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Abstract. This paper characterizes conditions for two-player bargaining problems and bargaining rules under which unilateral pre-donation always yields Pareto utility gains. The paper also computes the optimal pre-donation of each player under the class of proportional bargaining rules.

Keywords: Cooperative bargaining; pre-donation.

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1 Introduction

This paper characterizes conditions for two-player cooperative bargaining problems and bargaining rules under which unilateral pre-donation always yields Pareto utility gains. A model for cooperative bargaining was first introduced by Nash (1950). This model involves four elements: a fixed number of players, a bargaining set of payoff allocations over which players bargain, a disagreement (or threat) point in the bargaining set to be enjoyed by the players if they fail to reach an agreement, and a bargaining rule that selects a feasible payoff allocation for each bargaining problem (consisting of a bargaining set along with a disagreement point). The rule proposed, and also axiomatized, by Nash (1950) maximizes the product of the payoff gains of players with respect to their disagreement payoffs. Following Nash (1950), many bargaining rules were proposed and axiomatized in the literature. (See, for example, Raiffa (1953) and Kalai and Smorodinsky (1975) for the Kalai and Smorodinsky rule, Rawls (1972) for the egalitarian rule, and Kalai (1977) for the family of proportional rules.)

The idea of pre-donation in two-person cooperative bargaining was introduced by Sertel (1992). A one-sided pre-donation by a player is a commitment on his/her part to give a certain fraction of each payoff he/she may obtain in the bargaining set to the other player before the bargaining rule is applied to the bargaining problem. Sertel (1992) showed that in two-player bargaining problems with an affinely-linear Pareto frontier, the Nash bargaining rule can always be manipulated through pre-donations

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by the player with the higher valuation.[1] Akin et al. (2011) further showed that simple n-person bargaining problems with smooth Pareto frontiers, the manipulation of Kalai-Smorodisnky rule via pre-donation can yield Pareto gains. Very recently, Saglam (2022a) and Saglam (2022b) showed that such Pareto gains of pre-donation may also arise in industrial organization problems where the Pareto frontier is non-linear. Motivated by these results, we characterize in this paper conditions for two-player bargaining problems and bargaining rules under which unilateral pre-donation always leads to Pareto utility gains. We also compute in this general setup the optimal pre-donation of each player under the class of proportional bargaining rules. The remainder of the paper is organized as follows. Section 2 presents the basic structures, Section 3 contains the results, and Section 4 concludes.

2 Basic Structures

We consider a society, involving two individuals (players), and denote it by $N = \{1, 2\}$. Following Nash (1950), we define **a two-player bargaining problem** as a nonempty subset S of $\subset \mathbb{R}^2_+$, involving von Neumann-Morgenstern utility allocations. If the players do not agree on any point in S, then each of them gets zero utility. (Thus, we have normalized the disagreement utilities to zero).

We assume that the set S is the closed subgraph of a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, i.e.

$$S := subgraph\ of\ f = \{(s_1, s_2) \in \mathbb{R}^2_+ : s_2 \leq f(s_1)\},$$

where f is continuously differentiable (almost everywhere), decreasing, and concave. We denote by Σ^2_0 the set of (0-normalized) two-player bargaining problems each of which is a closed subgraph of some function satisfying the above assumptions. Given any $S \in \Sigma^2_0$, we denote by f^S the function such that the subgraph of f^S is S. One can observe that f^S is uniquely defined under the assumptions stated above.

A bargaining rule $\mu : \Sigma^2_0 \to \mathbb{R}^2_+$ is a mapping such that for each $S \in \Sigma^2_0$, $\mu(S) \in S$. Then, $\mu_i(S)$ is the bargaining utility of player $i \in N$. Below, we will describe some well-known bargaining rules.

The Nash rule (1950) selects for each problem $S \in \Sigma^2_0$ the solution

$$N(S) = \text{argmax}_{s \in S}\, s_1 s_2, \tag{1}$$

at which the (net) utility product of players is maximized.

The Kalai-Smorodinsky rule (Raiffa, 1953; Kalai and Smorodinsky, 1975) selects for each problem $S \in \Sigma^2_0$ the solution

$$KS(S) = \max \{s \in S : \; s_1/s_2 = a_1(S)/a_2(S)\}, \tag{2}$$

where $a_i(S) = \max\{s_i \; : \; s \in S \text{ and } s_{-i} = 0\}$ denotes the ideal utility that player i expects from S. The point $a(S) = (a_1(S), a_2(S))$ is called the ideal point for S. The

[1] For more on this issue, see Sertel and Orbay (1998), Orbay (2003), and Akyol (2008).

Kalai-Smorodinsky rule chooses the maximal point of S on the line segment between the points $(0,0)$ and $a(S)$.

A bargaining rule is **dictatorial for player** i, and denoted by D^i, if for each $S \in \Sigma_0^2$

$$D^i(S) = \max\{s \in S \mid \ s_i \geq 0 \ \text{ and } \ s_j = 0 \text{ for } j \neq i\}. \tag{3}$$

The rule D^i selects for player i the best point in S , while keeping player $j \neq i$ at its disagreement utility.

Given any $\alpha > 0$, a bargaining rule is called **the α-proportional rule** (Kalai, 1977), or simply P^α, if for each $S \in \Sigma_0^2$ it selects the solution

$$P^\alpha(S) = \Phi(S)(1,\alpha) \text{ and } \Phi(S) = \max\{k \ : \ \ k(1,\alpha) \in S\}. \tag{4}$$

The rule P^α finds the maximal point of S on the line passing through the points $(0,0)$ and $(1,\alpha)$. This rule converges to the dictatorial rule D^1 (D^2) when α approaches 0 (∞). When $\alpha = 1$, P^α coincides with **the egalitarian rule** (Rawls, 1972) that maximizes the utility of the worst-off player.

Given any x and y in $\mathbb{R}^2_+$, $x > y$ means $x^i > y^i$ for all $i = 1,2$ and $x \geq y$ means $x^i \geq y^i$ for all $i = 1,2$. For any $S \in \Sigma_0^2$, we define **the set of weakly Pareto optimal points** in S, i.e.,

$$WPO(S) = \{s \in S \ : \ \ t > s \text{ implies } t \notin S\}, \tag{5}$$

and **the set of Pareto optimal points** in S, i.e.,

$$PO(S) = \{s \in S \ : \ \ t \geq s \text{ implies } t \notin S \setminus \{s\}\}. \tag{6}$$

We define the following axioms.

Weak Pareto Optimality (WPO) If $S \in \Sigma_0^2$, then $\mu(S) \in WPO(S)$.

Pareto Optimality (PO) If $S \in \Sigma_0^2$, then $\mu(S) \in PO(S)$.

Clearly, PO implies WPO. The Kalai-Smorodinsky and Nash rules satisfy PO. For any $\alpha > 0$, P^α satisfies WPO, but not PO.[2] The same is true for rules D^1 and D^2.

Strong Individual Rationality (SIR) If $S \in \Sigma_0^2$, then $\mu_i(S) > 0$ for each $i = 1,2$.

SIR is satisfied by many bargaining rules, including the Kalai-Smorodinsky and Nash rules, and the proportional rule P^α for any $\alpha > 0$. On the other hand, rules D^1 and D^2 do not satisfy SIR.

[2] The reason is that given any $\alpha > 0$, one can always find a problem $S \in \Sigma_0^2$ with $WPO(S)$ containing horizontal or vertical line segments ($WPO(S) \setminus PO(S) \neq \emptyset$) such that $P^\alpha(S)$ will lie in $WPO(S) \setminus PO(S)$.

Given two bargaining solutions S and T in Σ_0^2 and a bargaining rule μ defined on Σ_0^2 we say that T **is a twist of** S **that is partially favorable for player** i if $\mu(S) \in T$ and for every $s \in S \setminus T,\ \ s_i \leq \mu_i(S)$, and for some $s \in T \setminus S,\ \ s_i \geq \mu_i(S)$.

Monotonicity in Partially Favorable Twists for Player i **(MON-PFT-i)** If T is a twist of S that is partially favorable for player i, then $\mu_i(T) \geq \mu_i(S)$.

Many bargaining rules, including the Nash and Kalai-Smorodinsky rules as well the α-proportional rule for any $\alpha > 0$, satisfy MON-PFT-i.[3]

Domination in Twists (DOM-T) If T is a twist of S, i.e., $T \setminus S \neq \emptyset$ and $S \setminus T \neq \emptyset$, then $\mu(T) \geq \mu(S)$ or $\mu(S) \geq \mu(T)$.

DOM-T says that if players go from a bargaining set to a twist of it, then both players gain or lose together.[4] One can easily check that DOM-T is satisfied by any positively-sloped ray rule, including the proportional rule P^α for any $\alpha > 0$.

3 Results

We will first introduce Sertel's (1992) concept of pre-donation. **A pre-donation from player** i **to player** $j \neq i$ is a mapping $\boldsymbol{\tau^{\theta,i}} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, associated with some parameter $\theta \in [0, 1)$. This mapping transforms each $s \in \mathbb{R}_+^2$ into $\boldsymbol{\tau^{\theta,i}}(s)$ such that $\boldsymbol{\tau_i^{\theta,i}}(s) = (1-\theta)s_i$ and $\boldsymbol{\tau_j^{\theta,i}}(s) = s_j + \theta s_i$ if $j \neq i$. For any bargaining problem $S \in \Sigma_0^2$ and any pre-donation $\boldsymbol{\tau^{\theta,i}}$, we can first calculate

$$\boldsymbol{\tau^{\theta,i}}(S) = \{\tau^{\theta,i}(s) \mid s \in S\} \tag{7}$$

and then its comprehensive closure

$$\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,i}}(S) = \{t \in \mathbb{R}_+^2 \mid t_i \leq s_i \text{ and } t_j \leq s_j \text{ if } j \neq i, \text{ for some } s \in \boldsymbol{\tau^{\theta,i}}(S)\}. \tag{8}$$

So, pre-donation $\boldsymbol{\tau^{\theta,i}}(S)$ transforms the bargaining problem S into $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,i}}(S)$. In the following two lemmas, we prove, along with several other results, that the transformed problems $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)$ and $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S)$ are inside Σ_0^2.

Lemma 1. *For any $S \in \Sigma_0^2$ and $\theta \in (0, 1)$, it is true that*

(i) $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S) \in \Sigma_0^2$ if and only if $\partial f^S(s_1)/\partial s_1 < -\theta$ for all $s_1 \in (0, a_1(S))$, and

[3] A stronger version of this axiom, called Twisting, or Monotonicity in Favorable Twists for Player i, was first introduced by Myerson and Thomson (1980). That axiom considers twists around the solution $\mu(S)$ for any bargaining problem S, whereas we allow twists around other points as well.

[4] This axiom relaxes the Domination axiom of Thomson and Myerson (1980) where the dominance comparison is valid for any two bargaining problems.

(ii) $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S) \supsetneq S \cap ([0,(1-\theta)a_1(S)] \times \mathbb{R}_+)$ *and* $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)) \cap PO(S) = \{(0, a_2(S))\}$ *if and only if* $-1 < \partial f^S(s_1)/\partial s_1$ *for all* $s_1 \in (0, a_1(S))$.

Proof. Pick any $S \in \Sigma_0^2$ and $\theta \in (0,1)$. Recall that S is the subgraph of some function f^S. To prove part (i), first recall that $\underline{\boldsymbol{\tau}}_1^{\boldsymbol{\theta,1}}(s) = (1-\theta)s_1$ and $\underline{\boldsymbol{\tau}}_2^{\boldsymbol{\theta,1}}(s) = s_2 + \theta s_1$ for all $s \in S$. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g((1-\theta)s_1) = f^S(s_1) + \theta s_1$ for all $s_1 \in [0, a_1(S)]$ or equivalently $g(s_1) = f^S(s_1/(1-\theta)) + \theta s_1/(1-\theta)$ for all $s_1 \in [0,(1-\theta)a_1(S)]$. Clearly, $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)$ is the subgraph of g. Notice also that g is continuously differentiable a.e. and concave. Moreover, g is decreasing everywhere, and thus $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S) \in \Sigma_0^2$, if and only if $g((1-\theta)s_1)$ is decreasing for all $s_1 \in (0, a_1(S))$ or equivalently $\partial f^S(s_1)/\partial s_1 < -\theta$ for all $s_1 \in (0, a_1(S))$.

Now, let us prove part (ii). We know that $(0, a_2(S)) \in PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)) \cap PO(S)$ by the definition of $\tau^{\theta,1}$. Then, we have $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S) \supsetneq S \cap ([0,(1-\theta)a_1(S)] \times \mathbb{R}_+)$ and $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)) \cap PO(S) = \{(0, a_2(S))\}$ if and only if $g(s_1) > f^S(s_1)$ for all $s_1 \in (0,(1-\theta)a_1(S)]$. So, pick any $s_1 \in (0,(1-\theta)a_1(S)]$. We know that $g(s_1) > f^S(s_1)$ if and only if $f^S(s_1/(1-\theta)) - f^S(s_1) > -\theta s_1/(1-\theta)$. Define $\gamma = 1/(1-\theta)$. (Note that when θ varies from 0 to 1, γ varies from 1 to ∞.) The above inequality becomes $f^S(\gamma s_1) - f^S(s_1) > -(\gamma-1)s_1$. Since this inequality is continuous in γ and must hold for all γ, it must be true that

$$\frac{\partial f^S(s_1)}{\partial s_1} = \lim_{\gamma \to 1} \frac{f^S(s_1 + (\gamma-1)s_1) - f^S(s_1)}{(\gamma-1)s_1} > -1.$$

This condition ensures that $f^S(\gamma s_1) - f^S(s_1) > -(\gamma-1)s_1$ holds when γ is arbitrarily close to 1. Also, notice that $g(s_1) - f^S(s_1) = f^S(\gamma s_1) + (\gamma-1)s_1 - f^S(s_1)$ is increasing in γ. Thus, we have established that $g(s_1) - f^S(s_1) > 0$ holds for any $\gamma \in (1,\infty)$, or equivalently for any $\theta \in (0,1)$, if and only if $\partial f^S(s_1)/\partial s_1 > -1$. ■

Lemma 1 characterizes a necessary and sufficient condition under which the pre-donation $\boldsymbol{\tau}^{\boldsymbol{\theta,1}}$ made by player 1 to player 2 leads to a bargaining problem in Σ_0^2 and another (necessary and sufficient) condition under which $\boldsymbol{\tau}^{\boldsymbol{\theta,1}}$ strictly expands the Pareto frontier $PO(S)$ of the bargaining problem S outward everywhere except for the point $(0, a_2(S))$. Similar conditions for $\boldsymbol{\tau}^{\boldsymbol{\theta,2}}$ are obtained in Lemma 2 below.

Lemma 2. *For any* $S \in \Sigma_0^2$ *and* $\theta \in (0,1)$, *it is true that*

(i) $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S) \in \Sigma_0^2$ *if and only if* $\partial(f^S)^{-1}(s_2)/\partial s_2 < -\theta$ *for all* $s_2 \in (0, a_2(S))$, *and*

(ii) $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S) \supsetneq S \cap (\mathbb{R}_+ \times [0,(1-\theta)a_2(S)])$, *and* $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S)) \cap PO(S) = \{(a_1(S), 0)\}$ *if and only if* $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2$ *for all* $s_2 \in (0, a_2(S))$.

Proof. The proof is similar to that of Lemma 1. Pick any $S \in \Sigma_0^2$ and $\theta \in (0,1)$. To prove part (i), recall that $\underline{\boldsymbol{\tau}}_1^{\boldsymbol{\theta,2}}(s) = s_1 + \theta s_2$ and $\underline{\boldsymbol{\tau}}_2^{\boldsymbol{\theta,2}}(s) = (1-\theta)s_2$ for all $s \in S$. Also recall that S is the subgraph of f^S. Since f^S is decreasing everywhere, it is invertible. Let $g := (f^S)^{-1}$. Also define $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h((1-\theta)s_2) =$

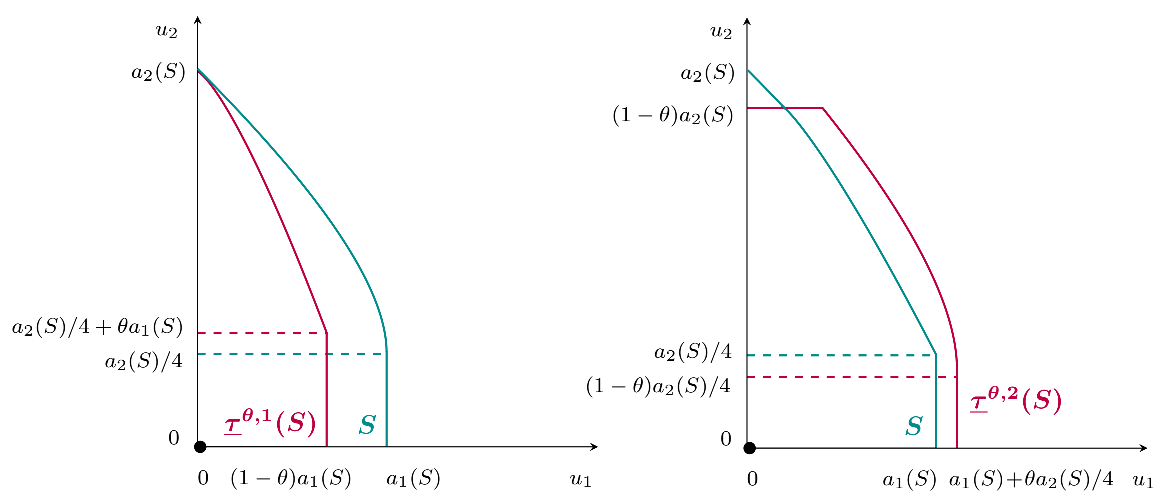
$g(s_2) + \theta s_2$ for all $s_2 \in [0, a_2(S)]$ or equivalently $h(s_2) = g(s_2/(1-\theta)) + \theta s_2/(1-\theta)$ for all $s_2 \in [0, (1-\theta)a_2(S)]$. Notice also that h is continuously differentiable a.e. and concave. Moreover, h is decreasing everywhere if and only if $h((1-\theta)s_2)$ is decreasing for all $s_2 \in (0, a_2(S))$ or equivalently $\partial(f^S)^{-1}(s_2)/\partial s_2 < -\theta$ for all $s_2 \in (0, a_2(S))$. In that case, h becomes invertible and $\underline{\boldsymbol{\tau}}_2^{\boldsymbol{\theta,2}}(S)$ becomes the subgraph of h^{-1}, implying $\underline{\boldsymbol{\tau}}_2^{\boldsymbol{\theta,2}}(S) \in \Sigma_0^2$.

Now, let us prove part (ii). We know that $(a_1(S), 0) \in PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S)) \cap PO(S)$ by the definition of $\tau^{\theta,2}$. Then, we have $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S) \supsetneq S \cap (\mathbb{R}_+ \times [0, (1-\theta)a_2(S)])$ and $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S)) \cap PO(S) = \{(a_1(S), 0)\}$ if and only if $h(s_2) > (f^S)^{-1}(s_2)$ for all $s_2 \in (0, (1-\theta)a_2(S)]$. So, pick any $s_2 \in (0, (1-\theta)a_2(S)]$. Then, $h(s_2) > (f^S)^{-1}(s_2)$ if and only if $(f^S)^{-1}(s_2/(1-\theta)) - (f^S)^{-1}(s_2) > -\theta s_2/(1-\theta)$. Define $\gamma = 1/(1-\theta)$. The above inequality becomes $(f^S)^{-1}(\gamma s_2) - (f^S)^{-1}(s_2) > -(\gamma - 1)s_2$. Since this inequality is continuous in γ and must hold for all γ, it must be true that

$$\frac{\partial(f^S)^{-1}(s_2)}{\partial s_2} = \lim_{\gamma \to 1} \frac{(f^S)^{-1}(s_2 + (\gamma - 1)s_2) - (f^S)^{-1}(s_2)}{(\gamma - 1)s_2} > -1.$$

This condition ensures that $(f^S)^{-1}(\gamma s_2) - (f^S)^{-1}(s_2) > -(\gamma - 1)s_2$ holds when γ is arbitrarily close to 1. Also, notice that $h(s_2) - (f^S)^{-1}(s_2) = (f^S)^{-1}(\gamma s_2) + (\gamma - 1)s_2 - (f^S)^{-1}(s_2)$ is increasing in γ. Thus, $h(s_2) - (f^S)^{-1}(s_2) > 0$ holds for any $\gamma > 1$, or equivalently for any $\theta \in (0, 1)$, if and only if $\partial(f^S)^{-1}(s_2)/\partial s_2 > -1$. ■

Figure 1. Possible Effects of One-Sided Pre-donation



(i) Player 1 Pre-donates

(ii) Player 2 Pre-donates

In Figure 1, we illustrate the possible effects of one-sided pre-donation on a particular bargaining problem borrowed from Saglam (2022a). In panel (i), the problem S shrinks to the modified problem $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta},\mathbf{1}}\boldsymbol{(S)}$ due to the pre-donation $\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{1}}$ by player 1. Panel (ii) shows that a pre-donation by player 2 has a different effect. The problem S shrinks inward at the top but expands outward elsewhere, forming the modified problem $\underline{\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{2}}\boldsymbol{(S)}}$. For the above problem S, the Pareto frontier is defined by $(f^S)^{-1}(s_2) = 2\sqrt{a_2(S)s_2} - 2s_2$ where $s_2 \in [a_2(S)/4, a_2(S)]$. One can check that S satisfies the condition in Lemma 1-(i), but not the condition in Lemma 1-(ii). On the other hand, S satisfies the conditions in both parts of Lemma 2. In Sertel (1992), the bargaining problems are restricted to S such that $f^S(s_1) = \alpha(a_1(S) - s_1)$, where $\alpha > 1$ is a constant and $s_1 \in [0, a_1(S)]$. These problems satisfy the condition in both Lemma 1-(i) and Lemma 2-(i). However, they do not satisfy the condition in Lemma 1-(ii), while they satisfy the condition in Lemma 2-(ii). Using the above lemmas, we can prove the following result.

Proposition 1. *Let μ be any bargaining rule that satisfies SIR and WPO. Pick any $S \in \Sigma_0^2$ and any $\theta \in (0, 1)$.*

(i) If f^S satisfies $-1 < \partial f^S(s_1)/\partial s_1 < -\theta$ for all $s_1 \in (0, a_1(S))$ and μ satisfies MON-PFT-2, then there exists $\theta_1 \in (0, \theta)$ such that $\mu_2(\boldsymbol{\tau}^{\boldsymbol{\theta_1},\mathbf{1}}(S)) > \mu_2(S)$.

(ii) If $(f^S)^{-1}$ satisfies $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2 < -\theta$ for all $s_2 \in (0, a_2(S))$ and μ satisfies MON-PFT-1, then there exists $\theta_2 \in (0, \theta)$ such that $\mu_1(\boldsymbol{\tau}^{\boldsymbol{\theta_2},\mathbf{2}}(S)) > \mu_1(S)$.

Proof. Let μ be any bargaining rule that satisfies SIR and WPO. Pick any $S \in \Sigma_0^2$ and any $\theta \in (0, 1)$. We will first prove part (i). Notice that if f^S satisfies $-1 < \partial f^S(s_1)/\partial s_1 < -\theta$ for all $s_1 \in (0, a_1(S))$, then Lemma 1 ensures that $\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{1}}(S) \in \Sigma_0^2$ and also $PO(\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{1}}(S))$ lies above $PO(S)$ at all $s \in S$ such that $s_1 \in (0, (1-\theta)a_1(S)]$. Clearly, $\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{1}}(S)$ contains $\mu(S)$ if θ is sufficiently close to 0. Thus, there exists $\theta_1 \in (0, \theta)$ such that $\boldsymbol{\tau}^{\boldsymbol{\theta_1},\mathbf{1}}(S)$ is a partially favorable twist for player 2. If μ satisfies MON-PFT-2, then $\mu_2(\boldsymbol{\tau}^{\boldsymbol{\theta_1},\mathbf{1}}(S)) \geq \mu_2(S)$. Moreover, since μ satisfies SIR and WPO, and $PO(\boldsymbol{\tau}^{\boldsymbol{\theta_1},\mathbf{1}}(S))$ lies above $PO(S)$ at all $s \in S$ such that $s_1 \in (0, (1-\theta_1)a_1(S)]$, we must have $\mu_2(\boldsymbol{\tau}^{\boldsymbol{\theta_1},\mathbf{1}}(S)) > \mu_2(S)$. The proof of part (ii) is similar. If $(f^S)^{-1}$ satisfies $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2 < -\theta$ for all $s_1 \in (0, a_2(S))$, then Lemma 2 ensures that $\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{2}}(S) \in \Sigma_0^2$ and also $PO(\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{2}}(S))$ lies above $PO(S)$ at all $s \in S$ such that $s_2 \in (0, (1-\theta)a_2(S)]$. Clearly, $\boldsymbol{\tau}^{\boldsymbol{\theta},\mathbf{2}}(S)$ contains $\mu(S)$ if θ is sufficiently close to 0. Thus, there exists $\theta_2 \in (0, \theta)$ such that $\boldsymbol{\tau}^{\boldsymbol{\theta_2},\mathbf{2}}(S)$ is a partially favorable twist for player 1. If μ satisfies MON-PFT-1, then $\mu_1(\boldsymbol{\tau}^{\boldsymbol{\theta_2},\mathbf{2}}(S)) \geq \mu_1(S)$. Moreover, since μ satisfies SIR and WPO, and $PO(\boldsymbol{\tau}^{\boldsymbol{\theta_2},\mathbf{2}}(S))$ lies above $PO(S)$ at all $s \in S$ such that $s_2 \in (0, (1-\theta_2)a_2(S)]$, we must have $\mu_1(\boldsymbol{\tau}^{\boldsymbol{\theta_2},\mathbf{2}}(S)) > \mu_1(S)$. ■

Proposition 1 shows that for any player $i = 1, 2$ and any bargaining rule that satisfies SIR, WPO, and MON-PFT-j with $j \neq i$, there always exists some pre-donation from player i to player j that is beneficial for player j, and thus should not be rejected or reversed, provided that the $PO(S)$ is sufficiently, but not extremely, flat in the (s_i, s_j) plane. The class of bargaining rules characterized in Proposition 1 admit, for example, Nash and Kalai-Smorodinsky rules as well as the α-proportional rule with any $\alpha > 0$. Below, we show that if any bargaining rule satisfies DOM-T in addition to the axioms stated in Proposition 1, then pre-donation from any player to the other player always yields Pareto welfare gains.

Proposition 2. *Let μ be any bargaining rule that satisfies SIR, WPO, and DOM-T. Pick any $S \in \Sigma_0^2$ and any $\theta \in (0, 1)$.*

(i) If f^S satisfies $-1 < \partial f^S(s_1)/\partial s_1 < -\theta$ for all $s_1 \in (0, a_1(S))$ and μ satisfies MON-PFT-2, then there exists $\theta_1 \in (0, \theta)$ such that $\mu_i(\boldsymbol{\tau}^{\boldsymbol{\theta_1,1}}(S)) > \mu_i(S)$ for each $i = 1, 2$.

(ii) If $(f^S)^{-1}$ satisfies $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2 < -\theta$ for all $s_2 \in (0, a_2(S))$ and μ satisfies MON-PFT-1, then there exists $\theta_2 \in (0, \theta)$ such that $\mu_i(\boldsymbol{\tau}^{\boldsymbol{\theta_2,2}}(S)) > \mu_i(S)$ for each $i = 1, 2$.

Proof. Directly follows from Proposition 1 and the fact that given any bargaining rule that satisfies DOM-T and any pair of bargaining problems $S, T \in \Sigma_0^2$ where T is a twist of S, we have $\mu_1(T) \geq \mu_1(S)$ if and only if $\mu_2(T) \geq \mu_2(S)$. ■

All axioms in Proposition 2 are satisfied by the class of proportional rules, P^α for any $\alpha > 0$. For this class of rules, we can also calculate the optimal pre-donation of each player when he/she maximizes its own payoff. Formally, we say that a pre-donation $\boldsymbol{\tau}^{\boldsymbol{\theta_i^*,i}}$ by player i is optimal for this player under the rule P^α for any $\alpha > 0$ if $\theta_i^* = \text{argmax}_{\theta\in[0,1)} P_i^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,i}}(S))$.

Proposition 3. *Let $S \in \Sigma_0^2$ be such that $f^S(a_1(S)) = 0$ and $-1 < \partial f^S(s_1)/\partial s_1 < 0$ for all $s_1 \in (0, a_1(S))$. Then, given any α-proportional rule with $\alpha > 0$, player 1's optimal pre-donation is $\boldsymbol{\tau}^{\boldsymbol{\theta_1^*,1}}$ where $\theta_1^* = \alpha/(1 + \alpha)$.*

Proof. Let $S \in \Sigma_0^2$ be such that $f^S(a_1(S)) = 0$ and $-1 < \partial f^S(s_1)/\partial s_1 < 0$ for all $s_1 \in (0, a_1(S))$ Then, by Lemma 1, for any $\theta \in (0, 1)$ it is true that $\boldsymbol{\tau}^{\boldsymbol{\theta,1}} \in \Sigma_0^2$, and also $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S) \supsetneq S \cap ([0, (1-\theta)a_1(S)] \times \mathbb{R}_+)$ and $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S)) \cap PO(S) = \{(0, a_2(S))\}$. The problem of player 1 is $\max_{\theta\in(0,1)} P_1^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,1}}(S))$. The solution θ_1^* implies $P_1^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_1^*,1}}(S)) = (1-\theta_1^*)a_1(S)$ and $P_2^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_1^*,1}}(S)) = f^S(a_1(S)) + \theta_1^* a_1(S)$. Then, using $f^S(a_1(S)) = 0$ and $P_2^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_1^*,1}}(S))/P_1^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_1^*,1}}(S)) = \alpha$, we obtain $\theta_1^*/(1 - \theta_1^*) = \alpha$, implying $\theta_1^* = \alpha/(1 + \alpha)$, which is always in $(0, 1)$.

Proposition 4. *Let $S \in \Sigma_0^2$ be such that $(f^S)^{-1}(a_2(S)) = 0$ and $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2$ < 0 for all $s_2 \in (0, a_2(S))$. Then, given any α-proportional rule where $\alpha > 0$, it is true* *that player 2's optimal pre-donation is* $\boldsymbol{\tau}^{\boldsymbol{\theta_2^*,2}}$ *where* $\theta_2^* = 1/(1+\alpha)$.

Proof. Let $S \in \Sigma_0^2$ be such that $(f^S)^{-1}(a_2(S)) = 0$ and $-1 < \partial(f^S)^{-1}(s_2)/\partial s_2 < 0$ for all $s_2 \in (0, a_2(S))$. Then, by Lemma 2, for any $\theta \in (0,1)$ it is true that $\boldsymbol{\tau}^{\boldsymbol{\theta,2}} \in \Sigma_0^2$, and also $\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S) \supsetneq S \cap (\mathbb{R}_+ \times [0, (1-\theta)a_2(S)])$ and $PO(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S)) \cap PO(S) = \{(a_1(S), 0)\}$. The problem of player 2 is $\max_{\theta\in(0,1)} P_2^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta,2}}(S))$. The solution θ_2^* implies $P_2^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_2^*,2}}(S)) = (1-\theta_2^*)a_2(S)$ and $P_1^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_2^*,2}}(S)) = (f^S)^{-1}(a_2(S)) + \theta_2^* a_2(S)$. Then, using $(f^S)^{-1}(a_2(S)) = 0$ and $P_2^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_2^*,2}}(S))/P_1^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_2^*,2}}(S)) = \alpha$, we obtain $(1-\theta_2^*)/\theta_2^* = \alpha$, implying $\theta_2^* = 1/(1+\alpha)$, which is always in $(0,1)$.

Propositions 3 and 4 imply that if the assumptions in the propositions are satisfied, then given any bargaining problem $S \in \Sigma_0^2$ and any proportional bargaining rule P^α where $\alpha \in (0,1)$, via an optimal pre-donation plan player 1 will always secure a bargaining utility of $a_1(S)\,\alpha/(1+\alpha)$ whereas player 2 will always secure a bargaining utility of $a_2(S)/(1+\alpha)$. Moreover, since P^α satisfies SIR, WPO, and DOM-T, we also know that for any $i = 1, 2$, it is true that player $j \neq i$ will also strictly benefit from player i's optimal pre-donation, i.e., $P_j^\alpha(\underline{\boldsymbol{\tau}}^{\boldsymbol{\theta_i^*,i}}(S)) > P_j^\alpha(S)$.

4 Conclusions

In this paper, we showed that in two-player bargaining problems a unilateral pre-donation from one of the players to the other always yields Pareto welfare gains if (i) the Pareto frontier of the bargaining problem is sufficiently, but not extremely, flat from the angle of the player who makes the unilateral pre-donation and (ii) the bargaining rule satisfies several axioms, including Strong Individual Rationality, Weak Pareto Optimality along with some monotonicity and dominance axioms defined on the set of twisted bargaining problems created by pre-donation. As these axiomatic conditions are satisfied by the class of proportional rules (introduced by Kalai, 1977), we also computed the optimal pre-donation of each player under these rules, and showed that each player who optimally pre-donates always secures a constant share of her ideal utility and the player who receives pre-donation strictly benefits from it.

We should note that pre-donation from one player to another, based on a multiplicative factor for the donator (i.e. $(1-\theta)$), leads to a balanced transformation of the set of payoffs before bargaining takes place. Another way to define such a balanced transformation can be via fixed lump-sum transfers. However, a fixed amount of lump-sum transfer cannot be feasible for the donator (transferer) at some payoff vectors where he/she is entitled to an extremely small payoff. Moreover, since the rate of transformation of the bargaining set is constant under lump-sum transfers, the (concave) Pareto frontier of the bargaining set can move outward (at some range of payoff vectors), leading to potential Pareto improvements, only if the transferer and the player with the highest ideal payoff are the same. In the case of pre-donation, however, a recent work of Saglam (2022b) shows that the Pareto frontier of the bargaining set can be moved outward by all players independently of their ideal payoffs. Thus, transformations of the bargaining problem via pre-donation present more possibilities for Pareto improvements than transformations via lump-sum transfers.

Future research may investigate conditions under which multilateral pre-donations occur in the equilibrium of two-stage strategic games where the players, after choosing in the first stage their pre-donations simultaneously or sequentially, can cooperatively calculate in the second stage the solution to their modified bargaining problem.

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