trade-off between efficiency and equity, since efficiency considerations are absent (the mean of after-tax incomes is fixed), the higher value $W(y')$ of W must be a consequence of higher equity. Equivalently, there must exist an alternative equal-yield tax vector $T' \in R_n^+$, which makes a readjustment of the taxes collected under T, that is, poorer pay less and richer pay more such that $y' = (x - T')$. Thus, $Q^r\left(x, t', t^{IM}\right) = c_2$. This completes the proof of the theorem. Δ

5. Concluding Remarks

This article presented an analytical discussion on local and global effective measures of tax progression. Consistency properties of global measures in terms of redistribution has also been examined. We also addressed the problem of judging the effective progression of a tax system in comparison with that of a 'norm', an inequality minimizing (welfare maximizing) tax program, from an ethical angle, under the constraints that the before-tax income distribution and the total tax size are fixed. A specific class of social welfare functions satisfying a minimal equity postulate becomes helpful in designing both relative and absolute indices for measuring such performances. The ethical effective progressivity indices we apply for this particular purpose are the Blackorby-Donaldson (1984) indices. Each social evaluation function belonging to the specific class generates one relative index and one absolute index. We also demonstrate analytically that our performance indices ensure the existence of a social planner's 'targeted transitional' tax system whose effective progression is higher than that of the current tax structure but less than that of the underlying norm. It will certainly be worthwhile to axiomatically characterize the performance indices. We leave this as a future research program.

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