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Abstract

This article introduces a general model for voting games with multiple weight vectors. Each weight vector characterises a parameter representing a distinct aspect of the voting mechanism. Our main innovation is to model the winning condition by an arbitrary dichotomous function which determines whether a coalition is winning based on the weights that the coalition has for the different parameters. Previously studied multi-parameter games are obtained as particular cases of the general model. We identify a new and interesting class of games, that we call hyperplane voting games, which are compactly expressible in the new model, but not necessarily so in the previous models. For the general model of voting games that we introduce, we describe dynamic programming algorithms for determining various quantities required for computing different voting power indices. Specialising to the known classes of multi-parameter games, our algorithms provide unified and simpler alternatives to previously proposed algorithms.

JEL Classification No.: C71

Keywords: weighted majority voting game, multi-parameter games, Boolean formula, voting power, dynamic programming.

1 Introduction

In a voting body, modeled as a game, the members are considered to be the players of the game and a subset of members is considered to be a coalition of the players. A coalition that can pass a resolution is called a winning coalition. A very well-known game is a weighted majority voting game in which each player has a non-negative weight. In such a game, a proposal is accepted if the weights of the players in the coalition supporting the proposal sum up to a number at least as high as a pre-determined quota. Such a voting system has a very simple structure because there is only one set of weights and one quota. A well known example of weighted majority voting game is the weighted voting procedure followed by the International Monetary Fund (IMF)[1].

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[1]See Article XII, Section 5(a) of the IMF for determination of the voting weights, and Leech, 2002a and Brandner and Grech, 2009 for analyses of the IMF voting game.

In real life situations, however, more complex forms of voting may arise. For instance, on 1 November 2014 the European Union (EU) council adopted the following double majority rule for a proposal to be accepted: (i) 55 per cent of the EU countries vote in favour, and (ii) the proposal gets support from countries representing at least 65 per cent of the total EU population. This is referred to as Qualified Majority Voting (QVM); in addition to requiring majority in one dimension it requires majority in another dimension. Further, a coalition of at least 4 members is required to block a resolution, a condition which is called blocking minority. As another (hypothetical) example one may consider voting over the change in the mode of payment of annuity to bondholders of an organization. For concreteness, let us suppose that a change in the mode of annuity payment needs approvals of at least 50% of the bondholders and 80% of the members of the Board of Directors of the organization, or it needs the approval of at least 80% of the bondholders and 50% of the members of the Board. The figures 50% and 80% in this example are somewhat arbitrary and can be substituted by other values.

To understand further variations in the simple structure of the weighted majority game, note that the weights represent the "say" of a player in a decision making scenario. For example in the context of a decision making body of sovereign nations, the "say" of a particular nation could depend, among others, on the GDP of the nation, its population size, and the amount of contribution it makes to the body. Capturing the effect of several such widely different parameters using a single numerical value (which is the weight assigned to a nation) is fraught with problems. It is difficult to justify the assignment of weights to the various players.

Games with multiple weight vectors have been studied in the literature. A weighted AND game with k weight vectors is the intersection of k different weighted majority voting games; a coalition in the extended game is said to be winning if it wins in every component game (Taylor and Zwicker (1993), Algaba et al. (2003), Aziz et al. (2009), Alonso-Meijide et al. (2009), Bolus (2011), Wilms (2020)). In a similar manner, a weighted OR game with k weight vectors is essentially the OR of the individual games; a coalition in the extended game is said to be winning if it wins in at least one of the component games (Aziz et al. (2007), Wilms (2020)). A more general kind of game considers a combination of the component games using a monotone Boolean formula. To determine whether a coalition in the extended game is winning, first it is required to determine the Boolean value (1 for winning and 0 for losing) of the coalition in the component games. Next these Boolean values are combined using the Boolean formula to determine whether the coalition is winning in the extended game. Such games have been studied by Algaba et al. (2007), Faliszewski et al. (2009), Kurz and Napel (2016) and Wilms (2020).

Given that there are natural weighted voting scenarios where considering more than one parameter makes sense, and the lack of a unified approach to the study of such games in the existing literature, it becomes worthwhile to formalise a general and unified definition of multi-parameter weighted voting games. Our main conceptual contribution is to introduce a new class of games using an appropriate abstraction. In the set-up that we consider, each player receives a weight corresponding to each of the parameters. A decision rule specifies whether a coalition of players is winning or not. We make the reasonable assumption that such a decision rule depends on the sum of the weights of the players for each of the parameters. So there are two key features in our model, namely weight vectors corresponding to multiple parameters and a winning rule which can depend arbitrarily on the weights that a coalition has for each of the parameters. The winning rule for a k-parameter weighted voting game is specified by a decision function f, a dichotomous function, taking values 0 and 1, defined on the set of all possible k-tuples of weights that may arise from the set of all possible coalitions. We refer to this function as a decision function since by taking the value 1 (respectively 0) it distinctively identifies a coalition as winning (respectively losing). By appropriately defining the decision function f, our model can be specialised to obtain any of the above mentioned classes of multi-parameter games which have been studied in the literature; see Section 3.1 for details. While the new class of voting games proposed in our article can be compactly modelled using our new framework, none of the existing frameworks, however, can succinctly do so.

Going beyond the known types of multi-parameter weighted voting games, our definition allows the modelling of a new and interesting class of games which we call *hyperplane voting games*[2]. We motivate hyperplane voting games by the following hypothetical example. A coalition of shareholders of a company has two kinds of weights. The first is the total number of shares that the coalition holds and the second is the total number of entities in the coalition. Let the proportion of shares held by a coalition S be denoted by $\rho_S^{(1)}$ (where $\rho_S^{(1)}$ is the ratio of the total number of shares held by S to the total number of shares of the company) and the proportion of the number of shareholders be denoted by $\rho_S^{(2)}$ (where $\rho_S^{(2)}$ is the ratio of the size of S to the total number of shareholders of the company). For taking certain decisions (such as those on corporate social responsibilties), suppose a company wishes to consider both the proportion of shares and the proportion of shareholders supporting a decision. This can be modelled by defining two non-negative numbers c_1 and c_2, with $c_1 + c_2 = 1$, and defining a coalition S to be winning if $c_1\rho_S^{(1)} + c_2\rho_S^{(2)} \geq q$, where $q \in (0, 1)$ is a pre-defined quota. The winning condition can be visualised as a straight line separating the two-dimensional Euclidean space into two parts, one corresponding to winning and the other corresponding to losing. More generally, in a hyperplane voting game with k weight vectors the decision function can be visualised as a hyperplane separating the k-dimensional Euclidean space into winning and losing subspaces; see Section 3.2 for details.

An important result on voting games shows that any monotone simple game can be expressed as a weighted AND game (Taylor and Zwicker 1993, 1995, 1999). The minimum number of component games required to express the original game is said to be the dimension of the game. It has been proved by Deĭneko and Woeginger (2006) that determination of the dimension of a game is NP-hard (see also Elkind et al. 2009). Consequently, even though in theory it is possible to express a simple monotone game as a weighted AND game with k weight vectors, determining the minimum value of k is in practice very difficult. Kurz and Napel (2016) introduced the notion of Boolean dimension of a game which is the minimum number of games such that the original game can be expressed as a monotone Boolean combination of the individual games.

General multi-parameter weighted voting games with monotone decision functions are certainly monotone simple games. So it follows that a k-parameter weighted voting game can also be modelled as a weighted AND game with k_1 weight vectors and as a monotone Boolean combination of k_2 weighted majority games for some positive integers k_1 and k_2. From the above mentioned complexity results, determination of k_1 is computationally difficult. While we do not know of any complexity results on the Boolean dimension of a game, it seems likely that obtaining k_2 is also computationally difficult. Moreover, one would expect both k_1 and k_2 to be exponential in k, although presently there are no results on the relation between k and k_1, k_2. So while general multi-parameter weighted voting games can in theory be captured using previous modelling techniques, in practice this is computationally difficult and even if possible it would likely result in a loss of compactness of representation. Since the compactness of representation determines the complexity of the algorithms for determining various properties of games, the exercise of translating general multi-parameter weighted voting games to previous models does not seem to be useful.

The fundamental notion related to a voting game is voting power. The voting power of a player quantifies a player's ability to affect the outcome. The literature contains a number of voting power measures, the most important among them being the Penrose-Banzhaf (proposed independently by

[2]A special class of non-transferable utility (NTU) games defines the feasible pay-off vectors of a coalition using a hyperplane; such games are called hyperplane games (Maschler and Owen (1989), Yu (2022)). In an NTU hyperplane game, for each coalition there is an associated hyperplane which defines its feasible pay-off vectors, whereas, in a hyperplane voting game, there is a single hyperplane which separates the winning from the losing coalitions. Consequently, NTU hyperplane games studied in the literature and the hyperplane voting games that we introduce are completely different concepts.

Penrose (1946) and Banzhaf (1965)) and the Shapley-Shubik (1954) measures[3]. To study a particular game, it is important to be able to compute the voting powers of all the players. This requires computing several basic values related to a voting game. Algorithms for computing various power measures for weighted voting games are known (see Matsui and Matsui 2000 and Chakravarty, Mitra, and Sarkar 2015). These algorithms become important tools for analysing practical games. As an example, we mention the work of Bhattacherjee and Sarkar (2019) which used such algorithms for studying the EU and IMF voting games with respect to the inequality in their voting powers[4],[5].

The general modelling of multi-parameter games that we introduce would be of no practical use unless there are algorithms for computing the various quantities of such games required to determine the important voting power measures. The bulk of the technical contribution of the present work is to show how these quantities can be computed for general multi-parameter weighted voting games. As a result, it is possible to compute the standard voting power measures that have been proposed in the literature. Consequently, these algorithms provide tools for studying various aspects of real-life multi-parameter situations.

Algorithms for computing voting powers of weighted AND games, weighted OR games and monotone Boolean combination of weighted games have been studied by several authors (Algaba et al. (2003), Faliszewski et al. (2009) and Wilms (2020)). These works provide separate analysis of the three different kinds of games. Further, the expressions for the various measures are quite complicated (see Wilms, 2020). Through the use of the abstraction of the decision function that we introduce, we are able to provide a unified *and* simple analysis of the various quantities related to the computation of voting power measures for the already known classes of multi-parameter weighted voting games. Further, we describe how to compute the number of minimal winning coalitions (and consequently power indices, such as the Deegan-Packel index) for general multi-parameter weighted voting games. To the best of our knowledge, the Deegan-Packel and other indices based on the number of minimal winning coalitions have not been considered in the literature for the three particular kinds of multi-parameter games that have been studied earlier.

In Section 2 we provide the necessary background. The new model of multi-parameter voting games is introduced in Section 3. Two basic recurrences and their computation using dynamic programming are described in Section 4. These recurrences are used in Section 5 to obtain formulas for the number of winning coalitions and the number of coalitions in which a player is a swing and in Section 6 to obtain the number of minimal winning coalitions.

2 Preliminaries

In the following, the cardinality of a finite set S will be denoted by $\#S$ and the absolute value of a real number x will be denoted by $|x|$.

2.1 Voting games

We provide some standard definitions arising in the context of voting games. For details the reader may consult Felsenthal and Machover (1998), Laurelle and Valenciano (2008) and Chakravarty, Mitra and Sarkar (2015).

[3]Alternatives and variations of these indices were suggested, among others, by Rae (1969), Coleman (1971), Deegan and Packel (1978), Johnston (1978) and Holler (1982).

[4]For an earlier analysis on the inequality of voting system in the EU, see Leech (2002b).

[5]For earlier but not closely related approaches to the measurement of inequality in voting games, see Einy and Peleg (1991), Laruelle and Valenciano (2004) and Weber (2016).

Let $N = \{1, \ldots, n\}$ be a set of n players. A subset of N is called a *coalition*. The set of all voting coalitions is denoted by 2^N. A voting game G is given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$, where a winning coalition is assigned the value 1 and a losing coalition is assigned the value 0. Below we recall some basic notions about voting games.

1. A voting game G is said to be *monotone* if for coalitions S, T with $S \subseteq T \subseteq N$, $\widehat{G}(S) = 1$ implies $\widehat{G}(T) = 1$.
2. For any $S \subseteq N$ and player $i \in N$, i is said to be a *swing* in S if $i \in S$, $\widehat{G}(S) = 1$, and $\widehat{G}(S \setminus \{i\}) = 0$. That is, if player i leaves the winning coalition S then the resulting coalition is a losing coalition.
3. The number of coalitions in which a player i is a swing will be denoted by m_i.
4. A player $i \in N$ is called a *null* (or often also *dummy*) if i is not a swing in any coalition, i.e., if $m_i = 0$.
5. A player $i \in N$ is called a *blocker* if i is present in every winning coalition.
6. For a voting game G, the set of all winning coalitions will be denoted by W_G.
7. A coalition $S \subseteq N$ is called a *minimal winning coalition* if $\widehat{G}(S) = 1$ and there is no $T \subset S$ for which $\widehat{G}(T) = 1$. That is, no proper subset of the winning coalition S can be winning.

2.2 Voting power

The notion of power is an important concept in a voting system. A *power measure* captures the capability of a player to influence the outcome of a vote. Given a game G on a set of players N and a player i in N, a power measure $\mathcal{P}$ associates a non-negative real number $v_i = \mathcal{P}_G(i)$ to the player i. The number v_i captures the power that i has in the game G. There are a number of voting power measures in the literature, the most prominent among them being the Shapley-Shubik and the Banzhaf measures. For a game G, the ability to compute the following quantities permit the computation of several of the important voting power measures.

1. Number of winning coalitions.
2. Number of coalitions in which a player is a swing.
3. Number of coalitions of a particular cardinality in which a player is a swing.
4. Number of minimal winning coalitions.
5. Number of minimal winning coalitions containing a player.
6. Number of minimal winning coalitions of a particular cardinality containing a player.

The Banzhaf measure can be computed from the number of coalitions in which a player is a swing; the Shapley-Shubik index can be computed from the number of coalitions of a particular cardinality in which a player is a swing; the two Coleman measures can be computed from the number of coalitions in which a player is a swing and the total number of winning coalitions in the game; the Deegan-Packel index can be computed from the total number of minimal winning coalitions in the game and the number of minimal winning coalitions of a particular cardinality for a player. On the other hand, the Johnston index can be regarded as an amalgam of the Banzhaf and Deegan-Packel indices; the Rae index relies on the total number of ways in which a voter agrees with the outcome of the voting system;

the Holler index, like the Deegan-Packel index, is also based on the minimal winning coalitions, though the argument for using minimal winning coalitions is distinct from that used in the Deggan-Packel index. See Felsenthal and Machovar (1998), Matsui and Matsui (2000) and Chakravarty, Mitra and Sarkar (2015) for further details.

3 General multi-parameter weighted voting games

We consider voting games with multiple weight vectors with each weight vector capturing some aspect of the background problem. Further, we consider the winning rule to be quite general which permits wide flexibility in modelling. We start with the formal definition of such a game.

Definition 1 (General k-parameter weighted voting game) *Consider a tuple* $(N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$*, where*

- $N = \{1, \ldots, n\}$ *is a set of players;*
- *for* $1 \leq j \leq k$, $\mathbf{w}^{(j)} = (w_1^{(j)}, w_2^{(j)}, \ldots, w_n^{(j)})$ *is a vector of non-negative weights with* $w_i^{(j)}$ *being the weight of player* i *in the* j*-th weight vector such that* $w_i^{(1)} + w_i^{(2)} + \cdots + w_i^{(k)} > 0$ *for* $i = 1, \ldots, n$*;*
- *and* $f : \Omega \rightarrow \{0, 1\}$, *is a decision function, where*

$$\Omega \;=\; \{(s_1, \ldots, s_k) : \text{ there is a coalition } S \subseteq N \text{ satisfying } w_S^{(1)} = s_1, \ldots, w_S^{(k)} = s_k\},$$

with $w_S^{(j)} = \sum_{i \in S} w_i^{(j)}$, $j = 1, \ldots, k$, *being the sum of the weights of all the players in the coalition* S *as given by the* j*-th weight vector.*

The tuple $(N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$ *defines a voting game* G *given by its characteristic function* $\widehat{G} : 2^N \rightarrow \{0, 1\}$ *in the following manner.*

$$\widehat{G}(S) = f(w_S^{(1)}, w_S^{(2)}, \ldots, w_S^{(k)}).$$

We will write

$$G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$$

to denote the voting game arising from the tuple $(N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$.

We note the following points regarding Definition 1.

1. While we allow weights to be zero, the condition $w_i^{(1)} + w_i^{(2)} + \cdots + w_i^{(k)} > 0$ ensures that any player i has a positive weight in at least one of the parameters.
2. The set Ω can be visualised as follows. The k-tuple $(s_1, \ldots, s_k)$ is in Ω if and only if there is some coalition S whose weight according to the j-th weight vector is s_j, $j = 1, \ldots, k$.
3. The decision function f determines the winning coalitions. A coalition S is winning if and only if f evaluates to 1 on input $(w_S^{(1)}, \ldots, w_S^{(k)})$.
4. If f is monotone on each component, then the game G is a monotone simple game and this is our main interest. We note that the formulas in Section 5 hold irrespective of whether f is monotone or not, while the formulas in Section 6 require f to be monotone.

For notational convenience, let $\omega^{(j)} = w_N^{(j)}$, for $j = 1, \ldots, k$. In other words, $\omega^{(j)}$ is the total weight of the j-th weight vector. Note that for $(s_1, \ldots, s_k) \in \Omega$, we have $0 \leq s_j \leq \omega^{(j)}$ for each $j = 1, \ldots, k$.

We illustrate the expressive power of the new model in two ways. Firstly, we show that all previously considered models of weighted voting can be obtained as particular cases of the new model. Secondly, we introduce an interesting new class of games which can be compactly modelled using the new framework, but cannot necessarily be done so using known models of weighted voting.

3.1 Particular cases of general multi-parameter weighted voting games

By suitably defining the decision function f, it is possible to obtain particular sub-classes of multi-parameter weighted voting games. Below we show how previously known classes of weighted voting games can be derived from general multi-parameter weighted voting games by suitably defining f.

Weighted majority voting games. Suppose $k = 1$ and $f(w_S^{(1)})$ is defined to take the value 1 if and only if $w_S^{(1)}/\omega^{(1)} \geq q$, where $q \in (0, 1)$ is a fixed real number. Then G is a weighted majority voting game.

Weighted AND games. Suppose $q_1, \ldots, q_k \in (0, 1)$ and $f(w_S^{(1)}, \ldots, w_S^{(k)})$ is defined to take the value 1 if and only if

$$\left(w_S^{(1)}/\omega^{(1)} \geq q_1\right) \wedge \cdots \wedge \left(w_S^{(k)}/\omega^{(k)} \geq q_k\right).$$

This class of multi-parameter games has been variously called vector-weighted system (Taylor and Zwicker, 1993), weighted k-majority voting game (Algaba et al., 2003), meet-multiple weighted voting game (Aziz et al., 2009), weighted multiple majority game (Alonso-Meijide et al., 2009), vector-weighted majority game (Bolus, 2011) and weighted k-tier AND-game (Wilms, 2020).

Weighted OR games. Suppose $q_1, \ldots, q_k \in (0, 1)$ and $f(w_S^{(1)}, \ldots, w_S^{(k)})$ is defined to take the value 1 if and only if

$$\left(w_S^{(1)}/\omega^{(1)} \geq q_1\right) \vee \cdots \vee \left(w_S^{(k)}/\omega^{(k)} \geq q_k\right).$$

This class of multi-parameter games has been called join-multiple weighted voting game (Aziz et al., 2007) and weighted k-tier OR-game (Wilms, 2020).

Monotone Boolean combination of weighted games. Let $X_1, \ldots, X_k$ be Boolean valued variables and $\phi(X_1, \ldots, X_k)$ be a monotone Boolean function[6] of the variables $X_1, \ldots, X_k$. Suppose $q_1, \ldots, q_k \in (0, 1)$, and for $j = 1, \ldots, k$, let X_j take the value 1 if and only if $w_S^{(j)}/\omega^{(j)} \geq q_j$. Define

$$f(w_S^{(1)}, \ldots, w_S^{(k)}) \quad = \quad \phi(X_1, \ldots, X_k). \tag{1}$$

This class of games has been studied by Algaba et al. (2007), Faliszewski et al. (2009), Kurz and Napel (2016) and Wilms (2020). The Lisbon voting rules of the EU Council can be modelled using a monotone Boolean combination of two weighted voting games (see Algaba et al. (2007) and Kurz and Napel (2016)). The previous two cases of weighted AND games and weighted OR games can be seen to be special cases of monotone Boolean combination of weighted games.

Previous papers which studied the above particular cases of general multi-parameter weighted voting games did not use the abstraction of the decision function f that we have introduced. This abstraction has several advantages. For one thing, it allows a unified definition of multi-parameter games, where

[6] A Boolean function of variables $X_1, \ldots, X_k$ is said to be monotone if it can be expressed by a Boolean algebra expression involving the variables $X_1, \ldots, X_k$ and the logical connectives $\wedge$ (AND) and $\vee$ (OR).

particular cases can be obtained by appropriately defining the function f as we have shown above. In contrast, previous works used separate definitions for each of the above types of games. Secondly, using the function f allows unified and reasonably simple expressions for the formulas for the number of winning coalitions and the number of coalitions where a player is a swing which determines some of the important power indices. The computation of power indices for the above types of games have been considered in the literature: for weighted AND games by Algaba et al. (2003) and Wilms (2020), for weighted OR games by Wilms (2020) and for monotone Boolean combination of weighted games by Faliszewski et al. (2009) and Wilms (2020). In particular, we note that Wilms (2020) provides separate expressions for weighted AND games, weighted OR games and monotone Boolean combination of weighted games. In comparison, we provide unified formulas that hold for all of the above games as also for more general games. The reader may compare the formulas given in Section 5 to those in Wilms (2020) to convince oneself of the simplicity of our approach. Thirdly, in Section 6 also using the function f, we provide formulas for the number of minimal winning coalitions for general multi-parameter weighted voting games and so in particular for the above types of games. To the best of our knowledge, formulas for the number of minimal winning coalitions for the above types of games have not appeared earlier in the literature.

3.2 Hyperplane voting games

Suppose the set N of players participate in k committees. The voting weights of the players in the various committees are not necessarily the same. So we have k weight vectors $\mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}$, with weight vector $\mathbf{w}^{(j)}$ specifying the weights of the n players in the j-th committee. Let as before $\omega^{(j)}$ be the sum of the j-th weight vector. A coalition S has weight $w_S^{(j)}$ in the j-th committee. The proportional weight of S in the j-th committee is $w_S^{(j)}/\omega^{(j)}$. All the committees do not necessarily have the same importance. We may assign weight c_j to the j-th committee, where $c_1, \ldots, c_k$ are real numbers in $[0, 1]$ such that $c_1 + \cdots + c_k = 1$. Let $q \in (0, 1)$ be a quota. The coalition S is defined to be winning if

$$c_1 \cdot \frac{w_S^{(1)}}{\omega^{(1)}} + \cdots + c_k \cdot \frac{w_S^{(k)}}{\omega^{(k)}} \quad \geq \quad q. \tag{2}$$

This defines a general k-parameter weighted voting game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, where $f(w_S^{(1)}, \ldots, w_S^{(k)})$ is 1 if and only if (2) holds. Taking $k = 2$ and the two weights of a coalition to be the number of shares held by the coalition and the size of the coalition, we obtain the example of 2-parameter hyperplane game described in the introduction.

The formal definition of hyperplane voting games is the following.

Definition 2 *A hyperplane voting game is a general multi-parameter weighted voting game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, where the decision function f is defined as follows:*

$$f(w_S^{(1)}, \ldots, w_S^{(k)}) = 1 \text{ if and only if } d_1 w_S^{(1)} + \cdots d_k w_S^{(k)} \geq q, \tag{3}$$

for fixed real numbers $d_1, \ldots, d_k$ and q.

Note that we obtain (2) by taking $d_j = c_j/\omega^{(j)}$, for $j = 1, \ldots, k$.

In Definition 2, the function f can be seen as a decision rule to partition the domain Ω of the decision function f into two parts by a hyperplane; a coalition $S \subseteq N$ is winning if and only if its corresponding weight vector $(w_S^{(1)}, \ldots, w_S^{(k)}) \in \Omega$ is on or above the hyperplane which defines the support of the function f. In view of this, we call such games to be hyperplane voting games. Note that in the formal definition given in Definition 2, we do not insist that the sum of the d_i's must be 1. The coefficients

$d_1, \ldots, d_k$ are defined to be real numbers. If we restrict the $d_1, \ldots, d_k$ to be non-negative, then we obtain a sub-class of hyperplane voting games which are montone. We will primarily be interested in monotone hyperplane voting games, though below we briefly mention an example of a non-monotonic hyperplane voting game.

The crucial difference between hyperplane voting games and monotone Boolean combination of weighted games arises from the manner in which the weights are used in defining the function f (compare (3) with (1)). In hyperplane voting games, the decision function f can be directly defined using the k possible weights of a coalition, while in a monotone Boolean combination of weighted games, the k weights are first used to determine winning/losing conditions individually in each of the k games and then these Boolean values are combined using a Boolean formula to obtain the value of f.

Non-monotonic games: The Boolean formula ϕ in (1) is a monotone Boolean formula. Faliszewski et al. (2009) have considered the more general case where ϕ is an arbitrary Boolean formula. The corresponding games are not necessarily monotone. Keeping this in mind, we discuss an interesting example of how hyperplane voting games can be used to model possibly non-monotonic decision making procedures arising in machine learning contexts.

Suppose in (3), the coefficients $d_1, \ldots, d_k$ are allowed to be negative. It follows that f is not necessarily monotone and so the hyperplane voting game arising from such an f is also not necessarily monotone. We provide a natural interpretation of such a hyperplane voting game in the context of the binary classification problem in machine learning. Suppose an object has k features. The requirement is to categorise it into one of two categories (say acceptable or unacceptable) based on the opinion of n persons. Each person classifies the object into one of the two categories. The opinions of the n persons have different weights for the k features which are given by k weight vectors. Suppose S is the set of persons who have classified the object as acceptable. The weights $w_S^{(1)}, \ldots, w_S^{(k)}$ of the coalition for the k features are then used to determine whether the object is finally acceptable or not. So the decision is based on the value of a function f applied to the weights $w_S^{(1)}, \ldots, w_S^{(k)}$. In machine learning context, a commonly used decision function is a hyperplane i.e., $f(w_S^{(1)}, \ldots, w_S^{(k)}) = 1$ if and only if $d_1 w_S^{(1)} + \cdots + d_k w_S^{(k)} \geq d_0$ for some real numbers $d_0, d_1, \ldots, d_k$. The coefficient d_j is the weight associated with the j-th feature. A positive d_j represents a positive feature of the object, i.e. a feature which makes the object acceptable, while a negative d_j represents a negative feature of the object, i.e. a feature which makes the object unacceptable. If some negative feature is present, then the resulting k-parameter hyperplane voting game is not monotone[7].

3.3 Compactness of representation

We note that any simple monotone game can be expressed as a weighted AND game (Tayler and Zwicker, 1993) and also as a monotone Boolean combination of weighted games (Kurz and Napel, 2016). General multi-parameter weighted voting games with monotone decision functions are simple monotone games. So it is indeed possible to express general multi-parameter weighted voting games as weighted AND games and also as a monotone Boolean combination of weighted games. The key issue here is the number of weight vectors that would be required for such representation. Suppose a general k-parameter weighted voting game is expressible as a weighted AND game with k_1 weight vectors and also as a monotone Boolean combination of k_2 weighted games. The NP-completeness result by Deĭneko and

[7]Alternatively, one may restrict $d_1, \ldots, d_k$ to be non-negative, and instead allow the weights of the persons to be negative, where a negative weight indicates a person's disapproval of the object for a particular feature. Negative weights also result in a non-monotonic voting game defined by a hyperplane. It does not, however, fit our definition of hyperplane voting games since our definition of general multi-parameter weighted voting games restricts the weight vectors to have non-negative components.

Woeginger (2006) on determining the dimension of a game suggests that obtaining k_1 is computationally intractable. Similarly, obtaining k_2 is also likely to be computationally intractable though we are not aware of any result on this problem. So while theoretically possible, in practice converting a general multi-parameter weighted voting game to either a weighted AND game or a monotone Boolean combination of weighted games is likely to be computationally infeasible. Further, it seems likely that both k_1 and k_2 will be exponential in k, though at present there are no results on the relation between k and k_1, k_2.

Taking the above into consideration, general multi-parameter voting games permit a more compact representation and this has importan t consequences to the computation of various power indices. As we show below, the time complexity for the computation of the various power indices using k weight vectors is linear in $\omega^{(1)} \cdots \omega^{(k)}$. So replacing k weight vectors by k_1 or k_2 weight vectors, where k_1 and k_2 are exponential in k will result in the computation time becoming exponential in k. Thus, the compactness of the expressive power of general multi-parameter weighted voting games is of fundamental importance to the computation of the various power indices for such games.

4 Basic recurrences

The crux of the methods to compute the quantities mentioned in Section 2.2 are two basic recurrences. In this section, we present these recurrences and in later sections, we show how the various quantities can be obtained from these two recurrences.

Consider a k-parameter game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$. From the weight vectors $\mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}$ we define a $k+1$ dimensional table T, where $T(i, s_1, \ldots, s_k)$, with $0 \leq i \leq n$, being the number of subsets S of $\{1, \ldots, i\}$ with $w_S^{(1)} = s_1, \ldots, w_S^{(k)} = s_k$.

Proposition 1 *For $i \geq 1$,*

$$T(i, s_1, \ldots, s_k) \;=\; T(i-1, s_1, \ldots, s_k) + T(i-1, s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}) \tag{4}$$

with boundary conditions as follows.

$$T(i, s_1, \ldots, s_k) \;=\; \begin{cases} 0 & \textit{if for any } j \in \{1, \ldots, k\}, \textit{ either } s_j < 0 \textit{ or } s_j > \omega^{(j)}; \\ 0 & \textit{if } i = 0, s_1, \ldots, s_k \geq 0 \textit{ and at least one of } s_1, \ldots, s_k \textit{ is positive}; \\ 1 & \textit{if } i \geq 0 \textit{ and } s_1 = \cdots = s_k = 0. \end{cases} \tag{5}$$

Computing the table T using dynamic programming requires time $O(n\omega^{(1)} \cdots \omega^{(k)})$.

Proof: First we consider the boundary conditions given by (5). There are three cases to consider.

1. Since the weights are non-negative, clearly if any s_j is negative, then there cannot be a coalition S with $w_S^{(j)} = s_j$. So it follows that $T(i, s_1, \ldots, s_k) = 0$ if $s_j < 0$. Since the sum of the j-th weight vector is $\omega^{(j)}$, no coalition can have weight greater than $\omega^{(j)}$ in the j-th component. So it follows that $T(i, s_1, \ldots, s_k) = 0$ if $s_j > \omega^{(j)}$.

2. If $i = 0$, then the only possible coalition S is the empty set $\emptyset$, and for $S = \emptyset$, $w_S^{(1)} = \cdots = w_S^{(k)} = 0$. So for $i = 0$, if any $s_i > 0$, then it again follows that $T(i, s_1, \ldots, s_k) = 0$.

3. If $s_1 = \cdots = s_k = 0$, then the only possible coalition is $S = \emptyset$. This is because any player has a positive weight in at least one of the k weight vectors, and so if S is non-empty, then $w_S^{(j)} > 0$ for at least one $j \in \{1, \ldots, k\}$. So for $i \geq 0$, there is exactly one coalition S, namely $S = \emptyset$, such that $w_S^{(1)} = \cdots, w_S^{(k)} = 0$. So it follows that $T(i, 0, \ldots, 0) = 1$.

The proof of (4) follows from the following fact. A subset S of $\{1,\dots,i\}$ has $w_S^{(1)} = s_1,\dots,w_S^{(k)} = s_k$ if and only if either S is a subset of $\{1,\dots,i-1\}$, with $w_S^{(1)} = s_1,\dots,w_S^{(k)} = s_k$, or $S\setminus\{i\}$ is a subset of $\{1,\dots,i-1\}$ with $w_S^{(1)} = s_1 - w_i^{(1)},\dots,w_S^{(k)} = s_k - w_i^{(k)}$.

For $1\leq j\leq k$, the value of s_j lies in the set $\{0,\dots,\omega^{(j)}\}$. So for a fixed value of i, the table T has $\omega^{(1)}\cdots\omega^{(k)}$ entries. A dynamic programming algorithm will fill up the entries of the complete table T in the following manner. Set $T(0,0,\dots,0)=1$ and $T(0,s_1,\dots,s_k)=0$ for all tuples $(s_1,\dots,s_k)$ with at least one s_j positive. This sets the values of $T(0,s_1,\dots,s_k)$. Next for $i>0$, set $T(i,0,\dots,0)=1$. Now for each $i\in\{1,\dots,n\}$, use (4) to obtain the value of $T(i,s_1,\dots,s_k)$, with $0\leq s_j\leq\omega^{(j)}$, $1\leq j\leq k$, from the already obtained values of the table for $i-1$. Thus, the time taken for each i is $O(\omega^{(1)}\cdots\omega^{(k)})$, and so the time taken to compute the entire table is $O(n\omega^{(1)}\cdots\omega^{(k)})$. □

Entries in the table T count the number of coalitions of given weights. Suppose that we also wish to obtain the cardinalities of these coalitions. To this end, we define a $k+2$ dimensional table C, where $C(i,c,s_1,\dots,s_k)$, with $0\leq i,c\leq n$, is the number of subsets S of $\{1,\dots,i\}$ having cardinality c with $w_S^{(1)} = s_1,\dots,w_S^{(k)} = s_k$.

Proposition 2 *For $i\geq 1$,*

$$C(i,c,s_1,\dots,s_k) = C(i-1,c,s_1,\dots,s_k) + C(i-1,c-1,s_1-w_i^{(1)},\dots,s_k-w_i^{(k)}) \tag{6}$$

with boundary conditions as follows.

$$C(i,c,s_1,\dots,s_k) = \begin{cases} 0 & \text{if for any } j\in\{1,\dots,k\},\ \text{either } s_j<0 \text{ or } s_j>\omega^{(j)}; \\ 0 & \text{if } i=0, 0\leq c\leq n, s_1,\dots,s_k\geq 0 \\ & \quad\text{and at least one of } s_1,\dots,s_k \text{ is positive}; \\ 0 & \text{if } 0\leq i\leq n, 1\leq c\leq n,\ \text{and } s_1=\cdots=s_k=0; \\ 1 & \text{if } 0\leq i\leq n, c=0,\ \text{and } s_1=\cdots=s_k=0. \end{cases} \tag{7}$$

Computing the table C using dynamic programming requires time $O(n^2\omega^{(1)}\cdots\omega^{(k)})$.

Proof: The proof of the boundary conditions is very similar to that in Proposition 1 and so we skip this proof.

The proof of (7) follows from the following fact. A subset S of $\{1,\dots,i\}$ with $\#S=c$ and $w_S^{(1)} = s_1,\dots,w_S^{(k)} = s_k$ if and only if either S is a subset of $\{1,\dots,i-1\}$ with $\#S=c$ and $w_S^{(1)} = s_1,\dots,w_S^{(k)} = s_k$, or $S\setminus\{i\}$ is a subset of $\{1,\dots,i-1\}$ with $\#S=c-1$ and $w_S^{(1)} = s_1-w_i^{(1)},\dots,w_S^{(k)} = s_k-w_i^{(k)}$.

The proof of the runtime is similar to that of Proposition 1. For a pair (i,c) there are $O(\omega^{(1)}\cdots\omega^{(k)})$ entries in the table C. The dynamic programming algorithm proceeds as follows. First fill up the entries of C determined by the boundary conditions. This determines the values of C for $i=c=0$. Next for each (i,c), with $i,c>0$, use (6) to determine the values of $C(i,c,s_1,\dots,s_k)$ from those found in the previous steps. Thus, for each pair (i,c), the time taken to fill up the table C is $O(\omega^{(1)}\cdots\omega^{(k)})$ and so the total time taken is $O(n^2\omega^{(1)}\cdots\omega^{(k)})$. □

Remark 1 *From the definitions of $T(i,s_1,\dots,s_k)$ and $C(i,c,s_1,\dots,s_k)$, it follows that if $(s_1,\dots,s_k)\notin\Omega$, then $T(i,s_1,\dots,s_k)=C(i,c,s_1,\dots,s_k)=0$, where Ω is the domain of the decision function f. In our analysis in Sections 5 and 6, we obtain sums over tuples $(s_1,\dots,s_k)$, where the individual terms of the sum are products of $f(s_1,\dots,s_k)$ with other quantities. While taking the sums, we will not restrict the tuples to be in Ω. This will mean applying f to tuples outside Ω. Formally, this is handled by* *extending the domain of f, where for $(s_1,\ldots,s_k)\notin\Omega$ the value of $f(s_1,\ldots,s_k)$ is set to be either* 0 *or* 1*. This will not cause any problem, since in all such cases we will show that the corresponding terms in the sum will turn out to be 0.*

5 Winning coalitions and swings

In this section, we use the results of Section 4 to show how the number of winning coalitions and the number of coalitions in which a player is a swing can be computed.

Proposition 3 *The number of winning coalitions in the game $G=(N,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)},f)$ is given by the following formula.*

$$\#W_G \;=\; \sum_{\substack{0\le s_1\le\omega^{(1)},\\ \cdots\\ 0\le s_k\le\omega^{(k)}}} f(s_1,\ldots,s_k)\cdot T(n,s_1,\ldots,s_n). \tag{8}$$

Proof: First note that if $(s_1,\ldots,s_k)\notin\Omega$, then $T(n,s_1,\ldots,s_k)=0$ and so the product $f(s_1,\ldots,s_k)\cdot T(n,s_1,\ldots,s_k)$ is also equal to 0 (see Remark 1).

$T(n,s_1,\ldots,s_k)$ is the number of coalitions $S\subseteq\{1,\ldots,n\}$ such that $w_S^{(1)}=s_1,\ldots,w_S^{(k)}=s_k$. Any such coalition S is a winning coalition if and only if $f(s_1,\ldots,s_k)=1$. So the right hand side of (8) counts the number of winning coalitions in G. □

For $i\in N$, let $W_G(i)$ be the set of winning coalitions containing the player i. Next we consider the cardinality of $W_G(i)$. The players and the weight vectors can be reordered so that the player under consideration can be considered to be n. So it is sufficient to find the number of winning coalitions containing the player n. This is given by the following result.

Proposition 4 *In the game $G=(N,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)},f)$, the number of winning coalitions containing the n-th player is given by the following relation.*

$$\#W_G(n) \;=\; \sum_{\substack{0\le s_1\le\omega^{(1)}-w_n^{(1)},\\ \cdots\\ 0\le s_k\le\omega^{(k)}-w_n^{(k)}}} f(s_1+w_n^{(1)},\ldots,s_k+w_n^{(k)})\cdot T(n-1,s_1,\ldots,s_k). \tag{9}$$

Proof: Let $\Omega_{|i}$ be the restriction of Ω to the k-tuples arising from coalitions of the players $\{1,\ldots,i\}$. If $(s_1+w_n^{(1)},\ldots,s_k+w_n^{(k)})\notin\Omega$, then it follows that $(s_1,\ldots,s_k)\notin\Omega_{|n-1}$ and so $T(n-1,s_1,\ldots,s_k)=0$ (see Remark 1).

The requirement is to count the number of coalitions $S\subseteq\{1,\ldots,n-1\}$ such that $S\cup\{n\}$ is winning. Since S is a subset of $\{1,\ldots,n-1\}$, we have $w_S^{(j)}\le\omega^{(j)}-w_n^{(j)}$, $1\le j\le k$. So it is sufficient to consider k tuples $(s_1,\ldots,s_k)$ with $0\le s_j\le\omega^{(j)}-w_n^{(j)}$, $j=1,\ldots,k$. The number of subsets S of $\{1,\ldots,n-1\}$ such that $w_S^{(1)}=s_1,\ldots,w_S^{(k)}=s_k$ is equal to $T(n-1,s_1,\ldots,s_k)$. These subsets are included in the count if $S\cup\{n\}$ is winning, i.e. if $f(s_1+w_n^{(1)},\ldots,s_k+w_n^{(k)})=1$. □

We next consider the characterisation of whether a player is a blocker or not. Player i is a blocker if and only if any coalition not containing i is a losing coalition. A coalition not containing i has weights $s_1,\ldots,s_k$ in the k parameters with $(s_1,\ldots,s_k)\in\Omega$ such that $s_1\le\omega^{(1)}-w_i^{(1)},\ldots,s_k\le\omega^{(k)}-w_i^{(k)}$. So

i is a blocker if and only if $f(s_1,\ldots,s_k)=0$ for all $(s_1,\ldots,s_k)\in\Omega$ such that $s_1\le\omega^{(1)}-w_i^{(1)},\ldots,s_k\le\omega^{(k)}-w_i^{(k)}$. This is equivalently stated as

$$\sum_{\substack{(s_1,\ldots,s_k)\in\Omega,\\ s_1\le\omega^{(1)}-w_i^{(1)},\\ \cdots\\ s_k\le\omega^{(k)}-w_i^{(k)}}} f(s_1,\ldots,s_k)=0. \tag{10}$$

The characterisation (10) is not computationally useful since the condition $(s_1,\ldots,s_k)\in\Omega$ needs to be checked. The table $T()$ provides this information. As before, by reordering players and weight vectors we may consider the player under consideration to be n. The player n is a blocker if and only if $\#W_G=\#W_G(n)$, which can be determined from Propositions 3 and 4. Alternatively, player n is a blocker if and only if the number of winning coalitions not containing player n is equal to 0. This is characterised by the following result.

Proposition 5 *In the game $G=(N,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)},f)$, the player n is a blocker if and only if*

$$\sum_{\substack{s_1\le\omega^{(1)}-w_n^{(1)},\\ \cdots\\ s_k\le\omega^{(k)}-w_n^{(k)}}} f(s_1,\ldots,s_k)T(n-1,s_1,\ldots,s_k)=0. \tag{11}$$

For $1\le j\le k$, we have the weight vector $\mathbf{w}^{(j)}=(w_1^{(j)},\ldots,w_n^{(j)})$. For $1\le i\le n$, define

$$\mathbf{w}^{(j,i)}=(w_1^{(j)},\ldots,w_{i-1}^{(j)},w_{i+1}^{(j)},\ldots,w_n^{(j)}).$$

In other words, $\mathbf{w}^{(j,i)}$ is obtained from $\mathbf{w}^{(j)}$ by dropping the component corresponding to i. Recall that the table T is prepared from the weight vectors $\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)}$. For $1\le i\le n$, define $T^{(i)}$ to be the table corresponding to (4) and (5) prepared from the weight vectors $\mathbf{w}^{(1,i)},\ldots,\mathbf{w}^{(k,i)}$. This corresponds to considering a game for the players $N\setminus\{i\}$ with the weight vectors $\mathbf{w}^{(1,i)},\ldots,\mathbf{w}^{(k,i)}$.

Proposition 6 *In the game $G=(N,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)},f)$, the number of coalitions in which player i is a swing is given by*

$$m_i = \sum_{\substack{0\le s_1\le\omega^{(1)}-w_i^{(1)},\\ \cdots\\ 0\le s_k\le\omega^{(k)}-w_i^{(k)}}} (1-f(s_1,\ldots,s_k))\cdot f(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\cdot T^{(i)}(n-1,s_1,\ldots,s_k). \tag{12}$$

Proof: We first argue that if either $(s_1,\ldots,s_k)\notin\Omega$ or $(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\notin\Omega$, then $T^{(i)}(n-1,s_1,\ldots,s_k)=0$ and so the corresponding terms in (12) are zero (see Remark 1). Let $\Omega^{(i)}=\{(s_1,\ldots,s_k)\in\Omega : (s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\in\Omega\}$. It follows that if $(s_1,\ldots,s_k)\notin\Omega^{(i)}$, then $T^{(i)}(n-1,s_1,\ldots,s_k)=0$. Now note that if either $(s_1,\ldots,s_k)\notin\Omega$ or $(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\notin\Omega$, then $(s_1,\ldots,s_k)\notin\Omega^{(i)}$ and so $T^{(i)}(n-1,s_1,\ldots,s_k)=0$.

In the modified game obtained by dropping player i, there are $n-1$ players and $T^{(i)}(n-1,s_1,\ldots,s_k)$ counts the number of coalitions $S\subseteq\{1,\ldots,n\}\setminus\{i\}$ in this modified game such that $w_S^{(1)}=s_1,\ldots,w_S^{(k)}=s_k$. Player i is a swing in the coalition $S\cup\{i\}$ in G if S is losing in G and $S\cup\{i\}$ is winning in G. So i is a swing in $S\cup\{i\}$ in G if $f(s_1,\ldots,s_k)=0$ and $f(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})=1$. The last condition is equivalent to $(1-f(s_1,\ldots,s_k))\cdot f(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})=1$. So the right hand side of (12) counts the number of coalitions in which player i is a swing in the game G. $\square$

Let $C^{(i)}$ be the table corresponding to (6) and (7) prepared from the weight vectors $\mathbf{w}^{(1,i)}, \ldots, \mathbf{w}^{(k,i)}$. The following result provides the cardinalities of the coalitions in which player i is a swing. The proof is similar to the proof of Proposition 6 and so we skip it.

Proposition 7 *In the game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, the number of coalitions of cardinality c in which player i is a swing is given by*

$$\sum_{\substack{0 \leq s_1 \leq \omega^{(1)} - w_i^{(1)}, \\ \ldots \\ 0 \leq s_k \leq \omega^{(k)} - w_i^{(k)}}} (1 - f(s_1, \ldots, s_k)) \cdot f(s_1 + w_i^{(1)}, \ldots, s_k + w_i^{(k)}) \cdot C^{(i)}(n-1, c-1, s_1, \ldots, s_k). \tag{13}$$

6 Minimal winning coalitions

In this section, we put a restriction on the weight vectors and the decision function f. The condition on the weight vectors is that they must be simultaneously decreasing, i.e.,

$$\begin{array}{c} w_1^{(1)} \geq \cdots \geq w_n^{(1)}, \\ \cdots \\ w_1^{(k)} \geq \cdots \geq w_n^{(k)}. \end{array}$$

The condition on f is that it must be monotone on each component, i.e., if $s_i' > s_i$, then

$$f(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_k) \geq f(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_k)$$

for all $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k$.

The above two conditions simplify the task of counting minimal winning coalitions. Suppose S is a winning coalition. To be a minimal winning coalition it is required that dropping any player from S results in a losing coalition. Let $i = \max S$ and suppose that dropping i from S results in a losing coalition. By the above two conditions, it follows that dropping ℓ from S, with $\ell < i$, instead of i also results in a losing coalition. So to check whether S is a minimal winning coalition, it is sufficient to check whether dropping the highest numbered player from S results in a losing coalition. While the two conditions simplify the task of counting minimal winning coalitions, we do not know whether they are necessary for being able to efficiently compute the number of minimal winning coalitions.

The above conditions on the weight vectors and the decision rule are satisfied by the Lisbon voting rules in the EU Council. This voting rule can be modelled as a 2-parameter game. The first parameter corresponds to the cardinality of a coalition and the weight vector $\mathbf{w}^{(1)}$ has the entry 1 for all the nations. The second parameter corresponds to the total size of the population of a coalition and the weight vector $\mathbf{w}^{(2)}$ records the populations of the nations. Let $\omega^{(2)}$ be the sum of all the components of $\mathbf{w}^{(2)}$. The function $f(w_S^{(1)}, w_S^{(2)})$ takes the value 1 if and only if $\left(w_S^{(1)}/28 \geq 0.55 \wedge w_S^{(2)}/\omega^{(2)} \geq 0.65\right) \vee \left(w_S^{(1)}/28 \geq 0.86\right)$. The last condition $w_S^{(1)}/28 \geq 0.86$ captures the blocking minority condition which states that at least 4 nations are required to block a resolution. This above formalisation of the Lisbon voting rule of the EU Council is from Kurz and Napel (2016). In this game $k = 2$ and $w_1^{(1)} = \cdots = w_n^{(1)} = 1$. So simply arranging the components of $\mathbf{w}^{(2)}$ in descending order ensures that the simultaneous decreasing condition on the weight vectors hold. Further, since f is a monotone Boolean combination of thresholding functions, the property of being monotone on each component also holds for f.

We now turn to the problem of determining the number of minimal winning coalitions in a general multi-parameter weighted voting game. In the game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, for $1 \leq i \leq n$, let E_i be the number of minimal winning coalitions S such that i is in S and $S \subseteq \{1, \ldots, i\}$. Then the number of minimal winning coalitions in G is equal to $E_1 + \cdots + E_n$. So to obtain the number of minimal winning coalitions in G it is sufficient to obtain E_i.

Proposition 8 *In the game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, the value of E_i is given by*

$$\sum_{\substack{0 \leq s_1 \leq \omega_i^{(1)}, \\ \cdots \\ 0 \leq s_k \leq \omega_i^{(k)}}} f(s_1, \ldots, s_k) \cdot \left(1 - f(s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)})\right) \cdot T(i-1, s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}), \tag{14}$$

where for $1 \leq j \leq k$, $\omega_i^{(j)} = w_1^{(j)} + \cdots + w_i^{(j)}$.

Proof: For the computation of E_i, it is sufficient to consider only the set of players $\{1, \ldots, i\}$. Let $\Omega_{|i}$ be as defined in the proof of Proposition 4. So in (14), f is applied to k-tuples in $\Omega_{|i}$. Now note that if either $(s_1, \ldots, s_k) \notin \Omega_{|i}$, or $(s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}) \notin \Omega_{|i}$, then $(s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}) \notin \Omega_{|i-1}$ which implies that $T(i-1, s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}) = 0$ and so the corresponding terms in (14) are 0 (see Remark 1).

The coalitions S such that $i \in S$, $S \subseteq \{1, \ldots, i\}$ and $w_S^{(1)} = s_1, \ldots, w_S^{(k)} = s_k$ are counted in the expression $T(i, s_1, \ldots, s_k) - T(i-1, s_1, \ldots, s_k) = T(i-1, s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)})$. Now suppose $f(s_1, \ldots, s_k) = 1$, i.e., S is a winning coalition and $f(s_1 - w_i^{(1)}, \ldots, s_k - w_i^{(k)}) = 0$, i.e., dropping i results in a losing coalition. By the simultaneous decreasing property of the weight vectors and the component-wise monotone property of f it follows that dropping any player ℓ with $\ell < i$ instead of i also results in a losing coalition. So such an S is a minimal winning coalition and is counted in E_i. From this we obtain the expression for E_i. □

Next we consider the number of minimal winning coalitions containing a fixed player i. The minimal winning coalitions counted in E_i certainly contain i and the minimal winning coalitions counted in E_ℓ for $\ell < i$ certainly do not contain i. It is, however, possible that for $\ell > i$, a minimal winning coalition counted in E_ℓ also contains i. For $\ell > i$, let $E_{i,\ell}$ be the number of minimal winning coalitions containing both i and ℓ. Then the number of minimal winning coalitions containing i is equal to $E_i + E_{i,i+1} + \cdots + E_{i,n}$. Proposition 8 already shows how to obtain E_i. The next result shows how to obtain $E_{i,\ell}$ for $\ell = i+1, \ldots, n$.

Proposition 9 *In the game $G = (N, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}, f)$, the value of $E_{i,\ell}$ for $i+1 \leq \ell \leq n$ is given by*

$$\sum_{\substack{0 \leq s_1 \leq \omega_\ell^{(1)} - w_i^{(1)}, \\ \cdots \\ 0 \leq s_k \leq \omega_\ell^{(k)} - w_i^{(k)}}} g(s_1, \ldots, s_k) h(s_1, \ldots, s_k) \cdot T^{(i)}(\ell - 1, s_1 - w_\ell^{(1)}, \ldots, s_k - w_\ell^{(k)}), \tag{15}$$

where

$$\begin{aligned} g(s_1, \ldots, s_k) &= f(s_1 + w_i^{(1)}, \ldots, s_k + w_i^{(k)}), \\ h(s_1, \ldots, s_k) &= \left(1 - f(s_1 + w_i^{(1)} - w_\ell^{(1)}, \ldots, s_k + w_i^{(k)} - w_\ell^{(k)})\right). \end{aligned}$$

Proof: For the computation of $E_{i,\ell}$ it is sufficient to restrict to the set of players $\{1,\ldots,\ell\}$. Let $\Omega_{|\ell}$ be defined as in the proof of Proposition 4 and $\Omega^{(i)}_{|\ell}$ be defined from $\Omega_{|\ell}$ in a manner similar to the definition of $\Omega^{(i)}$ from Ω in the proof of Proposition 6. We note that $(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\notin\Omega_{|\ell}$ if and only if $(s_1+w_i^{(1)}-w_\ell^{(1)},\ldots,s_k+w_i^{(k)}-w_\ell^{(k)})\notin\Omega_{|\ell-1}$. Further, if $(s_1+w_i^{(1)}-w_\ell^{(1)},\ldots,s_k+w_i^{(k)}-w_\ell^{(k)})\notin\Omega_{|\ell-1}$, then $(\ell-1,s_1-w_\ell^{(1)},\ldots,s_k-w_\ell^{(k)})\notin\Omega^{(i)}_{|\ell}$ implying that $T^{(i)}(\ell-1,s_1-w_\ell^{(1)},\ldots,s_k-w_\ell^{(k)})=0$. Consequently, if either $(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})\notin\Omega_{|\ell}$ or $(s_1+w_i^{(1)}-w_\ell^{(1)},\ldots,s_k+w_i^{(k)}-w_\ell^{(k)})\notin\Omega_{|\ell-1}$, then the corresponding terms in (15) are 0 (see Remark 1).

Recall that the table $T^{(i)}$ is constructed from the weight vectors obtained by dropping the player i. The number of coalitions $S\subseteq\{1,\ldots,\ell\}\setminus\{i\}$ containing ℓ such that $w_S^{(1)}=s_1,\ldots,w_S^{(k)}=s_k$ is given by $T^{(i)}(\ell,s_1,\ldots,s_k)-T^{(i)}(\ell-1,s_1,\ldots,s_k)=T^{(i)}(\ell-1,s_1-w_\ell^{(1)},\ldots,s_k-w_\ell^{(k)})$. By the simultaneous decreasing condition on the weights and the component-wise monotone property of f, such a coalition is counted in $E_{i,\ell}$ if and only if $S\cup\{i\}$ is winning and $(S\cup\{i\})\setminus\{\ell\}$ is losing. The last condition is equivalent to $f(s_1+w_i^{(1)},\ldots,s_k+w_i^{(k)})=1$ and $f(s_1+w_i^{(1)}-w_\ell^{(1)},\ldots,s_k+w_i^{(k)}-w_\ell^{(k)})=0$. □

Next we consider the cardinalities of the minimal winning coalitions containing a player i. Let $F_{i,c}$ be the number of minimal winning coalitions of cardinality c containing the player i. Similarly, for $\ell>i$, let $F_{i,\ell,c}$ be the number of minimal winning coalitions of cardinality c containing both the players i and ℓ. So by an argument similar to the one for the number of minimal winning coalitions containing i, we have that the number of minimal winning coalitions of cardinality c containing player i is

$$F_{i,c}+F_{i,i+1,c}+\cdots+F_{i,n,c}.$$

So to obtain the number of minimal winning coalitions of a particular cardinality containing a player, it suffices to obtain $F_{i,c}$ and $F_{i,\ell,c}$ for $\ell>i$. These expressions are obtained in a manner similar to that for E_i and $E_{i,\ell}$ and are stated in the following result.

Proposition 10 *In the game $G=(N,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(k)},f)$, the value of $F_{i,c}$ is given by*

$$\sum_{\substack{0\le s_1\le\omega_i^{(1)},\\ \cdots\\ 0\le s_k\le\omega_i^{(k)}}} f(s_1,\ldots,s_k)\cdot\Big(1-f(s_1-w_i^{(1)},\ldots,s_k-w_i^{(k)})\Big)\cdot C(i-1,c-1,s_1-w_i^{(1)},\ldots,s_k-w_i^{(k)}). \quad (16)$$

Further, the value of $F_{i,\ell,c}$ is given by

$$\sum_{\substack{0\le s_1\le\omega_\ell^{(1)}-w_i^{(1)},\\ \cdots\\ 0\le s_k\le\omega_\ell^{(k)}-w_i^{(k)}}} g(s_1,\ldots,s_k)h(s_1,\ldots,s_k)\cdot C^{(i)}(\ell-1,c-1,s_1-w_\ell^{(1)},\ldots,s_k-w_\ell^{(k)}), \quad (17)$$

where $g(s_1,\ldots,s_k)$ and $h(s_1,\ldots,s_k)$ are as defined in Proposition 9.

7 Conclusion

In this paper we have introduced a general class of simple voting games defined from multiple weight vectors. The classes of weighted AND games, weighted OR games and monotone Boolean combination of weighted games can be seen as particular cases of the general class that we introduce. Further, we introduce the notion of hyperplane voting games and show that such games can be compactly modelled using the new framework, but not necessarily so using the previously known frameworks. For the new class of games we have shown dynamic programming techniques to compute various important parameters of a game from which it is possible to compute the well known voting power indices. The unified approach and the ability to model new voting scenarios make the new class of games to be of both theoretical and practical interest. We hope that this class of games will attract further attention from researchers. In particular, it will be fruitful to investigate whether the techniques for improving the complexity of the dynamic programming algorithms for weighted majority voting games, such as those by Bolus (2011), Uno (2012) and Kurz (2016), can be generalised to apply to the new class of games.

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References

Algaba, E. J. M., Bilbao, J.M., García, F. and López, J. J.: Computing power indices in weighted multiple majority games. Mathematical Social Sciences 46, 63–80 (2003).

Algaba, E., Bilbao, J.M. and Fernández, J.: The distribution of power in the European constitution. European Journal of Operational Research 176 (3), 1752–1766 (2007).

Alonso-Meijide, J. M., Bilbao, J., Casas-Méndez, B. and Fernández, J.: Weighted multiple majority games with unions: generating functions and applications to the European Union. European Journal of Operational Research 198 (2), 530–544 (2009).

Banzhaf, J. F.: Weighted voting doesn't work: A mathematical analysis. Rutgers Law Review 19, 317-343 (1965).

Bhattacherjee, S. and Sarkar, P.: Correlation and inequality in weighted majority voting games, in Dasgupta, I, and Mitra, M.(eds.) Deprivation, Inequality and Polarization: Essays in Honor of S.R. Chakravarty, pp. 161-191. Springer, New York (2019).

Bolus, S.: Power Indices of simple games and vector-weighted majority games by means of binary decision diagrams. European Journal of Operational Research 210 (2), 258–272 (2011).

Chakravarty, S.R., Mitra, M. and Sarkar, P.: A Course on Cooperative Game Theory. Cambridge University Press, Cambridge (2015).

Coleman, J.S.: Control of collectives and the power of a collectivity to act, in: Lieberman, B. (eds.) Social Choice, pp. 269-298. Gordon and Breach, New York (1971).

Deĭneko, V. G. and Woeginger, G. J.: On the dimension of simple monotonic games. European Journal of Operational Research 170, 315-318 (2006).

Einy E. and Peleg B.: Linear measures of inequality for cooperative games. Journal of Economic Theory 53, 328-344 (1991).

Elkind, E., Goldberg, L. A., Goldberg, P. W. and Wooldridge, M. J.: On the computational complexity of weighted voting games. Annals of Mathematics and Artificial Intelligence 56, 109-131 (2009).

Faliszewski, P., Elkind E. and Wooldridge M.: Boolean Combinations of Weighted Voting Games. In Proceedings of 8th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), editors Sichmann, S. and Castelfranchi, 185–192 (2009).

Felsenthal, D.S. and Machover, M.: The Measurement of Voting Power. Edward Elgar (1998).

Holler, M.J.: Forming coalitions and measuring voting power. Political Studies 30, 262-271 (1982).

Johnston, R. J.: On the measurement of power: some reactions to Laver. Environment and Planning A 10, 907-914 (1978).

Kurz, S.: Computing the Power Distribution in the IMF. Arxiv, `https://doi.org/10.48550/arXiv.1603.01443`, (2016).

Kurz, S. and Napel, S.: Dimension of the Lisbon voting rules in the EU Council: a challenge and new world record. Optimization Letters 10, 1245-1256 (2016).

Laruelle A. and Valenciano F.: Inequality in voting power. Social Choice and Welfare 22, 413-431 (2004).

Laruelle, A. and Valenciano, F.: Voting and Collective Decision-Making: Bargaining and Power. Cambridge University Press, Cambridge (2008).

Leech, D.: Voting power in the governance of the International Monetary Fund. Annals of Operations Research 109, 375-397 (2002a).

Leech, D.: Designing the voting system for the council of the European Union. Public Choice 113, 437-464 (2002b).

Maschler, M. and Owen, G.: The consistent Shapley value for hyperplane games. International Journal of Game Theory 18, 389–407 (1989).

Matsui, T. and Matsui, Y.: A survey of algorithms for calculating power indices for weighted majority games. Journal of the Operations Research Society of Japan 43, 71-86 (2000).

Penrose, L.S.: The elementary statistics of majority voting. Journal of the Royal Statistical Society 109(1), 53–57 (1946).

Rae, D.: Decision rules and individual values in constitutional choice. American Political Science Review 63, 40-56 (1969).

Shapley, L .S. and Shubik, M, J.: A method for evaluating the distribution of power in a committee system. American Political Science Review 48, 787-792 (1954).

Strand, J.R., Rapkin, D.P, and Wood, D.B.: The political economy of a priori voting power in the World Bank and IMF. Mimeo (1997).

Taylor, A. D. and W. S. Zwicker: Weighted voting, multicameral representation, and power. Games and Economic Behavior 5, 170-181 (1993).

Taylor, A. D. and W. S. Zwicker: Simple games and magic squares. Journal of Combinatorial Theory, Series A, 71, 67-88 (1995).

Taylor, A. D. and W. S. Zwicker: Simple Games: Desirability Relations, Trading, and Pseudoweightings. Princeton University Press, New Jersey, USA (1999).

Uno, T.: Efficient Computation of Power Indices for Weighted Majority Games. In: Algorithms and Computation, editors Chao, K.-M., Sheng Hsu, T. and Lee, D.-T., Springer, 679–689, (2012).

Weber, M.: Two-tier voting: measuring inequality and specifying the inverse power problem. Mathematical Social Sciences 79, 40–45 (2016).

Wilms, I.: Dynamic programming algorithms for computing power indices in weighted multi-tier games. Mathematical Social Sciences 108, 175–192 (2020).

Yu, C.: Hyperplane games, prize games and NTU values. Theory and Decision 93, 359–370 (2022).