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The Market-Based Asset Price Probability

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Abstract

We consider the time-series records of the market trade values and volumes as the origin of the asset price stochasticity and describe random price properties through statistical moments of the market trade values and volumes. The market-based price probability differs from the conventional price probability proportional to number of trades at price *p*. We show that the market-based price probability results no correlations between *n-th* degrees of price and trade volume but doesn't cause statistical independence of price and trade volume. We derive the market-based correlation between price and squares of the trade volumes. Time-series records of the market trade values and volumes allow assess only finite number *m* of statistical moments and define first *m* price statistical moments. Approximations of the price characteristic function that match first *m* price statistical moments generate approximations of the market-based price probability. That approach unifies description of the asset price, returns, inflation and their volatilities as functions of the market trade values and volumes statistical moments. That describes the case when investor's market trades impact asset price probability. Market-based approach uncovers vital fault of the Value-at-Risk (VaR) as most conventional hedging tool. We show that accuracy of VaR assessment at horizon *T* is determined by precision of predictions of the market trade values and volumes statistical moments and depends on accuracy of macroeconomic forecasting. The market-based approach to price probability establishes direct economic ties between the asset pricing, market stochasticity and economic theory.

Keywords: asset price, price probability, returns, inflation, market trade

JEL : G12

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1. Introduction

The desires to predict the price for the next day, month or year are as old as market trade. Investors and traders for decades seek for reliable and precise price forecasts. However, it became clear that exact price guesses as well as tomorrow fortune forecasts are too fickle and variable. Exact price predictions for the next time term were replaced by set of probable price values. Centuries of asset pricing research (Dimson and Mussavian, 1999) track price probability up to Bernoulli's studies 1738, but possibly, Bachelier (1900) was one of the most influential paper that outlines probabilistic character of the price behavior and forecasting. "The probabilistic description of financial prices, pioneered by Bachelier." (Mandelbrot, et al., 1997). "in fact the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier." (Shiryaev, 1999). During last century the endless number of studies discussed asset pricing models and described different hypothesis, laws and properties of a random asset price (Kendall and Hill, 1953; Muth, 1961; Sharpe, 1964; Fama, 1965; Stigler and Kindahl, 1970; Black and Scholes, 1973; Merton, 1973; Tauchen and Pitts, 1983; Mackey, 1989; Friedman, 1990; Cochrane and Hansen, 1992; Campbell, 2000; Heaton and Lucas, 2000; Cochrane, 2001; Poon and Granger, 2003; Andersen et.al., 2005a; 2005b; 2006; Cochrane, 2005; Wolfers and Zitzewitz, 2006; DeFusco et.al., 2017; Weyl, 2019; Cochrane, 2022). Rigorous mathematical treatments of stochastic description and probabilistic modelling of asset price can be found in (Shiryaev, 1999; Shreve, 2004). We referred only a negligible part of endless studies on asset pricing.

Asset price dynamics is under impact of multiple factors that cause irregular or random price change during almost any time interval. That generates studies of price variations and dependence on market (Fama, 1965; Tauchen and Pitts, 1983; Odean, 1998; Poon and Granger, 2003; Andersen et.al., 2005b; DeFusco et.al., 2017), macroeconomic (Cochrane and Hansen, 1992; Heaton and Lucas, 2000; Diebold and Yilmaz, 2008), business cycles (Mills, 1946; Campbell, 1998), expectations (Muth, 1961; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014), trading volumes (Karpoff, 1987; Campbell et.al., 1993; Gallant et.al., 1992; Brock and LeBaron, 1995; Llorente et.al., 2001) and many other factors that for sure impact price trends and fluctuations. The line of factors and references can be continued (Goldsmith and Lipsey, 1963; Andersen et.al., 2001; Hördahl and Packer, 2007; Fama and French, 2015).

In this paper we consider the asset price probability density function (PDF). It seems to be one of the most common and well-studied issues of financial economics. Almost all standard probability distributions (Forbes, et.al., 1992; Walck, 2011) were tested to check how they can model, describe and predict price PDF. A lot was done but it is twice interesting to have a fresh look at the traditional matter. Indeed, since Bachelier (1900) the joint efforts of economists and statisticians were directed to uncover the "correct" model of price PDF and its change. May be credibility and domination of Bachelier and his famous followers focus research not to that side?

Conventional treatment (Shiryaev, 1999) of random price $p(t_i)$ time-series during the averaging interval Δ is based on frequency of trades at price p. Properties of a random variable are described by PDF or by set of statistical moments. If m_p – number of trades at price p and N- total number of trades during Δ then probability $P(p)$ of price p is assessed as:

$$P(p) \sim \frac{m_p}{N} \tag{1.1}$$

Price *n-th* statistical moments $E[p^n(t_i)]$ are determined as math expectations $E[..]$ of $p^n(t_i)$:

$$E[\, p^n(t_i)] = \frac{1}{N} \sum_{i=1}^{N} p^n(t_i) \tag{1.2}$$

That conventional *frequency-based* description of price (1.1; 1.2) as a random variable during Δ serves as ground for almost all asset-pricing models.

We don't critique notable studies but remind that the asset price is not a single, main and independent issue of economics and finance. Asset pricing is woven deeply into relations, laws and properties of the economics and finance. We demonstrate that the price probability determined by market stochasticity could take form far from (1.1; 1.2) and that results in significant distinctions valuable for investors and financial markets. We consider the asset price PDF problem as a result of market trades and not as a standing separately question. We take price determined by each market trade at time t_i by trade value $C(t_i)$, trade volume $U(t_i)$ and trade price $p(t_i)$ those match simple relations:

$$C(t_i) = p(t_i)U(t_i) \tag{1.3}$$

However, trivial equation (1.3) establishes important requirement on the PDF of time-series that match (1.3). Indeed, PDF of the time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ those match (1.3) cannot be determined *independently*. Given probabilities of the trade value and volume (1.3) determine the price probability that can be different from (1.2; 1.3). Given random properties of the market trade value $C(t_i)$ and volume $U(t_i)$ completely determine random properties of the market price $p(t_i)$.

That *market-based* approach to asset price probability establishes direct economic relations between stochasticity of the market trade value and volume on one hand and randomness of the market price on the other hand. Actually, we replace the problem: what is the "correct" form of the price probability determined by (1.2; 1.3) - by a different one. We consider how approximate description of the random market trade value and volume can approximate stochastic properties of the market price, but we don't study the specific properties of the market trade value and volume probabilities. In some sense our *market-based* description of the random price complements conventional assumption used by most asset pricing models (Cochrane, 2001, p.15): "the investor can freely buy or sell as much of the payoff x_{t+1} as he wishes, at a price p_t". This assumption states that any investor's trades don't impact market price probability. Such assumption is reasonable for a single investor who trades minor, negligible asset values and volumes to compare with market turnover. However, it is the investors, traders, market makers establish the market price. It is records of their market trade values and volumes determine random price properties. Big trades of investors make significant impact on market price. To take into account impact of investor's big trades on random price properties we consider price *n-th* statistical moments $p(t;n)=E[p^n(t_i)]$ determined by *n-th* statistical moments of market trade value $C(t;n)=E[C^n(t_i)]$ and volume $U(t;n)=E[U^n(t_i)]$. *N-th* statistical moments of trade value $C(t;n)$ and trade volume $U(t;n)$ with growth of n describe growing impact of big trades as *n-th* degree trade value $C^n(t_i)$ and volume $U^n(t_i)$. We describe asset price stochasticity during the averaging time interval Δ by price *n-th* statistical moments and introduce them as *n-th* degree of price $p^n(t_i)$ weighed average by *n-th* degree of trade volume $U^n(t_i)$ – generalization of the well-known volume weighted average price (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021). That definition of price *n-th* statistical moment equals ratio of sums of *n-th* degrees of trade values $\Sigma C^n(t_i)$ during interval Δ to sums of *n-th* degrees of trade volumes $\Sigma U^n(t_i)$ or equals ratio of *n-th* statistical moments of trade value $C(t;n)$ to *n-th* statistical moments of trade volume $U(t;n)$. That uncovers dependence of price statistical moments and price probability on the size of market trades and indicates impact of investor's big trades on price probability. Usage of *market-based* price probability model asset pricing beyond the conventional assumption: "the investor can freely buy or sell as much of the payoff x_{t+1} as he wishes, at a price p_t" (Cochrane, 2001, p.15).

In Sec.2 we describe how stochasticity of the market trade value and volume determine random properties of the asset price. In Sec. 3 and 4 we briefly consider consequences of our results on description of random properties of returns and inflation, asset pricing models and Value-at-Risk (VaR) as most conventional risk management tool. Sec. 5 – Conclusion. We assume that our readers are familiar with standard issues of asset pricing theory, probability theory, statistical moments, characteristic functions and etc. This paper for sure, is not for novices and we propose that readers already know or able find on their own definitions and explanations of the notions, terms and models that are not given in the text.

2. Market-based price probability

Let us consider the time-series of the market transactions with selected asset at moments t_i, $i=1,...$. Economic analysis of time-series has a long history and references (Davis, 1941; Anderson, 1971; Cochrane, 2005; Diebold, 2019) indicate author's preferences only. We take that time-series records describe the value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ of transaction at time t_i. The times t_i of the time-series records introduce the initial time division of the time axis. We study random properties of market trade using time-series records of performed transactions only. Thus, all possible factors those impact asset pricing are already imprinted into the time-series records of the market trade value $C(t_i)$ and volume $U(t_i)$. Let us study how stochasticity of the trade value $C(t_i)$ and volume $U(t_i)$ (1.3) determine the randomness of the price $p(t_i)$. For simplicity we take that transactions are performed at times t_i multiple of small interval ε:

$$t_i = \varepsilon \cdot i \;\; ; \;\; i = 0, 1, 2, ... \tag{2.1}$$

Time-series (2.1) establish time axis division multiple of ε. We study the market trade during time horizon T and assume that initial time division $\varepsilon << T$. Precise time division $\varepsilon << T$ results irregular random time-series and of little help for modelling price at long time horizon T. Description of market price at horizon T that can equal weeks, months or years requires aggregation of the initial market time-series during some reasonable time interval Δ that takes intermediate value (2.2):

$$\varepsilon < \Delta < T \tag{2.2}$$

Time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ aggregated or averaged during the interval Δ result time-series with time axis division multiple of Δ. For simplicity let take the interval Δ multiple of ε for some n as:

$$\Delta = 2n \cdot \varepsilon \;\; ; \;\; N = 2n + 1 \;\; ; \;\; \varepsilon \ll \Delta \ll T \tag{2.3}$$

Aggregation of time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ during Δ generates corresponding time-series at moments t_k and results time axis division multiple of Δ

$$t_k = \Delta \cdot k \;\; ; \;\; \Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \;\; ; \;\; k = 0, 1, 2, ... \tag{2.4}$$

Let us take that each member of time-series of the trade value $C(t_k)$ at time t_k (2.4) is determined by aggregation or averaging of time-series $C(t_i)$ during Δ_k (2.4). Total trade value $C(t_k)$ and total trade volume $U(t_k)$ during Δ_k are determined as

$$C(t_k) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k) = \sum_{i=1}^{N} U(t_i) \tag{2.5}$$

$$t_k - \frac{\Delta}{2} \le t_i \le t_k + \frac{\Delta}{2} \tag{2.6}$$

Due to our assumption (2.3) there are $N=2n+1$ members of time-series $C(t_i)$ or $U(t_i)$ in each time interval Δ_k. Let us consider time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ as random variables during Δ_k (2.4). We determine the mean trade value $C(t_k;1)$ and mean volume $U(t_k;1)$ at time t_k averaged during Δ_k using conventional *frequency-based* approach to probability similar to (1.1; 1.2) :

$$C(t_k;1) = E[C(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k;1) = E[U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{2.7}$$

We use *E[..]* to denote mathematical expectation. We underline that mean values of market trade (2.7) are determined during the interval Δ_k (2.2-2.4). Different choice of the interval Δ (2.2) may result in different average trade value and volume (2.7).

Probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ during Δ_k (2.6) are determined in a conventional way (Shiryaev, 1999; Shreve, 2004). Probability of trade value $P(C)$ and volume $P(U)$ are proportional to frequency of trades at value C and trades at volume U. If there are m_c trades at value C and m_u trades at volume U during Δ_k (2.6) then, due to (2.3) probabilities $P(C)$ and $P(U)$ are assessed as:

$$P(C) \sim \frac{m_c}{N} \quad ; \quad P(U) \sim \frac{m_u}{N} \tag{2.8}$$

Further we note conventional approach to probability (1.1; 2.8) as the *frequency-based* in contrary to the *market-based* definition of the price probability below. Statistical moments of the market trade value $C(t_i)$ and volume $U(t_i)$ for t_i during Δ_k (2.4; 2.6) are assessed as usual:

$$C(t_k;n) = E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t_k;n) = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \quad ; \; n = 1, \dots \tag{2.9}$$

For $n=1,2,\dots$ statistical moments (2.9) completely determine the trade value $C(t_i)$ and volume $U(t_i)$ as random variables during Δ_k (2.4).

Now let us consider random properties of the market price determined by stochasticity of market trade value and volume. As the mean price $p(t_k;1)$ or price 1-st statistical moment we take volume weighted average price (VWAP) that was introduced more than 30 years ago and is widely used now (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021). During the time interval Δ_k (2.6) VWAP $p(t_k;1)$ takes form:

$$p(t_k;1) = E[p(t_i)] = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \tag{2.10}$$

Using (1.3; 2.9) obtain equivalent form of VWAP $p(t_k;1)$ (2.10):

$$p(t_k;1) = E[p(t_i)] = \frac{\sum_{i=1}^{N} p(t_i)U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{\sum_{i=1}^{N} C(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{C(t_k;1)}{U(t_k;1)} \tag{2.11}$$

Relations (2.11) demonstrate that VWAP or 1-st price statistical moment $p(t_k;1)$ equals the ratio of total value $\Sigma C(t_i)$ during Δ_k (2.6) to total volume $\Sigma U(t_i)$ during Δ_k (2.6), or equally ratio of *1-st* statistical moments of the trade value $C(t_k;1)$ to *1-st* statistical moments of the trade volume $U(t_k;1)$ during Δ_k (2.6). These simple relations help us determine price *n-th* statistical moments $p(t_k;n) = E[p^n(t_i)]$.

Actually, just one VWAP $p(t_k;1)$ (2.10; 2.11) is not sufficient to define properties of price as a random variable during Δ_k (2.4; 2.6). To define random price properties one should determine *n-th* statistical moments for $n=1,2,3,\ldots$. Let us take *n-th* degree of (1.3) and obtain:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 1,2,3,\ldots \tag{2.12}$$

We extend VWAP (2.10) based on (1.3) and using (2.12) introduce price *n-th* statistical moments $p(t_k;n) = E[p^n(t_i)]$ for $n=1,2,\ldots$ as:

$$p(t_k;n) = E[p^n(t_i)] = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \tag{2.13}$$

Hence, due to (2.5; 2.9; 2.12) obtain that relations (2.13) equal:

$$p(t_k;n) = E[p^n(t_i)] = \frac{\sum_{i=1}^{N} p^n(t_i)U^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C(t_k;n)}{U(t_k;n)} \tag{2.14}$$

Relations (2.14) define price *n-th* statistical moment $p(t_k;n)$ as the ratio of total sum of *n-th* degree of trade values $\Sigma C^n(t_i)$ during Δ_k (2.6) to total sum of *n-th* degree of trade volumes $\Sigma U^n(t_i)$ during Δ_k (2.6), or equally as ratio of *n-th* statistical moments of the trade value $C(t_k;n)$ to *n-th* statistical moments of the trade volume $U(t_k;n)$ (2.9). Relations (2.13; 2.14) for all n determine price statistical moments $p(t_k;n)$ and hence determine properties of price as a random variable during Δ_k (2.4; 2.6).

Let us explain and justify the reasons in favor of our definition (2.13; 2.14) of the *market-based* price statistical moments $p(t_k;n)$. Relations (1.3; 2.12) mean that random properties of the time-series of trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ during Δ (2.6) are interdependent. Hence, given random properties of the market trade value $C(t_i)$ and volume $U(t_i)$ time-series should determine random properties of the asset price $p(t_i)$. However, current agent's habits, beliefs and traditions consider the price $p(t_i)$ as independent random variable regardless to stochasticity of trade value $C(t_i)$ and volume $U(t_i)$ time-series. As we mentioned above, such approximation of random price corresponds with conventional assumption (Cochrane, 2001) that any market deals of investor don't' impact market price. That assumption may be valid only for negligible size of market trades performed by investor to compare with market turnover during Δ. It is obvious, that description of big trades and description of market price as a whole must take into account the size of investor's market transactions, the size of trade value $C(t_i)$, volume $U(t_i)$. Thus conventional approximate description of the random market price $p(t_i)$ based on frequency-based $P(p)$ probability (1.1) should be complemented by description of random price properties as the result of market trade stochasticity.

To approximate random price properties during the averaging interval Δ_k (2.4) we determine price *n-th* statistical moments $p(t_k;n)=E[p^n(t_i)]$ through *n-th* statistical moments of market trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.9) that match (2.12). One can easy obtain that *n-th* statistical moments of the trade value $C(t_k;n)$ and volume $U(t_k;n)$ as well as sums of *n-th* degrees of trade values $\Sigma C^n(t_i)$ and volumes $\Sigma C^n(t_i)$ during Δ_k (2.6) with growing n more and more take into account the size of each big trade value $C^n(t_i)$ and volume $U^n(t_i)$. Thus relations (2.13; 2.14) project impact of big trades performed by investor on price statistical moments $p(t_k;n)$. Conventional VWAP introduces dependence of $p(t_k;1)=E[p(t_i)]$ (2.10; 2.11) on math expectations $E[C(t_i)]$ and $E[U(t_i)]$ of equation (1.3) or (2.12) for $n=1$. We take the similar interrelations (2.14) between math expectations of terms in (2.12) for all $n=2,3,\ldots$ Conventional VWAP $p(t_k;1)$ (2.10; 2.11) based on (1.3) or (2.12) for $n=1$ have sense of ratio of sum of all trade values $\Sigma C(t_i)$ to sum of all trade volumes $\Sigma U(t_i)$ during Δ. Our definitions of price *n-th* statistical moments $p(t_k;n)$ use equation (2.12) and determine math expectation of *n-th* degrees of price $E[p^n(t_i)]$ (2.13; 2.14) for all $n=2,3,..$ similar to (2.11) as ratio of sum of all *n-th* degrees of values $\Sigma C^n(t_i)$ to sum of all *n-th* degrees of volumes $\Sigma U^n(t_i)$ during Δ. We simply extend VWAP from (1.3) to (2.12) for all n=2,3,4,… The set of *n-th* statistical moments (2.13; 2.14) for all $n=1,2,3...$ completely describes price as a random variable during Δ and establishes direct dependence between statistical moments of the market trade value and volume and statistical moments of the market price.

Let us underline that definition of the VWAP $p(t_k;1)=E[p(t_i)]$ (2.11) hides important consequences. Actually, VWAP relations (2.11) result in zero correlations between time-series of the price $p(t_i)$ and volume $U(t_i)$ during Δ_k (2.4). Indeed, from (1.3; 2.7; 2.11) obtain:

$$E[C(t_i)] = E[p(t_i)U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) \equiv \frac{1}{\sum_{i=1}^{N} U(t_i)}\sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) \equiv E[p(t_i)]E[U(t_i)] \quad (2.15)$$

Hence (2.15) causes no correlations between time-series of market price and trade volume:

$$corr\{pU\} = E[p(t_i)U(t_i)] - E[p(t_i)]E[U(t_i)] = 0 \quad (2.16)$$

Zero correlation (2.15; 2.16) between VWAP and trade volume is a result of the price averaging procedure (2.7; 2.10; 2.11). Actually, numerous studies "observe" correlations between price and trading volume (Tauchen and Pitts, 1983; Karpoff, 1987; Gallant et.al., 1992; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). That underlines the different approaches to price averaging procedure. We repeat that usage of VWAP states no correlations between volume and price.

It is obvious that definitions of price *n-th* statistical moments $p(t_k;n)=E[p^n(t_i)]$ during Δ_k (2.13; 2.14) as *n-th* price degree $p^n(t_i)$ weighed by *n-th* degree of the trade volume $U^n(t_i)$ result in zero correlations between time-series of the *n-th* degree of the price $p^n(t_i)$ and trade volume $U^n(t_i)$. Using (2.9; 2.12; 2.13) one can easy obtain for all $n=1,2,3,\ldots$

$$E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) \equiv \frac{1}{\sum_{i=1}^{N} U^n(t_i)}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \equiv E[p^n(t_i)]E[U^n(t_i)] \equiv p(t_k;n)U(t_k;n) \quad (2.17)$$

Hence, for all n correlations $corr\{p^nU^n\}$ between time-series of *n-th* degree of price $p^n(t_i)$ and trade volume $U^n(t_i)$ during Δ_k (2.6) equal zero:

$$corr\{p^nU^n\} = E[p^n(t_i)U^n(t_i)] - E[p^n(t_i)]E[U^n(t_i)] = 0 \quad (2.18)$$

Zero correlations (2.18) provide a different way for derivation of price statistical moments $p(t_k;n)$. Indeed, from (2.14) follows:

$$C(t_k;n) = p(t_k;n)U(t_k;n) \quad (2.19)$$

Relations (2.9; 2.19) define price *n-th* statistical moments $p(t_k;n)$ during Δ_k (2.6) that can be derived directly form (2.12) and zero correlations (2.18). Taking math expectation of (2.12):

$$C(t_k;n) = E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = E[p^n(t_i)]E[U^n(t_i)] = p(t_k;n)U(t_k;n) \quad (2.20)$$

Thus zero correlations (2.18) can be treated as consequences of the definition (2.14) and as assumptions for deriving (2.19; 2.20) as mathematical expectation of (2.12).

The choice of price averaging procedure between the *frequency-based* and the *market-based* approaches determines different random properties of price. However, zero correlations (2.18) don't cause statistical independence between the trade volume and price random variables. As example, we derive correlation $corr\{pU^2\}$ between time-series of market price $p(t_i)$ and squares of trade volumes $U^2(t_i)$ during Δ_k (2.6):

$$E[pU^2] = E[p]E[U^2] + corr\{pU^2\} = E[CU] = E[C]E[U] + corr\{CU\}$$

$$corr\{pU^2\} = E[CU] - p(t_k;1)U(t_k;2) = corr\{CU\} - p(t_k;1)\sigma^2(U) \quad (2.21)$$

Thus correlations between time-series of price $p(t_i)$ and squares of trade volume $U^2(t_i)$ can be positive only if high positive correlations $corr\{CU\}$ between trade value $C(t_i)$ and volume $U(t_i)$ are bigger then mean price $p(t_k;1)$ multiplied by trade volume volatility $\sigma^2(U)$:

$$corr\{pU^2\} > 0 \;\Leftrightarrow\; corr\{CU\} > p(t_k;1)\sigma^2(U) > 0$$

$$\sigma^2(U) = U(t_k;2) - U^2(t_k;1)$$

Otherwise correlations are always negative: $corr\{pU^2\}$ <0. Correlation $corr\{CU\}$ between trade value $C(t_i)$ and volume $U(t_i)$ time-series is assessed using conventional *frequency-based* approach (2.9):

$$corr\{CU\} = E[CU] - E[C]E[U] = \frac{1}{N}\sum_{i=1}^{N} C(t_i)U(t_i) - C(t_k;1)U(t_k;1)$$

It is well known, that the set of *n-th* statistical moments of a random variable for all *n=1,2,...* determines its characteristic function as Taylor series (Shephard, 1991; Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). Price characteristic function $F(t_k;x)$ as Taylor series at moment t_k takes form:

$$F(t_k;x) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(t_k;n)\, x^n \tag{2.22}$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F(t_k;x)|_{x=0} \tag{2.23}$$

Market-based asset price characteristic function $F(t_k;x)$ (2.22; 2.23) depends on set of *n-th* statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ and the price PDF also depends on market trade statistical moments (2.9). Any predictions of the *market-based* asset price PDF at horizon T should match forecasts of *n-th* statistical moments of the market trade value and volume at same horizon T.

However, exact expressions of the market trade value and volume probability density functions (PDF) are unknown. Any records of market trade time-series for any given time interval Δ_k (2.4) permit assess only finite number m of the market trade statistical moments $C(t_k;n)$ and $U(t_k;n)$, $n=1,2,...m$. Hence, one can operate by finite number m of price statistical moments $p(t_k;n)$ only. Finite number m of price statistical moments describes *approximations* of the price characteristic function $F_m(t_k;x)$. Let us take m-approximations of the price characteristic function $F_m(t_k;x)$ (2.24; 2.25) that generate m-approximations of the price probability measure $\eta_m(t_k;p)$ (2.26; 2.27) (Olkhov, 2021b):

$$F_m(t_k;x) = \exp\left\{\sum_{j=1}^{m} \frac{i^j}{j!}\, a_j\, x^j - b\, x^{2n}\right\} \quad ; \quad m = 1,2,..;\; b \geq 0;\; 2n > m \tag{2.24}$$

In (2.24) $b=0$ can be zero if and only if m is even and $i^m a_m$ in (2.24) is negative. For each approximation $F_m(t_k;x)$ terms a_j , $j=1,..m$ in (2.24) depend on price statistical moments $p(t_k;j)$, $j\leq m$ and match relations (2.25). The term bx^{2n} >0 doesn't impact relations (2.25) but ensures integrability of approximation $F_m(t_k;x)$ of characteristic function and existence of the approximation of the price probability measures $\eta_m(t_k;p)$ as Fourier transforms (2.26). Uncertainty of the coefficient $b\geq 0$ and degree $2n$ of x^{2n}, $2n>m$ in (2.24) illustrates well-known fact that m statistical moments don't define characteristic function and probability measure of a random variable precisely. Expressions (2.24) present the set of approximations

$F_m(t_k;x)$ of the price characteristic functions with different $b \geq 0$ and $2n > m$ and corresponding set of approximations of the price probability measures $\eta_m(t_k;p)$ those match (2.25; 2.26).

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F_m(t_k;x)|_{x=0} \quad ; \quad n \leq m \tag{2.25}$$

Such m-approximation $F_m(t_k;x)$ of the characteristic function reproduces first m price statistical moments (2.23) and generates m-approximation of the PDF $\eta_m(t_k;p)$ at moment t_k during the interval Δ_k (2.4):

$$\eta_m(t_k;p) = \frac{1}{\sqrt{2\pi}} \int dx\, F_m(t_k;x) \exp(-ixp) \tag{2.26}$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \int dp\ p^n \eta_m(t_k;p) \quad ; \quad n \leq m$$

For $n=2$ approximation of the price characteristic function $F_2(t_k;x)$ takes simple form (2.27).

$$F_2(t_k;x) = \exp\left\{i\, p(t_k;1)x - \frac{\sigma_p^2(t_k)}{2} x^2\right\} \tag{2.27}$$

Fourier transform (2.26) of $F_2(t_k;x)$ generates simple Gaussian distribution $\eta_2(t_k;p)$ with the *market-based* asset price volatility $\sigma_p^2(t_k)$:

$$\eta_2(t_k;p) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_p(t_k)} \exp\left\{-\frac{\left(p - p(t_k;1)\right)^2}{2\sigma_p^2(t_k)}\right\} \tag{2.28}$$

$$\sigma_p^2(t_k) = E[(p(t_i) - p(t_k;1))^2] = p(t_k;2) - p^2(t_k;1) = \frac{C(t_k;2)}{U(t_k;2)} - \frac{C^2(t_k;1)}{U^2(t_k;1)} \tag{2.29}$$

We underline that Gaussian approximation of the asset price probability measure (2.28) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$. Prediction of Gaussian price probability $\eta_2(t_k;p)$ (2.28) at horizon T requires forecasts of the second statistical moments of the market trade value and volume (2.9) at the same horizon T.

Approximations $F_4(t_k;x)$ (2.30-2.33) match first four price statistical moments for different $b>0$, $2n>4$, (Olkhov, 2021b). Approximations $F_4(t_k;x)$ depend on the price volatility $\sigma_p^2(t_k)$ (2.29), price skewness $Sk(p)$ (2.31) and price kurtosis $Ku(p)$ (2.32)

$$F_4(t;x) = \exp\left\{i\, p(t;1)x - \frac{\sigma_p^2(t_k)}{2} x^2 - i\, \frac{a_3}{6} x^3 + \frac{a_4}{24} x^4 - bx^{2n}\right\} \; ; \; 2n > 4 \tag{2.30}$$

$$a_3 = E\left[\left(p - p(t;1)\right)^3\right] = Sk(p)\sigma_p^3(t_k) \tag{2.31}$$

$$Kr(p)\sigma_p^4(t_k) = E\left[\left(p(t_i) - p(t;1)\right)^4\right] \tag{2.32}$$

$$a_4 = [Kr(p) - 3]\sigma_p^4(t_k) \tag{2.33}$$

It is important that the market-based price volatility $\sigma_p^2(t_k)$ (2.29) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$ (2.9) (Olkhov, 2020). Usage of the *market-based* approach to price probability opens the way for description of *market-based* price autocorrelations (Olkhov, 2022a; 2022b).

Our consideration of the price statistical moments $p(t_k;n)$ through statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ complement well-known description of the random variable determined as difference of two random variables. Indeed, taking logarithm of (1.3) one easy obtains that logarithm of price $ln(p)$ equals logarithm of the trade value $ln(C)$ minus logarithm of the trade volume $ln(U)$. That case is described in many probability introductory notes (Papoulis and Pillai, 2002; p.181) and we refer there for details. However, time-series records of market trades allow assess only finite number of statistical moments and don't identify the exact form of the trade value C and volume U PDF as well as joint PDF of $ln(C)$ and $ln(U)$ that are required to derive log price PDF (Papoulis and Pillai, 2002; p.181). At that point usage of our approximate description of price random properties using the *market-based* price statistical moments (2.13; 2.14; 2.19) is more justified and preferable. Below we present some consequences of usage *market-based* price statistical moments (2.13; 2.14) as approximate description of price random properties during the averaging interval Δ_k (2.6).

3. Returns and inflation

The *market-based* asset price statistical moments permit describe statistical properties of returns via statistical moments of the market trade value and volume. Actually, returns $r(t_1,t_2)$ are determined as

$$r(t_1,t_2) = \frac{p(t_2)-p(t_1)}{p(t_1)} = \frac{p(t_2)}{p(t_1)} - 1 \qquad (3.1)$$

Let's take price index $d(t_1,t_2)$ (3.2) as:

$$p(t_2) = p(t_1)d(t_1,t_2) \qquad (3.2)$$

$$r(t_1,t_2) = d(t_1,t_2) - 1 \qquad (3.3)$$

In Sec.2 we already derived the *market-based* asset price statistical moments $p(t_k;n)$ (2.19). We use the same approach to assess the *market-based n-th* statistical moments of the price index $d(t_1,t_2)$ (3.2). We shall consider two simple cases. First, to describe returns we assume that the index $d(t_1,t_2)$ (3.2) is determined with respect to the fixed price $p(t_1)$ and consider statistical properties of the price index by time t_2 averaged during the interval Δ_k (2.4). In the second case we model inflation and consider random properties of the index $d(t_k,t_2)$ with respect to collective price level averaged during Δ_k taking price level $p(t_2)$ as random during interval Δ_{k+m} (2.4). Our model doesn't describe details of returns and inflation but displays dependence of returns of inflation on randomness of the market trade value and volume.

***1-st case - Returns*.**

Relations (3.1) define returns $r(t_1,t_2)$ at moment t_2 with price $p(t_2)$ with respect of previous moment t_1 with price $p(t_1)$. Price $p(t_2)$ is unpredictable and one assesses average returns $r(t_1,t_2)$ or its volatility taking price $p(t_2)$ as random variable during the averaging interval Δ_k. As usual one considers return's irregular time-series as initial data to assess statistical moment of returns, using conventional *frequency-based* probability of returns. However, assessments of returns on large asset sizes those propose big market deals that could impact market price during averaging interval Δ result that agents should consider *market-based* probability of returns determined by statistical moments of random market trade value and volume. We derive *market-based* assessment of returns statistical moments based on price statistical moments (2.13; 2.14). Consider (2.9; 2.19; 3.2) and for the *n-th* statistical moments of the price index $d(t_1,t_k;n)$ averaged by t_2 during Δ_k (2.4) obtain:

$$d(t_1,t_k;n) \equiv E[d^n(t_1,t_2)] \tag{3.4}$$

$$d(t_1,t_k;n)p^n(t_1) = p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} \tag{3.5}$$

E[...] – math expectation during Δ_k (2.4). From (3.4; 3.5) obtain expressions for *n-th* statistical moments of returns $r(t_1,t_k;n)$:

$$r(t_1,t_k;n) \equiv E[r^n(t_1,t_2)] \tag{3.6}$$

$$r(t_1,t_k;n) = E[(d(t_1,t_2)-1)^n] \tag{3.7}$$

Due to (3.4; 3.5; 3.7) *n-th* returns statistical moment $r(t_1,t_k;n)$ is a simple sum of *m-th* statistical moments of the price index $d(t_1,t_k;m)$, $m \le n$:

$$r(t_1,t_k;n) = \sum_{m=0}^{n}(-1)^{(n-m)}\frac{n!}{m!(n-m)!}d(t_1,t_k;m) \ ; \ d(t_1,t_k;0) = 1 \tag{3.8}$$

Due to (3.5) returns *n-th* statistical moments $r(t_1,t_k;n)$ can be presented through the market trade value and volume statistical moments (3.9):

$$r(t_1,t_k;n) = \sum_{m=0}^{n}(-1)^{(n-m)}\frac{n!}{m!(n-m)!}p^{-m}(t_1)\,\frac{C(t_k;m)}{U(t_k;m)} \ ; \ \frac{C(t_k;0)}{U(t_k;0)} = 1 \tag{3.9}$$

or equally through price *n-th* statistical moments (see (3.5)). In particular, one can easy derive relations (3.11) between the price volatility $\sigma_p^2(t_k)$ (2.29), price $p(t_1)$ at moment t_1 and volatility of returns $\sigma_r^2(t_1,t_k)$ (3.10)

$$\sigma_r^2(t_1,t_k) = r(t_1,t_k;2) - r^2(t_1,t_k;1) \equiv E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^2\right] \tag{3.10}$$

$$\sigma_p^2(t_k) = p^2(t_1)\,\sigma_r^2(t_1,t_k) \tag{3.11}$$

2-d case - Inflation.

Inflation is determined as ratio of collective price level of goods or services averaged during time interval Δ_{k+m} with respect to same price level averaged during earlier interval Δ_k. The interval Δ_k can be equal week, month, quarter or year. Assessment of inflation's price level takes into account collective values and volumes of market trade of goods, services etc. (Fox, et al., 2017). Large trade values and volumes impact assessment of inflation and hence should conduct the *market-based* assessment of inflation based on random properties of collective trade values and volumes. Let us consider collective market trade values and volumes that model the price level in a way similar to Sec.2. Let us take "instantaneous" inflation $In(t_i;t_k)$ of price level $p(t_i)$ during averaging interval Δ_{k+m} similar to (3.1):

$$In(t_i, t_k) = \frac{p(t_i)}{p(t_k;1)} - 1$$

Then inflation *n-th* statistical moments $In(t_{k+m},t_k;n)$ averaged during interval Δ_{k+m} take form:

$$In(t_{k+m}, t_k; n) = E[In^n(t_i, t_k)] = E[\left(\frac{p(t_i)}{p(t_k;1)} - 1\right)^n] \quad (3.12)$$

From (2.14) obtain:

$$\frac{p(t_{k+m};n)}{p^n(t_k;1)} = \frac{C(t_{k+m};n)}{C^n(t_k;1)} \frac{U^n(t_k;1)}{U(t_{k+m};n)}$$

Hence from (3.12) obtain *n-th* statistical moment of inflation $In(t_{k+m},t_k;n)$:

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \frac{p(t_{k+m};n-j)}{p^{n-j}(t_k;1)}$$

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \frac{C(t_{k+m};n-j)}{C^{n-j}(t_k;1)} \frac{U^{n-j}(t_k;1)}{U(t_{k+m};n-j)}$$

Let us introduce trade value index $c(t_{k+m};n|t_k,1)$ (3.13) as ratio of trade value *n-th* statistical moment $C(t_{k+m};n)$ (2.9) averaged during the interval Δ_{k+m} to *n-th* degree of mean trade value $C(t_k;1)$ (2.7) averaged during the earlier interval Δ_k. The similar meaning has trade volume index $u(t_{k+m};n|t_k,1)$ (3.13):

$$c(t_{k+m}; n|t_k; 1) = \frac{C(t_{k+m};n)}{C^n(t_k;1)} \quad ; \quad u(t_{k+m}; n|t_k; 1) = \frac{U(t_{k+m};n)}{U^n(t_k;1)} \quad (3.13)$$

Using (3.13) inflation *n-th* statistical moments $In(t_{k+m},t_k;n)$ take form:

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \frac{c(t_{k+m};n-j|t_k;1)}{u(t_{k+m};n-j|t_k;1)} \quad (3.14)$$

Mean inflation $In(t_k,t_{k+m};1)$ during Δ_{k+m} with respect to time term Δ_k takes form:

$$In(t_{k+m}, t_k; 1) = \frac{p(t_{k+m};1)}{p(t_k;1)} - 1 = \frac{C(t_{k+m};1)}{C(t_k;1)} \frac{U(t_k;1)}{U(t_{k+m};1)} - 1 \quad (3.15)$$

Volatility of inflation $\sigma^2_{In}(t_{k+m},t_k)$ during Δ_{k+m} with respect to time term Δ_k

$$\sigma^2_{In}(t_{k+m}, t_k) = In(t_{k+m}, t_k; 2) - In^2(t_{k+m}, t_k; 1)$$

$$\sigma^2_{In}(t_{k+m}, t_k) = \frac{C(t_{k+m};2)}{C^2(t_k;1)} \frac{U^2(t_k;1)}{U(t_{k+m};2)} - \frac{C^2(t_{k+m};1)}{C^2(t_k;1)} \frac{U^2(t_k;1)}{U^2(t_{k+m};1)}$$

Volatility of inflation $\sigma^2_{In}(t_{k+m},t_k)$ (3.16) takes form alike to returns volatility $\sigma_r^2(t_1,t_k)$ (3.11):

$$\sigma^2_p(t_{k+m}) = p^2(t_k; 1)\sigma^2_{In}(t_{k+m}, t_k) \quad (3.16)$$

However, volatility $\sigma^2_p(t_k)$ in (3.11) describes price fluctuations of selected asset but volatility $\sigma^2_p(t_{k+m})$ (3.16) describes fluctuations of collective price level that assesses inflation. It is reasonable that volatility of inflation $\sigma^2_{In}(t_{k+m},t_k)$ during Δ_{k+m} with respect to Δ_k equals the ratio of price volatility $\sigma^2_p(t_{k+m})$ during Δ_{k+m} to square of mean price $p^2(t_k,1)$ during Δ_k. The trade value index $c(t_{k+m};n|t_k,1)$ (3.13) and trade volume index $u(t_{k+m};n|t_k,1)$ (3.13) describe growth of the market trade value and volume during the interval $\Delta_{k+\mathrm{m}}$ with respect to Δ_k. Market trade is important indicator of the economic growth and development. Relations (3.13-3.16) link description of asset returns and inflations with economic growth during selected time intervals Δ_k and Δ_{k+m} (2.4). We leave further investigation of above relations between economic growth and market trade indexes for future.

4. Asset pricing and value-at-risk

Asset-pricing

Most asset-pricing models deal with prices averaged by some probability (Cochrane, 2001; Campbell, 2002). Predictions of the price probability at certain time horizon T play the core role for the assessments of price forecasts at horizon T. Asset pricing when assumption: “the investor can freely buy or sell as much of the payoff x_{t+1} as he wishes, at a price p_t” (Cochrane, 2001, p.15) – is not valid, requires taking into account impact of large market deals as well as investor’s trades on price statistical moments and thus on price probability. We introduce *market-based* approach that approximates the impact of market trade value and volume during averaging interval Δ_k (2.4) on price statistical moments and price probability. Introduction of the *market-based* price probability determined by statistical properties of the market trade value and volume (2.19 - 2.23) makes predictions of the price PDF one of the core problems of macroeconomics and finance. Indeed, any prediction of the price PDF at time horizon T for the given averaging interval Δ_k (2.4) requires forecasting of the market trade value and volume probabilities at the same horizon T and during the same Δ_k (2.4). In simple words, to predict price PDF one should forecast the market trade values and volumes probabilities during the interval Δ_k (2.4) at horizon T. That causes forecasting economic and financial factors those impact market trade at horizon T: supply and demand, production function and investment, economic development and growth and etc. Introduction of the *market-based* price probability ties up prediction of asset price probability with problems of forecasting of market trade, economic development and growth. One should take into account basic relations (2.19-2.23) those determine price statistical moments through statistical moments of the market trade value and volume. Approximations (2.24-2.27) that take into account first 2,3,4 statistical moments should check how forecasts of approximate price probability match predictions of the market trade value and volatility statistical moments.

Value-at-risk

Approximate predictions of the asset price probability determine accuracy and reliability of Value-at-Risk (VaR) – one of the most widespread tool for hedging risk of the market price change. Economic ground of VaR was developed more than 30 years ago (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Choudhry, 2013). "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon" (Longerstaey and Spencer, 1996). Nevertheless the great progress in VaR performance since then, the core features of VaR remain the same. To assess VaR at horizon T one should estimate integral of the left tail of the returns or price PDF predicted at horizon T to project "potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon". Such assessment limits the possible capital loss due to market price variations for selected time horizon T for given probability. VaR is used by largest banks and investment funds to hedge market price change of their AUM and portfolios valued at billions USD. Thus, banks and funds should consider impact of their large trades on market price probability. Hence, largest banks and investment funds should take into account and follow *market-based* approximation of price and returns probability.

Usage of any VaR version requires predictions of returns PDF at horizon T. As we show above, returns probability and returns statistical moments are completely determined by asset price probability and price statistical moments (3.1-3.14) and thus by market trade statistical moments (2.9; 3.9). Prediction of returns probability almost equals prediction of the asset price probability (3.5; 3.8). In Sec.2 we show that the *market-based* asset price probability and price statistical moments are determined by statistical moments of market trade value and volume. In other words: VaR as method to hedge large AUM from risks of market price change is based on prediction of price probability at horizon T and hence depends on forecasts of the market trade value and volume statistical moments at the same horizon T (Olkhov, 2021a). The accuracy of VaR assessment at horizon T is determined by accuracy of forecasting the market trade value and volume statistical moments. The more statistical moments of market trade are predicted, the higher accuracy of prediction of market trade PDF, the higher accuracy of prediction of price statistical moments and PDF. Simply put: VaR assessment almost equals prediction of market trade PDF. However, imaginable exact forecast of the market trade value and volume statistical moments or PDF at horizon T

would provide for that lucky man a unique opportunity to manage and beat the market alone. That is much more profitable then any VaR assessments. One who will succeed in exact prediction of the market PDF will forget about VaR assessments and will enjoy beating the market alone! However, there still remains a "negligible" problem – how *exactly* predict the market trade PDF? It is a good issue for further research.

Accuracy of any assessments of VaR at horizon T is reduced by the accuracy of predictions market trade value and volume statistical moments at horizon T. That arises the problem of accuracy of any price PDF predictions in compare with accuracy of market trade probability forecasts. Further research may help establish economic ground and introduce possible limits on reliability of usage of VaR based on the *market-based* price probability.

5. Conclusion

The asset price probability plays the core role in macroeconomics and finance. Introduction of the *market-based* approximation of price PDF through the statistical moments of the market trade value and volume establishes the unified description of the price statistical moments, price indices and returns statistical moments and ties up predictions of the price and returns probabilities with forecasting the market trade value and volume statistical moments. That approach describes the case when investor's trades as well as large deals of other market participants, impact random properties of the asset price during the averaging interval Δ. We approximate price statistical moments and price probability by statistical moments of the market trade value and volume. It helps study bounds of reliability of Value-at-Risk determined by the accuracy of forecasting of the market trade probability. Investigation of the market trade value and volume PDF and prediction of their statistical moments are the problems for future.

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