Network Perception in Network Games

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Abstract

People form cognitive maps about their networks from the information they have −mental representations of who is connected with whom in the network they are embedded in (Krackhardt, 1987). The aim of this paper is twofold. First, we develop a model of how people form mental representations about the network from the (incomplete) information they have. We relate them with notions of equivalence among nodes and identify a cognitive bias towards asymmetric network structures. We then explore the incidence that players' network perception has on their equilibrium behavior and payoffs in the induced Bayesian Games. A general condition for equilibrium existence under different setups of incomplete network information is derived. Such a condition uncovers the relevance of the order of the automorphism group of the cognitive networks as a main driver of behavior and welfare.

Keywords: Network Games, Graphical Games, Network Cognition, Incomplete Information, Structural Equivalence, Automorphic Equivalence
JEL: A14, D85, J60, J30.

1 Introduction

In many contexts, people act on the basis of others' choices. When deciding whether to get a vaccine individuals are influenced by the behavior of their friends and acquaintances. The same applies to decisions such as voting, starting a business, going on strike or engaging in criminal activities. Empirical evidence on these issues is vast (see Bramoulé et al. 2020 for a survey), and has motivated the development of theoretical models where the mutual influence that people exert on each other depends on the network topology (Bala and Goyal, 1998; Jackson and Yariv, 2007; Galeotti et al. 2010; Bramoullé et al. 2014; Bourlès et al. 2017, 2021). In such models, interactions among agents is represented through an adjacency matrix or graph, and the major interest is to characterize the impact that specific features of the social structure have on individuals' choices and welfare.

Attempts to study these issues have been approached in two different ways. One approach presumes that individuals have complete information about the network they are embedded in (e.g. Goyal and Moraga-González, 2001; Ballester et al. 2006; Bramoullé and Kranton, 2007, Bramoullé et al. 2014).[1] This is rather a strong assumption, which may not hold in reality, as people usually have limited information about the network they integrate. Moreover, even in contexts in which complete information is available, people exhibit cognitive limitations when encoding and remembering the network accurately (Dessi et al. 2016, Brashears, 2013; Brashears and Quintane, 2015; Brashears et al. 2016). Although this approach has allowed to derive interesting and novel results,[2] it presents an important drawback: even when one focuses on a particular network, a wide range of equilibrium outcomes are possible. Drawing conclusions about the impact of each specific network feature on behavior is therefore difficult. A second approach assumes that people do not have complete information about the network they are part of, but their information confines to a specific aspect of the network architecture (Jackson and Yariv, 2007; Galeotti et al. 2010; Feri and Pin, 2020;

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[1] This is the approach in the main bulk of the literature on network games.

[2] Ballester et al. (2006) for example, show that the equilibrium actions of players are proportional to their Bonacich centrality, while Bramoullé et al. (2014) prove that equilibrium outcomes depend on the lowest eigenvalue of the network matrix.

Ruiz-Palazuelos, 2021). Galeotti et al. (2010), for example, consider a setup where agents do not know the identity of the agents they are going to interact with. Players' network information limits to the number of connections they have (their degree) and the degree distribution of the population. Under these conditions, every symmetric equilibrium is proved to be monotone non-decreasing (non-increasing) in players' degrees under strategic complements (substitutes) when nodes have degrees either with independent probabilities or with probabilities that are positively (negatively) correlated.[3] Such a result evinces a fundamental premise of many seminal works: social connections create personal advantages.[4]

The information setup of Galeotti et al. (2010) applies to situations where the unique network feature that affects people's decisions is the number of individuals they expect to interact with, from a relatively unbounded population (e.g. a country). This applies to decisions such as learning a language or getting vaccinated. A natural way of modeling this type of situations (where the network influencing people's behavior is typically very large) is to identify players' beliefs about the network with a probability degree distribution, setting aside beliefs about other network aspects. This assumption is reasonable in environments that exhibit the following features: (i) people have a good sense of their volume of future interactions and (ii) they have neither information about the identity of their contacts nor about any other aspect of the network topology.

Notwithstanding this, in many real-life situations people have incomplete network information but they know how popular their opponents might be, whether they know each other, their own proclivity to occupy central positions in the network, etc. (Killduf et al. 2008; Simpson et al. 2011; Brands et al. 2013; Smith et al. 2020). Such more detailed information in turn allows network members to learn further properties of their local and global networks. Indeed, research in social psychology documents that people entering into a social group tend to form a cognitive map of the existing network−a mental picture of connections capturing who is connected with whom in the group (Kilduff and Tsai, 2003). For example, newcomers to a firm may have a certain idea about the network integrated by company members: they may notice the number of people working in each department, the connections among the company sections, the degree of hierarchy of the firm, who shares office with whom, who eats with whom, etc. All this information provides signals enabling to form representations about the firm network. This has profound implications for network analysis, as it is the cognitive network(s) (the social structure(s) that exists in people's minds), rather than the network they are actually embedded in, what drives people's choices. This raises novel research questions:

- From the perspective of network perception, how can we model people's beliefs about the network in this kind of contexts? What can people deduce about the underlying social structure from their incomplete network information? Which network features affect how people view their social networks?
- From the game-theoretical perspective, what can be said about equilibrium behavior and the aggregate welfare in the presence of network cognitive maps? What is the relation between degree, equilibrium actions and payoffs in this type of scenarios? Can we link the depth of network information to the multiplicity of equilibria?

This paper aims to answer these questions. We propose a theoretical model of how people form mental representations about the network from the (incomplete) information they have. We present different information settings which differ in the depth of information that agents have about the network they are embedded in, ranging from an extreme case of incomplete network information (a setup similar to that in Galeotti et al. 2010) to the complete information scenario. We analyse how, and to what extent, agents can infer different aspects of the interaction structure in function of the information that they possess even if they are not directly informed about these aspects. To that aim, we show how exploring the role of information requires to distinguish two types of information: information about links (related to connectivity) and information about nodes (relevant to learn who is connected with whom). Building up on this, we develop a theoretical framework linking incomplete network information to agents' beliefs about the properties of the underlying network architecture, and we use it to analyse the equilibrium behavior and payoffs in the induced Bayesian games played on networks.

[3] Games of strategic complements (substitutes) represent situations where individuals' incentives to take actions increase (decrease) with the number of people playing such actions. Examples of strategic complements include the decision of using a software, supporting a political party or learning a language. Strategic substitutes encompass decisions such as starting a new business in a geographical area, paying for a public good or experimenting with a novel technology.

[4] The monotonicity property of equilibria implies that, in both types of games, the expected payoffs of highly (poorly) connected agents are higher (lower).

We first show that even a minimal knowledge of the social environment enables people to learn both the network geometries that are compatible with their information as well as their probability distribution, which in turn allows them to calculate the probability distribution of any network feature (e.g. the clustering coefficient, the betweeness centrality of any node, the number of components, etc.). We provide conditions under which all agents have identical perceptions of the underlying network architecture: they have identical beliefs about the feasible network geometries and their probabilities. We show that, once we depart from such conditions, agents may have different beliefs about the set of feasible geometries and/or about its probability distribution. Furthermore, individuals with complete and incomplete network knowledge can coexist under any information setup−even when their level of network information is limited and equal across agents−and this may have important consequences on behavior.

Our model reveals a clear-cut result: when several networks are compatible with agents' information (i.e. different networks in individuals' minds can correspond to the network they are actually embedded in), the probability distribution of these feasible networks depends on the order of their automorphism group, a network property that captures the degree of similarity among network members in terms of how they are connected to others.[5] This result holds under any scenario in which individuals' information enables them to form mental representations about the underlying social network. We show that the probability that an agent assigns to each feasible network geometry increases monotonically as the order of its automorphism group decreases. As an implication, agents believe more likely to occupy positions in networks with more asymmetric structures (say, in hierarchical structures integrated by people occupying heterogeneous network positions) rather than in networks where individuals are connected to others in a more egalitarian way.

These results have implications for structural theory. In effect, the existence of an automophism between two nodes has been recognized as a criterion to determine whether individuals play the same role or have the same status in the social system (Lorrain and White, 1971; Sailer, 1978; Winship and Mandel, 1983; Burt, 1987; Faust, 1988; Müller and Brandes, 2022). In particular, the notions of *structural equivalence* and *automorphic equivalence*−both based on the existence of automorphisms among nodes− identify individuals that are related to others in a similar way (Wasserman and Faust, 1994; Easley and Kleinberg, 2010; Everett and Borgatti, 1992). Both concepts are conceived as a formalization of the notion of social position, and have been used to identify individuals playing the same role in the society.[6] Our model uncovers the relevance of these equivalence notions for network perception. By characterizing the specific incidence that these measures have in the order of the automorphism group of the feasible networks, we identify a perception bias towards less homogeneous social structures. Specifically, the probabilistic weight that an individual assigns to each feasible network increases as the degree of similarity among their members according to these notions decreases.

In the second part, we explore the implications of our network perception model on Bayesian games of strategic substitutes and strategic complements played on networks. For both games, we derive a condition that guarantees the existence of an equilibrium in different setups of incomplete network information. The condition shows that the degree of substitutability between players' actions and the ones of their feasible neighbors depends on the automorphism group of the cognitive networks to which such neighbors belong. The smaller the order of the automorphism group of a feasible network, the more likely the network is to correspond to the network in which players are actually embedded in, and the more influenced players are by their neighbors' actions in that network. This result has a twofold implication. First, regardless on the topology of the network in which players are actually immersed in, equilibrium actions are mostly influenced by the topology of the networks that are more asymmetric, among all those that are compatible with players' information. Second, when the returns of playing a certain equilibrium action differ across networks and equilibrium profiles, the equilibria in which such returns are higher in more asymmetric structures are sustainable for a wider range of parameter values. This suggest a positive relation between the degree of asymmetry of the network and the equilibrium welfare it yields. Along, we provide a sufficient condition for a network to be efficient−to be a network under which equilibrium welfare is at least as high as in any other network compatible with players' network information−. The condition suggests that, under certain conditions, the only networks that are efficient are the more asymmetric ones.

As for the impact of degree on equilibrium actions and payoffs, no general conclusions can be extracted.

[5] We provide a formal definition in Section 1.

[6] The idea is that what defines the role of a person in the social system, say, the role of a professor in a university, is the way (s)he is connected to students, professors, administrative staff, and people playing other roles. These equivalence notions have also been applied to identify sectors of an economy, as firms producing similar products have similar relations to providers, consumers, and other actors in the economy (see Burt 1977, 1978a. 1978b).

Equilibrium actions do not vary monotonically with players' degree as in Galeotti et al. (2010), but multiple equilibria with different patterns can exist, even when players' network information is restricted to their own degree and the degree distribution of the network. These results hold under all our incomplete information setups, but they are surprising specially when players have asymmetric beliefs about the underlying network. We show that people are able to reach different equilibria even in these contexts.[7]

Last, we explore the effects of manipulating players' network information on the equilibrium structure. Subtle variations in players' network information may induce abrupt variations in players' cognitive maps and equilibrium behavior. As a result, the equilibrium structure may change dramatically even if the network information varies only slightly, but the direction in which it changes is network-specific. Increasing players' network information has thus non-monotonic effects on the structure and number of equilibria.

The main innovation of this work is the development of a theoretical framework of network perception and its application to the study of network games. The proposed framework complements the large literature on network cognition in social psychology and sociology (Krackhardt; 1987, Carley; 1986; Michaelson and Contractor, 1992; Freeman, 1992; Kumbasar et al. 1994; Casciaro, 1998; Johnson and Orbach; 2002; Janicik and Larrick, 2005; Smith, 2020) and recently in economics (Dessi et al. 2016), by characterizing formally the formation of people's cognitive network maps. In this sense, we uncover a relation between some canonical notions of similarity among nodes (automorphical equivalence and structural equivalence) and network perception.[8] Although such notions have been fundamental in structural theory, they have not been theoretically related, to the best of our knowledge, to network perception.[9] The uncovered relation between these notions and agents' network perception reflects a cognitive bias towards asymmetric structures that may imply a misperception of agents' social environments, since empirical evidence shows that a certain degree of symmetry is ubiquitous in real-life social networks (MacArthur et al. 2008; Ball and Geyer-Schulz, 2018a, 2018b).[10]

As for network games, an important contribution of this work is to point out the cognitive dimensions of social networks as a crucial element for the study of strategic interactions. Such a dimension has been overlooked in the literature on network games. This paper provides a first step toward bridging the two main assumptions regarding network knowledge: extremely limited information (Galeotti, et al. 2010; Jackson and Yariv, 2005; 2007; Sundararajan, 2008) and complete information (Goyal and Moraga-González, 2001; Ballester et al. 2006; Bramoullé and Karton, 2007, 2014). We characterize the intermediate information setups and exploit their role in network interactions. The strengths of our model for the analysis of network games are two. First, a fundamental critique of the analysis of games under incomplete information is that the equilibrium achieved strongly depends on the information assumptions that are made (Weinstein and Yildiz, 2007). While this critique applies generally to all incomplete information games, we show that it is particularly relevant for those played on networks, given (i) the wide range of network elements that may shape players' behavior and welfare and (ii) the variety of network characteristics that players can learn from the knowledge of particular network aspects, as a consequence of the intrinsic interdependency among different network features. The proposed model overcomes these obstacles by providing equilibrium predictions that apply for different information setups. It further shows that the introduction of incomplete information is not panacea to solve the equilibrium selection problem. Second, the interest of the study of network games−and what differentiates them from other types of strategic interactions− is the fact that the way in which players influence each other is determined by the topology of the network. Yet, the impact of the network in agents' behavior can be blurred when players' network information is limited to a reduced set of network elements. Our predictions rely on a belief structure that includes a variety of aspects of the network topology. This allows to study the strategic interactions of people in realistic contexts of incomplete network information, while preserving the relevance of the role of the network in such interactions.

Last, while most network applications in economics focus on connectivity, centrality, and network density (see Jackson, et al. 2017), we uncover the role of one particular feature of the network structure−the net-

[7]Although these findings may be surprising, they are consistent with empirical evidence. Different studies show that small pieces of network information allow people to meet targets that could be expected to require a full knowledge of the network structure to be reached (see Milgram, 1967; Travers and Milgram, 1977, Dodds et al. 2003 or Backstrom et al. 2012).

[8]These notions are canonical in social network analysis (see e.g. Everett, 1985; Hanneman and Riddle, 2003; Newman, 2004, Leicht, et al. 2006; Casse et al. 2013; Jin et al. 2014; Prota and Doreian, 2016). Although they were initially studied in sociology (Boorman and White, 1976; White et al. 1976; Sailer, 1978; Doreian, 1988; Winship, 1988; Burt, 1976, 1990; Borgatti and Everett, 1992; Doreian et al. 2005), their study has extended to more general domains (see e.g. Rossi et al. 2014)

[9]Some empirical papers study how perceived similarity relates to these notions of equivalence among nodes (e.g. Michaelson and Contractor, 1992). Yet, their context is different to ours.

[10]Wang et al. (2009) also find a certain degree of symmetry in their analysis of the world trade network. On the contrary, almost all random graphs are asymmetric (Erdös-Renyi, 1963).

work automorphism group−in agents' perception, behavior, and welfare. This network property is receiving increasing attention in mathematics and physics (MacArthur and Anderson, 2006; Xiao, et al. 2008a, 2008b, 2008c; Wang et a. 2009; Dehmer et al. 2020), given its impact on the dynamics of processes that take place on networks (Golubitsky and Stewart, 2003), on the network's eigenvalue spectrum (Cvetkovíc et al. 1979) or because of its utility to simplify the network topology by collapsing redundant information (Xiao et al 2008b), among other applications.[11] This paper points out the importance of this network property on behavior and welfare, by showing that behavior of people may be shaped to a greater extent by the network characteristics associated to asymmetric structures in incomplete information contexts as a consequence of the greater probabilistic weight that they assign to these network architectures.

The paper is organized as follows. Section 2 presents some background definitions. Section 3 presents different setups of network information. Section 4 provides the results on network perception, which are the bases of our theoretical framework. Results on network games are presented in Section 5 and Section 6. Section 7 concludes.

2 Background Definitions

Let $g = (N, E)$ be a social network characterized by a set of nodes $N = \{1, .., n\}$ and a set of edges or links E between them. Each node in g represents one agent and there are $n = \big|N\big|$ agents in the network. Let g_{ij} denote the link between $i \in N$ and $j \in N$; $g_{ij} = 1$ if individuals $i \in N$ and $j \in N$ are directly linked in g and $g_{ij} = 0$ otherwise, with $g_{ij} = g_{ji}$, $\forall i, j \in N$. The network is represented by a n symmetric adjacency matrix $A = (g_{ij})_{i,j \in N}$, with $g_{ii} = 0$. A path between $i \in N$ and $j \in N$ is a sequence of links $i_1 i_2, ..., , i_{K-1}, i_K$ such that $g_{i_k} g_{i_{k+1}} = 1$ for each $k \in \{1, 2, ..., K - 1\}$, and such that each node in the sequence is distinct. The distance between i and j, denoted d_{ij}, is the length of the shortest path between them. A network is connected if there is a path connecting any two nodes.

The neighborhood of node i is the set of agents directly connected to i, $N_i(g) = \{j \in N : g_{ij} = 1\}$. The degree of node i is the cardinality of $N_i(g)$, $k_i(g) = |N_i(g)|$. Although both characteristics are similar, the level of network information that they capture is different: the degree reflects with how many other agents one interacts without providing any information about their identities, while the neighborhood reflects both their number and their identities. Considering one or another feature has important consequences in certain parts of our analysis. The set of i's second-order neighbors is $N_i^2(g) = \{s \in N : g_{ij} g_{js} = 1 \text{ for some } j \in N, i \neq s\}$.

The degree distribution of network g, denoted $\mathcal{F}_g(k)$, specifies, for all $k \in \{0, 1, ..., n - 1\}$, the fraction of nodes that have degree k in this network:[12]

$$\mathcal{F}_g(k) = \frac{1}{n} \left|\{i \in N : k_i(g) = k\}\right|$$

The degree counts in network g, denoted $D_g(k) = n * \mathcal{F}_g(k)$, are the numbers of nodes that have degree k in this network.

Let $\mathtt{k}_{\mathtt{N_i(g)}} = (k_1, k_2, ..., k_{k_i})$ be the vector of degrees of all agents in $N_i(g)$, where k_j is the degree of neighbor $j \in N_i(g)$ $(j = 1, 2, ..., k_i)$ and $k_j \geq k_{j+1}$. The joint degree distribution of g, denoted $\mathcal{F}_g(k, (k_1, k_2, ..., k_k)$, is the proportion of nodes in g that have degree k and neighbors with degrees given by the vector $k_{N_i(g)} = (k_1, k_2, ..., k_k)$,

$$\mathcal{F}_g\Big(k, (k_1, k_2, ...k_k)\Big) = \frac{1}{n} \left|\Big\{i \in N : \Big(k_i(g), \mathtt{k}_{\mathtt{N_i(g)}}\Big) = \Big(k, (k_1, k_2, ..., k_k)\Big)\Big\}\right|$$

for all $k \in \{0, 1, ..., n - 1\}$.

Analogously, we write $\mathcal{F}_g(t)$ to denote generally the frequency distribution of a network measure, where t is a specific value of such a measure and $t_i(g)$ the value that it has for node $i \in N$:

$$\mathcal{F}_g(t) = \frac{1}{n} \left|\{i \in N : t_i(g) = t\}\right|$$

for all $t \in \mathcal{T}$, where $\mathcal{T}$ is the set of feasible values of this measure.

[11] See also Soicher, (2004) and Kocay, (2007), for its application to simplify the computational complexity of network algorithms.

[12] In contexts of random networks, $\mathcal{F}_g(k)$ is naturally interpreted as the probability that a randomly selected node has degree k (Vega-Redondo, 2007), while here it is the distribution of degree frequencies in the network.

The geometry of a network correspond to its structure: the network architecture created by its edges. Two networks have the same geometry if and only if they are isomorphic: there exists a bijection (an isomorphism) $f : N \to N'$, such that $ij \in E$ if and only if $f(i)f(j) \in E'$ (see Borgatti and Everett, 1992). Thus, f just relabels the nodes, but their network structure is the same. For example, the four networks in Figure 1 are isomorphic. We use the symbol $\cong$ to denote an isomorphism; $g \cong g'$ means that g and g' are isomorphic.

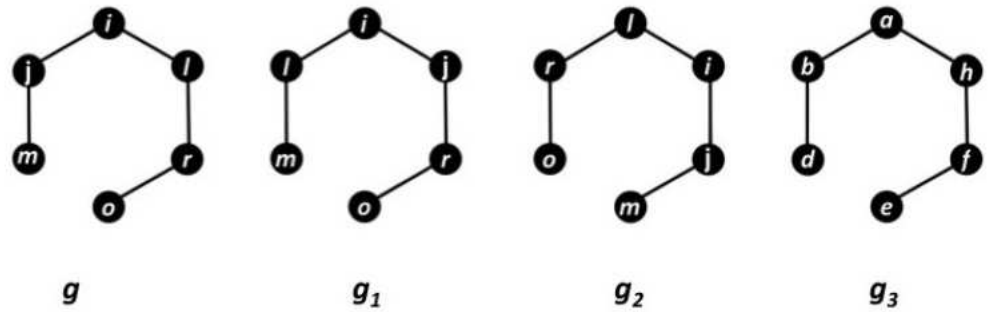

Figure 1: A set of isomorphic networks

Network $g = (N, E)$ is different from network $g' = (N', E')$ if and only if their respective adjacency matrices differ, i.e. if $g_{ij} \neq g'_{ij}$, for at least one $ij \in E \cup E'$. The adjacency matrix of a network depends on two aspects: (i) the network geometry and (ii) the distribution of labels among the nodes (how agents are distributed within the network). Hence, networks g and g' can be different if either (i) or (ii) (or both) differs in the two networks. For example, network g and g_1 in Figure 1 are isomorphic. However, since agents are distributed differently in g and in g_1 both networks are distinct, as reflected in their respective adjacency matrices:

$$g = \begin{pmatrix} g_{ii} & g_{ij} & g_{il} & g_{im} & g_{io} & g_{ir} \\ g_{ji} & g_{jj} & g_{jl} & g_{jm} & g_{jo} & g_{jr} \\ g_{li} & g_{lj} & g_{ll} & g_{lm} & g_{lo} & g_{lr} \\ g_{mi} & g_{mj} & g_{ml} & g_{mm} & g_{mo} & g_{mr} \\ g_{oi} & g_{oj} & g_{ol} & g_{om} & g_{oo} & g_{or} \\ g_{ri} & g_{rj} & g_{rl} & g_{rm} & g_{ro} & g_{rr} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \neq g_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Similarly, g_3 is different from the other three networks in Figure 1, since it is integrated by different agents. In our analysis, the only isomorphic networks to g that will play a role are those integrated by the agents in N.

An isomorphism f of a graph with itself that preserves the adjacency matrix is known as an *automorphism*, $f : N \to N$, where $ij \in E$ if and only if $f(i)f(j) \in E$. That is, an automorphism is a permutation of the labels of the nodes in g that results in a network $g' = g$. For example, $f : f(i) = l, f(j) = r, f(l) = i, f(m) = o, f(r) = j, f(o) = m$ is an automorphism of g, as it results in network $g_2 = g$ (see Figure 1). The set composed of all the automorphisms of g is the automorphism group of g, denoted $Aut(g)$ (Chartrand et al. 2010). Note that all graphs in $Aut(g)$ represent the same network, since they all have the same adjacency matrix. The order of the automorphism group of g is the number of elements in $Aut(g)$, $|Aut(g)| \geq 1$. It captures the degree of symmetry of the network: the greater $|Aut(g)|$, the more symmetric network g is (Xiao et al., 2008b).

Two nodes $i \in N$ and j are automorphically equivalent if they are identical in terms of all network measures (degree, second-order degree, centrality, number of cycles to which they belong, etc.). Namely, i and j are automorphically equivalent if and only if there exists an automorphism $f : N \to N$ such that $f(i) = l$. We write $i \equiv l$ to indicate that i and l are automorphically equivalent. In network g of Figure 1, $i \equiv l$, $j \equiv r$ and $m \equiv o$. Other pairs of agents are not automorphically equivalent, since their number of second-order neighbors is different. Two nodes occupy the same *position* in the network if and only if they are automorphically equivalent, regardless on whether the identity of their direct and indirect neighbors is distinct. We write $o_i(g)$ to refer to the position of i in network g. In network g in Figure 1 there are three different positions: $o_i(g) = o_l(g)$, $o_j(g) = o_r(g)$ and $o_m(g) = o_o(g)$.

An important property of automorphically equivalent nodes is that we can exchange their labels to form a new network that is identical to the original one (Friedkin and Johnsen, 1997). For example, starting from graph g in Figure 1, we can exchange the labels of i and l and relabel all other nodes to obtain network g_2 (in the same Figure) with the same adjacency matrix as g, $g_2 = g$. Notice that this is possible because $i \equiv l$. If we swap the labels of two nodes that are not automorphically equivalent, say, i and j in network g, we cannot obtain a network equal to g.

The orbit of node i is the set composed of all nodes that occupy the same position as i, $O_i(g) = \{l \in N : o_i(g) = o_l(g)\} = \{l \in N : l \equiv i\}$. In network g of Figure 1, $O_i(g) = O_l(g) = \{i, l\}, O_j(g) = O_r(g) = \{j, r\}, O_m(g) = O_o(g) = \{m, o\}$.

Structural equivalence is a particular form of automorphic equivalence. Node $i \in N$ is structurally equivalent to $l \in N$ if and only if both agents are connected to the same nodes, $N_i(g) \setminus \{l\} = N_l(g) \setminus \{i\}$.[13] We write $i \equiv_s l$ to indicate that i and l are structurally equivalent. Structural equivalence is more demanding than automorphic equivalence: it not only requires that the nodes occupy indistinguishable structural locations in the network, but also that the identities of the agents connected to them are the same. Thus, structural equivalent nodes must be automorphically equivalent, but the opposite is not true. For example, in network g in Figure 2, nodes s and m are structurally equivalent and therefore $s \equiv m$. However, there is no pair of structurally equivalent nodes in network g of Figure 1, despite the fact that $i \equiv l$, $j \equiv r$ and $m \equiv o$. The set of nodes that are structurally equivalent to i is denoted $S_i(g) = \{j \in N : j \equiv_s i\}$.

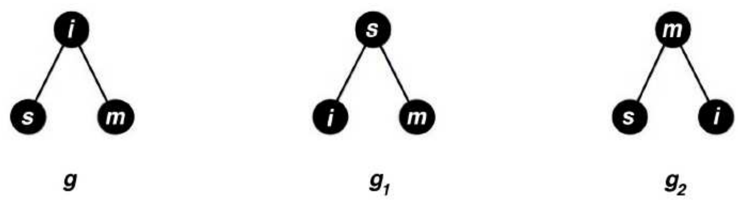

Figure 2: Networks in $G_{\mathcal{F}_g(k)}$ when $D_g(1) = 2$ and $D_g(2) = 1$

Both automorphically equivalence and structural equivalence are notions that identify actors who are connected in the same to other nodes in the network. These notions have been used to identify individuals that play the same *social role* in the social system (Faust, 1988; Borgatti and Everett, 1992). The underlying argument is that what defines the role that a person plays in a society, say, a CEO, is the distinctive set of links that people that are CEOs have to people that employees, directors, team leaders, administrative staff and so on, in the same way as employees are defined by their relationships to team leaders, providers, directors, CEOs and individuals playing other roles. According to this conception, people in the same orbit can be regarded as people playing the same role in the society, and thus the set of orbits reflects the degree of diversity of social roles in the network. Such a presence ultimately reflects in the order of the automorphism group of the network, as the following lemma highlights.

Lemma 1. *Let $g = (N, E)$ and $g' = (N', E')$ two networks such that $N = N'$. If $\left|O_i(g)\right| \leq \left|O_i(g')\right| \forall i \in N$ and for some $i \in N$ either (i) $\left|O_i(g)\right| < \left|O_i(g')\right|$ or (ii) $\left|S_i(g)\right| < \left|S_i(g')\right|$ or both, then $|Aut(g)| < |Aut(g')|$.*

Lemma 1 relates these equivalence notions with the order of the automorphism group of the network. By construction, the order of the automorphism group of a network increases as the number of automorphically equivalent nodes does, *ceteris paribus*. Yet, maintaining fixed the orbits in the network, $|Aut(g)|$ increases with the number of structurally equivalent nodes, as structural equivalence is more strict than automorphic equivalence. Consider for instance the networks in Figure 3. In $g_z \in \{g_2, g_3\}$ there are two different orbits of size 2, $O_r(g_z) = O_m(g_z) = \{r, m\}$ and $O_l(g_z) = O_o(g_z) = \{l, o\}$ and two orbits of size 1, $O_i(g_z) = \{i\}$ and $O_j(g_z) = \{j\}$. However, since $S_r(g_2) = S_m(g_2) = \{r, m\}$ and $S_l(g_2) = S_o(g_2) = \{l, o\}$, while $S_x(g_3) = \emptyset$ for $x \in \{i, j, r, m, l, o\}$ $|Aut(g_3)| = 2 < |Aut(g_2)| = 4$.[14]. Similarly, $|Aut(g_1)| = 2 < |Aut(g_2)| = 4$, as the orbit of

[13]According to the standard definition, i an l are structurally equivalent if and only if $N_i(g) = N_l(g)$ (Burt, 1976). Since this definition is too strict, different relaxations have been proposed (see e.g. Everett et al. 1990). Our relaxed definition increases the set of structurally equivalent nodes: while three nodes $\{i, j, k\}$ composing a network triangle are not structurally equivalent according to the standard definition, they are structurally equivalent according to ours. Observe that i and l with $N_i(g) = \{l\}$ and $N_l(g) = \{i\}$ are structurally equivalent, since $N_i(g) \setminus \{l\} = N_l(g) \setminus \{i\} = \emptyset$. However, neither i nor l are structurally equivalent to an isolated node m, since $N_i(g) \setminus \{m\} = \{l\} \neq N_m(g) \setminus \{i\} = \emptyset$, and $N_l(g) \setminus \{m\} = \{i\} \neq N_m(g) \setminus \{l\} = \emptyset$.

[14]In Section 4 (footnote 23), we explain the calculation of the order of the automorphism group of these networks.

each node is at least as high in g_2 as in g_1 and the number of structurally equivalent nodes to $r(m)$ is greater in g_2 than in g_1.[15]

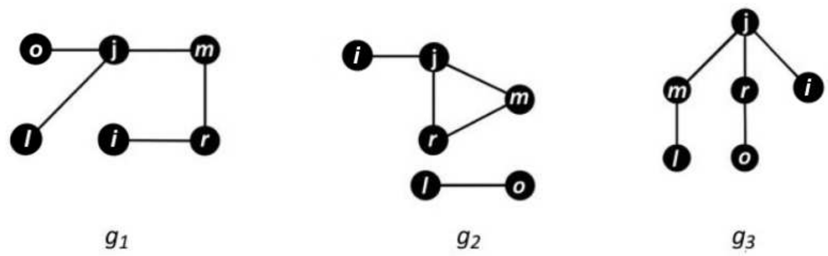

Figure 3: Networks g_1, g_2 and g_3

Let G be the set of all feasible networks integrated by the agents in N. The set $G_{\mathcal{F}_g(k)} \subseteq G$ is the subset of different networks in G with degree distribution $\mathcal{F}_g(k)$ and size n:

$$G_{\mathcal{F}_g(k)} = \{g \in G : |\{i \in N : k_i(g) = k\}| = D_g(k), \forall k\}$$

Suppose for example that $N = \{i, s, m\}$, $D_g(1) = 2$ and $D_g(2) = 1$. Then, $G_{\mathcal{F}_g(k)}$ is integrated by three networks in Figure 2, $G_{\mathcal{F}_g(k)} = \{g, g_1, g_2\}$. In this example, all networks in $G_{\mathcal{F}_g(k)}$ are isomorphic. However, $G_{\mathcal{F}_g(k)}$ can generally contain networks with different geometries.

The clustering coefficient of each $i \in N$, $C_i(g)$, is the proportion of agents in $N_i(g)$ that i are mutually linked (the proportion of triangles in i's neighborhood):

$$C_i(g) = \frac{\sum_{j \neq i; k \neq j; k \neq i} g_{ij} g_{ik} g_{jk}}{\sum_{j \neq i; k \neq j; k \neq i} g_{ik} g_{ij}}.$$

3 Network knowledge

Let $g = (N, E)$ be the network in which individuals are embedded in, and $I_i(g)$ the information set that each $i \in N$ has about network g. Consider a scenario where each $i \in N$ has two pieces of information about g: (i) a private information about a set of characteristics of i's position in g, denoted $t_i(g)$, and (ii) a common knowledge information about g as a whole, referred as $I^c(g)$. All agents have the same type of private information about their network position. However, the content of such an information may differ among individuals. For example, if $t_i(g) = k_i(g)$, then $t_j(g) = k_j(g)$ $\forall j \in N$. Nevertheless, it is possible that $t_i(g) \neq t_j(g)$. We further assume that $\{|E|, \theta, n\} \in I^c(g)$, where $\theta \in \{\emptyset, \theta\}$ is an information about the network formation process. However, agents do not have information about the identity of their direct and indirect neighbors. In our analysis, any information setup exhibiting these features is referred as Setting S, and $I_i(g) = \{t_i(g), I^c(g)\}$ $\forall i \in N$ under Setting S.

Setting S accomplishes to different information contexts. Below, we present three particular examples: Setting S(a), Setting S(b) and Setting S(c).

Setting S(a). The private information of each $i \in N$ is $t_i(g) = k_i(g)$ and $I^c(g) = \{\mathcal{F}_g(k), n, \theta\}$. That is, agents know their degree, the degree distribution of the network and its size, and may have some information about the mechanisms that drive the formation of links, if $\theta \neq \emptyset$.[16]. Setting S(a) is similar to that in Galeotti et al. (2010)[17] Such a setting applies, for example, to recently hired employees: they may know the number of people with whom they are going to interact in the company, but not necessarily who these people will be at the moment they are hired. Similarly, teachers may anticipate the number of students they will have in their classes, but not their identity at the beginning of the school year.

[15]Note that, although the set of orbits is greater in g_3 than in g_1, nodes in the same orbit are structurally equivalent in g_1 but not in g_3. Since structural equivalence is more strict than automorphical equivalence, $|Aut(g_1)| = |Aut(g_3)|$ in this example.

[16]The signal θ affects network perception as we explain in Section 4.

[17]The difference between our Setting S(a) and the setup in Galeotti et al. (2010) is that players know the distribution of degree frequencies in our setup, while in Galeotti et al. (2010) they know the probability degree distribution. Both assumptions lead to results on network perception that are essentially the same, as we explain later.

Figure 4(g) displays network g, while Figure 4(a) represents how i views the network under Setting S(a) conditioned on $I_i(g) = \{k_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], n, \theta\} = \{2, [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta\}$. The solid (dashed) lines in Figure 4(a) are the links in network g that are (not) observed by i. Note that all nodes but i are unlabelled in Figure 4(a) because i does not have information about the identity of these agents.

Setting S(b). Under Setting S(b), agents' private information corresponds to their degree and the degree of their neighbors, $t_i(g) = (k_i(g), \mathtt{k}_{\mathtt{N_i(g)}})$ $\forall i \in N$, while $I^c(g) = \{\mathcal{F}_g(k), n, \theta\}$. Suppose for instance that network g is the network in Figure 4(g). From $I_i(g) = \{k_i(g), (k_l, k_o), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], 6, \theta\} = \{2, (3,2), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta\}$, i knows that agents in $N_i(g)$ have degrees 3 and 2, but not their identities. Figures 4(b_1) and 4(b_2) represent the two possibilities in which links of i's neighbors can be disposed given $I_i(g)$. Since i does not know which of these two configurations is the true one, $I_i(g)$ is jointly represented by Figures 4(b_1) and 4(b_2). In Figure 4(b_2), the dashed line represents the link that is not directly observed by i.[18]

Setting S(b) applies for contexts where individuals may have information about the popularity of their future partners, but not about their identities. Think for instance about actors considering to take part of a movie. They may anticipate the popularity of other actors involved in the project, but not who these people will be when accepting the role. Likewise, organizers of a business event may anticipate the popularity of the people that are likely to attend, but not necessarily their identities at the time they organize it.

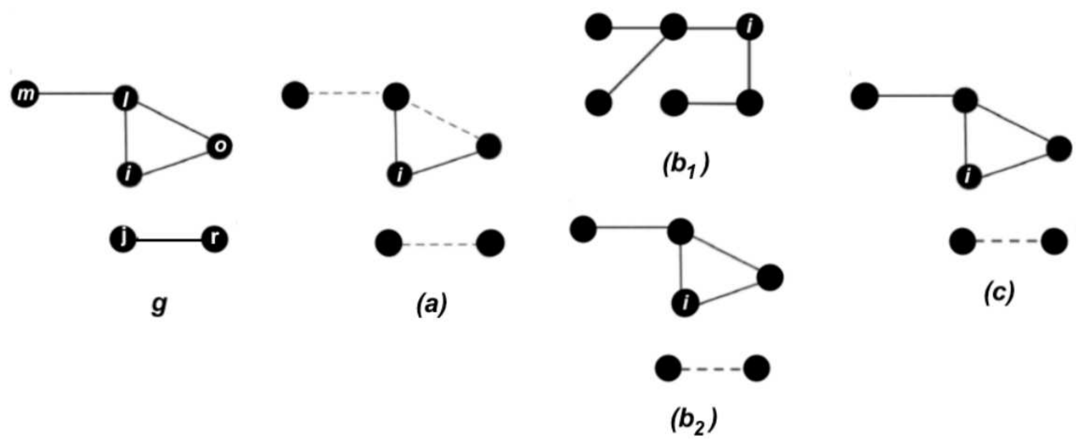

Figure 4: Network g and $I_i(g)$ under Setting $S(a)$, $S(b)$ and $S(c)$ Unlabelled nodes represent agents whose identity is unknown for i. Solid (dashed) lines represent links that are (not) directly observed by i.

Setting S(c). In this setup, $t_i(g) = \big(k_i(g), \mathtt{k}_{\mathtt{N_i(g)}}, C_i(g)\big)$ $\forall i \in N$ and $I^C = \{\mathcal{F}_g(k), n, \theta\}$. Consider for example network g in Figure 4(g). Figure 4(c) represents the information that i has about the network under Setting S(c), $I_i(g) = \{k_i(g), (k_l, k_o), C_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], 6, \theta\} = \{2, (3,2), 1, [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta\}$. Such an information structure applies for many environments. A prospective leader team of a firm may anticipate the popularity of the people that will integrate each of the teams, as well as the proportion of connections that are likely to form among their members. The same applies for a person that works as a coordinator of the company or as an intermediary among firms.

These setups are examples of information contexts exhibiting the characteristics of Setting S. Beyond these examples, there exist others. For example, in certain situations individuals may have information about their tendency to be close to others in the network, but not about the identity of their partners when choosing actions; or situations in which they know their proclivity to serve as bridges among others, but not the identity of their neighbors when deciding their behavior. To simplify the exposition, we present our network perception model on the basis of any information setup that present the characteristics of Setting S (i.e. under the assumption that agents do not have information about the identity of neighbors, neighbors' neighbors and so on). Such an approach enables us to obtain a simple analytical expression for many of our results, simplifying the exposition. In Section 7, we analyse scenarios where agents have information about the identity of their neighbors and neighbors' neighbors. As we show there, the implications of our network perception model maintain for such information setups.

[18] In a subsequent section, we show that i can learn such a link from $I_i(g)$ (i.e. she can deduce that the degree-one agents integrate a dyad), despite not being provided such an information directly.

4 Network Perception

4.1 Network beliefs

From $I_i(g)$, each $i \in N$ forms beliefs about the network (s)he is embedded in. Define $B_i(g) \subseteq G$ as the set of all networks that are compatible with $I_i(g)$ under Setting S, with $b_i(g) = |B_i(g)|$. Namely, $B_i(g)$ is the set of feasible networks in i's beliefs; each network in $B_i(g)$ could be network g according to the information of i. Depending on the information setup, $B_i(g)$ is different:

- Under Setting S(a), $B_i(g) = \{g' \in G_{\mathcal{F}_g(k)} : k_i(g') = k_i(g)\}$
- Under Setting S(b), $B_i(g) = \{g' \in G_{\mathcal{F}_g(k)} : k_i(g') = k_i(g), \, \mathtt{k}_{\mathtt{N_i(g')}} = \mathtt{k}_{\mathtt{N_i(g)}}\}$
- Under Setting S(c), $B_i(g) = \{g' \in G_{\mathcal{F}_g(k)} : k_i(g') = k_i(g), \, \mathtt{k}_{\mathtt{N_i(g')}} = \mathtt{k}_{\mathtt{N_i(g)}}, \, C_i(g') = C_i(g)\}$

Set $B_i(g)$ may differ across agents under any information setup. Furthermore, it is possible the coexistence of individuals with complete and incomplete network knowledge. This can occur even within one information setup (where the level of network information of all network members is the same), since the content of the the information of different agents can be distinct. The following example illustrates.

Example 1. Consider network g in Figure 2. Given $\mathcal{F}_g(1) = \frac{2}{3}$, $\mathcal{F}_g(2) = \frac{1}{3}$ and $n = 3$, $G_{\mathcal{F}_g(k)}$ is integrated by the three networks in Figure 2, $G_{\mathcal{F}_g(k)} = \{g, g_1, g_2\}$. Under Setting S(a), $I_i(g) = \{k_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2)], n, \theta\} = \{2, \left[\frac{2}{3}, \frac{1}{3}\right], 3, \theta\}$, thereby, $B_i(g) = \{g\}$. On the contrary, $B_s(g) = \{g, g_2\}$ and $B_m(g) = \{g, g_1\}$, since $I_s(g) = I_m(g) = \{1, \left[\frac{2}{3}, \frac{1}{3}\right], 3, \theta\}$ under this information setup. Hence, i knows the whole network, while the network knowledge of s and m is incomplete. ■

Distribution of the feasible networks. The common knowledge information of agents may include some information about the forces that drive network formation, denoted θ. Specifically, $\theta \in \{\emptyset, \theta\}$ is an information that each $i \in N$ may possess about the mechanisms that affect the formation of links. Such an information is obtained from the context. For example, θ can represent the knowledge that a prospective employee has about the number and frequency of meetings involving people working in different departments, the information that (s)he has about the type of networking activities that are likely to take place in the company or the sort of relationships that are likely to be established among company members. If i expects to integrate a company whose members work in a top-down manner, i may believe more likely to occupy the position of an agent in network g_3 in Figure 3 rather than the position of an individual in g_2 in the same figure. The opposite applies if i expects to integrate a company characterized by teamwork. Thus, the probability distribution of the networks in $B_i(g)$ depends on θ. We denote $\mu_{g_x}(\theta)$ to the probabilistic weight that i gives to $g_x \in B_i(g)$ on the basis of θ. As θ is informative about the network structure but not about the identity of neighbors, $\mu_{g_x}(\theta) = \mu_{g_y}(\theta)$ as long as i's position in these networks is the same ($o_i(g_x) = o_i(g_y)$).

When $\theta = \emptyset$, each $i \in N$ believes that links form randomly as in the Erdös-Renyi model and $\mu_{g_x}(\theta) = \frac{1}{b_i(g)}$ $\forall g_x \in B_i(g)$. If rather $\theta \neq \emptyset$, the forces that drive network formation differ from randomness in i's beliefs, and i assigns probability $\mu_{g_x}(\theta) = \frac{1}{b_i(g)}(1 + \kappa_{g_x})$ to each $g_x \in B_i(g)$. Until the end of Section 4.3 , we analyse network perception when the networks in $B_i(g)$ are uniformly distributed, $\theta = \emptyset$.[19]

Feasible network geometries. So far we have provided the example of a simple network (Example 1), where all networks in $B_i(g)$ have the same geometry, $\forall i \in N$. However, some networks in $B_i(g)$ may have different network geometries, while others may be different but isomorphic. We say that a particular network geometry is a feasible geometry if some graph in $B_i(g)$ have such a geometry. Let $\Omega_i(g) = \{1, 2, ..., \omega_i(g)\}$ be the set of feasible geometries in i's beliefs, with $\omega_i(g) = |\Omega_i(g)|$. The set of (isomorphic) networks in $B_i(g)$ with geometry $z \in \Omega_i(g)$ is denoted $B_i^z(g)$, and $b_i^z(g) = |B_i^z(g)|$. Networks in $B_i^z(g)$ differ in how agents are allocated within the network, but they all have the same network geometry. Throughout our analysis, we assume that $g_z \in B_i(g)$ has geometry z $\forall z \in \Omega_i(g)$ and geometries in $\Omega_i(g)$ are indexed according to their degree of symmetry in an increasing order, $|Aut(g_z)| < |Aut(g_{z+1})| \forall z \in \Omega_i(g) : z + 1 \leq \omega_i(g)$. Claim 1 provides conditions under which all agents have identical beliefs about the network geometry.

Claim 1. *If $I_i(g) = \{t_i(g), I^c\}$ and $\mathcal{F}_g(t) \in I^c$ $\forall i \in N$ under Setting S, then $\Omega_i(g) = \Omega_j(g)$ $\forall i, \, j \in N$*

[19]Example 5, at the end of Section 4.3, illustrates the difference between both scenarios ($\theta = \emptyset$ and $\theta \neq \emptyset$). Clearly, $\sum_{g_x \in B_i(g)} \mu_{g_x}(\theta) = 1$ in any case

In words, if the common knowledge information of agents includes the frequency distribution of the network aspect that is privately known by them, all individuals have identical beliefs about the set of feasible geometries. This happens for example under Setting S(a), as we ilustrate below. Note that the feasibility of a network geometry only depends on $|E|$ and on n, which is common knowledge. Since the private information of agents does not provide them any information about the network structure than that conveyed in the common knowledge one, $\Omega_i(g) = \Omega_j(g)\ \forall i, j \in N$.

Distribution of the feasible geometries ($\boldsymbol{\theta} = \boldsymbol{\emptyset}$). Each i can infer the probability that network g has a particular geometry z by counting the number of (isomorphic) networks in $B_i(g)$ with this geometry and dividing this number by the total number of feasible networks, $b_i(g)$. In other words, agent i believes that network g has geometry z with probability $\rho_i^z = \sum\limits_{z \in B_i^z(g)} \mu_{g_z}(\theta) = \frac{b_i^z(g)}{b_i(g)}$. Agent i may assign more probability to some network geometries than to others, as Example 2 illustrates.

Example 2. Suppose we are under Setting S(a), and $I_i(g) = \{k_i, [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], n, \theta\} = \left\{k_i(g), \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right], 6, \emptyset\right\}$. From $I_i(g)$, a fully rational i can infer that there are three feasible network geometries, depicted in Figure 5. Depending on how agents are allocated in the network, there are different networks with each of these geometries. In particular, there exist $b_i(g) = 450$ different networks that are feasible in the beliefs of an i with $k_i(g) = 1$: $b_i^1(g) = 180$ networks have geometry 1, $b_i^2(g) = 90$ have geometry 2, and $b_i^3(g) = 180$ have geometry 3, as we show in the following sections. ■

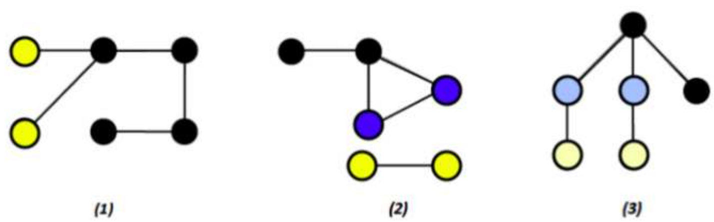

Figure 5: In each network, nodes in dark blue (yellow) are structurally equivalent. Nodes in light blue (yellow) are automorphically equivalent but not structurally equivalent

$k_i(g)$	$b_i^1(g)$	$b_i^2(g)$	$b_i^3(g)$	$b_i(g)$
$k_i(g)$=1	180	90	180	450
$k_i(g)$=2	120	60	120	300
$k_i(g)$=3	60	30	60	150
$\sum$	360	180	360	900

Table 1: Network beliefs conditioned on $I_i(g)$ (Example 2).

Notice that in Example 2 all agents have identical beliefs about the feasible geometries regardless of their degree: they assign probability $\frac{180}{450} = \frac{120}{300} = \frac{60}{150} = 0.4$ to geometry 1, probability $\frac{90}{450} = \frac{60}{300} = \frac{30}{150} = 0.2$ to geometry 2, and probability 0.4 to geometry 3. As we show below, this not by chance, but it is a general property of agents' beliefs for information setups exhibiting the conditions in Claim 1.

Under other information setups, network members may have different beliefs about the set of feasible geometries and their probabilities. Consider, for instance, network g in Figure 4(g). Under Setting S(b), $i \in N$ knows that (s)he has a degree-two neighbor and a degree-three one. Hence, the only geometries that are feasible given $I_i(g)$ are geometry 1 and geometry 2 in Figure 5.[20] Conditional on $I_l(g)$, on the contrary, the feasible geometries are geometries 2 and 3 in Figure 3, while conditional on $I_j(g)$ the only feasible one is geometry 2 in the same figure.

4.2 Isomorphisms of a graph

Let $\bar{N} \subseteq N$ be a subset of nodes in a network g, with $\bar{n} = |\bar{N}|$. We compute the number of distinct labelings of the nodes in $N \setminus \bar{N}$. In other words, we compute the number of distinct isomorphic networks to g that can

[20] Observe that no degree-two agent has a degree-two neighbor in the third network of Figure 3. Hence, geometry 3 is not a feasible geometry given $I_i(g)$.

be obtained by permuting exclusively the labels of the nodes in $N \setminus \bar{N}$. We denote this number $y(g \mid \bar{N})$.[21]

Labels of the nodes in $\bar{N}$ are not permuted, they are maintained fixed. Notice that in some cases we may permute the labels of some nodes in $N \setminus \bar{N}$ without any incidence in the adjacency matrix of the network: given g we may permute the labels of some nodes in $N \setminus \bar{N}$ and get a network $g' = g$. The set of different ways in which we can (exclusively) permute the labels of the nodes in $N \setminus \bar{N}$ without affecting the adjacency matrix of g is given by the stabilizer of $\bar{N}$, $stab_g(\bar{N})$. The stabilizer of a subset of nodes $\bar{N}$ in g is the set of all automorphisms that map each node in $\bar{N}$ into itself, $stab_g(\bar{N}) = \{f \in Aut(g) : f(v) = v, \forall v \in \bar{N}\}$ (Erwin and Harary, 2006). Consider for example network g_2 in Figure 6. Since $r \equiv_s m$ and $l \equiv_s o$, $Stab_g(i) = \{f, f', f'', f'''\}$, resulting in g_2, g_2', g_2'' and g_2''' in Figure 6, respectively.

Lemma 1. *Let $g = (N, E)$. The total number of distinct isomorphic networks to g that can be obtained by exclusively permuting the labels of the nodes in $N \setminus \bar{N}$ is:*

$$y(g \mid \bar{N}) = \left|\{g' \in G : o_i(g') = o_i(g)\ \forall i \in \bar{N}\}\right| = \frac{(n-\bar{n})!}{\left|stab_g(\bar{N})\right|}$$

Proof. There are $(n-\bar{n})!$ possible permutations of the labels of the nodes in $N \setminus \bar{N}$. For each of these $(n-n)!$ possible labelings, there are $\left|stab_g(\bar{N})\right|$ that are equal, as there are $\left|stab_g(\bar{N})\right|$ different ways in which we can permute the labels of the nodes in $N \setminus \bar{N}$ without any incidence in the adjacency matrix of the network. Thereby, $y(g \mid \bar{N}) = \frac{(n-\bar{n})!}{\left|stab_g(\bar{N})\right|}$. ■

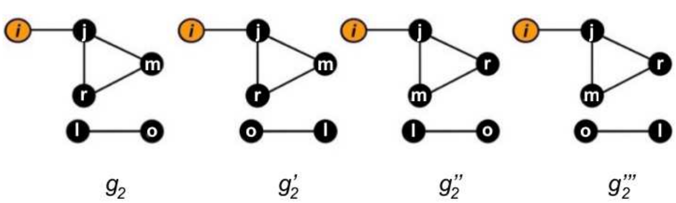

Figure 6: Actions of the automorphisms in $Stab(i)$ in g_2

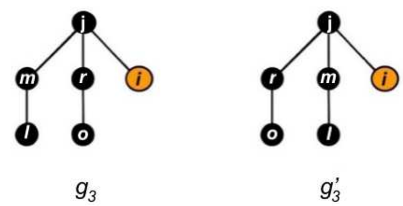

Figure 7: Actions of the automorphisms in $Stab(i)$ in g_3

Example 3. Suppose $\bar{N} = \{i\}$, $\bar{n} = 1$.

(a) Consider network g_1 in Figure 8. Since there is no pair of automorphically equivalent nodes in $N \setminus \bar{N}$, each permutation of the labels of the nodes in $N \setminus \bar{N}$ gives rise to a different network. Therefore, $Stab_{g_1}(i) = \{f\}$, where $f(i) = i\ \forall i$, and $|Stab_{g_1}(i)| = 1$. Hence, $y(g_1 \mid \{i\}) = \frac{(n-1)!}{1} = 120$. Similarly, $y(g_6 \mid \{i\}) = 120$.[22]

(b) Consider now network g_2 in Figure 8. Since $r \equiv_s m$ and $l \equiv_s o$, $Stab_{g_2} = \{f, f', f'', f'''\}$, resulting in g_2, g_2', g_2'', and g_2''' in Figure 6, respectively. Since $|Stab_{g_2}(i)| = 4, y(g_2 \mid \{i\}) = \frac{(n-1)!}{4} = 30$.

(c) As for network g_3 in Figure 8, since $m \equiv r$ and $l \equiv o$, $|Stab_{g_3}(i)| = 2$, as shown in Figure 7. Therefore, $y(g_3 \mid \{i\}) = \frac{(n-1)!}{2} = 60$. Analogously, $y(g_4 \mid \{i\}) = y(g_5 \mid \{i\}) = 60$.[23] ■

Example 3 calculates the number of distinct isomorphic networks that exist conditional on a given position of i making use of Lemma 1. For instance, there exist 120 different networks with the geometry of g_1 in Figure 8 in which i occupies the position $o_i(g_1)$; all these networks differ in how agents different from i are allocated. Lemma 1 is necessary to compute $b_i^z(g)$ and $b_i(g)$, as we show in the next section.

[21] Consider for example network g in Figure 2, and assume $\bar{N} = \{s\}$. Maintaining fixed the position of agent s in that network, there are two different networks that are isomorphic to g: network g and network g_2 in Figure 2. Hence, $y(g \mid \bar{N}) = y(g \mid \{s\}) = 2$.

[22] Recall from Section 2 that automorphically equivalent nodes have the property that their labels can be interchanged to form a new network that is identical to the original one. Since $j \equiv r$ in g_6, we could interchange the labels of these two nodes and relabel all other nodes in the network to obtain a network that is identical to g_6 . Nevertheless, there is no way to exchange the labels of any nodes different from i and obtain network g_6 maintaining fixed the label of i, since all nodes different from i are located at a different distance from i.

[23] Since $|Stab_{g_1}(i)| = 1$ and $|O_i(g_1)| = 2$, we can compute $|Aut(g_1)|$ applying the Orbit-Stabilizer Theorem (in the Appendix): $|Aut(g_1)| = |O_i(g_1)| * |Stab_{g_1}(i)| = 2$. Analogously, $|Aut(g_2)| = |O_i(g_2)| * |Stab_{g_2}(i)| = 4$ and $|Aut(g_3)| = |O_i(g_3)| * |Stab_{g_3}(i)| = 2$, as we introduced in Section 2.

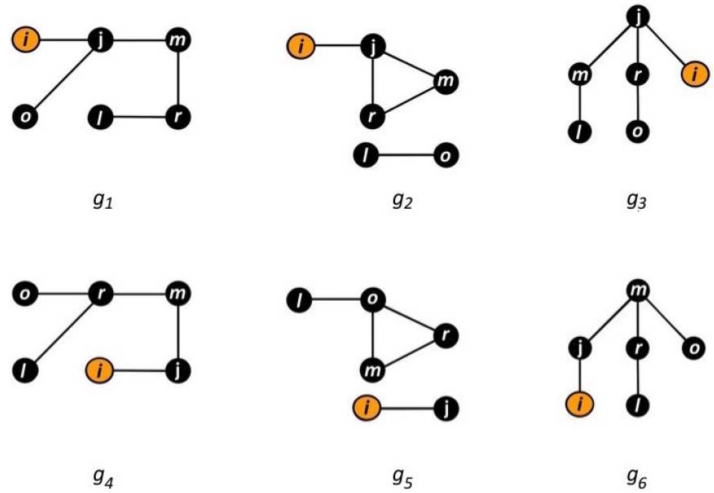

Figure 8: Networks in Example 3

4.3 Beliefs about the network geometry

In our incomplete information setups, agents do not have complete information about their network position. The information set of each $i \in N$ only describes some aspects of $o_i(g)$, but there may exist distinct positions compatible with such an information. We say that a position is feasible in i's beliefs if it is consistent with $I_i(g)$: if it exists a positive probability that i occupies this position according to i's beliefs. The set of feasible positions of i in g_z is $P_i^z(g) = \{o_i(g_z) : g_z \in B_i^z(g)\}$, and identifies the positions that i can occupy in network g if g has geometry $z \in \Omega_i$. The total number of agents in $g_z = (N^z, E^z)$ that occupy a position in $P_i^z(g)$ is $\alpha_i(g_z) = \{i \in N^z : o_i(g_z) \in P_i^z(g)\}$.

Proposition 1 calculates $b_i^z(g)$ and $b_i(g)$ under Setting S.

Proposition 1. *Let $g_z \in B_i^z(g)$. Under Setting S:*

$$b_i^z(g) = \frac{(n-1)!\alpha_i(g_z)}{|Aut(g_z)|} \quad \text{and} \quad b_i(g) = \sum_{z \in \Omega_i(g)} b_i^z(g)$$

Proof. Assume we are under Setting S. If $P_i(g_z) = \{o_i(g_z), o_i(g_s), ..., o_i(g_l)\}$, there exist at least one network in $B_i(g)$ in which i occupies the position $o_i(g_z)$. Precisely, there are $y(g_z \mid \{i\})$ (isomorphic) networks in $B_i^z(g)$ in which i occupies the position $o_i(g_z)$; all these networks differ in how agents different from i are allocated. Similarly, there are $y(g_s \mid \{l\})$ (isomorphic) networks in $B_i^z(g)$ where i occupies the position $o_i(g_s)$, and analogously for other positions in $P_i^z(g)$. Hence, if $P_i^z(g) = \{o_i(g_z), o_i(g_s), ..., o_i(g_l)\}$, $b_i^z(g) = y(g_z \mid \{i\}) + y(g_s \mid \{i\}) + ... + y(g_l \mid \{i\})$. By Lemma 1:

$$b_i^z(g) = y(g_z \mid \{i\}) + y(g_s \mid \{i\}) + ... + y(g_l \mid \{i\}) = \frac{(n-1)!}{|Stab_{g_z}(\{i\})|} + \frac{(n-1)!}{|Stab_{g_s}(\{i\})|} + ... + \frac{(n-1)!}{|Stab_{g_l}(\{i\})|}$$

Applying the Orbit-Stabilizer Theorem (in the Appendix) this is equal to:

$$\frac{(n-1)!|O_i(g_z)|}{|Aut(g_z)|} + \frac{(n-1)!|O_i(g_s)|}{|Aut(g_s)|} + ... + \frac{(n-1)!|O_i(g_l)|}{|Aut(g_l)|} = \frac{(n-1)!\ \alpha_i(g_z)}{|Aut(g_z)|}$$

■

Example 4 illustrates Proposition 1.

Example 4. Consider the information structure of Example 2, with $k_i(g) = 1$. Figure 5 represents $\Omega_i(g) = \{1, 2, 3\}$, while Figure 8 displays $g_z \in B_i$, $g_z \in \{g_1, g_2, g_3, g_4, g_5, g_6\}$. Since $P_i^1(g) = \{o_i(g_1), o_i(g_4)\}$,

$P_i^2(g) = \{o_i(g_2), o_i(g_5)\}$ and $P_i^3(g) = \{o_i(g_3), o_i(g_6)\}$:

$$b_i^1(g) = y\big(g_1 \mid \{i\}\big) + y\big(g_4 \mid \{i\}\big) = 120 + 60 = \frac{(n-1)!\alpha_i(g_1)}{|Aut(g_1)|} = \frac{5!\,3}{2} = 180$$
$$b_i^2(g) = y\big(g_2 \mid \{i\}\big) + y\big(g_5 \mid \{i\}\big) = 30 + 60 = \frac{(n-1)!\alpha_i(g_2)}{|Aut(g_2)|} = \frac{5!\,3}{4} = 90$$
$$b_i^3(g) = y\big(g_3 \mid \{i\}\big) + y\big(g_6 \mid \{i\}\big) = 60 + 120 = \frac{(n-1)!\alpha_i(g_3)}{|Aut(g_3)|} = \frac{5!\,3}{2} = 180$$

Hence, $b_i(g) = b_i^1(g) + b_i^2(g) + b_i^3(g) = 450$.

■

Example 4 shows that an agent i with information set $I_i(g) = \left\{1, \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right], 6, \theta(g) = \emptyset\right\}$ believes that there are 450 feasible networks: 180 have geometry 1 in Figure 5, 90 have geometry 2, and 180 have geometry 3 in the same Figure, as we introduced in Example 2. Following the same procedure as for the degree-one agent, it can be obtained the number of feasible networks in the beliefs of each $i \in N$ with $k_i \neq 1$ (see Table 1).

Proposition 1 provides an expression for $b_i^z(g)$ and another for $b_i(g)$. Dividing both expressions, we get the probabilistic weight that i assigns to geometry $z \in \Omega_i(g)$ when $\theta_i = \emptyset$, $\rho_i^z = \sum_{g_z \in B_i^z(g)} \mu_{g_z}(\theta) = \frac{b_i^z(g)}{b_i(g)}$. Corollary 1 gives such a probability, and shows its relation with the order of the network automorphism group.

Corollary 1. *Let $g_z \in B_i^z(g)$ be a network with geometry $z \in \Omega_i(g)$ and $\theta = \emptyset$. Under Setting S, each $i \in N$ believes that network g has geometry z with probability* $\rho_i^z = \frac{1}{1+\sum_{x \in \Omega_i(g)\setminus\{z\}} \frac{\alpha_i(g_x)|Aut(g_z)|}{\alpha_i(g_z)|Aut(g_x)|}}$

Corollary 1 shows that the probabilistic weight that each $i \in N$ assigns to each feasible geometry $z \in \Omega_i(g)$ depends on two aspects: (i) the number of agents that occupy a position in $P_i^z(g)$, $\alpha_i(g_z)$, and (ii) the order of the automorphism group of the network, $|Aut(g_z)|$. The probability that i assigns to z increases with $\alpha_i(g_z)$ and decreases with $|Aut(g_z)|$. As an implication, agents believe more likely to be immersed in a network integrated by agents playing a great diversity of social roles rather than a network with a more homogeneous structure, whenever $\alpha_i(g_z) = \alpha\ \forall z \in \Omega_i(g)$. As explained in Section 2, the order of the automorphism group of a network captures the variety of social roles their members play: the lower $|Aut(g_z)|$, the greater the number of distinct positional subgroups that integrate g_z and greater the diversity of social roles in the network. Note in Corollary 1 that ρ_i^z decreases as $|Aut(g_z)|$ increases, *ceteris paribus*, which means that $\rho_i^z < \rho_i^y$ if $|Aut(g_z)| > \big|Aut(g_y)\big|$ and $\alpha_i(g_z) \leq \alpha_i(g_y)$, where $g_z \in B_i^z(g)$ and $g_y \in B_i^y(g)$.

The second implication of Corollary 1 is that, when the conditions in Claim 1 hold, all network members assign the same probability to each feasible network geometry, even if $t_i(g) \neq t_j(g)$. When agents know the frequency distribution of the network aspect that is privately known by players and the network size, $\alpha_i(g_z) = \alpha_i(g_y)$, $\forall g_z, g_y \in B_i(g)$ and $\forall i \in N$. Consequently $\rho_i^z = \frac{1}{1+\sum_{x \in \Omega_i(g)\setminus\{z\}} \frac{|Aut(g_z)|}{|Aut(g_x)|}} = \rho_j^z\ \forall i, j \in N$. This occurs for example under Setting S(a).

Under other information setups (e.g. Setting S(b) and Setting S(c)), agents may have not only different beliefs about the feasible geometries, but also about their probabilities when the set of feasible geometries is equal across players. Since ρ_i^z depends on $\alpha_i(g_z)\ \forall i \in N$, $\rho_i^z(g)$ can be different from $\rho_j^z(g)$ if the number of agents that occupy a position in $P_i^z(g)$ and $P_j^z(g)$ is different ($\alpha_i(g_z) \neq \alpha_j(g_z)$).

Network beliefs when $\boldsymbol{\theta \neq \emptyset}$. So far we have focused on the case where i does not have any information about the network formation process, $\theta = \emptyset$. In absence of this information, each i assigns the same probability to each feasible network as links were formed randomly. Nevertheless, the context in which agents move may provide them information about the mechanisms that drive the creation of links. If such mechanisms differ from random selection, i may assign distinct probabilities to different networks in $B_i(g)$, as Example 5 illustrates.

Example 5. Consider the information assumptions in Example 2, with $\theta \in \{\theta_0, \theta_1\}$, $\theta_0 = \emptyset$ and $k_i(g) = 3$. Figure 5 displays $\Omega_i(g) = \{1, 2, 3\}$. Assume θ_1 is the probability that the network is connected, and $\theta_1 = 0.9$. Column 2(4) in Table 2 shows the probabilistic weight that each $i \in N$ assigns to networks g_1, g_2 and g_3 in Figure 8 conditioned on θ_0 (θ_1). Column 3(5) provides the probability that network g has geometry $z \in \Omega_i(g)$ conditioned on $\theta_0(\theta_1)$. ■

g_z in Figure 8	$\mu_{g_z}(\theta_0) = \frac{1}{b_i(g)}$	$\rho_i^z(\theta_0) = \rho_i^z = \frac{b_i^z(g)}{b_i(g)}$	$\mu_{g_z}(\theta_1) = \frac{1}{b_i(g)}(1+\kappa_{g_z})$	$\rho_i^z(\theta_1) = \frac{b_i^z(g)}{b_i(g)}(1+\kappa_{g_z})$
g_z, $z=1$	$\frac{1}{150}$	0.4	$\frac{1}{150}\left(1+\frac{0.1}{120}\right)$	0.45
g_z, $z=2$	$\frac{1}{150}$	0.2	$\frac{1}{150}\left(1-\frac{0.1}{30}\right)$	0.1
g_z, $z=3$	$\frac{1}{150}$	0.4	$\frac{1}{150}\left(1+\frac{0.1}{120}\right)$	0.45

Table 2: Network beliefs conditioned on $I_i(g)$ (Example 5)

We conclude this section by stressing that the implications of our network perception model can hold under information setups that do not exhibit the features of Setting S. Specifically, if players do not know $|E|$, but their beliefs about the population are captured by a probability degree distribution, the probability that each i assigns to each feasible geometry $z \in \Omega_i(g)$ depends on three aspects: (i) the probability that g has $|E|$ links, (ii) $\alpha_i(g_z)$ and (iii) $|Aut(g_z)|$. Yet, maintaining constant $|E|$ and $\alpha_i(g_z)$, ρ_i^z decreases with $|Aut(g_z)|$. Suppose for instance that, instead of knowing $|E|$, agents know that each link in the network forms with probability $p \in (0,1)$ as in the Erdös-Renyi network model. In this case, the probability that i assigns to a feasible geometry $z \in \Omega_i(g)$ with $|E|$ links is:

$$p^{|E|}(1-p)^{\binom{n}{2}-|E|} \;\; b_i^z(g)$$

where $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$ is the probability that the network has $|E|$ links. Then, if $g_z \in B_i(g)$ and $g_y \in B_i(g)$ have the same connectivity and $\alpha_i(g_z) = \alpha_i(g_y)$, $\rho_i^z > \rho_i^y$ whenever $|Aut(g_z)| < |Aut(g_y)|$.

4.4 Inference about network features

An important implication of our model is that each $i \in N$ can learn the probability distribution of any network characteristic from $I_i(g)$. For example, i can deduce the probability that the network aspect that is privately known by j, say, j's degree under Setting S(a), has a particular value, $t_j(g) = k$. Define $T_i = \{t_1, t_2, ..., t_x\}$ as a set of possible values for $t_i(g)$ under Setting S, with $T_i(g) = T_j(g)$ $\forall i, j \in N$. The subset of positions in $P_i^z(g)$ in which i has a neighbor whose private information belongs to T is $P_i^z(g \mid T) = \{o_i(g_z) \subseteq P_i^z(g) : \exists j \in N_i(g_z) : t_j(g_z) \in T\}$. For instance, under Setting S(a), $P_i^z(g \mid T)$ is the subset of positions in $P_i^z(g)$ in which i has a neighbor with a degree in T. The number of agents in $g_z \in B_i(g)$ that occupy a position in $P_i^z(g \mid T)$ is $\alpha_i(g_z \mid T) = |\{i \in N^z : o_i(g_z) \in P_i^z(g \mid T)\}|$, $g_z = (N^z, E^z)$.

Proposition 2 provides the probability that i has a neighbor j such that $t_j(g) \in T$ conditioned on $I_i(g)$ under Setting S.

Proposition 2. *Under Setting S, the probability that it exist a $j \in N_i(g)$ such that $t_j(g) \in T$ conditioned on $I_i(g)$ is:*

$$p[o_i(g) \subseteq P_i(g \mid T) \mid I_i(g)] = \sum_{z \in \Omega_i(g)} \mu_{g_z}(\theta) \frac{(n-1)!\,|\alpha_i(g_z \mid T)|}{|Aut(g_z)|}$$

Proof. Assume $P_i^z(g \mid T) = \{o_i(g_z), o_i(g_y), ..., o_i(g_m))\}$. If network g has geometry $z \in \Omega_i(g)$, i has a neighbor $j \in N_i(g) : t_j(g) \in T$ if i occupies any position in $P_i^z(g \mid T)$. There are $y(g_z \mid \{i\})$ feasible networks in which i occupies the position of $o_i(g_z)$, and i assigns probability $\mu_{g_z}(\theta)$ to each of them.[24] Analogously, there are $y(g_y \mid \{i\})$ feasible networks in which i occupies the position of i in g_y, each of them with probability $\mu_{g_y}(\theta) = \mu_{g_z}(\theta)$, and similarly for other positions in $P_i^z(g \mid T)$. Then:

$$\begin{aligned} p\big[o_i(g) \in P_i^z(g \mid T) \mid \; I_i(g)\big] &= \mu_{g_z}(\theta) y(g_z \mid \{i\}) + \mu_{g_s}(\theta) y(g_y \mid \{i\}) + ... + \mu_{g_m}(\theta) y(g_m \mid \{i\}) \\ &= \mu_{g_z}(\theta) \frac{(n-1)!\big|O_i(g_z)\big|}{\big|Aut(g_z)\big|} + \mu_{g_y}(\theta) \frac{(n-1)!\big|O_i(g_y)\big|}{\big|Aut(g_y)\big|} + ... + \mu_{g_m}(\theta) \frac{(n-1)!\big|O_i(g_m)\big|}{\big|Aut(g_m)\big|} = \\ &= \mu_{g_z}(\theta) \frac{(n-1)!\,|\alpha_i(g_z \mid T)|}{|Aut(g_z)|} \end{aligned}$$

where the second equality holds applying the Orbit-Stabilizer Theorem, and:

$$p\big[o_i(g) \in P_i(g \mid T) \mid \; I_i(g)\big] = \sum_{z \in \Omega_i(g)} p\big[o_i(g) \in P_i^z(g \mid T) \mid \; I_i(g)\big]$$

[24] Recall that θ is informative about the network structure, but not about the identity of i's neighbors. Then, $\mu_{g_z}(\theta) = \mu_{g_y}(\theta)$ whenever g_z and g_y have the same geometry.

■

Note that i can learn the probability of any network aspect under Setting S, even if it does not correspond to the private information of any player. Assume for example that $t_i(g)$ does not denote the private information of i under Setting S, but the value of a particular measure of i's position in g, say, $t_i(g) = C_i(g)$ under Setting S(a). If we assume that T is a set of feasible values for $C_i(g)$, and $P_i(g \mid T) = \{o_j(g_z) \subseteq P_i(g) : c_i(g) \in T\}$, Proposition 2 shows how i can learn the probability that $C_i(g) \in T$.

5 Network Games

We analyse the strategic interactions that take place in the network under Setting S. Network members are the players of a Bayesian Game in which they have incomplete information about the position of other players. We exploit our belief formation framework to analyse the equilibria that arise under such incomplete information framework.

5.1 The games

Our major interest is to characterize equilibrium outcomes when players base their actions on the private knowledge that they have about the characteristics of their network position, such as the knowledge of their degree under Setting S(a) or the knowledge of their degree and their neighbors' under Setting S(b). To bring out such a relation clearly, we follow Galeotti et al. (2010) and Feri and Pin (2020) and focus on the symmetric equilibria of the games in most part of our analysis (configurations where agents' choices are determined by their private network knowledge).[25] To that aim, the type of each $i \in N$ in g is denoted $\tau_i(g)$, and corresponds to the private information of i throughout Section 5: under Setting S(a), $\tau_i(g) = t_i(g) = k_i(g)\ \forall i \in N$, under Setting S(b), $\tau_i(g) = t_i(g) = (k_i(g), k_{N_i(g)})\ \forall i \in N$, and so on. To simplify notation, we write τ_i to denote generally the type of i. The common type space (the set of feasible types) is $\mathcal{T}$, while $\mathcal{F}_g(\tau)$ is the frequency distribution of types in network g. A symmetric strategy σ is a mapping that specifies which action is chosen as a function of each player's type $\sigma(\tau_i) \in X$, where X is the action set. The strategy σ is symmetric if and only if $\sigma(\tau_i) = \sigma(\tau_j)\ \forall \tau_i, \tau_j : \tau_i = \tau_j$.

We analyse network games of strategic substitutes and strategic complements similar to those in Bramoullé and Kranton (2007) and Jackson (2010), respectively. In both games, each player chooses simultaneously an action in $X = \{0, 1\}$. For strategic complements, playing 1 may be interpreted as adopting a software, attending to an event or engaging in a criminal activity and playing 0 as not doing so. For strategic substitutes, action 1 may be experimenting with a novel technology, contributing to a public good or collecting information, and 0 as not taking these actions. In both cases, playing 1 bears a cost $c \in (0, 1)$ and playing 0 is free.

The active players are the agents that play action 1. Let $A_\sigma = \{\tau_i \in \mathcal{T} : \sigma(\tau_i) = 1\ \ \forall i \in N\}$ denote the set of active types in both games, i.e. the set of types for which σ specifies action 1. The set of positions in $P_i^z(g)$ in which i has an active neighbor is $P_i^z(g \mid A_\sigma) = \{o_i(g_z) \in P_i^z(g) : \exists j \in N_i(g_z) : \tau_j(g_z) \in A_\sigma\}$, while $P_i(g \mid A_\sigma) = \{o_i(g_z) \in P_i(g) : \exists j \in N_i(g_z) : \tau_j(g_z) \in A_\sigma\}$ is the set of all feasible positions of i in which i is linked to an active player.

In both games, the utility function of each $i \in N$ when all players in g follow the strategy σ is denoted $u_i(\sigma, g)$, and $u_i(\sigma, g)$ depends on the action of i and on the sum of actions of i's neighbors. The expected utility of i of playing 0 when all agents follow the strategy σ is $E_{U_i}\big(0, \sigma, I_i(g)\big)$, while $E_{U_i}\big(1, \sigma, I_i(g)\big)$ is the expected utility of i of playing 1 conditioned on the same strategy.

[25] As discussed in Galeotti et al. (2010), this is further motivated by the fact that the set of all equilibria is roughly equal to the set of symmetric equilibria in large networks. This is because agents with the same private information (say, individuals with the same degree under Setting S(a)) have virtually identical beliefs about the underlying network. For instance, the only difference between i's and j's beliefs about the network under Setting S(a) when $k_i(g) = k_j(g)$ is that $j(i)$ is a feasible neighbor of $i(j)$, while $i(j)$ is not. Then, both agents face virtually the same probability over neighbors' actions in large networks and their best-responses to their neighbors' choices are, in the vast majority of the cases, identical. Thus, the range of parameter values for which players with identical types are best responding with different actions is negligible for large networks.

Strategic substitutes. The utility of each i is:

$$u_i(\sigma, g) = \begin{cases} 0 & \text{if} \quad \sigma(\tau_i) = 0 \quad \text{and} \quad \sum\limits_{j \in N_i(g)} \sigma(\tau_j) = 0 \\ 1 & \text{if} \quad \sigma(\tau_i) = 0 \quad \text{and} \quad \sum\limits_{j \in N_i(g)} \sigma(\tau_j) \geq 1 \\ 1 - c - \mu(\sigma) & \text{if} \quad \sigma(\tau_i) = 1 \end{cases}$$

where $\mu(\sigma) = 0$ if $\sum\limits_{j \in N_i(g)} \sigma(\tau_j) = 0$ and $\mu(\sigma) = \mu \in [0, 1 - c)$ otherwise.

Thus, players prefer that some of their neighbors play action 1 rather than taking this action themselves. However, if none of their neighbors plays 1, they prefer playing 1 rather than 0. The parameter μ represents players' regret when they incur in the cost of playing 1 and observe that they could have free ridden.

Under strategic substitutes, $E_{U_i}\big(0, \sigma, I_i(g)\big)$ is the probability that at least one of agent in $N_i(g)$ plays action 1. That is, the probability that i has a neighbor with a type in A_g given $I_i(g)$:

$$E_{U_i}\big(0, \sigma, I_i(g)\big) = p\big[o_i(g) \in P_i(g \mid A_\sigma) \quad \mid I_i(g)\big]$$

When $\theta = \emptyset$:

$$E_{U_i}\big(0, \sigma, I_i(g)\big) = \frac{|\{g \in B_i(g) : \exists j \in N_i(g) : \tau_j(g) \in A_\sigma\}|}{b_i(g)}$$

In equilibrium, each i plays 0 if and only if:

$$E_{U_i}\big(0, \sigma, I_i(g)\big) \geq E_{U_i}\big(1, \sigma, I_i(g)\big) = 1 - c - E_{U_i}\big(0, \sigma, I_i(g)\big)\mu$$

That is, if:

$$\frac{1-c}{1+\mu} \leq E_{U_i}\big(0, \sigma, I_i(g)\big) = p\big[o_i(g) \in P_i(g \mid A_\sigma) \quad \mid I_i(g)\big]$$

Strategic complements. In this case:

$$u_i(\sigma, g) = \begin{cases} -c & \text{if} \quad \sigma(t_i) = 1 \quad \text{and} \quad \sum\limits_{j \in N_i(g)} \sigma(\tau_i) = 0 \\ 1 & \text{if} \quad \sigma(t_i) = 1 \quad \text{and} \quad \sum\limits_{j \in N_i(g)} \sigma(\tau_j) \geq 1 \\ -\mu(\sigma) & \text{if} \quad \sigma(\tau_j) = 0 \end{cases}$$

for each $i \in N$, with $\mu(\sigma) = 0$ if $\sum\limits_{j \in N_i(g)} \sigma(t_j) = 0$ and $\mu(\sigma) = \mu \in [0, c)$ otherwise.

It is readily seen that players prefer playing 1 if some neighbor plays this action and playing 0 otherwise. In this case, μ represents the regret of players when they take action 0 and observe that they could have obtained greater payoffs by playing 1. Under strategic complements:

$$E_{U_i}\big(1, \sigma, I_i(g)\big) = -c + p\big[o_i(g) \in P_i(g \mid A_\sigma) \quad \mid I_i(g)\big]$$

When $\theta = \emptyset$:

$$E_{U_i}\big(1, \sigma, I_i(g)\big) = -c + \frac{|\{g \in B_i(g) : \exists j \in N_i(g) : \tau_j(g) \in A_\sigma\}|}{b_i(g)}$$

Each i is best responding with action 1 if and only if:

$$E_{U_i}\big(1, \sigma, I_i(g)\big) \geq E_{U_i}\big(0, \sigma, I_i(g)\big) = -p\big[o_i(g) \in P_i(g \mid A_\sigma) \quad \mid I_i(g)\big] \ \mu$$

or equivalently, if:

$$\frac{c}{1+\mu} \leq p\big[o_i(g) \in P_i(g \mid A_\sigma) \quad \mid I_i(g)\big]$$

Threshold. Hereafter, q denotes the threshold above which i is best responding with action 0(1) under strategic substitutes (complements): $q = \frac{1-c}{1+\mu}$ under strategic substitutes and $q = \frac{c}{1+\mu}$ under strategic complements.

5.2 Equilibrium

In this section, we characterize the Bayes-Nash equilibria of the games and analyse the effects of varying the perceived network on equilibrium behavior. We further explore the consequences of our network perception model on equilibrium welfare.

5.2.1 Equilibrium behavior

Let $g_z = (N^z, E^z)$. Under Setting S, the number of agents that occupy a feasible position of i, $\alpha_i(g_z)$, is number of individuals of type τ_i in $g_z \in B_i(g)$. Define $\alpha_i(g_z \mid A_\sigma) = |\{i \in N^z : o_i(g_z) \in P_i^z(g \mid A_\sigma)\}|$ as the number of agents that occupy a feasible position of i in g_z and have an active neighbor when all agents follow the strategy σ. Namely, $\alpha_i(g_z \mid A_\sigma)$ is the number of players of type τ_i in $g_z \in B_i(g)$ that have an active neighbor. Proposition 3 provides a condition under which a symmetric strategy σ constitutes an equilibrium under strategic substitutes (complements) under Setting S. The condition shows that, for each $z \in \Omega_i(g)$, the best response of each $i \in N$ depends on two aspects: (i) the proportion of type-τ_i agents that have an active neighbor in $g_z \in B_i(g)$ and (ii) the probability that i assigns to geometry z, ρ_i^z.

Proposition 3. *Let σ be a symmetric strategy under Setting S and $\theta = \emptyset$. The strategy σ constitutes a Bayes-Nash Equilibrium under strategic substitutes (complements) if and only if:*

$$\sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z \mid A_\sigma)}{\alpha_i(g_z)} \rho_i^z \geq q \quad \forall i : \tau_i \in X \qquad \textit{and} \qquad \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z \mid A_\sigma)}{\alpha_i(g_z)} \rho_i^z \leq q \quad \forall i : \tau_i \in Y$$

where $X = \mathcal{T} \setminus A_\sigma$ and $Y = A_\sigma$ under strategic substitutes and $X = A_\sigma$ and $Y = \mathcal{T} \setminus A_g$ under strategic complements.

Proof. See the Appendix.

Proposition 3 characterizes the influence that players' cognitive maps about network g have on their equilibrium actions. The greater the proportion of agents that occupy a feasible position of i and are linked to an active type in each of the feasible networks, the more likely i is to occupy the position of any of these individuals, and the greater are i's incentives of playing action 0(1) under strategic substitutes (complements). If $\alpha_i(g_z \mid A_\sigma)$ and $\alpha_i(g_z) = \alpha_i(g_y)$ for two networks $g_z \in B_i^z(g)$ and $g_y \in B_i^y(g_y)$ such that $|Aut(g_z)| < |Aut(g_y)|$, then the actions of i's neighbors in g_z have a greater impact on i's behavior than the action of i's neighbors in g_y, since $\rho_i^z > \rho_i^y$.[26] Thus, the degree of substitutability between players' actions and the actions of their feasible neighbors depends negatively on the order of the automorphism group of the network to which such neighbors belong.

Now we ask: is the equilibrium unique? Example 6 shows that different equilibria with varied patters are possible, even when players' network information is limited to their degree and the degree distribution of the network. These results are not exclusive of this information setup (Setting S(a)). In Section 7 we show that multiple and varied equilibria can exist even in information settings where players' have heterogeneous beliefs about the underlying social structure.

Example 6. Assume $\forall i \in N$: $I_i(g) = \{k_i(g), [F_g(1), F_g(2), F_g(3)], n, \theta = \emptyset\} = \left\{k_i(g), [\frac{1}{5}, \frac{3}{5}, \frac{1}{5}], 5, \theta = \emptyset\right\}$ and $\mathcal{T} = \{k, k', k''\} = \{1, 2, 3\}$. Figure 9 shows $\Omega_i(g) = \{1, 2\}$ $\forall i \in N$. Note that $|Aut(g_1)| = |Aut(g_2)| = 2$, where $g_z \in B_i(g)$ has geometry $z \in \Omega_i(g) = \{1, 2\}$.

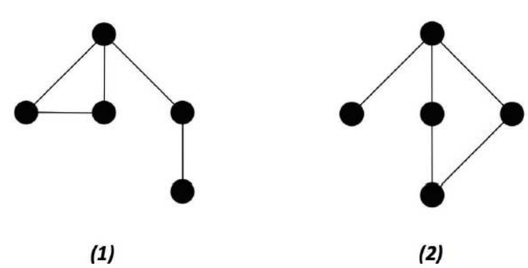

Figure 9: $\Omega_i(g)$ $\forall i \in N$ in Example 6

There are three strategies that constitute a pure-strategy symmetric equilibrium:

[26] Observe in Corollary 1 that, if $\alpha_i(g_z) = \alpha_i(g_y)$ and $|Aut(g_z)| < |Aut(g_y)|$, then $\rho_i^z > \rho_i^y$.

(a) $\sigma_1 : \sigma_1(2) = 0$ and $\sigma_1(k_i) = 1$ for $k_i \in \{1,3\}, \forall i \in N$ and $\frac{1}{2} \leq q \leq \frac{5}{6}$.
(b) $\sigma_2 : \sigma_2(3) = 1$ and $\sigma_2(k_i) = 0$ for $k_i \in \{1,2\}, \forall i \in N$ and $q \geq \frac{1}{2}$.
(c) $\sigma_3 : \sigma_3(1) = 1$ and $\sigma_3(k_i) = 0$ for $k_i \in \{2,3\}, \forall i \in N$ and $q \geq \frac{5}{6}$.

Table 3 lists the expected utility of playing 0 of each type of $i \in N$ when all agents follow each of these strategies.

$\mathcal{T}$	$E_{U_i}\big(0,\sigma_1,I_i(g)\big)$	$E_{U_i}\big(0,\sigma_2,I_i(g)\big)$	$E_{U_i}\big(0,\sigma_3,I_i(g)\big)$
$k=1$	1/2	1/2	0
$k'=2$	5/6	5/6	1/6
$k''=3$	1/2	0	1/2

Table 3: $E_{U_i}(1,\sigma_x,I_i(g))$ in Example 6, $x \in \{1,2,3\}$.

■

Galeotti et al. (2010) show that, in binary games of strategic substitutes (complements), the equilibrium is unique and monotone non-increasing (non-decreasing) in players' degrees when nodes have degrees either with independent probabilities or with probabilities that are negatively (positively) correlated and players are not indifferent between playing different actions. Their result is intuitive: under degree independence, the beliefs of i about the degree of each $j \in N_i(g)$ are identical as the beliefs of s about the degree of each $l \in N_s(g)$, even when $k_i > k_s$. This implies that i and s face the same probability distribution over the action of $j \in N_i(g)$ and $l \in N_s(g)$, respectively, since the action of each $j \in N$ is determined by $t_j(g) = k_j(g)$. However, given that i has a more neighbors, the probability that at least one agent in $N_i(g)$ plays 1 is greater than the probability that at least agent in $N_s(g)$ does. Then, for all $i, s \in N : k_i(g) > k_s(g)$, $E_{U_i}\big(0,\sigma,I_i(g)\big) \geq E_{U_s}\big(0,\sigma,I_s(g)\big)$ under strategic substitutes and $E_{U_i}\big(1,\sigma,I_i(g)\big) \geq E_{U_s}\big(0,\sigma,I_s(g)\big)$ under strategic complements. As a result, the equilibrium is characterized by a threshold: under strategic substitutes (complements) players with a degree lower than $\bar{k}$ play action 1(0) and other players action 0(1). If nodes have degrees with probabilities that are negatively (positively) correlated, the probability that i has a neighbor with a degree below (above) the threshold increases with $k_i(g)$, and the monotonicity property of equilibrium under strategic substitutes (complements) maintains.

Example 6 shows that, in contrast to Galeotti et al. (2010), equilibria are not necessarily monotone non-increasing in players' degrees under strategic substitutes under Setting S(a). The reason is that, in our model each i learns the probability of having neighbors with particular degrees from $I_i(g)$, and this probability is not always independent nor does necessarily vary monotonically with $k_i(g)$, even if links are formed randomly in agents' beliefs ($\theta = \emptyset$). This means that the probability that $s \in N$ has a degree-k neighbor conditioned on $I_s(g)$ may be greater than the probability that i has a degree-k neighbor conditioned on $I_i(g)$, even if $k_s < k_i$. Thus, if $\sigma(k) = 1$, s may be more likely to have an active neighbor than i, despite having a lower number of connections. If this occurs, $E_{U_s}\big(0,\sigma,I_s(g)\big)$ may be greater than $E_{U_i}\big(0,\sigma,I_i(g)\big)$, s may be responding with a lower action than i, and the equilibrium is not monotone non-increasing. Under strategic complements, similar results are obtained. If conditioned on $I_i(g)$, the probability that i has a degree-k neighbor is neither independent nor monotone increasing in $k_i(g)$, players with greater degrees may have lower incentives to play higher actions, and the monotonicity property of equilibrium may fail to hold.[27]

In a nutshell, unless players have some specific beliefs about the network assortativity patterns, there is not a systematic relationship between degree and equilibrium behavior, even if players' private information about the network confines to this network measure. However, there is a network feature that has a clear impact on behavior and payoffs in all setups of incomplete network information: the order of the automorphism group of the network. In the following section we show the influence that this network property has on equilibrium welfare.

5.2.2 Equilibrium welfare

.

[27] Suppose for instance that $I_i(g) = \{k_i(g), J_g\big(k,(k_1,k_2,...,,k)\big),n,\theta\} = \{k_i(g),[J_g\big(1,(1)\big),J_g\big(2,(2,2)\big)],8,\theta\} = \{k_i(g),[\frac{1}{4},\frac{3}{4}],8,\theta\}$. Conditioned on $I_i(g)$ there are two feasible geometries: one in which degree-two nodes form two different three-cycles and the degree-one nodes a dyad, and another in which degree-two nodes are connected a six-cycle and the degree-one nodes are linked. Under these conditions, there are four symmetric equilibria: $\sigma_1 : \sigma_1(k) = 1$ for $k = 1$ and $\sigma_1(k) = 0$ for $k = 2$, $\sigma_2 : \sigma_2(k) = 0$ for $k = 1$ and $\sigma_2(k) = 1$ for $k = 2$ and and two equilibria where all agents play action 0(1).

We now explore the implications of Proposition 3 on players' payoffs and the aggregate welfare. Under Setting S, the set of equilibria is the same across all networks that are compatible with players' information. However, the payoffs that players receive may differ across these networks, if their respective geometries are different. Consider, for example, any two networks g_1 and g_2 with geometries 1 and 2 in Figure 9. Under Setting S(a), the three strategies in Example 6 are equilibrium strategies. However, when agents play σ_1, the sum of their payoffs is $5-2c$ in network g_1, and $4-2(c+\mu)$ in network g_2. Thus, although players' choices are identical, the sum of the payoffs they obtain is not.

We want to understand how the equilibrium welfare relates to the geometry of the network. To that aim, we define $W_{\tau_i}(\sigma, g)$ as the sum of the payoffs of type-τ_i players in network g. Under strategic substitutes:

$$W_{\tau_i}(\sigma, g) = \sum_{i \in N: \tau_i(g)=\tau_i} u_i(\sigma, g) = \begin{cases} \alpha_i(g|A_\sigma) & \text{if} \quad \sigma(\tau_i) = \mathit{0} \\ \alpha_i(g)(1-c) - \alpha_i(g|A_\sigma)\mu & \text{if} \quad \sigma(\tau_i) = \mathit{1} \end{cases}$$

Under strategic complements:

$$W_{\tau_i}(\sigma, g) = \sum_{i \in N: \tau_i(g)=\tau_i} u_i(\sigma, g) = \begin{cases} \alpha_i(g|A_\sigma) - \big[\alpha_i(g) - \alpha_i(g|A_\sigma)\big]c & \text{if} \quad \sigma(\tau_i) = \mathit{1} \\ -\alpha_i(g|A_\sigma)\mu & \text{if} \quad \sigma(\tau_i) = \mathit{0} \end{cases}$$

Similarly, the average welfare of type-τ_i players in g is $\frac{W_{\tau_i}(\sigma,g)}{\alpha_i(g)}$, while $W(g, \sigma) = \sum\limits_{i \in N} u_i(\sigma, g)$ is the aggregate welfare in g.

Proposition 3 shows that the range of parameter values for which i is best responding with action $\sigma(\tau_i)$ increases as the average welfare of type-τ_i players in each of the feasible networks conditioned on σ does. The extent to which it increases depends on $|Aut(g_z)|$. Consider for instance the game of strategic substitutes. Conditioned on $\sigma(\tau_i) = 0$, $\frac{W_{\tau_i}(\sigma,g_z)}{\alpha_i(g_z)} = \frac{\alpha_i(g_z|A_\sigma)}{\alpha_i(g_z)}$ in each $g_z \in B_i(g)$. Action $\sigma(\tau_i) = 0$ is a best-response if and only if $E_{U_i}\big(0, \sigma, I_i(g)\big)$ is greater than a threshold $q = \frac{1-c}{1+\mu}$. As $E_{U_i}\big(0, \sigma, I_i(g)\big)$ increases with $\frac{\alpha_i(g_z|A_\sigma)}{\alpha_i(g_z)}$, the range of parameter values for which $\sigma(\tau_i) = 0$ is a best response increases as the average welfare of type-τ_i players conditioned on $\sigma(\tau_i) = 0$ in each $g_z \in B_i(g)$ does. Similarly, the range of parameter values for which $\sigma(\tau_i) = 1$ is a best response increases as $\frac{W_{\tau_i}(\sigma,g_z)}{\alpha_i(g_z)}$ conditioned on $\sigma(\tau_i) = 1$ is greater ($\frac{\alpha_i(g_z|A_\sigma)}{\alpha_i(g_z)}$ conditioned on $\sigma(\tau_i) = 1$ is lower). In both cases, the geometry of g_z determines the extent to which the range of parameter values changes, since ρ_i^z depends negatively on $|Aut(g_z)|$. As this holds for all $i \in N$, Proposition 3 suggests that, under certain conditions, the equilibria that provide greater payoffs in more asymmetric structures are sustainable for a greater range of parameter values. Corollary 2 clarifies this result.[28]

Corollary 2. *If $\Omega_i(g) = \Omega_j(g)$, $\rho_i^z = \rho_j^z$ $\forall i, j \in N$, and $\sum_{z=x}^{\omega_i(g)} \frac{W_{\tau_i}(g_z,\sigma')}{\alpha_i(g_z)} \geq \sum_{z=x}^{\omega_i(g)} \frac{W_{\tau_j}(g_z,\sigma)}{\alpha_j(g_z)}$ $\forall i, j \in N:$ $\sigma(\tau_i) = \sigma'(\tau_j)$ and $\forall x \in \Omega_i(g)$, with strict inequality for some $i, j \in N: \sigma(\tau_i) = \sigma'(\tau_j)$ and some $x \in \Omega_i(g)$, then σ' is sustainable for wider range of parameter values than σ.*

Corollary 2 applies for setups in which all players have identical beliefs about the network geometry. Such a symmetry in beliefs is present when the conditions in Claim 1 hold, for example. Recall that, if the conditions in Claim 1 satisfy, $\alpha_i(g_z) = \alpha_i(g_y)$ $\forall g_z, g_y \in B_i(g)$ and $i \in N$, and all agents have identical beliefs about the feasible network geometries and their probabilities.

Proposition 4 provides a condition that guarantees that equilibrium welfare is at least as high in g as in any other network composed of the same types of players as g and compatible with the information of some player. Formally, network g is *efficient* if $\nexists g' \in \bigcup\limits_{i \in N} B_i(g) : \mathcal{F}_g(\tau) = F_{g'}(\tau)$ such that $W(\sigma, g') > W(\sigma, g)$, for any σ that constitutes a (symmetric) equilibrium under Setting S.

Proposition 4. *Assume we are under Setting S, network g has geometry $z \in \Omega_i(g)$, and $\mu = 0$. Network g is efficient if:*

1. $c < \frac{\rho_i^z}{\alpha_i(g_z)}$ *under strategic substitutes.*
2. $1 - c < \frac{\rho_i^z}{\alpha_i(g_z)}$ *under strategic complements.*

[28]Recall from Section 4 that geometries in $\Omega_i(g)$ are indexed according to their degree of symmetry, $|Aut(g_z)| < |Aut(g_{z+1})|$, $\forall z \in \Omega_i(g) : z + 1 \leq \omega_i(g)$, $\forall i \in N$.

with $\frac{\rho_i^z}{\alpha_i(g)} \leq \frac{\rho_j^z}{\alpha_j(g)}$ $\forall j \in N$.

To illustrate Proposition 4, consider the information structure in Example 6, and let g_1 and g_2 be two networks with geometry 1 and 2 in Figure 9, respectively. Given that $|Aut(g_1)| = |Aut(g_2)| = 2$, all agents in g_1 believe that the network they are immersed in has geometry 1 with probability $\rho_i^1 = \frac{1}{2}$, and geometry 2 with the same probability, $\rho_i^2 = \frac{1}{2}$. Since $\alpha_i(g_1) = \alpha_i(g_1) = 1$ when $k_i(g) \in \{1, 3\}$ and $\alpha_i(g_1) = \alpha_i(g_2) = 3$ when $k_i(g) = 2$, Proposition 4 implies that network g_1 is efficient for $c < \frac{1}{6}$ under strategic substitutes and analogously for g_2. Note that for $c \geq \frac{1}{6}$, network g_2 is not efficient: when all agents play the equilibrium strategy σ_1−which is an equilibrium strategy for $\frac{1}{6} \leq c \leq \frac{1}{2}$ when $\mu = 0$− $W(g_1, \sigma_1) = 5 - 2c > W(g_1, \sigma_2) = 4 - 2c$.[29]

6 Network information effects

This section explores the effects of varying the depth of players' network information on equilibria. We first explore the impact of increasing players' network information on the structure and number of equilibria. Then, we analyse the consequences of assuming that players know the identity of other network members on the way they perceive the network as well as on their equilibrium behavior.

6.1 Information and structure of equilibria

Research on network games has shown how the introduction of incomplete network information can solve the problem of equilibrium multiplicity. The question that arises is the following: can we link the depth of network information to the number and/or structure of equilibria? Example 7 shows that increasing players' network information has non-monotone effects on the structure and number of equilibria.[30]

Example 7. Assume players are embedded in network g in Figure 2. Figure 10 shows the pure-strategy symmetric equilibria that arise under strict strategic substitutes under different information setups. The green (black) nodes in Figure 10 represent agents playing action 1(0).

a) **Galeotti et al. (2010)**. For all $i \in N$, let $I_i(g) = \{k_i(g), P_g(k)\}$, where $P_g(k)$ is the probability degree distribution. Suppose that according to $P_g(k)$ each node has degree $k = 1$ with probability p_1 and degree $k = 2$ with probability p_2, independently on the degree of other nodes. Under strict substitutes, the only equilibrium strategy is $\sigma_1 : \sigma_1(k_i) = 1$, $\forall k_i(g) = 1$ and $\sigma_i(k_i) = 0$, $\forall k_i(g) = 2$, for $p_1^2 < 1 - q < p_1$. Clearly,

$$E_{U_i}\big(0, \sigma_1, I_i(g)\big) = 1 - p_1^{k_i(g)}$$

which is greater than q for $k_i(g) = 2$ and $p_1^2 < 1 - q < p_1$, and lower than q for $k_i(g) = 1$ and the same parameter values. On the contrary, strategy $\sigma_2 : \sigma_2(k_i) = 0, \forall k_i = 1$ and $\sigma_2(k_i) = 1, \forall k_i = 2$. is not an equilibrium strategy: since the expected utility of playing zero is always greater for the degree-two players than for the degree-one ones, for any cost value for which the second agents are best responding with action 0, the former ones must be best-responding with this action as well. Then, σ_1 is the unique equilibrium strategy under strict strategic substitutes.[31]

b) Setting S(a). For all $i \in N$, $I_i(g) = \{k_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2)], n, \theta\} = \{k_i(g), \left[\frac{2}{3}, \frac{1}{3}\right], 3, \emptyset\}$.[32] Hence, $B_s(g) = \{g, g_2\}$, $B_m(g) = \{g, g_1\}$ and $B_i(g) = \{g\}$, where g, g_1 and g_2 are depicted in Figure 2. Since each i can infer the unique feasible geometry from $I_i(g)$, both σ_1 and σ_2 in (a) are equilibrium strategies for any cost value.

Suppose all players follow an asymmetric strategy $\sigma_3 : \sigma_3(k_s) = 1$ for k_s and $\sigma_3(k_m) = \sigma_3(k_i) = 0$. Agent i knows that $s \in N_i(g)$, and since $E_{U_i}\big(0, \sigma_3, I_i(g)\big) = 1$, i is best responding with action 0. The opposite occurs for s: when all agents play σ_3, $E_{U_s}\big(0, \sigma_3, I_s(g)\big) = 0$, and s is best responding with action

[29] σ_2 and σ_3 are equilibria for $c \geq \frac{1}{2}$ and $c \geq \frac{5}{6}$, respectively, and $W(g_1, \sigma_2) = W(g_2, \sigma_2) = 4 - c$ and $W(g_1, \sigma_3) = W(g_2, \sigma_3) = 2 - c$.

[30] As we analyse the effects of varying network information on the (total) number of equilibria that arise under different information setups, we have to consider all types of equilibria, included those involving asymmetric strategies. As mentioned above, the range of parameter values for which asymmetric equilibria sustain is negligible in large networks.

[31] Note that there is not equilibrium where all players play the same actions. If all agents play 1, $E_{U_i}\big(0, \sigma, I_i(g)\big) = 1 > q$ for all i, so each player wants to deviate and play action 0. The same applies if all they play 0: since $E_{U_i}\big(0, \sigma, I_i(g)\big) = 0 < q$, each i wants to change to action 1.

[32] We can also assume that $I_i(g) = \{k_i(g), P_g(k)\,[\mathcal{F}_g(1), \mathcal{F}_g(2)], n, \theta\}$ $\forall i \in N$, i.e. players know the probability degree distribution $P_g(k)$ and the particular realization of degrees in the network, $\mathcal{F}_g(k)$.

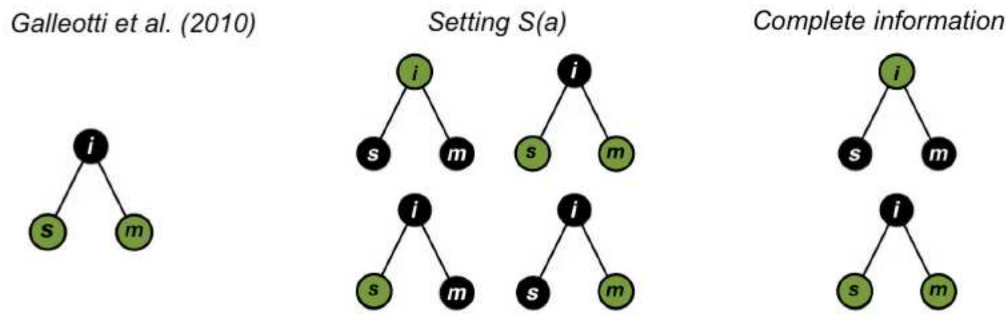

Figure 10: Pure-strategy equilibria under strict substitutes (Example 7)

1. Last, the probability that at least one neighbor of m plays 1 when all agents follow the strategy σ_3 is $E_{U_m}\big(0, \sigma_3, I_m(g)\big) = \frac{|\{g_1\}|}{b_m(g)} = \frac{1}{2}$. Then, under strict strategic substitutes ($q > \frac{1}{2}$), σ_3 and σ_4 are equilibrium strategies, where $\sigma_4 : \sigma_4(k_m) = 1$ for k_m and $\sigma_4(k_s) = \sigma_4(k_i) = 0$.

b) Complete information. Each i is best responding with action 0 if there is at least one neighbor playing 1, and with action 1 otherwise. Then, the only possible equilibrium strategies are σ_1 and σ_2. ■

Example 7 illustrates how reducing players' information does not necessarily solve the equilibrium selection problem: although the shift from the complete information setup to the information setting in Galeotti et al. (2010) eliminates any ambiguity in behavior, the shift from the complete information scenario to Setting S(a) does not. Notice that the set of equilibria under our Setting S(a) and the information setup in Galeotti et al. (2010) is markedly different even when the difference between both setups is subtle (under our Setting S(a), players' know the distribution of degree frequencies in the network while in Galeotti et al. (2010) they know the probability degree distribution).

6.2 Richer network information

So far we have focused on the case where players know their own identities but they do not have information about the identity of other network members. However, in many circumstances, people know the identity of their people they interact with, as well as the identity of their neighbors' neighbors. We now explore the effects that such a network knowledge has on players' network perception and behavior.

Define Setting S' as an information setup such that $I_i(g) = \{t_i(g), I^c(g)\}$ $\forall i \in N$, where $t_i(g)$ is the private information of i about i's network position and $I^c(g)$ a common knowledge information about network g. As in Setting S, $\{|E|, \theta, n\} \in I^c(g)$. However, Setting S and Setting S' differ in one important aspect: $t_i(g)$ includes information about the identity of the people that are located at a distance equal or lower than $d \geq 1$ from i under Setting S' while it does not under Setting S. Let $N_{I_i(g)} = \{j \in N : d_{ij} \leq \ d\}$ be the set of agents whose identity is known by i under Setting S', with $n_{I_i(g)} = |N_{I_i(g)}| > 1$ and $N_{I_i(g)} \in t_i(g)$. Setting S'(b) and Setting S'(c) are two particular cases of Setting S'.

Setting $\mathbf{S'(b)}$. Under Setting S'(B), $I_i(g) = \big\{N_i(g), \mathtt{k}_{N_i(g)}, \mathcal{F}_g(k), n, \theta\big\}$, $\forall i \in N$. For example, if g is the network in Figure 11(g), $I_i(g) = \{N_i(g), (k_l, k_o), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], 6, \theta\} = \big\{\{l, o\}, (3, 2), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta\big\}$. Figures $11(b_1)$ and $11(b_2)$ represent the two possible ways in which links can be allocated conditioned on $I_i(g)$.

Setting S'(c). The information set of each i is $I_i(g) = \big\{N_i(g), \mathtt{N}_{N_i(g)}, \mathcal{F}_g(k), n, \theta\big\}$, where $\mathtt{N}_{N_i(g)} = (N_j(g), ..., N_m(g))$ is the vector integrated by the neighborhoods of i's neighbors, $N_i(g) = \{j, ..., m\}$. The only difference between Setting S'(b) and Setting S'(c) is that under Setting S'(c) each i knows the identity of the agents in $N_i^2(g)$. The fact that i is informed about the identity of the agents in $N_i^2(g)$ implies that i knows the three- and four-cycles among the agents in $N_i(g) \cup \{i\}$, what in turn provides i information about the degree of the agents in $N_i^2(g)$. Assume network g is the network in Figure 11(g). Figure 11(c) represents the network knowledge of i given $I_i(g) = \big\{\{l, o\}, (N_l(g), N_o(g)), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta(g)\big\} = \big\{\{l, o\}, \big(\{i, m, o\}, \{i, l\}\big), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6, \theta(g)\big\}$. From $I_i(g)$, i knows that l and o are linked. Moreover, since neighbors l and o are simultaneously second-order neighbors of i, the degree of two agents in N_i^2 is also known by

i.[33]

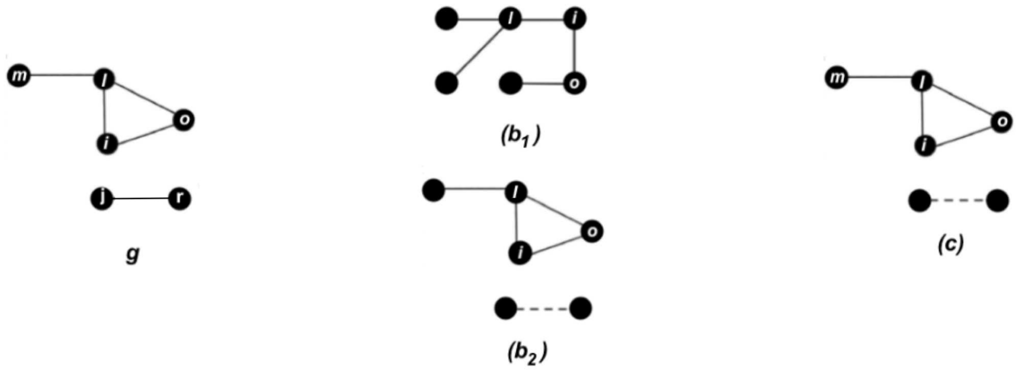

Figure 11: Network g and $I_i(g)$ under Setting S'(b) and S'(c). Unlabelled nodes represent agents whose identity is unknown for i. Solid (dashed) lines represent links that are (not) directly observed by i.

These examples illustrate how a subtle change in agents' network information− assuming that agents know their neighbors' degrees or they know their neighbors' neighborhoods− can have important consequences on agents' network knowledge. As Example 8 shows, such a difference between settings can entail abrupt changes in behavior, even when players' condition their behavior on the same network aspect under both information setups. But first we ask: are the predictions of our network perception model valid when players have information about other agents' identities?

The answer is positive. Let $\bar{B}_i^z(g) \subseteq B_i^z(g)$ be a subset of $B_i^z(g)$ that satisfies the following condition: if $g_z \in \bar{B}_i^z(g)$ and $g_y \in \bar{B}_i^z(g)$, then $o_i(g_z) \neq o_i(g_y)$ for some $i \in N_{I_i}$. Assume $\bar{B}_i^z(g)$ is not a subset of any other set $\bar{B}_i'(g) \subseteq B_i^z(g)$ satisfying such a condition. If $\bar{B}_i^z(g) = \{g_z, g_y, ..., g_l\}$, there are $y(g_z \mid \bar{n}_{I_i})$ isomorphic networks that could be network g; these networks only differ in how agents in $N \setminus N_{I_i}$ are allocated. Analogously, there are $y(g_y \mid \bar{n}_{I_i})$ different feasible networks that (only) differ in the position occupied by the agents in N_{I_i}. Following the same reasoning for all networks in $\bar{B}_i^z(g)$, the total number of networks with geometry $z \in \Omega_i(g)$ under Setting S' is:

$$b_i^z(g) = y(g_z \mid N_{I_i}) + y(g_z \mid N_{I_i}) + ... + y(g_y \mid N_{I_i}) = \sum_{g_l \in \bar{B}_i^z(g)} \frac{(n - n_{I_i})\,!}{|Stab_{g_z}(n_{I_i})|}$$

and $b_i(g) = \sum_{z \in \Omega_i(g)} b_i(g)$.

Observe that the perception bias towards assymmetric network structures maintains under Setting S'. As $|Aut(g_z)|$ decreases, either $|\bar{B}_i^z(g_z)|$ is greater or $|Stab_{g_z}(n_{I_i})|$ is lower (or both). This means that $b_i^z(g)$ increases as $|Aut(g_z)|$ decreases. Consequently, $\frac{b_i^z(g)}{b_i(g)}$ increases as the degree of asymmetry of g_z grows.

Example 8. Consider the networks[34] in Figure 13. The three networks have the same degree distribution, whereas only networks (b) and (c) have the same joint degree distribution. We analyse: (i) whether players' behavior may change depending on whether they have information about their second-order neighbors' identities (Setting S'(b) vs. Setting S'(c)) and (ii) whether players with the same degree and neighbors' degrees may behave differently depending on the observed geometry of links in their local network (Setting S(c)). To that aim, the type of each $i \in N$ defines as $\tau_i = (k_i(g), \mathtt{k}_{\mathtt{N_i}}(\mathtt{g}))$ both under Setting S'(b) and under Setting S'(c), and we compare the symmetric equilibria that arise in different networks within and across these setups.

Table 4 contains the symmetric equilibrium strategies in the networks of Figure 12 for the game of strategic substitutes and $\mu = 0$. As Table 4 shows, there are 5 feasible types, $\tau_i \in \{\tau^1, \tau^2, \tau^3, \tau^4, \tau^5\}$. The range of cost values for which each of these strategies constitutes an equilibrium in each network is displayed in Figure

[33]In a similar way, i could obtain some information about the degree of the agents in $N_i^2(g)$ by observing the four-cycles among the agents in $\{i\} \cup N_i(g)$: if i observes that a second-order neighbor z is linked both to $j \in N_i(g)$ and to $k \in N_i(g)$, i knows that $k_z \geq 2$. Analogously, if z is a common neighbor of x neighbors of i, i knows that $k_z \geq x$.

[34]To simplify the exposition, nodes in these networks are unlabeled. This allows us to provide all the calculations in terms of node i (see the Appendix).

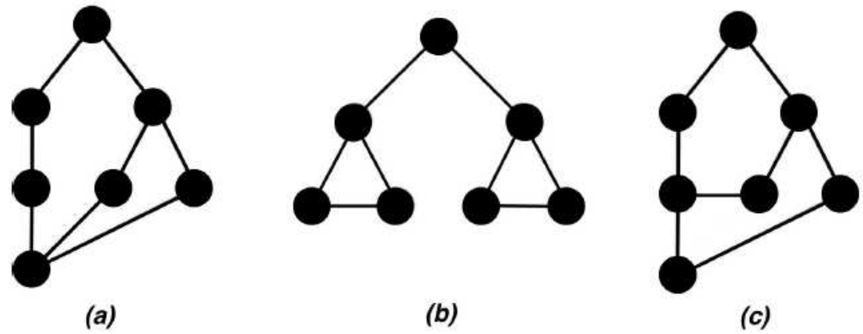

Figure 12: Networks in Example 8

13. As can be seen in this figure, equilibria can be different within and across setups, since the different information of players traduce in different beliefs about the network and this, in turn, in different equilibrium choices. ■

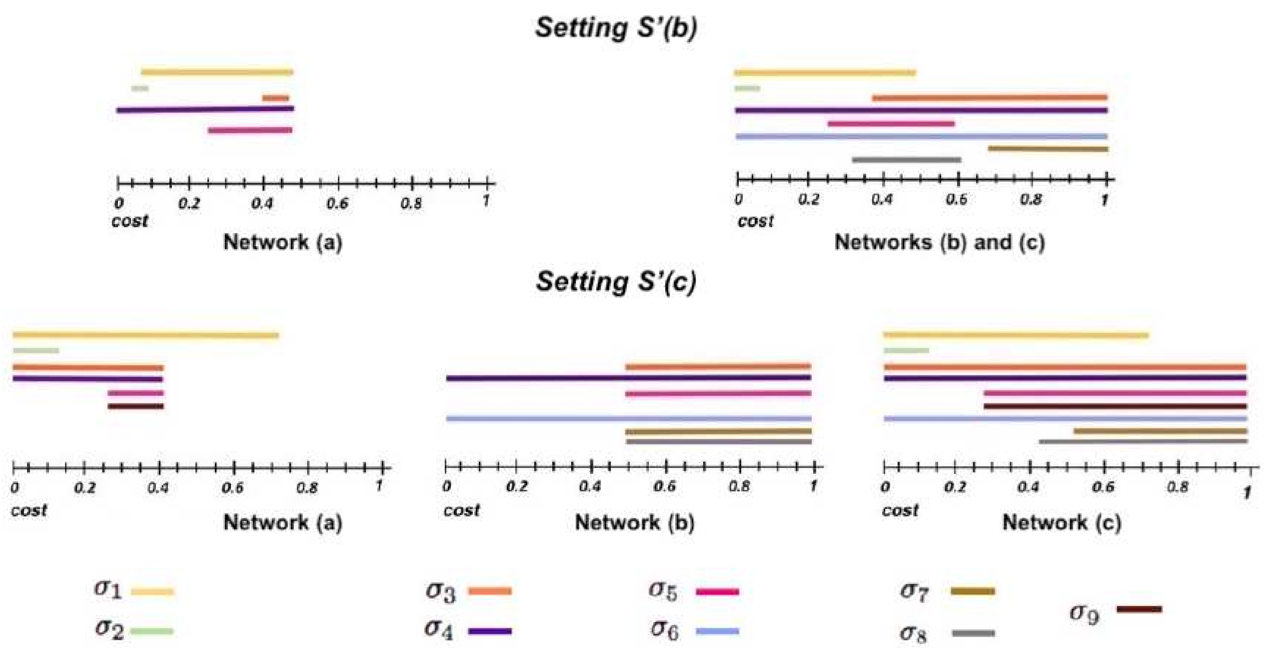

Figure 13: Symmetric equilibria in Example 8

Type of $i \in N$	$\sigma_1(\tau^x)$	$\sigma_2(\tau^x)$	$\sigma_3(\tau^x)$	$\sigma_4(\tau^x)$	$\sigma_5(\tau^x)$	$\sigma_6(\tau^x)$	$\sigma_7(\tau^x)$	$\sigma_8(\tau^x)$	$\sigma_9(\tau^x)$
$\tau^1 = \big(2;(2,2)\big)$	0	0	1	1	1	0	0	0	1
$\tau^2 = \big(2;(3,2)\big)$	1	1	0	0	0	0	0	0	0
$\tau^3 = \big(2;(3,3)\big)$	1	1	0	0	1	0	0	1	1
$\tau^4 = \big(3;(2,2,2)\big)$	0	0	1	1	0	1	1	0	0
$\tau^5 = \big(3;(3,2,2)\big)$	0	1	0	1	1	1	0	1	0

Table 4: Symmetric equilibrium strategies in Example 8

To summarize, the examples in this section provide a crucial message: although the framework of Galeotti et al. (2010) eliminates the problem of multiplicity of equilibria in network games, their uniqueness and monotonicity results are largely not robust to relaxing their information assumptions. As a result, their approach fails to refine the set of predictions in many situations in which people still have local network information. Such situations abound.

The introduction of incomplete information as a way of solving the problem of multiplicity of equilibria has faced a major critique: the equilibrium achieved depends on the way incomplete information is introduced (Weinstein and Yildiz, 2007). While this critique applies generally to all incomplete information games, it seems particularly relevant for those played on networks, given the variety of network aspects that players can infer from the information they are given.[35] Examples 7 and 8 illustrate that behavior is particularly sensitive to subtle changes in players' network information. This implies that, even when it is clear the network aspects that are unknown for players in specific contexts (e.g. the degree of agents that are two-link separated from them), fine differences in the information assumptions (e.g. assuming that players know the identity of their second-order neighbors or not) can bias the results in a particular direction.

7 Conclusion

Cognitive network research has showed how people form mental representations about their networks that influence their behavior (Brewer, 2011, Smith et al. 2021). Literature on network games has primarily opted for a simplification of people's network perception by making exogenous assumptions on players' network knowledge, abstracting from the fact that rational agents form beliefs about the network structure from the information they are given (e.g. Galeotti et al. 2010). However, network beliefs influence behavior. A strong contribution of this work is to show the cognitive dimensions of social networks as a key element for the study of network games. We derive the probability distribution of players' cognitive networks when agents' network information is incomplete, but enough for them to form mental representations about the underlying social structure.

To the best of our knowledge, this is the first paper that models people's cognitive maps of their networks and studies their relevance in network games. Our model uncovers the impact that canonical notions of equivalence among nodes have on network perception. Although these notions have broadly been studied by sociologists, they have not yet been theoretically related to network perception

We identify a bias in people's perception towards asymmetric network structures. If we order a set of networks with the same degree distribution according to the size of their automorphism group, such an order reflects a likelihood ranking of network geometries in people's beliefs. As an implication, the degree of substitutability between players' actions and the actions of their (feasible) neighbors−assumed to be exogenous in previous works[36]−is shown to depend on a particular network feature: the order of the automorphism group of the network to which such neighbors belong.

Our theoretical framework provides a way of capturing players' beliefs about a variety of network features that are absent in canonical models of network analysis (e.g. in random-graph models), which allows to analyse their incidence on behavior in incomplete information contexts. Our model uncovers two major challenges for network analysis. First, since players' infer a variety of network features from the information they are given, a subtle variation in players' network information can change completely the spectrum of equilibria. This requires a deeper understanding of the network knowledge that people actually have in different contexts, and calls for experimental research to analyse this issue. Second, the great range of equilibria that emerge under each information setup makes it hard to draw conclusions on the incidence that finer feature has on behavior. A prospective way to address this matter might be to impose some *ceteris paribus* restrictions on the set of feasible geometries (similarly as in Espinosa et al. 2020) in such a way that players' are only uncertain about one specific network feature, while they have a founded knowledge of its probability distribution. On the other hand, the relationship between the order of the network automorphism group and the set of eigenvalues of the adjacency matrix (see Cvetkovíc et al. 1979) requests further analysis of the implications of our model for network games, as the lowest eigenvalue has been identified as a major driver of social and economic outcomes (Bramoullé and Kranton, 2014). We leave this for future research.

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[35] The interdependency between different network features implies that players' information does not end in their information set, but they may learn different network aspects from the network information they are given. Observe that such a learning process may not occur in other Bayesian games.

[36] see for instance, Bramoullé and Kranton, 2014

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Appendix

The Orbit Stabilizer Theorem. Let $Stab_v(g) = \{g \in Aut(g) : f(v) = v\}$ be the stabilizer of node $v \in N$. Then,

$$|Aut(g)| = |O_v(g_z)| * |Stab_v(g)|$$

Proof of Lemma 1. Suppose $|O_i(g)| \leq |O_i(g')| \forall i \in N$. Then, for each automorphism $f : N \to N$ exists there is an identical automorphism $f : N' \to N'$. If $\exists i \in N : \big|O_i(g)\big| < \big|O_i(g')\big|$, then there exist at least one automorphism in g' that does not exist in g. Thereby, $\big|Aut(g)\big| < \big|Aut(g')\big|$.

We still have to prove that if $\big|O_i(g)\big| = \big|O_i(g')\big| \forall i \in N$, and $\exists m \in N' : \big|S_m(g)\big| < \big|S_m(g')\big|$, then $\big|Aut(g)\big| < \big|Aut(g')\big|$. Suppose $\exists m \in N' : |S_m(g')| > |S_m(g')|$. For each $r \in S_m(g') \setminus S_m(g)$, there exists an automorphism $f : N' \to N'$ such that

$$f(w) \begin{cases} r & if\ w = m \\ m & if\ w = r \\ w & otherwise \end{cases}$$

as $N_m(g') \setminus \{r\} = N_r(g') \setminus \{m\} \in S_m(g')$. On the contrary, it does not exist such an automorphism between m and r in network g, since $r \notin S_m(g)$. As a result, $\big|Aut(g)\big| < \big|Aut(g')\big|$. ■

Proof of Proposition 3. Suppose all agents follow the strategy σ. The expected utility of i of playing 0 under strategic substitutes is the probability that i occupies a position in $P_i(g \mid A_\sigma) = \{o_i(g_z), o_i(g_y), ..., o_i(g_r)\}$. Conditioned on $I_i(g)$ and $\theta = \emptyset$, each i assigns probability $\mu_{g_z}(\theta) = \frac{1}{b_i(g)}$ to each feasible network $g_z \in B_i(g)$. Then:

$$\begin{aligned} E_{U_i}\big(0, \sigma, I_i(g)\big) &= p\big[o_i(g) \in P_i(g \mid A_\sigma) \mid I_i(g)\big] \\ &= \mu_{g_z}(\theta) y(g_z \mid \{i\}) + \mu_{g_y}(\theta) y(g_y \mid \{i\}) + ... + \mu_{g_r}(\theta) y(g_r \mid \{i\}) \\ = \frac{1}{b_i(g)}\left[\frac{(n-1)}{|Stab_{g_z}(\{i\})|} + \frac{(n-1)}{|Stab_{g_y}(\{i\})|} + ... + \frac{(n-1)}{|Stab_{g_r}(\{i\})|}\right] &= \frac{(n-1)!}{b_i(g)} \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z|A_\sigma)}{|Aut(g_z)|} \end{aligned} \tag{1}$$

where the penultimate and last equality hold applying Lemma 1 and the Orbit-Stabilizer Theorem, respectively. By Proposition 1, $b_i(g) = \sum_{z \in \Omega_i(g)} b_i^z(g) = (n-1)! \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z)}{|Aut(g_z)|}$. Substituting $b_i(g) = (n-1)! \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z)}{|Aut(g_z)|}$ into (1) and operating, $E_{U_i}\big(0, \sigma, I_i(g)\big) = \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z|A_\sigma)}{\alpha_i(g_z)} \rho_i^z$. Playing 0 is a best response for i if and only if $E_{U_i}\big(0, \sigma, I_i(g)\big) \geq \frac{1-c}{1+\mu}$. Following the same reasoning, action 1 is a best response for i if and only if $E_{U_i}\big(1, \sigma, I_i(g)\big) = \sum_{z \in \Omega_i(g)} \frac{\alpha_i(g_z|A_\sigma)}{\alpha_i(g_z)} \rho_i^z \geq \frac{c}{1+\mu}$ under strategic complements. ■

Example 6. The set of types that play 1 according to the strategy σ_1 is $A_{\sigma_1} = \{1, 3\}$. Assume $k_i(g) = 2$. The set of feasible positions of i in which i is linked to a neighbor with a degree in $k \in \{1, 3\}$ is $P_i(g \mid A_{\sigma_1}) = \{o_i(g_1), o_i(g_2), o_i(g_3)\}$ where $\{g_1, g_2, g_3, g_4\} \in B_i(g)$ are represented in Figure 14. Given that $P_i^1(g \mid A_{\sigma_1}) = \{o_i(g_1), o_i(g_3)\}$ and $P_i(g_1) = \{o_i(g_1), o_i(g_3)\}$, $\frac{\alpha_i(g_1|A_{\sigma_1})}{\alpha_i(g_1)} = \frac{|O_i(g_1)|+|O_i(g_3)|}{|O_i(g_1)|+|O_i(g_3)|} = 1$. Similarly, since $P_i^2(g \mid A_{\sigma_1}) = \{o_i(g_2)\}$ and $P_i(g_2) = \{o_i(g_2), o_i(g_4)\}$, $\frac{\alpha_i(g_2|A_{\sigma_1})}{\alpha_i(g_2)} = \frac{|O_i(g_2)|}{|O_i(g_2)|+|O_i(g_4)|} = \frac{2}{3}$. By Corolary 1, $\rho_1(\theta) = \frac{1}{1+\frac{|Aut(g_1)|}{|Aut(g_2)|}} = \frac{1}{2}$ and $\rho_1(\theta) = \rho_2(\theta)$, since $|Aut(g_1| = |Aut(g_2|$. Then,

$$E_{U_i}\big(0, \sigma_1, I_i(g)\big) = \rho_1(\theta) + \frac{2}{3}\rho_1(\theta) = \frac{5}{6}$$

and analogously for other values in Table 3.
Table 5 provides the probability that i has a degree-k neighbor as a function of $I_i(g)$. As can be seen, the probability that i has a degree-k neighbor does not vary monotonically on k_i.

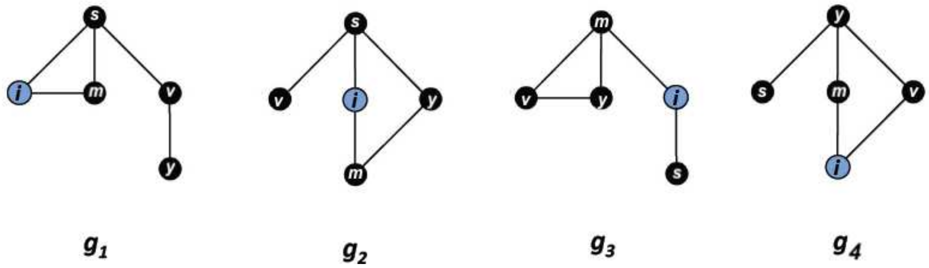

Figure 14: $P_i(g)$ in Example 6 for $k_i(g) = 2$

$k_i(g)$	$k_i(g) = 1$	$k_i(g) = 2$	$k_i(g) = 3$
$k = 1$	0	1/2	1/2
$k = 2$	1/6	5/6	5/6
$k = 3$	1/2	1	0

Table 5: Probability that i has a degree-k neighbor, conditioned on $I_i(g)$

■

Proof of Proposition 4. Let $g = g_1$. By construction, $g_1 \in B_i(g)\ \forall i \in N$.

Strategic substitutes. Let $g_2 \in B_i(g) : \mathcal{F}_{g_1}(\tau) = \mathcal{F}_{g_2}(\tau)$. Then, $\alpha_i(g_1) = \alpha_i(g_2)$. Imagine that $\sigma(\tau_i) = 0$ and $W_{\tau_i}(g_2, \sigma) > W_{\tau_i}(g_1, \sigma)$. That is, $\alpha_i(g_1, |A_\sigma) = \alpha_i(g_2, |A_\sigma) - \pi$. Action $\sigma(\tau_i) = 0$ is a best response for i if and only if:

$$\begin{aligned} E_{U_i}\big(0, \sigma, I_i(g)\big) &= \frac{\alpha_i(g_1, |A_\sigma)}{\alpha_i(g_1)}\rho_i^1 + \frac{\alpha_i(g_2, |A_\sigma)}{\alpha_i(g_2)}\rho_i^2 + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i} \\ &= \frac{\alpha_i(g_2, |A_\sigma) - \pi}{\alpha_i(g_1)}\rho_i^1 + \frac{\alpha_i(g_2, |A_\sigma)}{\alpha_i(g_2)}\rho_i^2 + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i} \geq 1 - c \end{aligned}$$

or equivalently iff.:

$$c \geq 1 - \left[\frac{\alpha_i(g_2 \mid A_\sigma)}{\alpha_i(g_1)}(\rho_i^1 + \rho_i^2) + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i}\right] + \frac{\pi}{\alpha_i(g_1)}\rho_i^1 \tag{2}$$

Since $1 - \left[\frac{\alpha_i(g_2|A_\sigma)}{\alpha_i(g_1)}(\rho_i^1 + \rho_i^2) + ... + \frac{\alpha_i(g_{\omega_i}|A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i}\right]$ is always positive, (2) does not satisfy if $\frac{\rho_i^1}{\alpha_i(g_1)} > c$. Then, i is not best responding with action 0 if $\frac{\rho_i^1}{\alpha_i(g_1)} > c$. Given that $\frac{\rho_i^1}{\alpha_i(g_1)} \leq \frac{\rho_j^1}{\alpha_j(g_1)}\ \forall j \in N$, then (2) does not satisfy for any $i \in N$ when $\frac{\rho_i^1}{\alpha_i(g_1)} > c$. Then, there is not a symmetric equilibrium strategy σ: $\sigma(\tau_j) = 0$ and $W_{\tau_j}(g_2, \sigma) > W_{\tau_j}(g_1, \sigma)$ for any $j \in N$, where $g_2 \in B_j(g) : \alpha_j(g_1) = \alpha_j(g_2)$.

When $\sigma(\tau_i) = 1$ and $\mu = 0$, $W_{\tau_i}(g_z, \sigma) = \alpha_i(g_z)(1 - c) = \alpha_i(g_y)(1 - c)$, $\forall g_z, g_y \in B_i(g) : \alpha_i(g_z) = \alpha_i(g_y)$ and $\forall i \in N$. Then, there is not an equilibrium strategy σ such that $\sigma(\tau_i) = 1$ and $W_{\tau_i}(g_2, \sigma) > W_{\tau_i}(g_1, \sigma)$, for some $g_2 \in B_i(g) : \alpha_i(g_1) = \alpha_i(g_2)$ and some $i \in N$. As a result, $W(g_1, \sigma) = \sum_{i \in N} W_{\tau_i}(g_1, \sigma) \geq W(g_2, \sigma) = \sum_{i \in N} W_{\tau_i}(g_2, \sigma)$.

Strategic complements. Reasoning is analogous under strategic complements. Suppose $\sigma(\tau_i) = 1$ and $\exists g_2 \in B_i(g) : W_{\tau_i}(g_2, \sigma) > W_{\tau_i}(g_1, \sigma) \wedge \alpha_i(g_1) = \alpha_i(g_2)$. In equilibrium:

$$\frac{\alpha_i(g_1 \mid A_\sigma)}{\alpha_i(g_1)}\rho_i^1 + \frac{\alpha_i(g_2 \mid A_\sigma)}{\alpha_i(g_2)}\rho_i^2 + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i} = \frac{\alpha_i(g_2 \mid A_\sigma) - \pi}{\alpha_i(g_1)}\rho_i^1 + \frac{\alpha_i(g_2 \mid A_\sigma)}{\alpha_i(g_2)}\rho_i^2 + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i} \geq c$$

which implies that:

$$1 - c \geq 1 - \left[\frac{\alpha_i(g_2 \mid A_\sigma)}{\alpha_i(g_1)}(\rho_i^1 + \rho_i^2) + ... + \frac{\alpha_i(g_{\omega_i} \mid A_\sigma)}{\alpha_i(g_{\omega_i})}\rho_i^{\omega_i}\right] + \frac{\pi\rho_i^1}{\alpha_i(g_1)} \tag{3}$$

If $\frac{\rho_i^1}{\alpha_i(g_1)} > 1-c$, (3) does not hold and $\sigma(t_i)=1$ is not a best response for i. As $\frac{\rho_i^1}{\alpha_i(g_1)} \leq \frac{\rho_j^1}{\alpha_j(g_1)}$ $\forall j \in N$, (3) does not satisfy for any $j \in N$ whenever $\frac{\rho_i^1}{\alpha_i(g_1)} > 1-c$. Then, applying the same reasoning as for strategic substitutes the result follows. ■

Example 8.

A) Symmetric equilibria under Setting S′(b).

Network (a). Network (a) is integrated by types 1, 2, 3 and 4. Hence, there is an equilibrium if all these types are playing their best response to their neighbors' actions.

When all agents play the strategy σ_1, the expected utility of playing 0 of each type of i is the probability that at least one agent in $N_i(g)$ plays 1:

$$E_{U_i}(0,\sigma_1,I_i(g)) = \frac{\big|\{g \in I_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(3,2\big) \vee \tau_j(g) = \big(2;(3,3\big)\}\big|}{b_i(g)}$$

Similarly,

$$E_{U_i}(0,\sigma_2,I_i(g)) = \frac{\big|\{g \in I_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(3,2\big) \vee \tau_j(g) = \big(2;(3,3\big) \vee \tau_j(g) = \big(3;(3,2,2\big)\}\big|}{b_i(g)}$$

$$E_{U_i}(0,\sigma_3,I_i(g)) = \frac{\big|\{g \in I_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(2,2\big) \vee \tau_j(g) = \big(3;(2,2,2\big)\big|}{b_i(g)}$$

$$E_{U_i}(0,\sigma_4,I_i(g)) = \frac{\big|\{g \in I_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(2,2\big) \vee \tau_j(g) = \big(3;(2,2,2\big) \vee \tau_j(g) = \big(3;(3,2,2\big)\big|}{b_i(g)}$$

$$E_{U_i}(0,\sigma_5,I_i(g)) = \frac{\big|\{g \in I_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(2,2\big) \vee \tau_j(g) = \big(2;(3,3\big) \vee \tau_j(g) = \big(3;(3,2,2\big)\big|}{b_i(g)}$$

Table 6 and Table 7 provides $E_{U_i}\big(0,\sigma_x,I_i(g)\big)$ for $\sigma_x \in \{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}$. Each of these strategies is an equilibrium strategy for the range of cost specified in Table 8.

Note that players of types 1 and 2 must play different actions in equilibrium. If both types play 1(0), the expected utility of playing 0 of a type-one i is 1(0). Then, i is not best responding with action 1(0) but with action 0(1).

Similarly, there cannot exist an equilibrium in which type-two players and type-four players take both action 1: if type-two players play 1, best response of type-four players is playing 0, since type-four players have a type-two neighbor with probability 1 (see Figure 18). Analogously, there is not an equilibrium such that type-three players and type-four players play both action 1, as type-four players are linked to a type-three neighbor with probability 1.

Last, there is neither an equilibrium in which types 3, 4 and 5 take action 0(1): if all these types play 0(1), the expected utility of playing 0 of each type-three player is 0(1), since each neighbor of i is either type 4 or type 5 with probability 1. Hence, best response of each type-three player is not playing 0(1) but playing 1(0).

Considering this, the only additional equilibrium strategies that could exist are the followings:

- $\sigma_9 : \sigma_9(\tau^x) = 0$, for $x \in \{2,4,5\}$, and $\sigma_9(\tau^x) = 1$ for $x \in \{1,3\}$.
- $\sigma_{10} : \sigma_{10}(\tau^x) = 0$, for $x \in \{1,3,4\}$ and $\sigma_{10}(\tau^x) = 1$ for $x \in \{2,5\}$.
- $\sigma_{11} : \sigma_{11}(tau^x) = 0$, for $x \in \{1,3,4\}$ and $\sigma_{11}(\tau^x) = 1$ for $x \in \{1,5\}$.

where:

$$E_{U_i}(0,\sigma_9,I_i(g)) = \frac{\big|\{g \in B_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(2,2)\big) \wedge \tau_j(g) = \big(2;(3,3)\big)\}\big|}{b_i(g)}$$

$$E_{U_i}(0,\sigma_{10},I_i(g)) = \frac{\big|\{g \in B_i(g) : \exists j \in N_i(g) : \tau_j(g) = \big(2;(3,2)\big) \wedge \tau_j(g) = \big(3;(3,2,2)\big)\}\big|}{b_i(g)}$$

for all $i \in N$.

Table 9 shows $E_{U_i}(0, \sigma_9, I_i(g))$, $E_{U_i}(0, \sigma_{10}, I_i(g))$ and $E_{U_i}(0, \sigma_{11}, I_i(g))$ $\forall i \in N$. Observe that, when all agents play the strategy σ_9, type-one players are best responding with action 1 if $1 - c \geq 0.529$, while type-two players are best responding with action 0 if $1 - c \leq 0.5$. Since both things cannot occur simultaneously, strategy σ_9 is not an equilibrium strategy. Analogously, when all agents play the strategy σ_{10}, type-two players are best responding with action 1 if $1 - c \geq 0.916$, while type-three players are best responding with action 0 if $1 - c \leq 0.4$. Hence, strategy σ_{10} is not an equilibrium strategy. The same happens for strategy σ_{10}: in this case type-one players are best responding with action 1 if $1 - c \geq 0.529$, while type-three players are best responding with action 0 if $1 - c \leq 0.4$. Then, σ_{10} is not an equilibrium strategy.

Networks (b) and (c). Networks (b) and (c) are composed of types 2, 3 and 4. Hence, there exist an equilibrium if these types are playing their best response to their neighbors' actions.

As explained above, there cannot exist an equilibrium such that $\sigma(\tau_1) = \sigma(t_2) = 1$, $\forall i \in N$. However, since there are not type-one players neither in network (b) nor in network (c), strategies σ_6, σ_7 and σ_8 are equilibrium strategies for certain values of c in both networks. Table 10 provides $E_{U_i}(0, \sigma_6, B_i(g))$, $E_{U_i}(0, \sigma_7, B_i(g))$ and $E_{U_i}(0, \sigma_8, B_i(g)) \in N$. Table 11 provides the symmetric equilibria that exist for each range of c.

Table 6: Example 8

Type of i	$\underset{z\in\Omega_i(g)}{\cup}\overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_1,I_i(g))$	$E_{U_i}(0,\sigma_2,I_i(g))$	$E_{U_i}(0,\sigma_3,I_i(g))$
τ^1	Figure 15	$y(g_1 \mid \{i,j,l\}) = 6$	$\frac{\sum_{x=2}^{6} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{6} y(g_x\mid\{i,j,l\})} = \frac{96}{102}$		$\frac{\sum_{x\in\{1,2,6\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{6} y(g_x\mid\{i,j,l\})} = \frac{54}{102}$
		$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{2,6,4\}$			
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{3,5\}$.			
τ^2	Figure 16	$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{1,3,5,6\}$	$\frac{\sum_{x\in\{4,5,6,8\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{8} y(g_x\mid\{i,j,l\})} = \frac{72}{144}$	$\frac{144-y(g_7\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{8} y(g_x\mid\{i,j,l\})} = \frac{132}{144}$	$\frac{144-\sum_{x\in\{5,8\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{8} y(g_x\mid\{i,j,l\})} = \frac{108}{144}$
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{2,4,7,8\}$			
τ^3	Figure 17	$y(g_1 \mid \{i,j,l\}) = 4$	0	$\frac{\sum_{x=1}^{2} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{5} y(g_x\mid\{i,j,l\})} = \frac{28}{70}$	$\frac{\sum_{x=3}^{5} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{5} y(g_x\mid\{i,j,l\})} = \frac{42}{70}$
		$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{2,4\}$			
		$y(g_3 \mid \{i,j,l\}) = 6$			
		$y(g_5 \mid \{i,j,l\}) = 12$			
τ^4	Figure 18	$y(g_x \mid \{i,j,l,m\}) = 6$ for $x \in \{1,3,4,5,6,7\}$	1	1	0
		$y(g_x \mid \{i,j,l,m\}) = 3$ for $x \in \{2,8,9\}$			

Table 7: Example 8

Type of i	$\underset{z\in\Omega_i(g)}{\cup}\overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_4,I_i(g))$	$E_{U_i}(0,\sigma_5,I_i(g))$
τ^1	Figure 15	$y(g_1 \mid \{i,j,l\}) = 6$	$\frac{\sum_{x\in\{1,2,6\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{6} y(g_x\mid\{i,j,l\})} = \frac{54}{102}$	
		$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{2,4,6\}$		
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{3,5\}$.		
τ^2	Figure 16	$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{1,3,5,6\}$	1	$\frac{144-\sum_{x\in\{4,6\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x1}^{8} y(g_x\mid\{i,j,l\})} = \frac{108}{144}$
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{2,4,7,8\}$		
τ^3	Figure 17	$y(g_1 \mid \{i,j,l\}) = 4$	1	$\frac{\sum_{x=1}^{2} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{5} y(g_x\mid\{i,j,l\})} = \frac{28}{70}$
		$y(g_2 \mid \{i,j,l\}) = 24$ for $x \in \{1,3\}$		
		$y(g_3 \mid \{i,j,l\}) = 6$		
		$y(g_5 \mid \{i,j,l\}) = 12$		
τ^4	Figure 18	$y(g_x \mid \{i,j,l,m\}) = 6$ for $x \in \{1,3,4,5,6,7\}$	0	1
		$y(g_2 \mid \{i,j,l,m\}) = 3$ for $x \in \{2,8,9\}$		

Table 8: Symmetric equilibria in network (a) under Setting S'(b)

Equilibrium strategies	c
σ_1	$0{,}059 \leq c \leq 0{,}5$
σ_2	$0{,}059 \leq c \leq 0{,}084$
σ_3	$0{,}4 \leq c \leq 0{,}471$
σ_4	$c \leq 0{,}471$
σ_5	$0{,}25 \leq c \leq 0{,}471$

Table 9: Example 8

Type of i	$\underset{z \in \Omega_i(g)}{\cup} \overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0, \sigma_9, I_i(g))$	$E_{U_i}(0, \sigma_{10}, I_i(g))$	$E_{U_i}(0, \sigma_{11}, I_i(g))$
τ^1	Figure 15	$y(g_1 \mid \{i,j,l\}) = 6$	$\frac{\sum_{x \in \{1,2,6\}} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{6} y(g_x \mid \{i,j,l\})} = \frac{54}{102}$	$\frac{\sum_{x=2}^{6} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{5} y(g_x \mid \{i,j,l\})} = \frac{96}{102}$	$\frac{\sum_{x \in \{1,2,6\}} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{6} y(g_x \mid \{i,j,l\})} = \frac{54}{102}$
		$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{2,4,6\}$			
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{3,5\}$.			
τ^2	Figure 16	$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{1,3,5,6\}$	$\frac{\sum_{x \in \{1,2,3,7\}} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{8} y(g_x \mid \{i,j,l\})} = \frac{72}{144}$	$\frac{144 - \sum_{x \in \{7\}} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{8} y(g_x \mid \{i,j,l\})} = \frac{132}{144}$	$\frac{144 - \sum_{x \in \{4,6\}} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{8} y(g_x \mid \{i,j,l\})} = \frac{108}{144}$
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{2,4,7,8\}$			
τ^3	Figure 17	$y(g_1 \mid \{i,j,l\}) = 4$	0	$\frac{\sum_{x=1}^{2} y(g_x \mid \{i,j,l\})}{b_i(g) = \sum_{x=1}^{5} y(g_x \mid \{i,j,l\})} = \frac{28}{70}$ $= 0{,}4$	
		$y(g_1 \mid \{i,j,l\}) = 24$ for $x \in \{2,4\}$			
		$y(g_3 \mid \{i,j,l\}) = 6$			
		$y(g_5 \mid \{i,j,l\}) = 12$			
τ^4	Figure 18	$y(g_x \mid \{i,j,l,m\}) = 6$ for $x \in \{1,3,4,5,6,7\}$	1	1	0
		$y(g_x \mid \{i,j,l,m\}) = 3$ for $x \in \{2,8,9\}$			

Table 10: Example 8

Type of i	$\underset{z\in\Omega_i(g)}{\cup}\overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_6,I_i(g))$	$E_{U_i}(0,\sigma_7,I_i(g))$	$E_{U_i}(0,\sigma_8,I_i(g))$
τ^2	Figure 16	$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{1,3,5,6\}$	1	$\frac{\sum_{x\in\{4,6,7\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{8} y(g_x\mid\{i,j,l\})} = \frac{48}{144}$	$\frac{144-\sum_{x\in\{4,6,7\}} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{8} y(g_x\mid\{i,j,l\})} = \frac{96}{144}$
		$y(g_x \mid \{i,j,l\}) = 12$ for $x \in \{2,4,8,7\}$			
τ^3	Figure 17	$y(g_1 \mid \{i,j,l\}) = 4$	1	$\frac{\sum_{x=3}^{5} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{5} y(g_x\mid\{i,j,l\})} = \frac{42}{70}$	$\frac{\sum_{x=1}^{2} y(g_x\mid\{i,j,l\})}{b_i(g)=\sum_{x=1}^{5} y(g_x\mid\{i,j,l\})} = \frac{28}{70}$
		$y(g_x \mid \{i,j,l\}) = 24$ for $x \in \{2,4\}$			
		$y(g_3 \mid \{i,j,l\}) = 6$			
		$y(g_5 \mid \{a,b,c\}) = 12$			
τ^4	Figure 18	$y(g_x \mid \{i,j,l,m\}) = 6$ for $x \in \{1,3,4,5,6,7\}$	0	0	1
		$y(g_2 \mid \{i,j,l,m\}) = 3$ for $x \in \{2,8,9\}$			

Table 11: Symmetric equilibria in networks (b) and (c) under Setting S'(b)

Equilibrium strategies	c
σ_1	$c \leq 0{,}5$
σ_2	$c \leq 0{,}084$
σ_3	$c \geq 0{,}4$
σ_4	$0 \leq c \leq 1$
σ_5	$0{,}25 \leq c \leq 0{,}6$
σ_6	$0 \leq c \leq 1$
σ_7	$c \geq 0{,}67$
σ_8	$0{,}33 \leq c \leq 0{,}67$

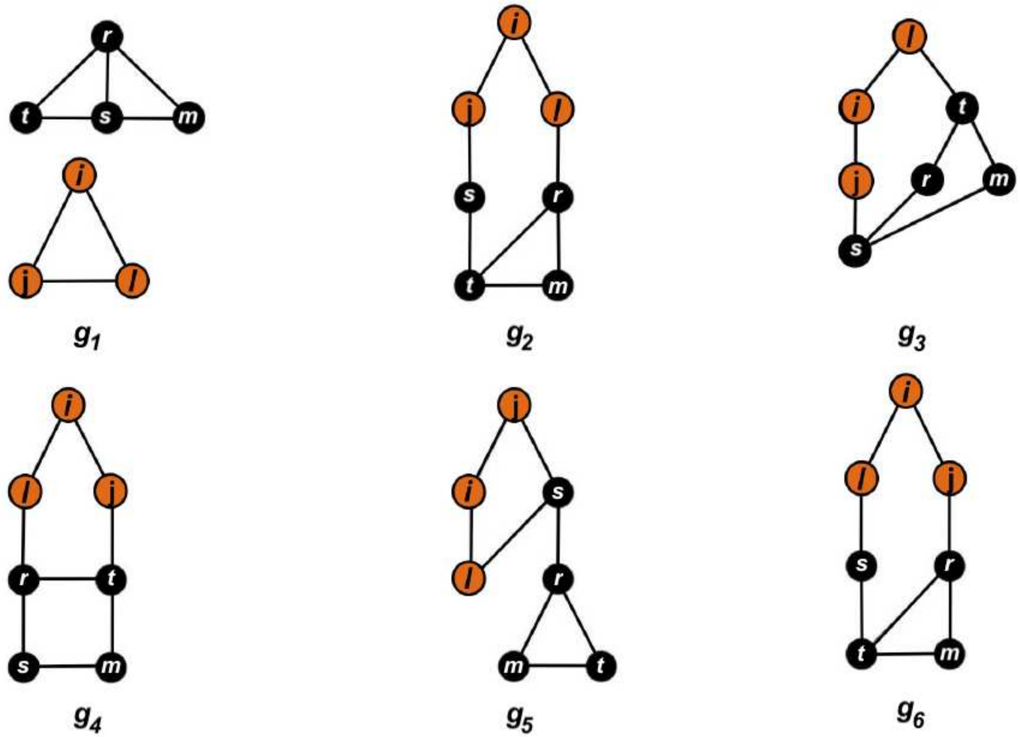

Figure 15: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(2,2)\big)$ under Setting B

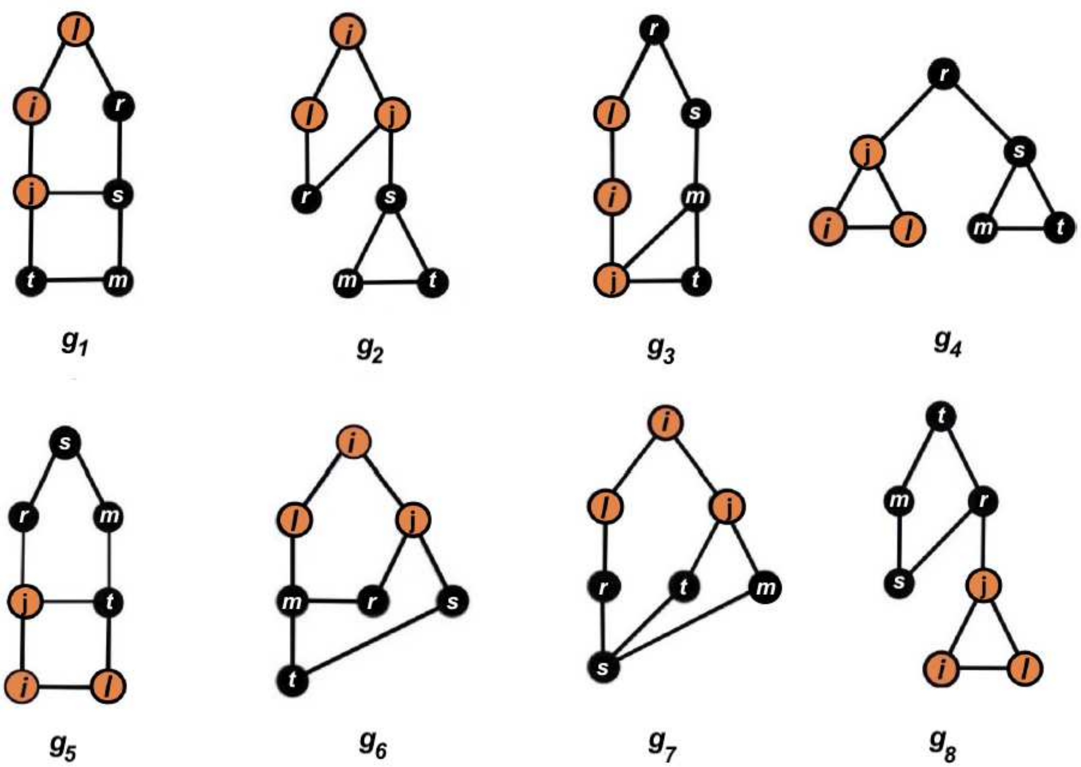

Figure 16: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(3,2)\big)$ under Setting B

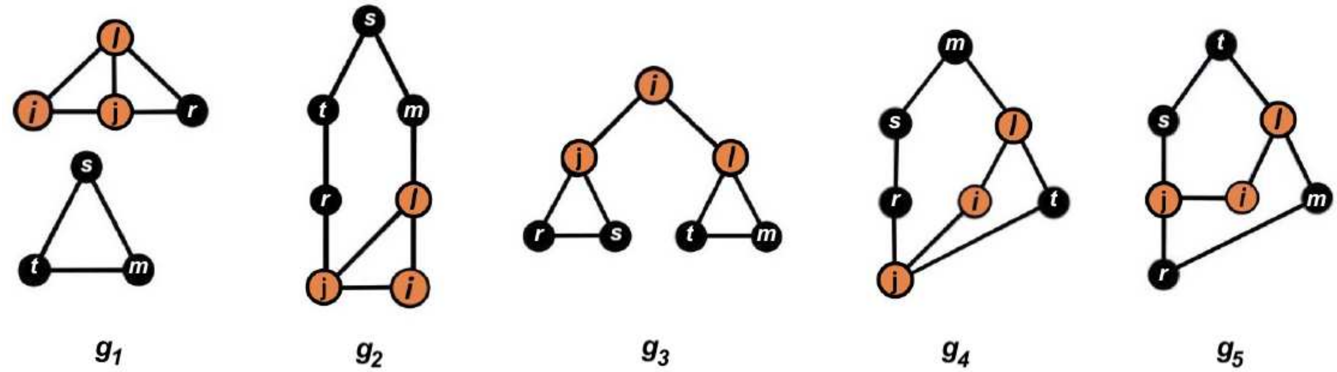

Figure 17: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(2,2)\big)$ under Setting B

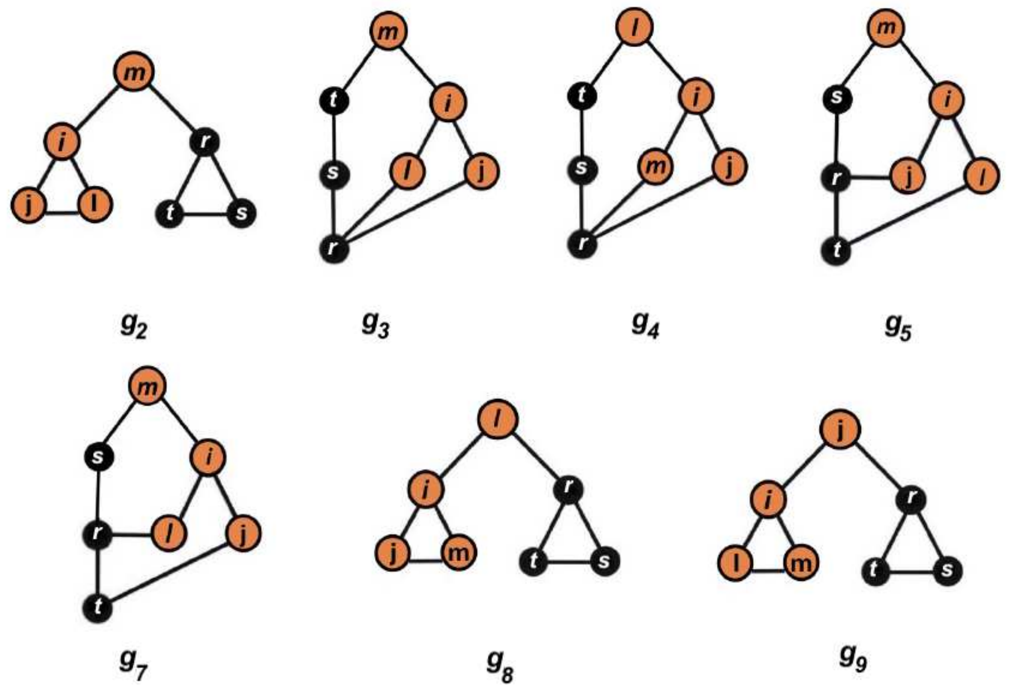

Figure 18: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(3,2)\big)$ under Setting B

B) Symmetric equilibria under Setting S(c).

Network (a). Table 11 and Table 12 show the expected utility of playing 0 of types 1 and 2 for each equilibrium strategy. Under Setting S'(c), type-three players observe that their two neighbors are not linked and consequently are type 4. If all type-four agents play 1, best response of type-three players is playing 0 and viceversa. Consequently, the strategies σ_{10} and σ_{11} defined in point A are not equilibrium strategies for any value of c.

Each type-four i can deduce the whole network from the $I_i(g)$. Hence, i can learn that two agents in $N_i(g)$ are type 2 and o is type 3. Then, a type-four i is best responding with action 1 if neither type-two players nor type-three players play action 1, and with action 0 if either type-two players or type-three players (or both) play 1. Table 13 shows the symmetric equilibria for each range of c. Strategies σ_6, σ_7 and σ_8 are not equilibrium strategies: if all players of type 1 and 2 play 0, the best response of a type-one player is playing 1.

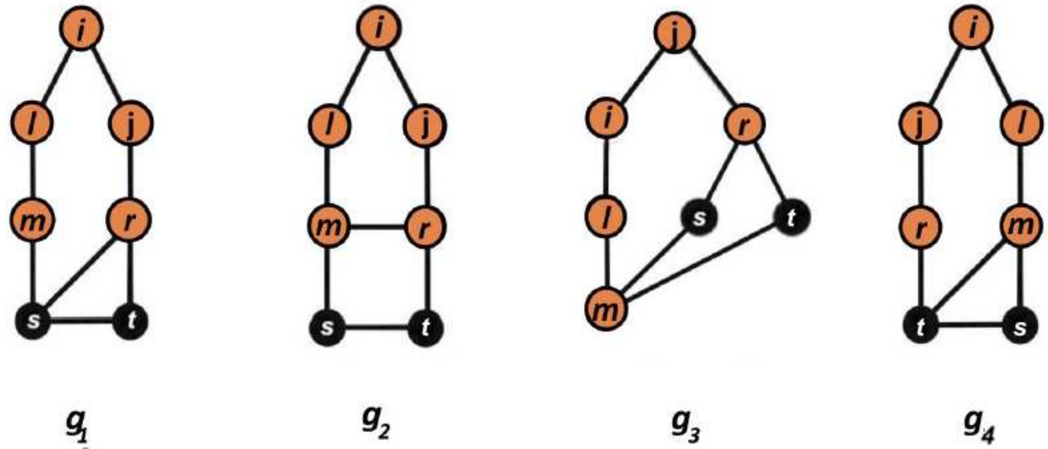

Figure 19: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(2,2)\big)$ under Setting S'(c)

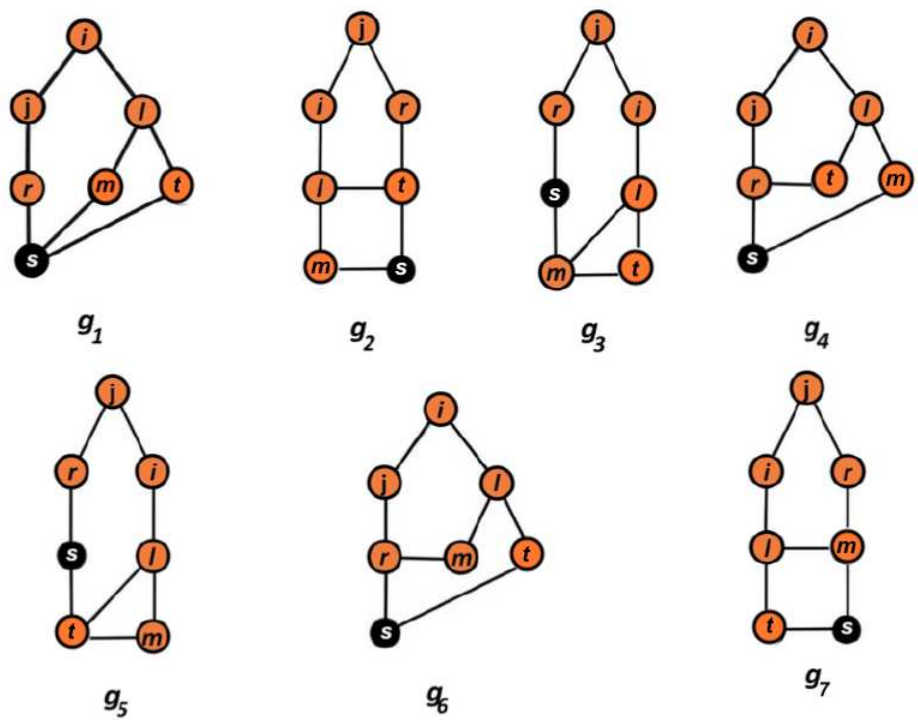

Figure 20: Set $\underset{z\in\Omega_i(g)}{\cup}\bar{B}_i^z(g)$ of each i of type $\big(2;(3,2)\big)$ under Setting S'(c)

Table 11: Example 8

Type of i	$\underset{z\in\Omega_i(g)}{\cup}\overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_1,I_i(g))$	$E_{U_i}(0,\sigma_2,I_i(g))$	$E_{U_i}(0,\sigma_3,I_i(g))$
τ^1	Figure 19	$y(g_x \mid \{i,j,l,m,r\}) = 2$ for $x \in \{1,2,4\}$ $y(g_3 \mid \{i,j,l,m,r\}) = 1$	1		$\frac{\sum_{x\in\{1,4\}} y(g_x\mid\{i,j,l,m,r\})}{b_i(g)=\sum_{x=1}^{4} y(g_x\mid\{i,j,l,m,r\})} = \frac{4}{7}$
τ^2	Figure 20	$y(g_x \mid \{i,j,l,m,r,t\}) = 1$ for $x \in \{1,2,4,5,6,7\}$	$\frac{\sum_{x=6}^{7} y(g_x\mid\{i,j,l,m,r,t\})}{b_i(g)=\sum_{x=1}^{7} y(g_x\mid\{i,j,l,m,r,t\})} = \frac{2}{7}$	$\frac{7-y(g_4\mid\{i,j,l,m,r,t\})}{b_i(g)=\sum_{x=1}^{7} y(g_x\mid\{i,j,l,m,r,t\})} = \frac{6}{7}$	1

Table 12: Example 8

Type of i	$\underset{z\in\Omega_i(g)}{\cup}\overline{B}_i^z(g)$	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_4,I_i(g))$	$E_{U_i}(0,\sigma_5,I_i(g))$	$E_{U_i}(0,\sigma_9,I_i(g))$
τ^1	Figure 19	$y(g_x \mid \{i,j,l,m,r\}) = 2$ for $x \in \{1,2,4\}$ $y(g_3 \mid \{i,j,l,m,r\}) = 1$	$\frac{\sum_{x\in\{1,4\}} y(g_x\mid\{i,j,l,m,r\})}{b_i(g)=\sum_{x=1}^{4} y(g_x\mid\{i,j,l,m,r\})} = \frac{4}{7}$		
τ^2	Figure 20	$y(g_x \mid \{i,j,l,m,r,t\}) = 1$ for $x \in \{1,2,4,5,6,7\}$	1	$\frac{7-\sum_{x\in\{4,6\}} y(g_x\mid\{i,j,l,m,r,t\})}{b_i(g)=\sum_{x=1}^{7} y(g_x\mid\{i,j,l,m,r,t\})} = \frac{5}{7}$	

Table 13: Symmetric equilibria in network (a) under Setting S'(c)

Equilibrium strategies	c
σ_1	$c \leq 0{,}714$
σ_2	$c \leq 0{,}143$
σ_3	$c \leq 0{,}429$
σ_4	$c \leq 0{,}429$
σ_5	$0{,}286 \leq c \leq 0{,}429$
σ_9	$0{,}286 \leq c \leq 0{,}429$

Network (b). Recall that network (b) is exclusively integrated by players of types 2, 3 and 4, and there exist an equilibrium as long as these types are playing their best response to their neighbors' actions.

Each type-two i observes that the degree-two agent in $N_i(g)$ is type 3. Then, there is not any symmetric equilibrium in which type-two players play 1: if all players of type 2 play 1, the expected utility of playing 0 of each type-two player is 1, so this player is best responding with action 1. The strategy σ_9 is not an equilibrium strategy : since $E_{U_i}(0, \sigma_9, I_i(g)) = 0$ for each type-two i, best response of i is playing action 1. Considering the networks in Figure 21:

$$E_{U_i}(0, \sigma_3, I_i(g)) = E_{U_i}(0, \sigma_7, I_i(g)) = \frac{y(g_0 \mid \{i, j, l, m\})}{b_i(g) = \sum_{x=0}^{1} y(g_0 \mid \{i, j, l, m\})} = \frac{3}{6} = \frac{1}{2}$$

$$E_{U_i}(0, \sigma_5, I_i(g)) = E_{U_i}(0, \sigma_8, I_i(g)) = \frac{y(g_1 \mid \{i, j, l, m\})}{b_i(g) = \sum_{x=0}^{1} y(g_0 \mid \{i, j, l, m\})} = \frac{3}{6} = \frac{1}{2}$$

and

$$E_{U_i}(0, \sigma_4, I_i(g)) = E_{U_i}(0, \sigma_6, I_i(g)) = 1$$

for each i of type 2.

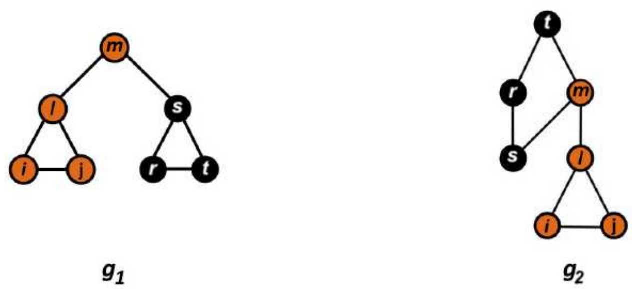

Figure 21: Set $\bigcup_{z \in \Omega_i(g)} \bar{B}_i^z(g)$ of each i of type $\big(2; (3, 2)\big)$ under Setting S'(c)

As for network (a), players of type 3 and 4 learn their neighbors' types from $I_i(g)$. Players of type 3 are best responding with action 0(1) if type-4 agents play 1(0) while type-four players 4 are best responding with action 1(0) if neither players with type 2 nor players with type 3 take this action.

Equilibrium strategies	cost
σ_3	$c \geq 0.5$
σ_4	$c \in [0, 1]$
σ_5	$c \geq 0.5$
σ_6	$c \in [0, 1]$
σ_7	$c \geq 0.5$
σ_8	$c \geq 0.5$

Table 14. Symmetric equilibria in network (b) under Setting S'(c)

Network (c). As in the previous case, network (c) is exclusively integrated by players of types 2, 3 and 4. If these types are playing their best response to their neighbors' actions there exist an equilibrium.

Under Setting S'(c), type-two players have identical beliefs about their neighbors' types in network (c) and in network (a), since the geometry created by their neighbors' links is identical in both networks. Then, Table 11 and Table 12 provide the expected utility of playing 0 of a type-two i when all agents play a strategy in $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_9\}$. As in network (a), players of type 3 and type 4 learn their neighbors' types from $I_i(g)$.

Table 14 lists the expected utility of playing 0 of each type-two i when all agents play σ_6, σ_7 or σ_8. Tables 15 provides the symmetric equilibrium strategies in network (c) for each cost value.

Figure	$y(g_x \mid n_{I_i})$	$E_{U_i}(0,\sigma_6,I_i(g))$	$E_{U_i}(0,\sigma_7,I_i(g))$	$E_{U_i}(0,\sigma_8,I_i(g))$
20	$y(g_x \mid \{i,j,l,m,r,o\}) = 1$ for $x \in \{1,2,4,5,6,7\}$	1	$\frac{\sum_{x\in\{1,4,6\}} y(g_x\mid\{i,j,l,m,r,t\})}{b_i(g)=\sum_{x=1}^{7} y(g_x\mid\{i,j,l,m,r,t\})} = \frac{3}{7}$	$\frac{7-\sum_{x\in\{1,4,6\}} y(g_x\mid\{i,j,l,m,r,t\})}{b_i(g)=\sum_{x=1}^{7} y(g_x\mid\{i,j,l,m,r,t\})} = \frac{4}{7}$

Table 14: Example 8

Network (c)	
	c
σ_1	$c \leq 0{,}714$
σ_2	$c \leq 0{,}143$
σ_3	$0 \leq c \leq 1$
σ_4	$0 \leq c \leq 1$
σ_5	$c \geq 0{,}286$
σ_6	$0 \leq c \leq 1$
σ_7	$c \geq 0{,}572$
σ_8	$c \geq 0{,}429$
σ_9	$c \geq 0{,}286$

Table 15: Example 8