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> *"A well regulated militia being necessary to the security of a free state, the right of the people to keep and bear arms shall not be infringed."*
>
> Second Amendment to the American Constitution

Abstract

This work constructs a non-cooperative, static game of gun control between the citizen and a pacifistic society characterised by law enforcement imperfection, by which the retention of firearms and the certitude of punishment against all crimes emerges both as a strict Nash equilibrium, in pure strategies, and as a strict dominant strategy equilibrium. The reason is that ratified by the Second Amendment to the American Constitution, discerning the necessity of a militia to the individual and societal security of a free state, by which the right of the people to keep and bear arms cannot be infringed.

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1. Gun control game: building blocks

1.1 Contribution and literature. This work derives a Nash equilibrium against gun control (i.e. gun laws) or in favour of gun ownership (i.e. gun rights), profiling the retention of firearms accompanied by the certainty of due penalties against any crime.

The closest work to it, in findings and methodology, is that by [16] Mailon and Wiseman. Strictly related literature also embeds [18] Taylor, [13] McDonald and [5] Chaudhri and Geanakoplos, as well as [1] Becker and [8] Ehrlich, by extension.

Broader literature related to it instead encompasses the following authors: [2] Bouton *et alii;* [3] Braga *et alii;* [4] Cerqueira and Coelho; [6] Cook and Ludwig; [7] Duggan; [9] Gius; [10] Hayo *et alii;* [11] Hesley and O'Sullivan; [12] Lee; [14] McQuoid *et alii;* [15] Mailon and Rubin; [17] Moorhouse and Wanner; [19] Zhukov.

Insofar as this work's result stem from a non-cooperative, static game, as hereby constructed, between the citizen and a pacifistic society characterised by law enforcement imperfection, it is a methodological and theoretical novelty or contribution across all kinds of pertinent academic literature (i.e. economic, political, sociological, historical, philosophical).

1.2 Game elements. As anticipated, the Nash equilibrium arises from a non-cooperative, static game with two players, being the citizen and society: $I = \{C,\ S\}$. Players are assumed to be rational and

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rationality itself is assumed to be common knowledge, that is, transfinite knowledge of reciprocal rationality is additionally assumed.

The citizen's strategies are "Guns" and "No guns", in reference to the ownership of guns or the retention of firearms: $S_C = \{G,\ \neg G\}$. It is crucial to stress that such two strategies are the legislation by society's state on gun ownership or its prohibition, being there no black market for guns by assumption (more anon).

Society's strategies are "Punishment" and no "No punishment", in reference to the penalty against the commission of a crime: $S_S = \{P,\ \neg P\}$. Strategy "Punishment" specifically models the credibility of punishment, nay, the certainty or certitude of inflicting an appropriate penalty, against any crime.

The strategy set product contains the combinations of all strategies, that is, all strategy profiles: $(s_C,\ s_S) \in \prod_{i=C}^{S} S_i = S_C \times S_S = \{(G,\ P),\ (G,\ \neg P),\ (\neg G,\ P),\ (\neg G,\ \neg P)\}$.

The payoff function is a bijection of the strategy set product into the real line: $\pi : \prod_{i=C}^{S} S_i \to \mathbb{R}$. As a consequence, the pure strategy game is a quadruple: $\Gamma_{PR} = \{I,\ \{S_i\}_{i=C}^{S},\ \pi\} = \{I,\ S_C,\ S_S,\ \pi\}$. Strategy profile payoffs are described below.

1.3 Simultaneous game. A static game is a simultaneous game, by which players do not know each other's actions, but only each other's strategies. The game thus features imperfect information. In practice, albeit, society, being the constituent assembly, plays before the citizen, who observes society's actions in relation to gun rights or laws.

Society can also play before the citizen by demarcating the judicial or legislative power of the state from a certain point in time onwards, by which an enforceable sentence of the state's supreme court of justice, hinging on constitutional gun rights or laws, explicit or implicit, or a law by the state's parliament, in view of such a sentence or in the absence thereof, legislates on firearm retention.

A move by society on gun ownership or prohibition, in a simultaneous or sequential context, is however captured by that of the citizen, as stressed above. As far as criminal sanctions may be concerned, since they are inflicted after the citizen has played society would by contrast pertinently play after the citizen.

Does such then suggest a dynamic game with perfect information instead, namely, a sequential game in which secondary players know the actions played by primary players? It does not, because it is a static game which derives a normative prescription on firearm retention, before time were to begin, once and for all.

More clearly, provided a society of pacifists in which neither law abiding citizens nor criminals, including regular felons, suicides and mass murderers, are to enjoy access to illegal firearms, but in which law enforcement is to be so imperfect as to fail in crime prevention, but not punishment, is such a society's (constituting) state to allow or to forbid the retention of firearms by citizens? The answer is supplied by the static game in question.

In addition, complete information is such that all players know each other's types, but because citizen types are of no interactive interest to society, which presents no types itself, they can be materially modelled through the citizen's strategy profile payoffs, being formally disregarded thereby together with the issue of complete or incomplete information.

Citizen types are of no interactive interest to society because a society legislating on fundamental rights deals with the natural person of the citizen in all of its abstraction, by which the summary character of the said citizen is the most such a society can envisage.

1.4 Model assumptions. For clarity, the model's assumptions are:

(i) a society characterised by pacifism, thereby being inherently, but not irrationally, averse to firearm retention;

(ii) law enforcement imperfection, by which the executive power of the state fails in preventing crime, but not in punishing it, on due intervention by the judicial power;

(iii) the threefold presence of criminals, comprising of regular felons, suicides and mass murderers, by which regular felons are the ordinary reason in favour of firearm retention, in the presence of law enforcement imperfection thus described, and suicides and mass murderers the ordinary reason against it;

(iv) the unrealistic exclusion of a black market for firearms even to the detriment of criminals, thereby reinforcing the assumption of utopian pacifism;

(v) the unrealistic possibility of successful self-defence from all felons, even mass murderers, without the use of firearms, innocuous as law abiding citizens have grown to be;

(vi) the unrealistic probability of suicide reduction, individually and societally, absent firearms, the tool for suicides being accessory and the despairing drive behind them being sheerly metaphysical.

A pacifistic society's inherent aversion to firearm retention is effectively irrational insomuch as it be based upon apriorism rather than ratiocination, as a consequence, the attribution of rationality even to such a society, functional to the present game's construction, can only fortify the upcoming Nash equilibrium against gun control.

The sole assumption which utopian pacifism can object to is that of law enforcement imperfection, which cannot however be conceded if realism is to be sufficiently preserved. While the unrealistic assumptions (i) of an outright exclusion of a black market for firearms, (ii) of transversal self-defence without the use of firearms and (iii) of suicidal diminution in the absence of firearms are to fortify the upcoming Nash equilibrium against gun control, law enforcement perfection would not merely destroy the upcoming Nash equilibrium against gun control but with it the sufficient realism in order for the model to be expedient, alongside common sense.

The contradictory concessions, nay, demands, of average post-modern pacifists in the regards of the suicides of euthanasia and the mass murders of abortion are ignored for scopes of a more successful interaction, for their consideration would only worsen the credibility of said pacifists' benevolence.

1.5 Citizen payoffs. The citizen's payoff under any strategy on the part of the citizen and society yields the sum of the individual payoffs pertinent to three citizen types, being (i) the non-criminal, (ii) the regular felon criminal and (iii) the mass murderer criminal.

The suicidal citizen type can be subsumed under the regular felon criminal citizen type, for the gain of the suicide criminal from gun ownership under strategy "Guns" by the citizen is homogeneous in scope to that of the regular felon criminal, regardless of society's strategy; in other words, both are advantaged by a gain of the same instrument towards the accomplishment of their unlawful ends.

Accordingly, whatever strategy may the citizen play, the suicide criminal's losses under strategy "Punishment" by society alter not those of the regular felon criminal, since by the very event of his decease the death of the suicide criminal adds nothing to that which the regular felon is already losing while still alive. Under strategy "No punishment" by society and either strategy on the part of the citizen the regular felon criminal instead incurs no loss and suffers no loss on account of the suicide criminal for the same reason.

In terms of sub-payoffs one consequently treats of non-criminal sub-payoff $\pi_{\neg CR}$, regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$. Formally: $\pi_C(s_C,\ s_S) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}}$.

The fact that the coefficients to all three sub-payoffs are unitary reflects an egalitarian consideration of all citizens, irrespective of their actions, being a principle especially characteristic of a pacifistic society. Inductively, the (pacifistic) assignment of a greater coefficient to the non-criminal type should reinforce the upcoming Nash equilibrium against gun control, but one does wonder whether the true spirit of pacifists may effectively crave for a smaller one after all; coefficient unity is therefore a good compromise. The differentiation between the citizen's payoffs contingent on the four strategy profiles nonetheless eventuates by means of the following cardinal transformations.

Starting from zero, the citizen's payoff under strategy profile "Guns, Punishment" is such that (i) non-criminal sub-payoff $\pi_{\neg CR}$ increases by one on account of the greater defence potential enjoyed, (ii) regular felon criminal sub-payoff $\pi_{CR_{RF}}$ increases by one on account of the greater offence potential enjoyed, but decreases by two on account of the punishment to be incurred, reflecting a net loss, and (iii) mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ increases by one on account of the greater offence potential enjoyed, but decreases by three on account of the punishment to be incurred, reflecting a net loss, greater than that of the regular felon: $\pi_C(G,\ P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(G,\ P) = 1 + (1-2) + [1-(2+1)] = 1 - 1 - 2 = -2$.

The absence of weapons other than firearms on the part of the citizen under strategy profile "Guns, Punishment" would cause non-criminal sub-payoff $\pi_{\neg CR}$, regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ to decrease by one half each on account of the relatively smaller defence or offence potential enjoyed by each: $\pi_C(G,\ P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(G,\ P) = (1-0.5) + (1-2-0.5) + [1-(2+1)-0.5] = 0.5 - 1.5 - 2.5 = -3.5$.

Starting from zero, the citizen's payoff under strategy profile "Guns, No punishment" is such that (i) non-criminal sub-payoff $\pi_{\neg CR}$ increases by one on account of the greater defence potential enjoyed, but decreases by one half on account of the absence of societal punishment, (ii) regular felon criminal sub-payoff $\pi_{CR_{RF}}$ increases by one on account of the greater offence potential enjoyed, without decreasing by two on account of the punishment to be otherwise incurred, and (iii) mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ increases by one on account of the greater offence potential enjoyed, without decreasing by three on account of the punishment to be otherwise incurred, greater than that of the regular felon: $\pi_C(G,\ \neg P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(G,\ \neg P) = (1 - 0.5) + 1 + 1 = 2.5$.

The absence of weapons other than firearms on the part of the citizen under strategy profile "Guns, No punishment" would cause non-criminal sub-payoff $\pi_{\neg CR}$, regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ to decrease by one half each on account of the relatively smaller defence or offence potential enjoyed by each: $\pi_C(G,\ \neg P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(G,\ \neg P) = (1 - 0.5 - 0.5) + (1 - 0.5) + (1 - 0.5) = 0.5 + 0.5 = 1$.

Starting from zero, the citizen's payoff under strategy profile "No guns, Punishment" is such that (i) non-criminal sub-payoff $\pi_{\neg CR}$ increases by one half on account of the regular felon criminal's partial disarmament and by another half on account of the mass murderer criminal's partial disarmament, commensurate with the latter's punishment relative gain, at a constant defence potential enjoyed, (ii) regular felon criminal sub-payoff $\pi_{CR_{RF}}$, at a constant offence potential enjoyed, decreases by two on account of the punishment to be incurred, reflecting a gross loss, and (iii) mass murderer criminal sub-payoff $\pi_{CR_{MM}}$, at a constant offence potential enjoyed, decreases by two and a half on account of of the punishment to be incurred in the absence of firearms, reflecting a gross loss, yet greater than that of the regular felon: $\pi_C(\neg G,\ P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(\neg G,\ P) = (0.5+0.5)+(0-2)+(0-2.5) = 1-2-2.5 = -3.5$.

While the offence potential of both the regular felon and the mass murderer be relatively thwarted by the absence of firearms, the punishment to be incurred by the regular felon under strategy profile "No guns, Punishment" remains unvaried and that of the mass murderer falls, although not sufficiently to match or overtake that of the regular felon.

The punishment to be incurred by the mass murderer falls while that of the regular felon does not, under strategy profile "No guns, Punishment", because the regular felon is assumed to succeed in his crime all the same, although with less ease, whereas the mass murderer's crime is to be substantially diminished by the absence of firearms.

The further absence of weapons other than firearms on the part of the citizen under strategy profile "No guns, Punishment" would cause non-criminal sub-payoff $\pi_{\neg CR}$, regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ to further decrease by one half each on account of the even smaller defence or offence potential enjoyed by each: $\pi_C(\neg G,\ P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(\neg G,\ P) = (0.5 + 0.5 - 0.5) + (0 - 2 - 0.5) + (0 - 2.5 - 0.5) = 0.5 - 2.5 - 3 = -6$.

Starting from zero, the citizen's payoff under strategy profile "No guns, No punishment" is such that (i) non-criminal sub-payoff $\pi_{\neg CR}$ increases by one half on account of the regular felon criminal's partial disarmament and by another half on account of the mass murderer criminal's partial disarmament, commensurate with the latter's punishment relative gain under strategy profile "No guns, Punishment", at a constant defence potential enjoyed, but also decreases by one half on account of the absence of societal punishment, (ii) regular felon criminal sub-payoff $\pi_{CR_{RF}}$ remains zero on account of the constant offence potential enjoyed, without decreasing by two on account of the punishment to be otherwise incurred, and (iii) mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ remains zero on account of the constant offence potential enjoyed, without decreasing by two and a half on account of the punishment to be otherwise incurred, yet greater than that of the regular felon: $\pi_C(\neg G,\ \neg P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(\neg G,\ \neg P) = (0.5 + 0.5 - 0.5) + 0 + 0 = 0.5$.

Non-criminal sub-payoff $\pi_{\neg CR}$ could not increase commensurately with the mass murderer criminal's punishment relative gain under strategy profile "No guns, No punishment", for there is not effectively any. The two and a half loss which the mass murderer criminal avoids under strategy profile "No guns, No punishment" is not a punishment gain, let alone relative, but a punishment non-loss due to the absence of punishment.

A punishment relative gain on the part of the mass murderer criminal under strategy "No guns" would by contrast be demarcated by a punishment reduction *per se* because of the absence of firearms (i.e. strategy profile "No guns, Punishment"). As a consequence, the mass murderer criminal's punishment non-loss is disjoined from the gain of the non-criminal under strategy profile "No guns, No punishment".

The further absence of weapons other than firearms on the part of the citizen under strategy profile "No guns, No punishment" would cause non-criminal sub-payoff $\pi_{\neg CR}$, regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ to further decrease by one half each on account of the even smaller defence or offence potential enjoyed by each: $\pi_C(\neg G,\ \neg P) = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C(\neg G,\ \neg P) = (0.5 + 0.5 - 0.5 - 0.5) + (0 - 0.5) + (0 - 0.5) = -0.5 - 0.5 = -1$.

Whatever the strategy played by society, all else unvaried, if a black market for firearms were not excluded then regular felon criminal sub-payoff $\pi_{CR_{RF}}$ and mass murderer criminal sub-payoff $\pi_{CR_{MM}}$ under strategy "No guns" on the part of the citizen would increase by one half, on account of the greater offence potential enjoyed as opposed to an otherwise constant one, albeit inferior to that enjoyed under strategy "Guns" on the same part: *ceteris paribus,* $\pi_C[\neg G,\ (P \veebar \neg P)] = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C[\neg G,\ (P \veebar \neg P)] = [(0.5+0.5)+(0.5-2)+(0.5-2.5)] \veebar [(0.5+0.5-0.5)+0.5+0.5] = (1-1.5-2) \veebar (0.5+0.5+0.5) = (-2.5 \veebar 1.5)$.

1.6 Society payoffs. Society's payoffs are generally characterised by two positive sub-payoffs, being that of punishment presence and that of gun presence: $\pi_P,\ \pi_G \in \mathbb{R}_{++}$. In detail, society's payoff under strategy profile "Guns, Punishment" yields negative sub-payoff $-\pi_G$ on account of the retention of firearms within a pacifistic society, being intrinsically averse to it: $\pi_S(G,\ P) = -\pi_G$. Such a payoff undergoes the cardinal transformation of negative one: $\pi_S(G,\ P) = -\pi_G \mapsto \pi_S(G,\ P) = -1$.

Society's payoff under strategy profile "Guns, No punishment" yields negative sub-payoff $-\pi_P - \pi_G$ on account of (i) the absence of punishment against the commission of crimes within a (pacifistic) society and (ii) the retention of firearms within a pacifistic society, being intrinsically averse to it: $\pi_S(G,\ \neg P) = -\pi_P - \pi_G$. While a non-pacifistic society deem the absence of punishment against the commission of crimes a loss, it is to be doubted whether a pacifistic society may effectively do so as well.

At face value, however, a pacifistic society is concerned about the absence of punishment against the commission of crimes no less than as it about the retention of firearms. In order to fortify the upcoming Nash equilibrium against gun control the pacifistic society at hand therefore admits as losses and attributes the same weight to the absence of punishment against the commission of crimes and to the retention of firearms, by which their respective coefficients are unitary. Such a payoff undergoes the cardinal transformation of negative two: $\pi_S(G,\ \neg P) = -\pi_P - \pi_G \mapsto \pi_S(G,\ \neg P) = -1 - 1 = -2$.

Derivatively, society's payoff under strategy profile "No guns, Punishment" yields a sub-payoff of zero on account of (i) the presence of punishment against the commission of crimes within a (pacifistic) society and (ii) the absence of firearm retention within pacifistic society: $\pi_S(\neg G,\ P) = 0$.

Society's payoff under strategy profile "No guns, No punishment" accordingly yields negative sub-payoff $-\pi_P$ on account of the absence of punishment against the commission of crimes within a (pacifistic) society: $\pi_S(\neg G,\ \neg P) = -\pi_P$. Such a payoff undergoes the cardinal transformation of negative one: $\pi_S(\neg G,\ \neg P) = -\pi_P \mapsto \pi_S(\neg G,\ \neg P) = -1$.

2. Nash equilibria and dominant strategies

2.1 Mixed strategies and best responses. John Forbes Nash Junior[1] made use of the Kakutani fixed point theorem to prove that every game with multiple, finite players and mixed strategies presents an equilibrium, a Nash equilibrium.

Therefore, by contraposition a game without of a Nash equilibrium is one of pure strategies, provided finite players, but a pure strategy game can feature a Nash equilibrium: assuming finite players, $(Mixed\ strategy\ game \longrightarrow Nash\ equilibrium) = (No\ Nash\ equilibrium \longrightarrow Pure\ strategy\ game)$, but $Pure\ strategy\ game \not\longrightarrow No\ Nash\ equilibrium$.

Mixed strategies are continuous probability assignments to pure strategies, as a consequence, they are uncountably infinite and their sets are thereby compact and convex, meeting Nash's use of the Kakutani fixed point theorem: $\forall i \in I,\ p : S_i \to [0,\ 1] \subset \mathbb{R}_+$, where p is a probability density function, such that, $\forall j \in [1,\ n] \subset \mathbb{N}_+,\ p(s_{ij}) = p_{ij} \in [0,\ 1] \subset \mathbb{R}_+$ and $\sum_{j=1}^{n} p_{ij} = 1$; $\forall i \in I,\ f : S_i \times [0,\ 1] \to \Sigma_i \subseteq \mathbb{R}_+$, where f is a probability assignment function, such that, $\forall j \in [1,\ n] \subset \mathbb{N}_+,\ f(s_{ij}p_{ij}) = \sigma_{ij} \in \Sigma_i \subseteq \mathbb{R}_+$ and $\sum_{j=1}^{n} s_{ij}p_{ij} = \sigma_i$. Mixed strategies are understood as randomisations over pure strategies. Alternatively,

[1] John Forbes Nash Junior, *Equilibrium Points in N-Person Games,* Proceedings of the National Academy of Sciences 36(1): 48-49, 1950.

pure strategies are understood as mixed strategies wherein particular pure strategies are played with a probability of one.

The mixed strategy sets of the citizen and of society are respectively denoted Σ_C and Σ_S. For notational simplicity, additionally: $p \equiv p_C$ and $q \equiv p_S$. The citizen's strategies are "Guns" and "No guns" and are respectively assigned probabilities p_1 and $p_2 = 1 - p_1$. Society's strategies are "Punishment" and "No punishment" and are respectively assigned probabilities q_1 and $q_2 = 1 - q_1$. The mixed strategy game is therefore a quadruple: $\Gamma_{MX} = \{I,\ \{\Sigma_i\}_{i=C}^{S},\ \pi\} = \{I,\ \Sigma_C,\ \Sigma_S,\ \pi\}$.

A best response function is a bijection of other players $\neg i$'s mixed strategy set into player i's mixed strategy set such that player i's mixed strategy is the best mixed strategy given other players $\neg i$'s mixed strategies, that is, a best response: $\forall i \in I,\ \rho_i : \Sigma_{\neg i} \to \Sigma_i$ such that $\sigma_i^* = \rho_i(\sigma_{\neg i}) = \sum_{j=1}^{n} s_{ij} p_{ij}^*$.

2.2 Nash equilibria. A Nash equilibrium is a strategy profile such that its payoff features player i's best response given other players $\neg i$'s best responses; it is thus a strategy profile of matching best responses: $\forall i \in I,\ NE := (\sigma_i^*,\ \sigma_{\neg i}^*)$ such that $\pi(\sigma_i^*,\ \sigma_{\neg i}^*)$.

A weak Nash equilibrium is a Nash equilibrium in which player i's best response is one or more: $\forall i \in I,\ NE_{WK} := (\sigma_i^*,\ \sigma_{\neg i}^*)$ such that $\pi_i(\sigma_i^*,\ \sigma_{\neg i}^*) \geq \pi_i(\sigma_i,\ \sigma_{\neg i}^*)$, whereby $\sigma_i^* \neq \sigma_i$ or $\sigma_i^* = \sigma_i$.

A strict Nash equilibrium is a Nash equilibrium in which player i's best response is one: $\forall i \in I,\ NE_{ST} := (\sigma_i^*,\ \sigma_{\neg i}^*)$ such that $\pi_i(\sigma_i^*,\ \sigma_{\neg i}^*) > \pi_i(\sigma_i,\ \sigma_{\neg i}^*)$, whereby $\sigma_i^* \neq \sigma_i$.

Strictly speaking, a Nash equilibrium in pure strategies is one in mixed strategies too, owing to its definition. For simplicity, however, a Nash equilibrium in mixed strategies is redefined such that all of its strategies are not pure: $\forall i \in I,\ NE_1 := (\sigma_i^*,\ \sigma_{\neg i}^*) \neq (s_i^*,\ s_{\neg i}^*)$, *ceteris paribus.*

Whenever strategy profile $(s_{ij},\ s_{\neg ij})$ be played with probabilities $p_{ij}^*,\ p_{\neg ij}^* \in (0,\ 1) \subset \mathbb{R}_{++}$, that is, in an open, real interval between zero and one, the mixed strategy Nash equilibrium is still delineated by strategy profile $(\sigma_i^*,\ \sigma_{\neg i}^*)$, wherein mixed strategies $\sigma_i^* = \sum_{j=1}^{n} s_{ij} p_{ij}^*$ and $\sigma_{\neg i}^* = \sum_{j=1}^{n} s_{\neg ij} p_{\neg ij}^*$.

A Nash equilibrium in semi-mixed strategies is correspondingly defined such that at least one of its strategies is mixed and the others are pure: $\forall i \in I,\ NE_2 := (\sigma_i^*,\ \sigma_{\neg i}^*),\ \exists \sigma_i^* \neq s_i^*$ and $\forall \sigma_{\neg i}^* = s_{\neg i}^*$, *ceteris paribus.*

Whenever strategy profile $(s_{ij},\ s_{\neg ij})$ be played with probabilities $p_{ij}^* \in (0,\ 1) \subset \mathbb{R}_{++}$ and $p_{\neg ij}^* = 1$ the semi-mixed strategy Nash equilibrium is still delineated by strategy profile $(\sigma_i^*,\ \sigma_{\neg i}^*)$, wherein mixed strategies $\sigma_i^* = \sum_{j=1}^{n} s_{ij} p_{ij}^*$ and $\sigma_{\neg i}^* = s_{\neg ij} p_{\neg ij}^* = s_{\neg ij} = s_{\neg i}^*$. A Nash equilibrium in pure strategies is lastly defined such that all of its strategies are pure: $\forall i \in I,\ NE_3 := (\sigma_i^*,\ \sigma_{\neg i}^*) = (s_i^*,\ s_{\neg i}^*)$, *ceteris paribus.*

2.3 Calculable probabilities and non-redundant strategies. A Nash equilibrium in mixed, semi-mixed or pure strategies, thus (re)defined, is moreover possible only if the cardinality of player i's "calculable probability" set $\bar{P}_i$, representing unknowns, is no smaller than that of other players $\neg i$'s "non-redundant strategy equation" set $\tilde{S}_{\neg i}$, representing equations, being itself no smaller than one.

In other words, if a Nash equilibrium in mixed, semi-mixed or pure strategies, thus (re)defined, is possible then the cardinality of other players $\neg i$'s "non-redundant strategy equation" set $\tilde{S}_{\neg i}$ is an element of the closed, natural interval between one and the cardinality of player i's "calculable probability" set $\bar{P}_i$.

Formally: $\forall i \in I,\ NE_{1,\,2,\,3} \longrightarrow n(\bar{P}_i) \geq n(\tilde{S}_{\neg i}) \geq 1$ or $n(\tilde{S}_{\neg i}) \in [1,\ n(\bar{P}_i)] \subset \mathbb{N}_+$, where $\bar{P}_i \subseteq P_i,\ n(P_i) = n(S_i),\ n(\bar{P}_i) = n(S_i \backslash \{s_{ij}\})$ and $\tilde{S}_{\neg i} \vdash \bar{S}_{\neg i} \subseteq S_{\neg i},\ \bar{S}_{\neg i}$ being other players $\neg i$'s "non-redundant strategy" set.

Otherwise, one would run into inconsistent overdetermination; but, in fact, a negation of such a syntactic implication's consequent is impossible by construction: $n(\tilde{S}_{\neg i}) \not< 1$ of necessity and therefrom $n(\bar{P}_i) \not< n(\tilde{S}_{\neg i}) \geq 1$, as to be seen. The additional constructional reasons are as follows.

Firstly, for clarity, player i's "calculable probabilities" are probabilities $\{p_{ij}\}_{j=1}^{n-1}$, thereby excluding player i's probability $p_{in} = 1 - \sum_{j=1}^{n-1} p_{ij}$. Likewise, other players $\neg i$'s "non-redundant strategy equations" are those identifications between other players $\neg i$'s expected payoffs originating from their non-redundant strategies, contained in other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$.

Player probabilities are moreover independent and, whenever analytically derived, can thus be multiplied to yield the probabilities of their respective strategy profiles, thence discerning mixed, semi-mixed or pure strategy Nash equilibria.

Secondly, if the cardinality of other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$ exceeds that of player i's "calculable probability" set $\bar{P}_i$ then other players $\neg i$'s expected payoffs originating from their non-redundant strategies, contained in other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$, are identified with one another in order for the cardinality of other players $\neg i$'s resulting "non-redundant strategy equation" set $\tilde{S}_{\neg i}$ to correspond to that of player i's "calculable probability" set $\bar{P}_i$: $n(\bar{S}_{\neg i}) > n(\bar{P}_i) \longrightarrow n(\tilde{S}_{\neg i}) = n(\bar{P}_i)$, by construction.

Indeed, the cardinality of player i's "calculable probability" set $\bar{P}_i$ cannot be exceeded by that of other players $\neg i$'s "non-redundant strategy equation" set $\tilde{S}_{\neg i}$, for further identifications with one another of other players $\neg i$'s expected payoffs originating from their non-redundant strategies, contained in other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$, would allow the two cardinalities to correspond: $n(\bar{P}_i) \not< n(\tilde{S}_{\neg i})$, by construction.

By reciprocally exploiting such a correspondence one can yield, although not always, the analytical derivation of player probabilities, thereby guaranteeing a Nash equilibrium in mixed, semi-mixed or pure strategies, thus (re)defined.

Thirdly, if the cardinality of player i's "calculable probability" set $\bar{P}_i$ is equal or greater than that of other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$ then the cardinality of player i's "calculable probability" set $\bar{P}_i$ exceeds that of other players $\neg i$'s "non-redundant strategy equation" set $\tilde{S}_{\neg i}$, because of the said identification with one another of other players $\neg i$'s expected payoffs originating from their non-redundant strategies, contained in other players $\neg i$'s "non-redundant strategy" set $\bar{S}_{\neg i}$, and player i's probabilities $\{p_{ij}\}_{j=1}^{n}$ are consequently underdetermined: $n(\bar{P}_i) \geq n(\bar{S}_{\neg i}) \longrightarrow n(\bar{P}_i) > n(\tilde{S}_{\neg i}) \longrightarrow \{p_{ij}\}_{j=1}^{n} \subset [0,\ 1] \subset \mathbb{R}_+$ are underdetermined.

2.4 Dominant strategies and sub-game perfect equilibria. A weak dominant strategy is at least one mixed strategy such that its payoffs feature player i's best mixed strategy regardless of other players $\neg i$'s mixed strategies; in other words, player i's best mixed strategy can be one or more: $\forall i \in I,\ DS_{WK} := \tilde{\sigma}_i$ such that $\pi_i(\tilde{\sigma}_i,\ \sigma_{\neg i}) \geq \pi_i(\sigma_i,\ \sigma_{\neg i})$, whereby $\tilde{\sigma}_i \neq \sigma_i$ or $\tilde{\sigma}_i = \sigma_i$.

A strict dominant strategy is a mixed strategy such that its payoffs feature player i's best mixed strategy regardless of other players $\neg i$'s mixed strategies; in other words, player i's best mixed strategy is exactly one: $\forall i \in I,\ DS_{ST} := \tilde{\sigma}_i$ such that $\pi_i(\tilde{\sigma}_i,\ \sigma_{\neg i}) > \pi_i(\sigma_i,\ \sigma_{\neg i})$, whereby $\tilde{\sigma}_i \neq \sigma_i$.

A weak dominant strategy equilibrium is the strategy profile of players i and $\neg i$'s weak dominant strategies: $\forall i \in I,\ DSE_{WK} := (\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i})$ such that $\pi_i(\tilde{\sigma}_i,\ \sigma_{\neg i}) \geq \pi_i(\sigma_i,\ \sigma_{\neg i})$ and $\pi_{\neg i}(\sigma_i,\ \tilde{\sigma}_{\neg i}) \geq \pi_{\neg i}(\sigma_i,\ \sigma_{\neg i})$.

A strict dominant strategy equilibrium is the strategy profile of players i and $\neg i$'s strict dominant strategies: $\forall i \in I,\ DSE_{ST} := (\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i})$ such that $\pi_i(\tilde{\sigma}_i,\ \sigma_{\neg i}) > \pi_i(\sigma_i,\ \sigma_{\neg i})$ and $\pi_{\neg i}(\sigma_i,\ \tilde{\sigma}_{\neg i}) > \pi_{\neg i}(\sigma_i,\ \sigma_{\neg i})$.

If a strategy profile is a dominant strategy equilibrium then it is a Nash equilibrium, but not *vice versa*. The reason is that other players $\neg i$'s mixed strategies, relative to player i's best mixed strategy, can be best responses and player i's mixed strategy, relative to other players $\neg i$'s best mixed strategies, can be a best response, yielding a Nash equilibrium, but players $\neg i$ and i's best responses are not all their other mixed strategies, excluding a dominant strategy equilibrium.

Formally: *ceteris paribus,* $\forall i \in I,\ (\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i}) \longrightarrow (\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i}) = (\sigma_i^*,\ \sigma_{\neg i}^*)$, since $\Diamond\pi(\tilde{\sigma}_i,\ \sigma_{\neg i}) = \pi(\tilde{\sigma}_i,\ \sigma_{\neg i}^*)$, $\Diamond\pi(\sigma_i,\ \tilde{\sigma}_{\neg i}) = \pi(\sigma_i^*,\ \tilde{\sigma}_{\neg i})$ and thus $\Diamond(\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i}) = (\sigma_i^*,\ \sigma_{\neg i}^*)$, but $(\sigma_i^*,\ \sigma_{\neg i}^*) \not\longrightarrow (\sigma_i^*,\ \sigma_{\neg i}^*) = (\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i})$, since $\pi(\tilde{\sigma}_i,\ \sigma_{\neg i}^*) \neq \pi(\tilde{\sigma}_i,\ \sigma_{\neg i}')$ and $\pi(\sigma_i^*,\ \tilde{\sigma}_{\neg i}) \neq \pi(\sigma_i',\ \tilde{\sigma}_{\neg i})$, respectively failing $\pi(\tilde{\sigma}_i,\ \sigma_{\neg i})$ and $\pi(\sigma_i,\ \tilde{\sigma}_{\neg i})$ for $(\tilde{\sigma}_i,\ \tilde{\sigma}_{\neg i})$.

The Nash equilibria of dynamic games with perfect information were termed sub-game perfect equilibria, by Reinhard Selten[2]. Sub-game perfect equilibria, which always exist, also arise in static games: for a given game, the set of sub-game perfect equilibria is a subset of the set of Nash equilibria. Consequently, the dynamic representations with perfect information of the game at hand do not need to be studied analytically.

3. Nash and dominant strategy equilibria: guns and punishment

Proposition 3.1 (Pure strategy Nash equilibria) *The game features one pure strategy Nash equilibrium, namely, strategy profile "Guns, Punishment". Formally:*

[2]Reinhard Selten, *Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit [Game Theory Treatment of an Oligopoly Model with Demand Inertia],* Zeitschrift für die Gesamte Staatswissenschaft 121: 301-24, 667-89, 1965.

$$(s^*_C,\ s^*_S) = (G,\ P). \tag{1}$$

Proof. Best responses in pure strategies are elaborated in relation to both players. Their matches are subsequently acknowledged as the game's pure strategy Nash equilibria.

Lemma 3.1.1 The citizen's best responses are the following. If society plays strategy "Punishment" the citizen's best response is then strategy "Guns", his payoff being relatively higher thereby: $s_S = P \longrightarrow s^*_C = G$, since $\pi_C(G,\ P) > \pi_C(\neg G,\ P) \longleftrightarrow -2 > -3.5$.

If society plays strategy "No punishment" the citizen's best response is then strategy "Guns", his payoff being relatively higher thereby: $s_S = \neg P \longrightarrow s^*_C = G$, since $\pi_C(G,\ \neg P) > \pi_C(\neg G,\ \neg P) \longleftrightarrow 2.5 > 0.5$.

Lemma 3.1.2 Society's best responses are the following. If the citizen plays strategy "Guns" society's best response is then strategy "Punishment", his payoff being relatively higher thereby: $s_C = G \longrightarrow s^*_S = P$, since $\pi_S(G,\ P) > \pi_S(G,\ \neg P) \longleftrightarrow -1 > -2$.

If the citizen plays strategy "No guns" society's best response is then strategy "Punishment", his payoff being relatively higher thereby: $s_C = \neg G \longrightarrow s^*_S = P$, since $\pi_S(\neg G,\ P) > \pi_S(\neg G,\ \neg P) \longleftrightarrow 0 > -1$.

Lemma 3.1.3 The matches of the two players' best responses in pure strategies yield strategy profile "Guns, Punishment", being the game's sole and strict pure strategy Nash equilibrium: $(s^*_C,\ s^*_S) = (G,\ P)$, since $s_C = (G \veebar \neg G) \longrightarrow s^*_S = P$ and $s_S = (P \veebar \neg P) \longrightarrow s^*_C = G$. *QED*

Table 1: Static gun control game

		(q_1)	$(1-q_1)$
	$C\backslash S$	P	$\neg P$
(p_1)	G	$(-2,\ -1)^*$	$(2.5,\ -2)$
$(1-p_1)$	$\neg G$	$(-3.5,\ 0)$	$(0.5,\ -1)$

Note. This is a static gun control game between the citizen and society. The citizen's strategies are "Guns" and "No guns". Society's strategies are "Punishment" and "No punishment". The sole and strict pure strategy Nash equilibrium, marked by an asterisk, is strategy profile "Guns, Punishment": $(s^*_C,\ s^*_S) = (G,\ P)$. There exist no Nash equilibria in mixed or semi-mixed strategies.

Pure strategy Nash equilibrium "Guns, Punishment" is a resounding refutation of the argument by which a pacifistic society as that yearned by certain parts of the American left must restrict the ownership of guns by citizens.

To be sure, the pacifistic society yearned by certain parts of the American left is delineated by regular felons, suicides and apparently even recurring mass murders, in which crime cannot be systematically prevented, but only systematically punished, and in which criminals are absurdly presumed not to obtain firearms in the event they were outlawed.

Does the game albeit present any Nash equilibria in mixed or semi-mixed strategies? The answer is found in the proposition below.

Proposition 3.2 (Mixed and semi-mixed strategy Nash equilibria) *The game features no Nash equilibria in mixed and semi-mixed strategies, namely, it features Nash equilibria only in pure strategies, being strategy profile "Guns, Punishment". Formally:*

$$(\sigma^*_C,\ \sigma^*_S) \overset{!}{=} (s^*_C,\ s^*_S) = (G,\ P). \tag{2}$$

Proof. Strategy expected payoffs are elaborated in relation to both players, feasibly solving for probabilities. Contingent on the obtainment of probabilities in relation to both players, best responses are subsequently elaborated. The game's Nash equilibria, in mixed, semi-mixed or pure strategies are finally acknowledged on account of all feasible probabilities and the matches of said best responses.

Lemma 3.2.1 The citizen's expected payoff by playing strategy "Guns" is the probabilistic sum of his payoffs across society's pure strategies: $\mathbb{E}[\pi(G)] = -2q_1 + 2.5(1-q_1) = -4.5q_1 + 2.5$.

The citizen's expected payoff by playing strategy "No guns" is the probabilistic sum of his payoffs across society's pure strategies: $\mathbb{E}[\pi(\neg G)] = -3.5q_1 + 0.5(1-q_1) = -4q_1 + 0.5$.

The two expected payoffs are expressed in terms of explicit probabilities. Such probabilities can be calculated by allowing the expected payoffs to correspond: $\mathbb{E}[\pi(G)] = \mathbb{E}[(\neg G)] \longleftrightarrow -4.5q_1 + 2.5 = -4q_1 + 0.5 \longrightarrow 2 = 0.5q_1 \longrightarrow q_1 = 4 \longrightarrow q_2 = 1 - q_1 = 1 - 4 = -3$ such that $\sum_{j=1}^{2} q_j = 1$, but

$\{q_j\}_{j=1}^2 \not\subset [0,\ 1] \subset \mathbb{R}_+$. Therefore, while sequence $\{q_j\}_{j=1}^2 \subset [0,\ 1] \subset \mathbb{R}_+$ and sum $\sum_{j=1}^2 q_j = 1$ exist by construction, the game cannot present Nash equilibria in mixed or semi-mixed strategies.

Lemma 3.2.2 For completeness, society's expected payoff by playing strategy "Punishment" is the probabilistic sum of his payoffs across the citizen's pure strategies: $\mathbb{E}[\pi(P)] = -1(p_1) + 0(1 - p_1) = -p_1$.

Society's expected payoff by playing strategy "No punishment" is the probabilistic sum of his payoffs across the citizen's pure strategies: $\mathbb{E}[\pi(\neg P)] = -2p_1 + (-1)(1 - p_1) = -p_1 - 1$.

The two expected payoffs are expressed in terms of explicit probabilities. Such probabilities can be calculated by allowing the expected payoffs to correspond: $\mathbb{E}[\pi(P)] = \mathbb{E}[(\neg P)] \longleftrightarrow -p_1 = -p_1 - 1 \longrightarrow 0 = -1$, which is a contradiction. Thus, while sequence $\{p_j\}_{j=1}^2 \subset [0,\ 1] \subset \mathbb{R}_+$ and sum $\sum_{j=1}^2 p_j = 1$ exist by construction, *a fortiori,* the game cannot present Nash equilibria in mixed or semi-mixed strategies.

Lemma 3.2.3 For completeness, the citizen's conditional best responses would be the following. If probability q_1 were greater than probability q_2 then probability p_1 would be unitary. More clearly, society would be more likely to play strategy "Punishment" and the citizen would respond by playing strategy "Guns", for his payoff would thereby be greater: $q_1 > q_2 \longrightarrow p_1 = 1$, *ceteris paribus,* since $\pi_C(G,\ P) > \pi_C(\neg G,\ P)$.

If probability q_2 were greater than probability q_1 then probability p_1 would be unitary. More clearly, society would be more likely to play strategy "No punishment" and the citizen would respond by playing strategy "Guns", for his payoff would thereby be greater: $q_2 > q_1 \longrightarrow p_1 = 1$, *ceteris paribus,* since $\pi_C(G,\ \neg P) > \pi_C(\neg G,\ \neg P)$.

If probabilities q_1 and q_2 were equal then probability p_1 would be unitary. More clearly, society would be equally likely to play one amongst strategies "Punishment" and "No punishment" and the citizen would respond by playing strategy "Guns", for his payoffs would thereby be greater: $q_1 = q_2 \longrightarrow p_1 = 1$, *ceteris paribus,* since (i) $\pi_C(G,\ P) > \pi_C(\neg G,\ P)$ and (ii) $\pi_C(G,\ \neg P) > \pi_C(\neg G,\ \neg P)$.

Lemma 3.2.4 For completeness, society's conditional best responses would be the following. If probability p_1 were greater than probability p_2 then probability q_1 would be unitary. More clearly, the citizen would be more likely to play strategy "Guns" and society would respond by playing strategy "Punishment", for his payoff would thereby be greater: $p_1 > p_2 \longrightarrow q_1 = 1$, *ceteris paribus,* since $\pi_S(G,\ P) > \pi_S(G,\ \neg P)$.

If probability p_2 were greater than probability p_1 then probability q_1 would be unitary. More clearly, the citizen would be more likely to play strategy "No guns" and society would respond by playing strategy "Punishment", for his payoffs would thereby be greater: $p_2 > p_1 \longrightarrow q_1 = 1$, *ceteris paribus,* since $\pi_S(\neg G,\ P) > \pi_S(\neg G,\ \neg P)$.

If probabilities p_1 and p_2 were equal then probability q_1 would be unitary. More clearly, the citizen would be equally likely to play one amongst strategies "Guns" and "No guns" and society would respond by playing strategy "Punishment", for his payoffs would thereby be greater: $p_1 = p_2 \longrightarrow q_1 = 1$, *ceteris paribus,* since (i) $\pi_S(G,\ P) > \pi_S(G,\ \neg P)$ and (ii) $\pi_S(\neg G,\ P) > \pi_S(\neg G,\ \neg P)$.

Lemma 3.2.5 The consideration of all feasible probabilities yields no mixed or semi-mixed strategy Nash equilibrium, because there lack feasible probabilities, and the hypothetical matches of the two players' conditional best responses would in fact yield strategy profile "Guns, Punishment", being the game's sole and strict pure strategy Nash equilibrium: $(\sigma_C^*,\ \sigma_S^*) \overset{!}{=} (s_C^*,\ s_S^*) = (G,\ P)$, by which $p_1 > p_2 \longrightarrow q_1 = 1$ and $q_1 > q_2 \longrightarrow p_1 = 1$ hypothetically as well. *QED*

Strategy profile "Guns, Punishment" is the game's sole and strict pure strategy Nash equilibrium and it thereby reinforces the omission of dynamic representations with perfect information of the same game, by which the only sub-game perfect equilibrium is necessarily strategy profile "Guns, Punishment".

An especial insight of the above propositions is that the sole and strict pure strategy Nash equilibrium "Guns, Punishment", together with the best responses of the citizen and society attendant on such a Nash equilibrium, holds irrespective of the punishment kind, as long as it be subjectively or societally commensurate with the potential crime, rather than objectively or meta-societally.

Indeed, the sub-payoffs proper to the citizen types of the regular felon criminal and the mass murderer criminal under strategy "Punishment" and strategy "No punishment", whatever the strategy of the citizen, have been modelled precisely on account of a subjective commensuration with the potential crime, not an objective one, fortifying the Nash equilibrium against gun control yet again.

In more detail, even if the punishment inflicted on regular felons and mass murderers were not an objectively commensurate penalty (e.g. fine, prison, death, especially for mass murderers), abiding not by natural and eternal law, but to the stage of corruption of the society in question, the citizen and society would still find it irrational or suboptimal to deviate from a condition of interactive decisions presenting firearm retention and punishment against the commission of crimes. Noteworthily, they would find it irrational or suboptimal reciprocally and not, that is, in terms of both Nash equilibria and dominant strategy equilibria, as the following proposition is to derive.

Proposition 3.3 (Dominant strategy equilibria) *The game features one dominant strategy equilibrium, namely, strategy profile "Guns, Punishment". Formally:*

$$(\tilde{\sigma}_C,\ \tilde{\sigma}_S) \stackrel{!}{=} (\tilde{s}_C,\ \tilde{s}_S) = (G,\ P). \tag{3}$$

Proof. Dominant strategies are elaborated in relation to both players. Their strategy profiles are then acknowledged as the game's dominant strategy equilibria.

Lemma 3.3.1 The citizen's mixed strategies are a probabilistic sum of his pure strategies: $\forall p_1 \in [0,\ 1] \subset \mathbb{R}_+,\ \sigma_C = Gp_1 + \neg G(1-p_1) = Gp_1 + \neg Gp_2$. Specifically, the citizen can play pure strategy "Guns", pure strategy "No guns" or a combination of the two: $\sigma_{C1} = G(1) + \neg G(1-1) = G,\ \sigma_{C2} = G(0) + \neg G(1-0) = \neg G$ or, $\forall p_1 \in (0,\ 1) \subset \mathbb{R}_{++},\ \sigma_{C3} = Gp_1 + \neg G(1-p_1)$.

Society's mixed strategies are a probabilistic sum of its pure strategies: $\forall q_1 \in [0,\ 1] \subset \mathbb{R}_+,\ \sigma_S = Pq_1 + \neg P(1-q_1) = Pq_1 + \neg Pq_2$. Specifically, society can play pure strategy "Punishment", pure strategy "No punishment" or a combination of the two: $\sigma_{S1} = P(1) + \neg P(1-1) = P,\ \sigma_{S2} = P(0) + \neg P(1-0) = \neg P$ or, $\forall q_1 \in (0,\ 1) \subset \mathbb{R}_{++},\ \sigma_{S3} = Pq_1 + \neg P(1-q_1)$.

Lemma 3.3.2 The citizen's expected payoffs under mixed strategy σ_{C3} and pure strategies by society are these: $\nexists\mathbb{E}[\pi_C(\sigma_{C3},\ P)] = -2p_1 + (-3.5)(1-p_1) = 1.5p_1 - 3.5$ and $\nexists\mathbb{E}[\pi_C(\sigma_{C3},\ \neg P)] = 2.5p_1 + 0.5(1-p_1) = 2p_1 + 0.5$, since $p_1 \notin (0,\ 1) \subset \mathbb{R}_{++}$, as shown above, thus, $\nexists\mathbb{E}[\pi(\sigma_{C3},\ P)]$ and $\nexists\mathbb{E}[\pi(\sigma_{C3},\ \neg P)]$.

Society's expected payoffs under mixed strategy σ_{S3} and pure strategies by the citizen are these: $\nexists\mathbb{E}[\pi_S(G,\ \sigma_{S3})] = -q_1 + (-2)(1-q_1) = q_1 - 2$ and $\nexists\mathbb{E}[\pi_S(\neg G,\ \sigma_{S3})] = 0(q_1) + (-1)(1-q_1) = q_1 - 1$, since $q_1 \notin (0,\ 1) \subset \mathbb{R}_{++}$, as shown above, thus, $\nexists\mathbb{E}[\pi(G,\ \sigma_{S3})]$ and $\nexists\mathbb{E}[\pi(\neg G,\ \sigma_{S3})]$.

The citizen's expected payoffs under pure strategies "Guns" and "No guns" and mixed strategy σ_{S3} by society are these: $\nexists\mathbb{E}[\pi_C(G,\ \sigma_{S3})] = -2q_1 + 2.5(1-q_1) = -4.5q_1 + 2.5$ and $\nexists\mathbb{E}[\pi_C(\neg G,\ \sigma_{S3})] = -3.5q_1 + 0.5(1-q_1) = -4q_1 + 0.5$, since $q_1 \notin (0,\ 1) \subset \mathbb{R}_{++}$, as shown above, thus, $\nexists\mathbb{E}[\pi(G,\ \sigma_{S3})]$ and $\nexists\mathbb{E}[\pi(\neg G,\ \sigma_{S3})]$.

Society's expected payoffs under pure strategies "Punishment" and "No punishment" and mixed strategy σ_{C3} by the citizen are these: $\nexists\mathbb{E}[\pi_S(\sigma_{C3},\ P)] = -p_1 + 0(1-p_1) = -p_1$ and $\nexists\mathbb{E}[\pi_S(\sigma_{C3},\ \neg P)] = -2p_1 + (-1)(1-p_1) = -p_1 - 1$, since $p_1 \notin (0,\ 1) \subset \mathbb{R}_{++}$, as shown above, thus, $\nexists\mathbb{E}[\pi(\sigma_{C3},\ P)]$ and $\nexists\mathbb{E}[\pi(\sigma_{C3},\ \neg P)]$.

The expected payoffs under strategy profile $(\sigma_{C3},\ \sigma_{S3})$ are finally these: $\nexists\mathbb{E}[\pi(\sigma_{C3},\ \sigma_{S3})] = [(-4.5q_1 + 2.5)p_1 + (-4q_1 + 0.5)(1-p_1),\ (q_1 - 2)p_1 + (q_1 - 1)(1-p_1)] = [(1.5p_1 - 3.5)q_1 + (2p_1 + 0.5)(1-q_1),\ -p_1q_1 + (-p_1 - 1)(1-q_1)] = [2p_1 - (4 + 0.5p_1)q_1 + 0.5,\ p_1 - q_1 - 1]$, since $p_1, q_1 \notin (0,\ 1) \subset \mathbb{R}_{++}$, as shown above.

Lemma 3.3.3 If society plays pure strategy "Punishment" the citizen's highest payoff is then found in pure strategy "Guns", relatively higher thereby: $s_S = P = \sigma_{S1} \longrightarrow \pi_C(G,\ P) = \pi_C(\sigma_{C1},\ P) = -2 > \pi_C(\neg G,\ P) = \pi_C(\sigma_{C2},\ P) = -3.5$ and $\nexists\mathbb{E}[\pi_C(\sigma_{C3},\ P)] = 1.5p_1 - 3.5$.

If society plays pure strategy "No punishment" the citizen's highest payoff is then found in pure strategy "Guns", relatively higher thereby: $s_S = \neg P = \sigma_{S2} \longrightarrow \pi_C(G,\ \neg P) = \pi_C(\sigma_{C1},\ \neg P) = 2.5 > \pi_C(\neg G,\ \neg P) = \pi_C(\sigma_{C2},\ \neg P) = 0.5$ and $\nexists\mathbb{E}[\pi_C(\sigma_{C3},\ \neg P)] = 2p_1 + 0.5$.

If society played mixed strategy σ_{S3} the citizen's highest payoff would then be found in accord with probabilities p_1 and q_1 : $s_S = \sigma_{S3} \longrightarrow \mathbb{E}[\pi_C(G,\ \sigma_{S3})] = \mathbb{E}[\pi_C(\sigma_{C1},\ \sigma_{S3})] = -4.5q_1 + 2.5 \gtreqless \mathbb{E}[\pi_C(\neg G,\ \sigma_{S3})] = \mathbb{E}[\pi_C(\sigma_{C2},\ \sigma_{S3})] = -4q_1 + 0.5 \gtreqless \mathbb{E}[\pi_C(\sigma_{C3},\ \sigma_{S3})] = 2p_1 - (4 + 0.5p_1)q_1 + 0.5$. Consequently, the citizen's dominant strategy is "Guns": $\tilde{\sigma}_C = G$.

Lemma 3.3.4 If the citizen plays pure strategy "Guns" society's highest payoff is then found in pure strategy "Punishment", relatively higher thereby: $s_C = G = \sigma_{C1} \longrightarrow \pi_S(G,\ P) = \pi_S(G,\ \sigma_{S1}) = -1 > \pi_S(G,\ \neg P) = \pi_S(G,\ \sigma_{S2}) = -2$ and $\nexists\mathbb{E}[\pi_S(G,\ \sigma_{S3})] = q_1 - 2$.

If the citizen plays pure strategy "No guns" society's highest payoff is then found in pure strategy "Punishment", relatively higher thereby: $s_C = \neg G = \sigma_{C2} \longrightarrow \pi_S(\neg G,\ P) = \pi_S(\neg G,\ \sigma_{S1}) = 0 > \pi_S(\neg G,\ \neg P) = \pi_S(\neg G,\ \sigma_{S2}) = -1$ and $\nexists\mathbb{E}[\pi_S(\neg G,\ \sigma_{S3})] = q_1 - 1$.

If the citizen played mixed strategy σ_{C3} society's highest payoff would then be found in accord with probabilities p_1 and q_1 : $s_C = \sigma_{C3} \longrightarrow \mathbb{E}[\pi_S(\sigma_{C3},\ P)] = \mathbb{E}[\pi_S(\sigma_{C3},\ \sigma_{S1})] = -p_1 \gtreqless \mathbb{E}[\pi_S(\sigma_{C3},\ \neg P)] = \mathbb{E}[\pi_C(\sigma_{C3},\ \sigma_{S2})] = -p_1 - 1 \gtreqless \mathbb{E}[\pi_S(\sigma_{C3},\ \sigma_{S3})] = p_1 - q_1 - 1$. Consequently, society's dominant strategy is "Punishment": $\tilde{\sigma}_S = P$.

Lemma 3.3.5 In sum, the matches of the two players' dominant strategies yield strategy profile "Guns, Punishment", being the game's strict dominant strategy equilibrium: $(\tilde{\sigma}_C,\ \tilde{\sigma}_S) \overset{!}{=} (\tilde{s}_C,\ \tilde{s}_S) = (G,\ P)$, since $s_C = [(G = \sigma_{C1}) \veebar (\neg G = \sigma_{C2})] \longrightarrow \tilde{\sigma}_S = \tilde{s}_S = P$ and $s_S = [(P = \sigma_{S1}) \veebar (\neg P = \sigma_{S2})] \longrightarrow \tilde{\sigma}_C = \tilde{s}_C = G$. *QED*

Table 2: Static gun control game with mixed strategies

$C \backslash S$	P	$\neg P$	$Pq_1 + \neg P(1 - q_1)$
G	$(-2,\ -1)^*$	$(2.5,\ -2)$	$[-4.5q_1 + 2.5,\ q_1 - 2]$
$\neg G$	$(-3.5,\ 0)$	$(0.5,\ -1)$	$[-4q_1 + 0.5,\ q_1 - 1]$
$Gp_1 + \neg G(1 - p_1)$	$[1.5p_1 - 3.5,\ -p_1]$	$[2p_1 + 0.5,\ -p_1 - 1]$	$[2p_1 - (4 + 0.5p_1)q_1 + 0.5,\ p_1 - q_1 - 1]$

Note. This is the static gun control game between the citizen and society with specified mixed strategies, which are although unsound. Citizen mixed strategy $Gp_1 + \neg G(1 - p_1) = Gp_1 + \neg Gp_2$ would be such that probability $p_1 \in (0,\ 1) \subset \mathbb{R}_{++}$, which is however false. Society mixed strategy $Pq_1 + \neg P(1 - q_1) = Pq_1 + \neg Pq_2$ would be such that probability $q_1 \in (0,\ 1) \subset \mathbb{R}_{++}$, which is however false. The strict dominant strategy equilibrium, marked by an asterisk, is strategy profile "Guns, Punishment": $(\tilde{\sigma}_C,\ \tilde{\sigma}_S) \overset{!}{=} (\tilde{s}_C,\ \tilde{s}_S) = (G,\ P)$.

The fact that strategy profile "Guns, Punishment" is both a strict Nash equilibrium, in pure strategies, and a strict dominant strategy equilibrium of the static gun control game at hand profoundly evidences the wisdom infused in the Second Amendment to the American Constitution, yea, the foresightedness of its authors in the Bill of Rights and of the advocates thereof.

It is albeit rather evident that, for any strategy played by society, if non-criminal sub-payoff $\pi_{\neg CR}$ under strategy "No guns" on the part of the citizen increased in a fashion more than commensurate with the mass murderer criminal's partial disarmament, however unjustifiably, given a constant defence potential enjoyed by the non-criminal and the regular felon criminal's partial disarmament, then the citizen's best response could become strategy "No guns": $s_S = (P \veebar \neg P) \longrightarrow s_C^* = \neg G$, provided $\pi_C[\neg G,\ (P \veebar \neg P)] > \{\pi_C[G,\ (P \veebar \neg P)] = (-2 \veebar 2.5)\}$; for instance, $\pi_C[\neg G,\ (P \veebar \neg P)] = \pi_{\neg CR} + \pi_{CR_{RF}} + \pi_{CR_{MM}} \mapsto \pi_C[\neg G,\ (P \veebar \neg P)] = [(0.5 + 3) + (0 - 2) + (0 - 2.5)] \veebar [(0.5 + 3 - 0.5) + 0 + 0] = [(3.5 - 2 - 2.5) \veebar 3] = (-1 \veebar 3)$, to be juxtaposed to $\pi_C[G,\ (P \veebar \neg P)] = (-2 \veebar 2.5)$, whence $\pi_C[\neg G,\ (P \veebar \neg P)] > \pi_C[G,\ (P \veebar \neg P)]$.

All else equal, the strict Nash equilibrium, in pure strategies, would become "No guns, Punishment", which would also be the strict dominant strategy equilibrium: $(s_C^*,\ s_S^*) = (\neg G,\ P)$, since $s_C = (G \veebar \neg G) \longrightarrow s_S^* = P$ and $s_S = (P \veebar \neg P) \longrightarrow s_C^* = \neg G$; $(\tilde{\sigma}_C,\ \tilde{\sigma}_S) \overset{!}{=} (\tilde{s}_C,\ \tilde{s}_S) = (\neg G,\ P)$, since $s_C = [(G = \sigma_{C1}) \veebar (\neg G = \sigma_{C2})] \longrightarrow \tilde{\sigma}_S = \tilde{s}_S = P$ and $s_S = [(P = \sigma_{S1}) \veebar (\neg P = \sigma_{S2})] \longrightarrow \tilde{\sigma}_C = \tilde{s}_C = \neg G$.

3.4 Gun rights. Sir William Blackstone[3] had held the right to keep and bear arms as one stemming from natural law, (i) entitling the citizen to self-defence and enabling him to (ii) resistance against oppression and (iii) the civic duty to act in concert in defence of the state. The first motivation is an entitlement of immediate comprehension and has acted as the basis for the payoff of the non-criminal citizen type in the static gun control game constructed above.

The third motivation hinges on a natural extension of the right to self-defence to the juridical person of the state, acting as a supplement to the armed forces of the same, for the state is none other than a composition of citizens through the familial unit, passing through the further, cultural channel of the village, town, city, province, region or federal state, man being the rational and social animal which Aristotle had highlighted him to be.

Indeed, insofar as renouncing to self-defence in principle may be discerned to be unjustifiable and unjust, as opposed to being occasionally justified in fact (e.g. martyrdom) and on such grounds even more justifiable and just in principle as an extraordinary exception to the principle, individual and societal self-defence is not only a natural right, but a natural duty too.

The second motivation, subsuming the other two, is by contrast the most controversial, owing to the delusively populistic or demagogic spirit of utopian pacifism having overwhelmed what used to called Christendom and, as of humanism, has gradually morphed into Western civilisation.

[3] https://avalon.law.yale.edu/18th_century/blackstone_bk1ch1.asp

Such a transmutation commenced with humanism and the Renaissance, it passed through the Reformation and the Age of Enlightenment, it continued in modern subjectivism and nihilism and it began finding its ineludible epilogue in postmodern structuralism and trans-humanism.

Be that as it may, whichever the form of government, were it a monarchy, an aristocracy or a timocracy, deviations from it (and resistance thereagainst) have ever been recognised as possible; such deviations are oppressive in character and are respectively itemised as tyrannies, oligarchies and democracies (e.g. plutocracies).

3.5 State tyranny. Resistance against oppression is most proximate to or in fact embryonal in the regards of a revolution against oppression, abiding by the Schoolmen's preceptive acknowledgement by which all positive law no longer participating of natural and eternal law ceases to bind: whenever widespread injustice become law and thereby inimical to moral law resistance and a revolution become an obligation.

The conditions presented by the Schoolmen for armed resistance and a revolution against tyranny, to be otherwise tolerated for a greater cause, are indeed (i) consistent tyranny, (ii) its evaluation as such by the timocratic members of society, (iii) the probability of success in overturning it and (iv) the expectation of a superior outcome in relation to the extant situation. The same principles transitively apply to armed resistance and revolutions against oligarchies and democracies.

James Madison notoriously argued that a federal army which had betrayed the mandate of society (and the federal government) would have been kept in check and in truth opposed by state militias, which would have thereby been able to repel the danger: "It may well be doubted, whether a militia thus circumstanced could ever be conquered by such a proportion of regular troops." (Source: https://guides.loc.gov/federalist-papers/text-41-50#s-lg-box-wrapper-25493411).

It is said that Thomas Jefferson was spuriously reported to have uttered the following words: "The strongest reason for the people to retain the right to keep and bear arms is, as a last resort, to protect themselves against tyranny in government." (Source: https://www.monticello.org/research-education/thomas-jefferson-encyclopedia/strongest-reason-people-to-retain-right-keep-and-bear-arms-spurious).

Regardless of their authenticity, their substance fully reflects the rationale of the Second Amendment to the American Constitution, which, besides subsuming self-defence of the individual and the state, in spite of the volume of case law in favour or against it, bears foundation in natural law and the use of sound logic and human reason, as this work has shown. Indeed, a virtuous man is a true philosopher, who is in turn defined as he who knows in light of the First Cause and judges straightly, ordaining everything to his end, especially his life, thereby living virtuously.

The shooting massacres, also referred to as mass shootings, occurred in the United States of America (USA) all through the past decade in particular (e.g. Las Vegas, Orlando nightclub, Sandy Hook Elementary School, Sutherland Springs church), whatever may be said of their veridicality, inevitably emerge as instrumental to the fomentation of emotion and to the obfuscation of reason, which instead lucidly spells the right to firearm retention accompanied by the credibility and certainty of a due punishment against any and all crimes.

The shooting massacres of 1996 Australia (i.e. Port Arthur Massacre) and 2019 New Zealand (i.e. Christchurch Mosque Shootings) for instance did just that, pulverising no less than a decisive portion of the cultural resistance against the abolition of firearm retention.

4. Conclusion

This work constructed a non-cooperative, static game of gun control between the citizen and a pacifistic society characterised by law enforcement imperfection, by which the retention of firearms and the certitude of punishment against all crimes emerges both as a strict Nash equilibrium, in pure strategies, and as a strict dominant strategy equilibrium.

The reason is that ratified by the Second Amendment to the American Constitution, discerning the necessity of a militia to the security of a free state. Such a security is both individual and societal, it is to be for crimes against the citizen whenever and wherever the state may fail to defend him and for crimes against the juridical person of the state, overt and covert, be they international wars or all forms of internal oppression. Consequently, the right of the people to keep and bear arms can and should not be infringed.

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