Contagion in debt and collateral markets

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Abstract

This paper investigates contagion in financial networks through both debt and collateral markets. Payment from a collateralized debt contract depends not only on the borrower's balance sheet, but also on the price of the underlying collateral. If the negative liquidity shock is small, then having more connections makes the network safer as contagion through the debt channel is minimized by diversified exposures. Even if the negative liquidity shock is large, collateral can mitigate counterparty exposures and reduce contagion through the debt channel. However, if collateral is not enough (leverage is high) and agents in the network are too interconnected, then collateral price can also plummet to zero and the whole network can collapse. Therefore, we show the importance of the interaction between the level of collateral and interconnectedness across agents. The model also provides the minimum collateral-debt ratio (haircut) to attain a robust macroprudential state for a given network structure and aggregate shock.

Keywords: collateral, financial network, fire sale, systemic risk

JEL Classification Numbers: D49, D53, G01, G21, G33

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1. Introduction

This paper studies how initial shocks propagate through a network of counterparties in collateralized debt markets. The Global Financial Crisis (GFC) is exacerbated by a failure of collateralized debt markets, which is the most common form of short-term financing among financial institutions, including repurchase agreement (repo) and asset-backed commercial paper (ABCP) markets (Gorton and Metrick, 2012). The collapse in prices of subprime mortgages in 2008 had a direct effect on many financial institutions. However, the initial shock was exacerbated by the resulting bankruptcy of Lehman Brothers, which spread the initial losses to Lehman's counterparties (Singh, 2017). Currently, the volume of collateralized debt markets is very large. The average daily amount of cash and collateral traded in bilateral repo markets in the U.S. is more than \$4 trillion for 2019, 2020, and 2021.[1] Also, the global market for securities lending is totaling more than \$3 trillion in outstanding contracts, with U.S. loans accounting for about half of the worldwide market in 2021 (Financial Stability Oversight Council, 2021). Hence, understanding how such markets can be vulnerable to contagion is important for both academics and policy makers.

A typical collateralized debt takes the form of a one-to-one relationship between a borrower and a lender because of customization (bespoke) and counterparty-specific contract terms, such as margins and rates. If the value of the collateral is greater than the face value of the debt, then the payment is always made in full even if the borrower becomes insolvent. However, if the value of the collateral is less than the face value of the debt, then the payment depends on both the price of the collateral and the cash balance of the borrowing counterparty.

Therefore, a collateralized debt network has two transmission channels of shocks—the collateral price channel and the debt counterparty channel. If the asset price declines, then the net wealth of all agents decreases through the collateral price channel. If an agent defaults and spreads losses to its counterparties, then the counterparties' net wealth decreases because of the counterparty channel. The counterparty losses can continue to depress the asset price, resulting in further losses.

In this paper, we develop a network model with collateral featuring the two channels of propagation, which is the first attempt to endogenize asset prices while the debt contracts are full recourse.[2] Typical financial network models in the literature investigate only either

[1] See https://www.sifma.org/wp-content/uploads/2022/02/SIFMA-Research-US-Repo-Markets-Chart-Book-2022.pdf.

[2] In Chang (2021), nonrecourse contracts are considered to solve for endogenous network formation. In this paper, we do not attempt to endogenize network formation, however we solve for the networks with full-recourse contracts. The full-recourse property complicates the propagation structure and makes the problem highly intractable because of higher-degree of interactions across the network.

contagion of liquidity shortages through the counterparty channel or price-mediated losses of common asset holdings. However, payments from collateralized debt contracts depend on the interaction between the debt counterparty channel and the collateral price channel because the collateral price changes endogenously and simultaneously. This paper explores the implications of this additional interaction of the two channels of propagation.

The model is based on an economy of n agents, who trade an asset that can be used as collateral in a competitive market in each period. The price of the asset is endogenously determined in a competitive Walrasian market, and the fundamental value of the collateral, which is realized in the final period, is common knowledge to everyone in the economy. In the initial period, agents borrow from each other using bilateral collateralized debt contracts, specifying the amount of debt and collateral. The structure of these liabilities can be summarized as the collateralized debt (financial) network. All the debt contracts mature in the interim period. Agents also have invested in a long-term project that generates return at the final period, and the return from the long-term project is not pledgeable. Liquidating the long-term project is costly, so it is socially inefficient to liquidate the project. However, if agents are under negative liquidity shocks (e.g. lower than expected short-term returns, sudden increase in deposit withdrawals, wage expenses, taxes, or other senior creditors), then they may have to liquidate their long-term investment projects to pay their debt. If an agent's net wealth is still negative after completely liquidating the long-term project, then the agent defaults, which may trigger additional defaults through the network.

This model is consistent with the literature as it extends and reproduces the results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), who develops a model of unsecured debt networks. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) show that there exists a phase transition property of contagion, also called "robust yet fragile," depending on the size of the liquidity shocks. If the shocks are small, then the complete network is the most stable and resilient network because more lending relationships can diversify the spread of losses. However, if the shocks are large, each lending relationship becomes a source for contagion. Therefore, the complete network, having the most linkages, becomes the least stable and resilient network. The same result holds in our model when the ratio between collateral and debt, collateral-debt ratio, is at or close to zero.

The co-existence of collateral and bilateral debt relationships (financial network) in our model generates many different equilibrium outcomes depending on the amount of collateral and the network structure as well as the size of the shock.

If the collateral-debt ratio is high enough, the equilibrium is in fully-insulated regime in which any borrower default can be fully covered by the market value of the collateral. The value of collateral is transferred to the lenders of a defaulting agent, effectively transferring payments from liquidity shocks to the lenders in the network. This is in line with the practice in real world markets, for example, repo collateral is exempt from automatic stay of bankruptcy provisions. Hence, a shock to an agent does not propagate to other agents regardless of the network structure and size of the liquidity shocks. Then, any network is completely insulated from liquidity shocks of any size, as the collateral provides proper "secureness" of the market by adding a guaranteed payment amount regardless of the borrower's balance sheet.

Even if the collateral-debt ratio is not high enough to guarantee full payments, as long as the collateral-debt ratio is above a threshold, the collateral price is at its fundamental value regardless of the network structure and the size of the liquidity shocks. As in the previous case, collateral limits the drainage of cash flow from the network, so the remaining agents can buy the assets on fire sale (from defaulting agents) at the asset's fundamental value. Even though the collateral price is at its fundamental value, the degree of contagion in the network depends on the network structure in this case. One can consider this as a robust regime because the complete network is the most stable and resilient network, implying that having more links is beneficial.

However, if the collateral-debt ratio is not high enough, a large negative liquidity shock may trigger a cascade of defaults and the collateral price can go well below its fundamental value. A liquidity shock causes counterparty defaults and collateral is not enough to cover the default amount, so additional defaults occur. Such counterparty losses reduce available cash to buy the assets on fire sale at the asset's fundamental value. The collateral asset price declines, increasing the amount of defaults and fire sales, which further decrease the asset price, and so on. Hence, once the network hits the threshold, collateral cannot serve its proper role of reducing the counterparty exposures due to the dual feedback loop between counterparty contagion and collateral price decline due to fire sales. Nevertheless, a network with limited interconnectedness across agents can prevent full collapse of the network and collateral price, as a liquidity shock can be contained within the part of the network that was hit by the shock.

Our results highlight the fragility of collateral's role on reducing counterparty exposures. When defaults occur, collateral can act as a buffer as long as the counterparty exposures themselves are limited. In most cases, either the collateral is enough or the external shocks are small, the collateral price will be its fair value, and no further contagion occurs. However, as soon as the counterparty exposures exceed the threshold and collateral is not enough, then the collateral price will plunge to zero, leaving all the contracts unsecured. Therefore, the equilibrium collateral price shows a bang-bang property.

Finally, the model provides insights on macro-prudential policy, such as leverage restrictions, as the economy faces aggregate shock, in particular, changes in the fundamental value of the collateral asset. We obtain the levels of collateral-debt ratio required to attain a robust or fully-insulated regime, depending on the fundamental value of the collateral asset. If the payoff of the asset decreases, the threshold level of collateral-debt ratio required for the robust and fully-insulated regimes increases.

Besides providing insights on financial contagion, this paper also provides insights on the role of explicit collateral and settlements in lending networks on mitigating contagion. Contrary to the many models in the macro literature, which uses the total amount of capital or going-concern value as collateral, contracts in our model specifies explicitly designated collateral, which is in line with the collateralized debt markets in the real world. In our model, even if the total amount of assets or individual asset holdings are the same, an economy with higher collateral-debt ratios would have lower systemic risk than an economy with lower collateral-debt ratios. This is because explicit collateral limits the transmission of idiosyncratic liquidity shocks to other agents in the network, minimizing the negative spillovers. Hence, reuse (rehypothecation) of collateral can improve financial stability for a given debt amount and asset holdings. Indeed, reuse of collateral can alter the total cash holdings and network structure as well as the debt amount (Chang, 2021). Such endogenous network formation is beyond the scope of this paper.

Our results have direct real world and policy implications. One of the most relevant applications is the contagion between the digital assets (e.g. crypto-assets) markets and the (traditional) financial markets. Decentralized Finance (DeFi) and other lending platforms of digital assets markets typically use collateralized debt contracts (Azar et al., 2022). If issuers of stablecoins, which are often used as collateral for DeFi lending, have typical safe assets in the financial markets as their reserve assets, then both the financial markets and digital assets markets share the same underlying collateral base. Our results suggest that if the two different markets have more than a small degree of interconnectedness, a shock to the digital assets markets can also propagate to the financial market, resulting in a systemic event.

1.1. Related Literature

The first contribution of this paper is developing a model with both the debt and collateral channels of contagion with full-recourse contracts, which is the first attempt in the literature. No major institution failed because of losses on its direct exposures to Lehman Brothers; thus, developing a model that combines different shock transmission channels in financial networks is important in understanding contagions through interconnectedness (Upper, 2011;

Glasserman and Young, 2016). The model in this paper incorporates default cascades and price-mediated losses, and the interaction of the two channels leads to novel properties of contagion.

The literature on financial networks usually focuses on the tradeoff between diversification and the contagion channel of having more links. This paper suggests that the tradeoff can change depending on the collateral-debt ratio and the aggregate shocks, because the contracts are collateralized, and the collateral asset price is endogenous. Eisenberg and Noe (2001) introduced an exogenous network model as a financial network with propagation through the payments, which Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) extended with a debt financial network without collateral. The payment equilibrium concept employed in such literature is used in this paper as well. The model of this paper is based on Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and extends their model further by incorporating collateral and collateral markets. The key difference is that we take the excess cash inflows of non-defaulting agents into account to track the changes in the collateral price that affects defaults through the collateralized debt contract structure. Elliott, Golub, and Jackson (2014) include discontinuous jumps in the payoffs of agents in the case of bankruptcy, which are also incorporated in this paper. The key difference is that jumps in their model are caused by a lump sum cost of bankruptcy, whereas the jumps in our model are caused by the additional fire sales of assets caused by bankruptcy. Therefore, multiplicity in this paper is almost confined to non-generic cases. Allen and Gale (2000) studied liquidity coinsurance through networks, which is also the case under the small shock case in our model.

The feedback from agents' net wealth to collateral price is crucial in this paper. Other papers consider the interaction between counterparty and price channels, such as Capponi and Larsson (2015); Cifuentes, Ferrucci, and Shin (2005); Di Maggio and Tahbaz-Salehi (2015); Gai, Haldane, and Kapadia (2011); and Rochet and Tirole (1996). This paper differs by incorporating the debt contracts with explicit collateral and the endogenous price channel of contagion for the underlying collateral.

The endogenous price determination in this paper is based on the literature on general equilibrium with collateralized debt, as in Geanakoplos (1997), Geanakoplos (2010), and Fostel and Geanakoplos (2015). This paper contributes to this literature by linking these features into the network contagion, which is crucial in bilateral repo and ABCP markets, and analyzing the effect of counterparty risks on prices.

Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or a common correlation structure (Cabrales, Gottardi, and Vega-Redondo, 2017). Moreover, the literature documents fire sales in financial markets, which implies that sales of an asset manager or a bank can depress asset prices and lead to more sales from others, depressing both the market price and balance sheets further (Coval and Stafford, 2007; Chen, Goldstein, and Jiang, 2010; Jotikasthira, Lundblad, and Ramadorai, 2012; Greenwood, Landier, and Thesmar, 2015; Goldstein, Jiang, and Ng, 2017; Duarte and Eisenbach, 2021). However, these models typically assume linear price impact from fire sales. Our model incorporates more complicated effects of fire sales by analyzing how asset prices can affect counterparty payments and vice versa.

Finally, this paper is also related to the literature on the role of collateral. Geanakoplos (2010) argues that collateralized debt makes the market more complete by being an enforcement device. However, only the aggregate level of collateral matters in the general equilibrium literature because contracts are fully anonymized and diversified. This paper shows how individual collateral matters when counterparty risk is involved. Demarzo (2019) addresses that collateral can be a cost-efficient commitment device and Donaldson, Gromb, and Piacentino (2020) argue that secured debt prevents debt dilution. This paper adds additional roles of collateral as mitigating and amplifying channels of counterparty contagion.

The closest to this paper is Chang (2021). As in this paper, Chang (2021) analyzes the interaction between the counterparty and price channel of spillovers. The main difference is that the model in Chang (2021) simplifies the borrower default contagion by assuming non-recourse contracts in order to focus on lender default and network formation. This paper focuses only on borrower default and exogenous networks while generalizing the contract structure to full-recourse contract to analyze more general and realistic borrower default contagion.

2. Model

The model builds on the model of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). The key new feature is the existence of assets that can be used as collateral, whose price is endogenously determined. The main heterogeneity of interest comes from how agents are connected to each other through collateralized debt relationships.

2.1. Agents and Goods

There are three periods $t = 0, 1, 2$ and two goods, cash and asset, denoted as e and h, respectively. Cash is the only consumption good and storable. Asset can be used as collateral at $t = 0$ and yields s amount of cash at $t = 2$. Agents gain no utility from just holding the asset. All agents learn the true value of the asset payoff s at $t = 1$; however, the asset payoff is realized at $t = 2$.

There are n different agents, and the set of all agents is $N = \{1, 2, \ldots, n\}$. Agents are risk neutral, and their utility is determined by how much cash they consume at $t = 2$. Each agent is investing in a long-term investment project that will give ξ amount of cash at $t = 2$ if it is held by maturity. The payoff from this long-term investment project is not pledgeable. Agent j can partially liquidate the project by $l_j \in [0, \xi]$ amount at $t = 1$ to receive scrap value of ζl_j in terms of cash, where $0 \leq \zeta < 1$ represents the liquidation efficiency.

All information is common knowledge and the markets for both goods are competitive Walrasian markets. Thus, agents are price-takers and there is no asymmetric information. The price of cash is normalized to 1 at any period, and the price of asset is p_t for $t = 0, 1, 2$. From now on, we use p instead of p_1 for the price of the asset at $t = 1$, as our main focus is analyzing the contagion in $t = 1$.

2.2. Collateralized Debt Network

At $t = 0$, each agent $j \in N$ holds e_j amount of cash and h_j amount of asset, which are exogenously given, to store it until $t = 1$. Also at $t = 0$, agents borrow or lend cash using assets as collateral. At $t = 1$, agents can buy or sell the asset in a competitive market. All borrowing contracts are a one-period contract between $t = 0$ and $t = 1$, which are exogenously determined. A borrowing contract consists of the total amount of promised cash payment, the ratio of collateral posted per one unit of promised cash, and the identities of the borrower and the lender. Denote d_{ij} as the total promised cash amount to pay at $t = 1$ to lender i by borrower j. Denote c_{ij} as the collateral-debt ratio per one unit of promised cash, which we refer to as collateral ratio from now on. If borrower j pays back the full amount of promised d_{ij}, then the lender returns the collateral in the amount of $c_{ij}d_{ij}$. Otherwise, the lender keeps the collateral, and the cash value of the collateral is $c_{ij}d_{ij}p$. Normalize $c_{ii} = d_{ii} = 0$ for all $i \in N$ without loss of generality.

Define $C = [c_{ij}]$ and $D = [d_{ij}]$ as the matrices of collateral ratios and promised debt payment amount, respectively. A collateralized debt network is a weighted directed multiplex graph that is formed by the set of vertices N and links with two layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = \left(\mathcal{G}^{[1]}, \mathcal{G}^{[2]}\right)$, where $\mathcal{G}^{[\alpha]} = \left(N, L^{[\alpha]}\right)$, $L^{[1]} = C$, and $L^{[2]} = D$. A (collateralized) debt network can be summarized by a double (C, D) given at $t = 0$ with the set of vertices N. A debt network describes how much each agent borrows from or lends to other agents at what margin (collateral ratio). Denote the total inter-agent liabilities of agent j as $d_j \equiv \sum_{i \in N} d_{ij}$.

We assume that *collateral constraints* and *resource constraints* hold, which imply

$$\sum_{k \in N} c_{jk} d_{jk} + h_j \geq \sum_{i \in N} c_{ij} d_{ij} \qquad \forall j \in N \tag{1}$$

$$\sum_{i \in N} h_i \geq \sum_{i \in N} c_{ij} d_{ij} \qquad \forall j \in N. \tag{2}$$

Collateral constraint means the total amount of collateral a borrowing agent j posts cannot exceed the amount of asset the agent has—either from other people's collateral that agent j has received as a lender or the amount of asset agent j purchased outright. This collateral constraint implies the model allows re-use (rehypothecation) of collateral.[3] Resource constraint means the total amount of assets an agent is posting cannot exceed the total amount of assets in the economy.[4]

Each agent can be hit by a negative liquidity shock in cash in the absolute value of $\epsilon > 0$ at $t = 1$. Agents should pay the liquidity shock first before paying other agents. We interpret ϵ as senior debt payment to external creditors, who also have linear utility. A realized state of the liquidity shocks is $\omega \equiv (\omega_1, \omega_2, \dots, \omega_n)$, and the set of all possible states is Ω. For example, $\omega_j = 1$ if agent j is under liquidity shock and $\omega_j = 0$ otherwise. However, there can be a more general liquidity shock weight structure, such as $\Omega = [0, \infty]^n$.

Liabilities other than the liquidity shock are all equal in seniority. Hence, any net wealth left after paying the liquidity shocks will be distributed across all agents on a pro rata basis. Without loss of generality, there are no additional endowments of goods at $t = 2$.

2.3. Timeline

The timeline of the model, depicted in figure 1, is the following. Agents' cash and asset holdings are determined at $t = 0$, and agents form a collateralized debt network, using the assets they own or collateral they received as collateral. The resulting network and cash and asset holdings are exogenously given. At the beginning of $t = 1$, asset payoff s is publicly revealed. Also at the beginning of $t = 1$, liquidity shocks (senior debt) of ϵ are realized for each agent. Agents may liquidate their long-term projects if they are short of liquidity. Each agent's debt is paid back, and collateral is returned to the borrower, if not defaulted. If an agent does not pay the debt in full, then the agent is defaulting on that contract, and any

[3]The same collateral can be reused for an arbitrary number of times as in Chang (2021), which is in contrast to other models of reuse of collateral, as in Gottardi, Maurin, and Monnet (2019); Infante and Vardoulakis (2021); Infante (2019); and Park and Kahn (2019).

[4]If resource constraint is not present, then there can be a spurious cycle of collateral justifying any arbitrary amount of collateral circulating in the economy. For example, $c_{21} = c_{32} = \cdots = c_{1n}$ can be a very large number and satisfy the collateral constraints while no one is actually owning the asset as $\sum_i h_i = 0$.

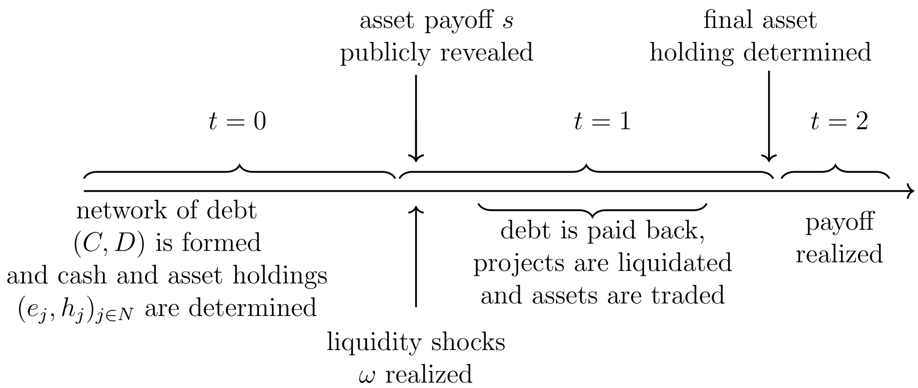


Figure 1: Timeline of the model

remaining assets in the agent's balance sheet will be distributed to all other creditors in a pro rata basis. At the end of $t = 1$, all agents' final asset holdings are determined. At $t = 2$, the payoff of the asset is realized and agents consume all the cash they have and gain utility from it.

3. Full Equilibrium

In this section, we define the equilibrium concept and its relevant elements.

3.1. Liquidation and Payment Rules

Let $x_{ij}(p)$ denote the actual payment net of collateral to agent i from agent j when the asset price is p at $t = 1$. This payment will be defined later in equation (6). The argument p is often omitted from now on. Denote

$$a_j(p) \equiv e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p) \tag{3}$$

as the total *cash inflow* of agent j before liquidating the project, where the first term, e_j, is j's cash holding, the second term, $h_j p$, is the market value of j's direct asset holdings, the third term, $\sum_{k \in N} c_{jk} d_{jk} p$, is the market value of collateral assets posted by j's borrowers, the fourth term, $\sum_{i \in N} c_{ij} d_{ij} p$, is the market value of collateral assets posted to j's lenders, and the fifth term, $\sum_{k \in N} x_{jk}(p)$, is the actual payment net of collateral from j's borrowers. The total amount of liabilities net of collateral posted for agent j is

$$b_j(p,\omega) \equiv \sum_{i\in N}(d_{ij} - c_{ij}d_{ij}p) + \omega_j\epsilon, \tag{4}$$

which can be considered as the required total *cash outflow*. Note that the first term of the right-hand side can be negative if the contract is over-collateralized. The function argument ω is often omitted for simplicity from now on.

If $a_j(p) > b_j(p)$—that is, cash inflow exceeds cash outflow—then $x_{ij} = d_{ij} - c_{ij}d_{ij}p$ for any $i \neq j$. If $a_j(p) \leq b_j(p)$—that is, cash outflow exceeds inflow—then agent j liquidates the long-term project to meet liabilities to others. Moreover, if the price of the asset is very low, the return from purchasing the under-priced asset, s/p, can be greater than the long-term return of the project, $1/\zeta$, and all agents will liquidate all of their projects regardless of their obligations. Mathematically, agent j's liquidation decision, $l_j(p) \in [0,\xi]$, is

$$l_j(p) = \begin{cases} \left[\min\left\{\frac{1}{\zeta}\left(b_j(p) - a_j(p)\right), \xi\right\}\right]^+ & \text{if} \quad p \geq s\zeta \\ \\ \xi & \text{if} \quad p < s\zeta, \end{cases} \tag{5}$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$, which guarantees that an agent does not liquidate the long-term project in a negative amount if one can meet its liabilities with total cash inflow. The liquidation decision follows the *liquidation rule* if equation (5) holds. The early liquidation of the long-term project is the primary source of inefficiency in the economy, as it is the only source of deadweight loss whereas all other payments (including defaults) are zero sum.

Given the liquidation rule, the actual payment to lender i from borrower j is determined as $x_{ij}(p)$. If agent j can pay all of the obligations (possibly by liquidating all or part of the project), then j can pay the original promised amount so $x_{ij}(p) = d_{ij} - c_{ij}d_{ij}p$ as the lender returns the collateral to the borrower, as depicted in the top-right figure of figure 2. Also, if the total value of collateral $c_{ij}d_{ij}p$ is greater than the face value of the debt d_{ij}, then actual payment $x_{ij}(p)$ can be negative because the more valuable collateral sitting in the lender's balance is returned to the borrower, as depicted in the bottom-right figure of figure 2. Agent j *defaults* on inter-agent debt if the payment net of collateral is less than the promised payment—that is, $x_{ij}(p) < d_{ij} - c_{ij}d_{ij}p$— for some $i \in N$. In the extreme case, if agent j cannot even pay the liquidity shock after full liquidation and the collateral value is less than the promised debt, then the actual payment will be $x_{ij}(p) = 0$ and the lender keeps the collateral, as depicted in the middle-right figure of figure 2. In an intermediate case,

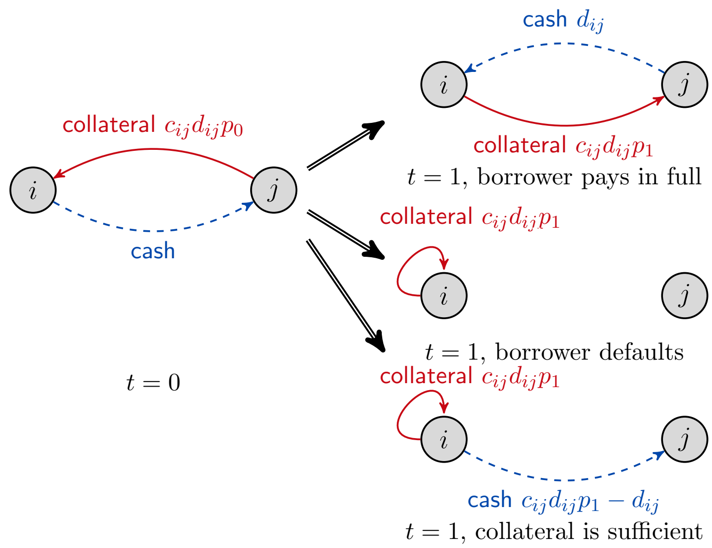


Figure 2: Flows of cash and collateral for three cases

Note: The two nodes, i and j, represent the lender and borrower of a contract, respectively. The blue dashed arrows represent flows of cash, and the red arrows represent flows of collateral. The left figure shows the flows in $t=0$. The top-right figure shows the flows in the case the borrower pays in full in $t=1$, the middle-right figure shows the flows in the case with borrower default in $t=1$, and the bottom-right figure shows the flows in the case in which the collateral value exceeds the payment in $t=1$.

agent j can pay the liquidity shock in full but cannot pay the inter-agent debt in full. Under such case, agent j's cash after the senior debt of liquidity shock is paid out on a pro rata basis. This interaction can be mathematically formulated as the following *payment rule*:

$$x_{ij}(p) = \min\left\{ d_{ij} - c_{ij}d_{ij}p \ , \quad q_{ij}(p)\left[a_j(p) + \zeta l_j + \sum_{i\in N}[c_{ij}d_{ij}p - d_{ij}]^+ - \omega_j\epsilon \right]^+ \right\}, \tag{6}$$

where $q_{ij}(p)$ is a weight under the *weighting rule*

$$q_{ij}(p) = \frac{[d_{ij} - c_{ij}d_{ij}p]^+}{\sum_{k\in N}[d_{kj} - c_{kj}d_{kj}p]^+} \tag{7}$$

for pro rata basis of the actual payment. Note that if the weighting rule is not defined—that is, $\sum_{k\in N}[d_{kj} - c_{kj}d_{kj}p]^+ = 0$—then, the weighting rule is never used in the payment rule, because any lender k will be paid in full, d_{kj}, because the market value of collateral exceeds the promised payment—that is, $d_{kj} < c_{kj}d_{kj}p$.

3.2. Fire Sales and Market Clearing

For a given collateralized debt network and state realization $(N, C, D, e, h, s, \omega)$, where $e \equiv (e_1, e_2, \ldots, e_n)$ and $h \equiv (h_1, h_2, \ldots, h_n)$, the *net wealth* of agent j is

$$\begin{aligned} m_j(p) \equiv & \, a_j(p) + \zeta l_j(p) - b_j(p) \\ = & e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p) + \zeta l_j(p) - \omega_j \epsilon + \sum_{i \in N} (c_{ij} d_{ij} p - d_{ij}) \\ = & e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in N} x_{jk}(p) \end{aligned} \tag{8}$$

under the liquidation and payment rules. Equation (8) is consisted of the following: cash holdings from $t = 0$, the market value of the asset holdings from $t = 0$, the market value of collateral received, cash from liquidating the long-term project, negative liquidity shock, the total payment to be paid, and the actual net payment received. If $m_j(p) < 0$, then agent j defaults.

If an agent has to sell all or part of the asset holdings, then the agent has to fire-sell the asset. Denote the *fire-sale* amount of agent j as

$$\phi_j(p) = \min \left\{ [h_j p - m_j(p)]^+ , h_j p \right\}. \tag{9}$$

If agent j's net wealth subtracted by j's asset holdings, $m_j(p) - h_j p$, is enough to cover all of the payments (positive), then $\phi_j(p) = 0$—that is, no fire sales. If agent j's net cash flow is not enough without the sales of asset holdings, then $\phi_j(p) > 0$. If the cash shortage exceeds the total asset holdings (i.e. $h_j p - m_j(p) > h_j p$), then the fire-sale amount reaches its upper bound $\phi_j(p) = h_j p$. Note that a defaulting agent would always have $\phi_j(p) = h_j p$.

The market for the asset is a perfectly competitive Warlasian market. Unless there is not enough cash to purchase all of the asset sales in the market in the asset's fundamental value s, the market price will always be its fair value s. However, if there is not enough cash in the market, then the asset price can go below its fundamental value as $p < s$, which is a *liquidity constrained price*. Under such a case, the market clearing condition becomes a cash-in-the-market pricing condition. The *market clearing condition* can be summarized as

$$\begin{aligned} \sum_{j \notin \mathcal{D}(p)} [m_j(p) - h_j p]^+ &= \sum_{i \in N} \phi_i(p) \qquad \text{if } \; 0 \le p < s \\ \sum_{j \notin \mathcal{D}(p)} [m_j(s) - h_j s]^+ &\ge \sum_{i \in N} \phi_i(s) \qquad \text{iff } \quad p = s, \end{aligned} \tag{10}$$

where $\mathcal{D}(p)$ is the set of agents who default under price p.

For the given rules, the definition of the equilibrium is as follows.

Definition 1. *For given $(N, C, D, e, h, s, \omega)$, if liquidation decisions $\{l_j(p)\}$ satisfy the liquidation rule (5), payments $\{x_{ij}(p)\}$ satisfy the payment rule (6), $\{m_j(p)\}$ is determined by net wealth equation (8), fire sale amount $\{\phi_j(p)\}$ determined by equation (9), and price p clears the market as in (10), then $(\{x_{ij}\}, \{l_j\}, \{m_j\}, \{\phi_j\}, p)$ is a full equilibrium.*

The notion of this full equilibrium is a generalization of the payment equilibrium in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), which is based on Eisenberg and Noe (2001). In contrast to these papers, agents in our model not only have financial liabilities and liquidation of projects, but also have posted collateral, and the price of the collateral asset is determined endogenously. Therefore, both the debt and collateral markets have spillovers to each other. Furthermore, unlike Chang (2021), the debt contract is full recourse. So the defaulting borrowers should still pay any discrepancy between the face value of the debt and the value of the collateral, making the borrower default contagion more general and realistic.

In line with the literature we refer to the interim equilibrium of our full equilibrium without fire sales and market clearing conditions—that is, the payment and liquidity decisions, $\{x_{ij}(p)\}$ and $\{l_j(p)\}$, which satisfy payment and liquidation rules, for a given asset price p—as the payment equilibrium of $(N, C, D, e, h, s, \omega)$ and p. Appendix A contains an alternative definition with matrix notation, which is useful for analysis.

Because the liquidation amount should cover the discrepancy $a_j(p) - b_j(p)$, if there is any, the payments should be between the collateral value or the full debt amount. Given that and the collateral constraints, we can show that the sum of positive net wealth is increasing in the asset price p.

Lemma 1. *The aggregate positive net wealth $\sum_{j \in N} [m_j(p)]^+$ is increasing in the asset price p. Moreover, if $\sum_{j \in N} [m_j(p)]^+ > 0$, then it is strictly increasing in the asset price p.*

All proofs are relegated to the appendix. The main intuition of the proof is that assets should be either owned by the non-defaulting agents as confiscated collateral or the increase in p increases the cash inflow of defaulting agents, resulting in more payments to the non-defaulting agents.

The default set, $\mathcal{D}(p)$, becomes larger as p decreases by Lemma 1. The left-hand sides of the market clearing conditions are the total amounts of cash in the market net of marked-to-market values of asset holdings. Also by Lemma 1, the aggregate net wealth increases as p increases. The market clearing price can be simplified as the following lemma.

Lemma 2. *The market clearing asset price can be represented as*

$$p = \min\left\{\frac{\sum_{j\in N}[m_j(p)]^+}{\sum_{j\in N} h_j}, s\right\}. \tag{11}$$

Even though we have the complex and realistic accounting for available cash and required amount of assets on sales, the resulting computation of the market clearing price is surprisingly simple. This is because a wealth that is not buying the asset, $m_j(p) - h_j p$, will only contribute to the fire sales, $h_j p - m_j(p)$, and the market value of fire sales can be simply represented as negative demand $m_j(p) - h_j p < 0$.

The following proposition shows that the full equilibrium always exists.

Proposition 1. *For any given collateralized debt network, cash and asset holdings, asset payoff, and realization of shocks $(N, C, D, e, h, s, \omega)$, a full equilibrium always exists and is generically unique for a given equilibrium price. Furthermore, there exists a full equilibrium with the highest price among the set of full equilibria.*

The multiplicity of equilibria is similar to that of Elliott, Golub, and Jackson (2014). Because the market clearing price jumps down after an additional default due to a jump in fire sales, the additional default could be caused by the decline in price followed by the default itself. Nevertheless, each equilibrium price has a (generically) unique payment equilibrium, and there exists a maximum full equilibrium that has the highest market clearing price among the set of full equilibria. From now on, we will focus on the results of the maximum full equilibrium as in Elliott, Golub, and Jackson (2014).

3.3. Discussion

The model incorporates the role of explicit collateral, as agents can settle the payments by giving up their collateral to their lenders. This is in line with the standard repo contracts such as Securities Industry and Financial Markets Association's (SIFMA) Master Repurchase Agreement (MRA) used by most U.S. dealers and SIFMA/International Capitam MArket Association (ICMA) Global Master Repurchase Agreement (GMRA) used for non-U.S. repos (Baklanova, Copeland, and McCaughrin, 2015). According to both SIFMA MRA and SIFMA/ICMA GMRA, after determining the market value of the collateral, all repo exposures between the two counterparties are netted off and whoever owns the residual sum must pay it by the next business day, including the interest on late payment.[5] Hence, the lender

[5] See https://www.sifma.org/wp-content/uploads/2017/08/MRA_Agreement.pdf for MRA and https://www.sifma.org/wp-content/uploads/2017/08/Global-Master-Repurchase-Agreement.pdf for GMRA.

has recourse to the borrower's balance sheet and claim any payment due net of the market value of the collateral (Gottardi, Maurin, and Monnet, 2019). The nondefaulting party may either immediately sell in a recognized market at prices as the nondefaulting party may reasonably deem satisfactory or give the defaulting party credit for collateral in an amount equal to the price obtained from a generally recognized source.[6] Hence, the default and settlement procedure in our model closely follows how they actually work in the financial markets in the real world.

The property of collateral covering the debt payment is crucial for our results. For example, if netting the debt with collateral was not possible, then after a negative liquidity shock to the system, all assets posted as collateral would have been put on fire sale, as all agents should liquidate their collateral to raise cash for payment obligations. Then, the model will be equivalent to having no collateral at all as in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) because the collateral is just a side-show of the debt contracts. Indeed, this is not the case in the real world, as financial institutions typically designate particular collateral and can effectively pay their obligations by giving up their collateral. In other words, collateral plays the role of money across the collateralized debt network when agents are paying their liabilities.

This role of collateral as money highlights the importance of reuse (rehypothecation) of collateral. As long as the market value of collateral is large enough, agents can pay each other by using the collateral. Hence, reuse of collateral can facilitate and secure more payments, enhancing financial stability of the network. However, such role of collateral exists only if the collateral price is high enough. In Section 4, we show that this role of collateral as money can break down when the liquidity shock is large. Furthermore, reuse of collateral can also undermine financial stability because reusing collateral multiple times can increase the leverage of the system and the asset price in $t = 0$, as highlighted by Chang (2021). In this paper, we do not analyze this directly, as we do not have endogenous networks.

4. Contagion and System Risk

In this section, we study how the structure of the collateralized debt network determines the extent of contagion and systemic risk of the market, which is the risk related to the total loss in values summed over all entities in the system (Glasserman and Young, 2016). For any given collateralized debt network and the corresponding full equilibrium, we define the

[6]The Master Securities Loan Agreement for securities lending transactions in the U.S. also states similar procedures (Baklanova, Copeland, and McCaughrin, 2015).

utilitarian social surplus in the economy as the sum of the returns to all agents at $t = 2$ as

$$U = \sum_{i \in N} (\pi_i + T_i),$$

where $T_i \leq \epsilon$ is the transfer from agent i to its senior creditors (liquidity shock), which simply transfers to $t = 2$, and π_i is the agent's long-term profit evaluated at $t = 2$. This definition of social surplus is consistent with that of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015).

We focus on regular debt networks in which the total inter-agent claims and liabilities of all agents are equal. In other words, a debt network is *regular* if $\sum_{i \in N} d_{ij} = \sum_{i \in N} d_{ji} = d$ for all $j \in N$ and for some $d \in \mathbb{R}^+$. Also, assume that all agents hold the same amounts of cash and asset as $e_i = e_0$ and $h_i = h_0$ for all $i \in N$. This normalization guarantees that any variation in the systemic risk is due to the interconnectedness of agents while abstracting away from potential effects from size, balance sheet heterogeneity, or hierarchical heterogeneity. Similarly, we assume that all agents have the same uniform collateral ratio—that is, $c_{ij} = c$ for all $i, j \in N$. For simplicity and following the benchmark case in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), assume $\zeta = 0$, which is the limit case of $\zeta \to 0$ because otherwise, the liquidation rule is not well defined.

Finally, in what follows, we assume that the liquidity shocks are randomly hit to κ number of agents and the size of the shock is potentially $\epsilon \in (e_0 + h_0 s + \zeta\xi, \infty)$. The lower bound on ϵ guarantees that without any payments by other agents, an agent under liquidity shock would be unable to pay its senior debt from the liquidity shock even with the fire sales of collateral at the best price.

The social surplus can be simplified as the following lemma.

Lemma 3. *For any full equilibrium, the social surplus in the economy is equal to*

$$U = n(e_0 + h_0 s + \xi) - (1 - \zeta) \sum_{i \in N} l_i.$$

Lemma 3 clarifies that the source of social inefficiency is coming from the early liquidation of the long-term project. This liquidation is due to either insufficient liquidity or low asset price that makes asset purchase more profitable than the long-term project, as in Brainard-Tobin's Q-theory. Denote E as the expectation operator on ω.

Definition 2. *For a fixed (N, e, h, s, Ω), consider two debt networks (C, D) and $(\tilde{C}, \tilde{D})$.*

1. *(C, D) is more stable than $(\tilde{C}, \tilde{D})$ if $EU \geq E\tilde{U}$.*

2. *(C, D) is more resilient than $(\tilde{C}, \tilde{D})$ if $\min_{\omega \in \Omega} U \geq \min_{\omega \in \Omega} \tilde{U}$.*

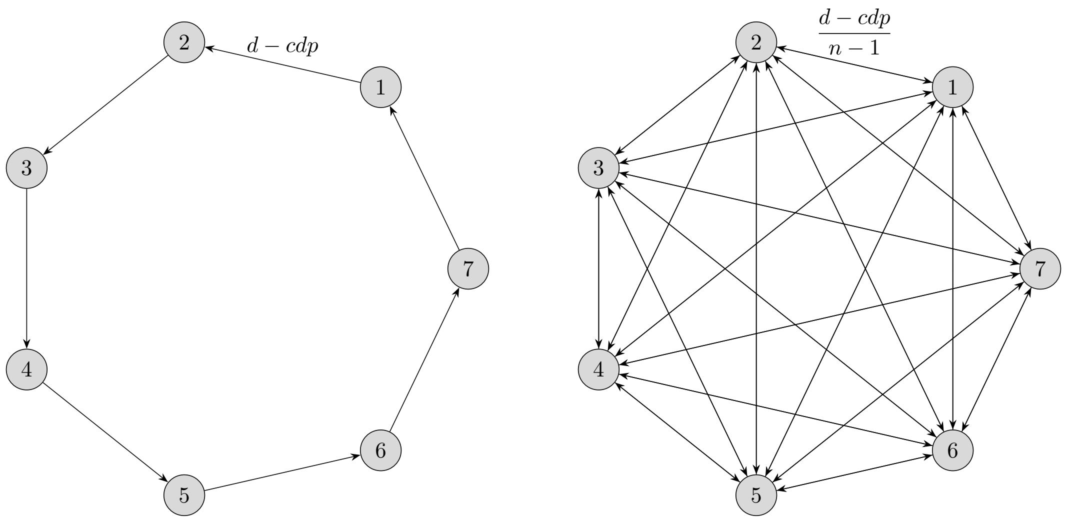


Figure 3: The ring network and the complete network

The two notions compare the expected and worst-case social surplus of the given collateralized debt network, respectively. For simplicity of exposition, assume that $\omega_j \in \{0, 1\}$ and $\kappa \equiv \sum_{j \in N} \omega_j = 1$ for any $\omega \in \Omega$ for now unless noted otherwise. This simple set-up allows us to examine the financial contagion over the two markets for each network structure in the most intuitive way. Relaxing these assumptions will be discussed in Section 5.

Before we describe the results, we define a few important concepts related to the results. First, we define the complete and ring networks. The complete network is a network in which every agent owes the same amount to each other, $d/(n-1)$. Thus, the number of links is the highest under the complete network. Second, the ring network is a network in which every agent borrows all the amount from one other agent. For example, agent 1 has to pay d to agent 2, who has to pay d to agent 3, who has to pay d to agent 4, and so forth. Agent n has to pay d to agent 1 to make the ring network a regular network. The ring network has the least number of links for a connected regular network. Figure 3 illustrates the ring and complete networks. Note that the total payment after netting the total collateral posted is $d - cdp$.

Definition 3. *A (collateralized) debt network* (C, D) *is a* δ*-connected network if there exists* $\mathcal{S} \subset N$ *such that* $\max\{d_{ij}, d_{ji}\} \leq \delta d$ *for any* $i \in \mathcal{S}, j \notin \mathcal{S}$.

A δ-connected network implies that a network can be separated into two subsets of vertices with the cross-subset links being relatively small as δ or less.

Definition 4. *A debt network* $(C, \tilde{D})$ *is a* γ*-convex combination of two networks* $(C, D), (C, D')$ *if and only if* $\tilde{d}_{ij} = \gamma d_{ij} + (1-\gamma) d'_{ij}$ *for any* $i, j \in N$.

The concept of γ-convex combination of two networks is basically the same as a typical convex combination of matrices. Note that the previous two network definitions are well defined under the uniform collateral ratio assumption.

4.1. Unsecured Debt Case

Suppose the uniform collateral ratio is $c = 0$ and $h_0 = 0$, so there is no collateral or asset in the market. This case encompasses the main model setting of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). When there are no collateral or asset holdings, the only remaining channel of contagion is the debt channel. The following result summarizes the main results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) related to the phase transition property of financial contagion depending on the size of the shock.

Proposition 2. *(Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015) For $\epsilon^* = ne_0$, there exists $d^* = (n-1)e_0$ such that for any $\epsilon < \epsilon^*$ and $d > d^*$ the following holds:*

1. *The complete network is the most resilient and stable.*
2. *The ring network is the least resilient and stable.*
3. *The γ-convex combination of the ring and complete networks becomes more stable and resilient as γ increases.*

Furthermore, for any $\epsilon > \epsilon^*$ and δ sufficiently small,

1. *Both the complete and ring networks are the least resilient and stable.*
2. *A δ-connected network is more resilient and stable than the complete network.*

The results and proofs are almost identical to those in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015).

4.2. Insulation by Collateral

Suppose the uniform collateral ratio is now $c > 0$. Note that we are still fixing each agent's endowment as e_0 of cash and h_0 of assets. Hence, we focus on the role of collateral for the fixed amount of debt and endowments. If the contracts are fully covered by collateral—that is, any face value of the debt is exceeded by the value of collateral—then all of the payments will be made in full and no agent will default on their inter-agent debt. This full insulation shuts down any possible network propagation, and any network will become the most stable and resilient network because of this insulation property of collateral.

Proposition 3. *(Collateral Insulation)*
Suppose that $\kappa < n$ number of agents are hit with liquidity shock. If $\kappa d \leq n(n-\kappa)e_0$ and

$$c^*(s,n) \equiv \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\} \leq \frac{s}{\zeta},$$

then, for any $c \geq c^*$, any network is the most resilient and stable network for any ϵ.

The first condition $\kappa d \leq n(n-\kappa)e_0$ is necessary to satisfy the economy's resource constraint. Otherwise, the network requires an exceedingly large amount of collateral circulating the economy. One way to interpret this condition is that the leverage is at a realistic level. For example, if $\kappa = 1$, the condition implies that an agent's total liabilities cannot exceed $(n-1)$ times of the total amount of cash in the economy, ne_0. The second constraint, $c^*(s,n) \leq s/\zeta$, is needed to prevent price-induced total fire sales stemming from the disproportional return of the asset compared with that of the long-term project. This result is also realistic if the long-term project is at least as profitable as the collateralizable asset. For example, yields of U.S. Treasury securities, which are commonly used as collateral, typically do not exceed average yield of firms' equity.

Under this case, enough collateral are in the market relative to other cash sources. Therefore, all debt can be covered by the collateral, and the market as a whole is insulated from any contagion regardless of the network structure. From now on, assume that $\kappa d \leq n(n-\kappa)e_0$ holds.

4.3. Contagion under Collateralized Debt

If the collateral ratio is not enough to provide full insulation, then the propagation still occurs as in the unsecured debt case. However, the implied network propagation changes as a result of the collateral contagion shifting the payments.

4.3.1. Limited Collateral Contagion

In this section, we show and analyze the cases in which collateral contagion is limited. First, we show that the upper bound of the number of defaulting agents is decreasing in the collateral ratio c for a given equilibrium price p. Then, we show that when the liquidity shock is small or the collateral ratio is relatively high, the asset price in $t = 1$ is always $p = s$, the fundamental value of the asset, so there is no fire-sale-induced liquidity constrained price. In other words, there is no contagion through the collateral price channel when the liquidity shock is small or the collateral ratio is high.

Lemma 4. *Suppose that κ number of agents are under liquidity shock. Let $\mathcal{D}$ and p be the set of agents defaulting on their inter-agent debt and the price in full equilibrium, respectively, and $cp < 1$. Then, the number of defaults is bounded above and below as the following:*

$$\kappa \leq |\mathcal{D}| < \frac{\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\}}{e_0 + h_0 p}.$$

Therefore, the upper bound of the number of defaults is decreasing in the collateral ratio c and the equilibrium asset price p.

The more interesting part of the lemma is the upper bound of defaults. The numerator represents the total liquidity outflow from the system. If the liquidity shock is small, then an agent under shock can pay this liquidity shock by the total inflow of cash, provided that other agents are paying their debt in full. Then, the total outflow from the system is simply the size of the shock, ϵ. However, if the shock is large, the total cash inflow, $e_0+h_0p+d-cdp$, will be drained out of the system. The denominator represents the individual endowment of each agent. Therefore, if the total outflow from the system can be covered by individual endowments of defaulting agents, then there will be no further defaults. Hence, we obtain the upper bound of the total number of defaulting agents.

Lemma 4 highlights the relationship between the individual endowments ($e_0 + h_0 p$), total debt amount (d), and the value of collateral (cdp). The upper bound is decreasing in the endowments, as having more endowments implies agents have more cash to pay for the liquidity shortfall. The upper bound is increasing in the total debt amount, as the liquidity shocks can trigger more inter-agent defaults. Finally, the upper bound is decreasing in the total value of collateral, as collateral guarantees that at least part of the debt is paid through collateral. Hence, collateral is effectively transferring some amount of liquidity to the lenders in case of default.

The next proposition shows that the asset price, p, in $t = 1$ is always its fundamental value, s, whenever the collateral ratio is high enough or the liquidity shock is small.

Proposition 4. *(No Contagion through Collateral Price Channel) Suppose that the collateral ratio c is $c < c^*(s,n)$ and $\kappa \in \{1,\ldots,n\}$. Let $\epsilon^* = ne_0/\kappa$. Then, if $\epsilon < \epsilon^*$, the equilibrium asset price in $t = 1$ is always $p = s$ regardless of the network structure. Furthermore, if $c \geq c_* \equiv \dfrac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$, then the equilibrium asset price in $t = 1$ is always $p = s$ regardless of the network structure for any ϵ. Finally, the threshold collateral ratio preventing the asset price going below the fundamental value is (weakly) lower than the threshold collateral ratio for full insulation—that is, $c_* \leq c^*$.*

Interestingly, the threshold of the liquidity shock size is exactly the same threshold for the phase transition of contagion under the unsecured debt case. This is because every agent is indirectly connected to all other agents through the collateral market. Even if an agent is completely isolated from all other agents, the agent will participate in the asset market. Therefore, the asset market will remain intact—that is, agents trade the assets in its fair value— as long as the total liquidity in the market is sufficient. This coincides with the case in which all agents are connected to each other—that is, the complete network. Hence, the collateral market is not the source of contagion channel as similar to the complete network preventing any further contagion through full diversification.

Also, high collateral ratio, which is still below the full insulation level c^*, can guarantee the asset price to remain $p = s$, regardless of the size of the liquidity shock. The main force is that the collateral can limit the drainage of liquidity as discussed in the description of Lemma 4. Any collateral ratio $c \geq c_*$ limits the total liquidity outflow from the system, so the remaining agents can buy the assets on fire sale at the highest price possible, which is s.

Combining this Proposition 4 and Proposition 2 leads to the following result.

Proposition 5. *Suppose that the collateral ratio c is $c < c^*(s, n)$, only one agent is hit with negative liquidity shock, $\kappa = 1$. If either the size of the shock is $\epsilon < \epsilon^*$ or the collateral ratio is $c \geq c_*$, then, the complete network is the most stable and resilient network while the ring network is the least stable and resilient network. The number of defaults in the ring network decreases as c increases. Furthermore, the γ-convex combination of the complete and ring networks becomes more stable and resilient as γ increases.*

Because there is no contagion through the collateral channel, there is only the debt channel of contagion. By using the technique of collateral netting, which we introduce in the Appendix A, we can convert the collateralized debt network into an unsecured debt network after netting out all the collateral movements. Then, the same argument used by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) applies to our setup. Furthermore, because the upper bound of defaults is decreasing in c by decreasing the total cash outflow from the network, the maximum number of defaults, which is the number of defaults in the ring network, is decreasing in c. In conclusion, if the liquidity shock is small or the collateral ratio is high, there is no collateral channel of contagion, and the debt channel of contagion is minimized by having the most links possible.

4.3.2. Active Collateral Contagion

If the liquidity shock is large and the collateral ratio is not high enough, the liquidity shock exceeds the total amount of available cash in the network, changing the nature of debt contagion as in the unsecured debt case. Furthermore, there can be contagion through the collateral price channel, which adds additional dimension to the overall contagion in the collateralized debt network. The following proposition highlights what happens when collateral contagion is active.

Proposition 6. *(Phase Transition under Collateralized Debt) Suppose that the collateral ratio c is $c < c_*(s,n)$, $\kappa = 1$, and $\epsilon > \epsilon^*$. Then, there exist d^* such that for any $d > d^*$ the following holds:*

1. *The complete and ring networks are the least stable and least resilient networks and the equilibrium asset price is $p = 0$.*

2. *For a small enough δ, a δ-connected network is more resilient and stable than the complete network and the equilibrium asset price is $p > 0$.*

One interesting implication of Proposition 6 and 5 is that the equilibrium asset price is either 0 or s under the ring and complete networks. As all agents are interconnected to each other within one component, a liquidity shock to one agent will reach to all agents in the network. If the price of collateral was high, then it would have mitigated the total outflow of cash (due to liquidity shock) from the network. However, the remaining net wealth of the network is small after the outflow of cash even when $p = s$, therefore the price will go down following the price equation (11). A decline in the price of collateral will increase the need for additional liquidity of each and every link, as the discrepancy between the debt and collateral, $d - cdp$, increases as p decreases. This effect further increases the fire sales of each agent, and the price decreases even further. This dual loop of contagion between the collateral market and debt payment leads the asset price to $p = 0$ and defaults of all agents in the network.

Similar to the unsecured debt networks, a δ-connected network is more resilient and stable than the complete network when the collateral ratio is not high enough. The key is to have two or more separate components, which are not heavily connected to each other in the network. Hence, it is the interconnectedness between the two components in the network that determines whether the contagion reaches to the other side, not the collateral ratio, provided that it is not high enough—that is, $c < c_*$. In other words, monitoring the average level of leverage is not enough, as having more interconnections is also crucial in determining whether the asset market will crash or not.

Lastly, recall that we are changing the amount of collateral while fixing each agent's endowment and total debt amount. Hence, our analysis highlights how specifying explicit collateral could change the contagion drastically ranging from full insulation to total collapse of the market and maximum number of defaults. This is because collateral decreases the

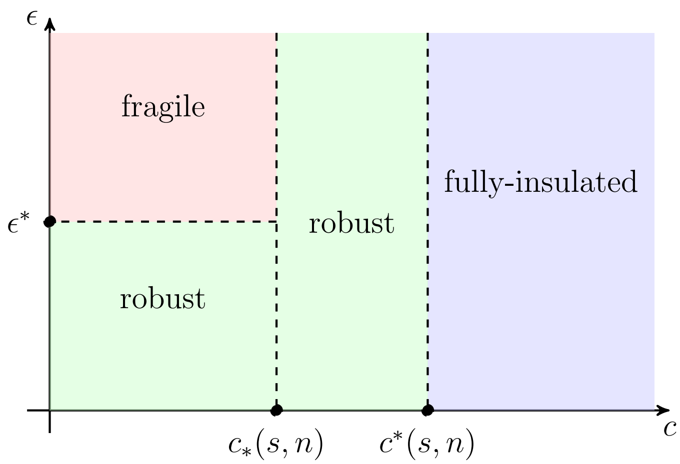


Figure 4: Different regimes depending on the collateral ratio and liquidity shock

influence of idiosyncratic shocks to other agents in the network by guaranteeing the market value of collateral to be paid to the remaining agents in the network. Such payment by collateral feeds back into the network and can support the high price of collateral in the first place. Therefore, our results highlight the importance of considering explicit collateral instead of treating an agent's total asset holdings or going-concern value as collateral.

Figure 4 summarizes our results. The horizontal axis represents the collateral ratio, c, and the vertical axis represents the size of the liquidity shock, ϵ. If c is above the threshold $c^*(s,n)$, then the equilibrium is under the fully-insulated regime, which is shaded in blue. If c is above the threshold $c_*(s,n)$ but below $c^*(s,n)$, then the equilibrium is under the robust regime, which is shaded in green. Under the robust regime, having more links makes the network more stable and resilient. If c is below $c_*(s,n)$, then the regimes depend on the size of the liquidity shock. If $\epsilon < \epsilon^*$, then the equilibrium is under the robust regime. However, if $\epsilon > \epsilon^*$, the equilibrium is under the fragile regime, which is shaded in red. Under the fragile regime, having more links can make the network less stable and resilient.

5. Extensions

In this section, we discuss extensions of the baseline model and the ways in which the results of the baseline model would change or remain.

5.1. Aggregate Shock and Collateral Phase Transition

We have assumed a fixed fundamental value of the asset s so far. The payoff of the asset in the future can also fluctuate at $t = 1$. Changes in value s can be considered as an aggregate shock to the economy because it changes the return (productivity) of the entire economy.

Proposition 7. *(Aggregate Shock and Vulnerability) If $s > s'$, then the phase transition collateral ratios of the two cases are $c_*(s, n) < c_*(s', n)$ and the full insulation collateral ratios are $c^*(s, n) \leq c^*(s', n)$. Similarly, if $n > n'$, then $c_*(s, n) < c_*(s, n')$ and $c^*(s, n) \leq c^*(s, n')$. Thus, a given network with collateral ratio c under s' or n' is more vulnerable to liquidity shocks than that under s or n is.*

The result implies that as the aggregate shock increases—that is, s decreases—then the overall safe region decreases as depicted in Figure 5. For the same collateral ratio, the complete network might be fully insulated, but after the aggregate shock, the network might become the least stable and resilient network if it is over the liquidity shock threshold. Thus, an aggregate shock on s can entail a different systemic risk implication for the same collateral amount and the same network structure. Therefore, Proposition 7 provides the required level of c for the given network structure and the desired level of financial stability—whether to shut down collateral price contagion by setting $c > c_*(s, n)$ or to shut down any contagion by setting $c > c^*(s, n)$.

5.2. Changes in the Total Supply of Assets

The results in Propositions 3 and 4 imply that an increase in h_0, the total supply of assets, would increase both c^* and c_*. This is because

$$c_* \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$$
$$c^* \equiv \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\},$$

which are increasing in h_0. First, this relationship implies that the required collateral ratio to fully insulate contagion is higher under a greater supply of assets holding all else equal. In other words, the total supply of assets would increase the amount of fire sales and put more downward pressure on prices. Thus, more collateral is needed to guarantee a contract to be fulfilled with collateral. Second, this relationship also implies that the ring network is more likely to be the least stable and resilient network as the required collateral to guarantee the

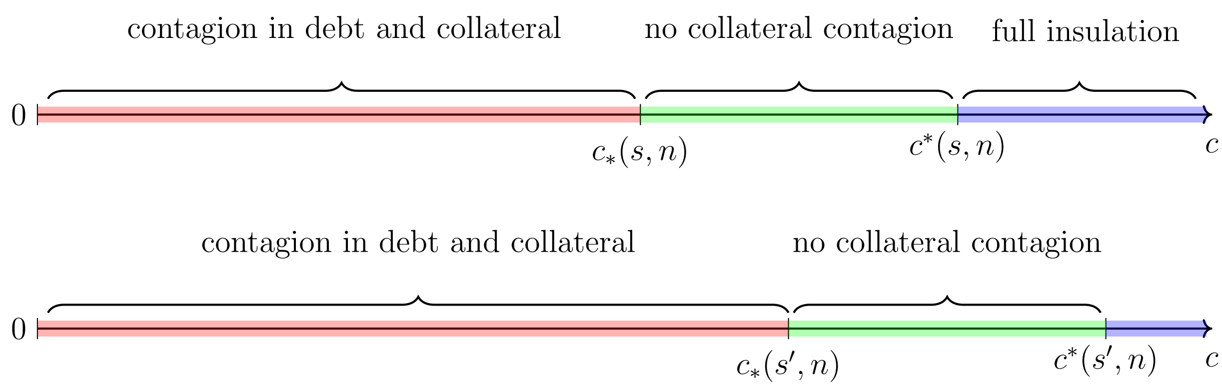


Figure 5: Aggregate shock and vulnerability

Note: Each line represents the collateral ratio. Red areas represent the region with both contagion in debt and collateral, green areas represent the region without collateral contagion, and blue areas represent the region with full insulation by collateral. The top line illustrates the three different regions under the baseline parameters (s, n). The bottom line illustrates the three different regions under the negative aggregate shock with $(s'n)$ such that $s' < s$. The figure shows that the required collateral ratios to attain the desired level of stability (no collateral contagion or full insulation) increase when there is a negative aggregate shock.

payments within the ring network increases. Therefore, the leverage that makes a financial network stable would heavily depend on the total supply of assets due to the collateral price channel of contagion.

This result is quite surprising and counter-intuitive as technically agents have more endowments with higher h_0. However, actual payments involve cash as agents under liquidity shocks should sell their assets to pay their liquidity shock (senior debt). Since the total amount of cash remains the same, more assets under fire sale would only depress the price of assets, decreasing the value of collateral, which increases the outflow of cash from the network. Hence, the network suffers from more defaults, depressing the price further. This result again highlights the importance of the role of explicit collateral on mitigating contagion in debt and collateral markets.

5.3. Generalized Fire Sales

The model can be extended to incorporate generalized fire sales as $\phi_i(p) \in [0, h_0 p]$ for all $i \in N$. One peculiar issue related to this generalization is the reversed effect of fire sales under the current equilibrium concept with instant simultaneous market clearing. For example, consider the complete network and suppose that ϵ is large enough to wipe out the agent under the liquidity shock, say agent 1. Then, the market clearing condition becomes

$$(n-1)e_0 - d + cdp = h_0 p + \sum_{i \in N \setminus \{1\}} \phi_i(p),$$

when $0 < p < s$. The price is positive only if

$$cdp > h_0 p + \sum_{i \in N \setminus \{1\}} \phi_i(p)$$

because $d > (n-1)e_0$. However, then the instant market clearing condition implies

$$p = \frac{d - (n-1)e_0}{cd - h_0 - \sum_{i \in N \setminus \{1\}} (\phi_i(p)/p)},$$

and an increase in $\phi_i(p)$ will rather increase the price p. This reversed effect of fire sales is there only when the collateral is large enough to dominate the fire-sale amount. This effect exists because the market clearing is instantaneous, so the increase in the right-hand side of the market clearing condition can be countered by an increase in the left-hand side of the market clearing condition with an increase in p.

This reversed effect is not unique to the model in this paper. In fact, this simultaneous reaction is the reason why empirical models of fire sales in the literature, such as Greenwood, Landier, and Thesmar (2015) and Duarte and Eisenbach (2021), use an iterative model of fire sales. Agents in such models hold other things fixed and react by the decision of fire sales, which will create spillovers in the next step and so on. The model in this paper can also be extended to this iterative fire-sale procedure by determining the fire-sale amount holding the previous price fixed:

$$\begin{aligned}
(n-1)e_0 - d + cdp^0 &= \sum_{i \in N} \frac{\phi_i^0(p^0)}{p^0} p^1 \\
(n-1)e_0 - d + cdp^k &= \sum_{i \in N} \frac{\phi_i^k(p^k)}{p^k} p^{k+1} \qquad \text{for any } k \leq K,
\end{aligned}$$

where the fire-sale amount is determined by the new price as

$$\phi_i^{k+1}(p^{k+1}) = \min\left\{ \left[h_j p^{k+1} - m_j(p^{k+1})\right]^+, h_j p^{k+1} \right\}.$$

This iterative procedure guarantees the negative effect of fire sales on price p. The maximum number of iteration K can be different across different contexts. For example, Greenwood, Landier, and Thesmar (2015) assume $K = 1$.

5.4. Fire Sales with External Traders

In the baseline model, the only source of demand is the agents within the debt network. Now suppose that there exist external traders, who are not directly involved with the debt network or its relevant payments but are buying and selling the assets. Indeed, the existence of the external traders can mitigate or amplify the severity of the fire-sale externalities in the market depending on parameters. Following the literature on fire sales, we assume that the external traders can amplify the problem of fire sales, as it can involve information asymmetry, liquidity hoarding, and margin spirals, which all accelerate the sales of the asset. Suppose that there are external traders with linear demand—that is, $\alpha - \beta p$. Without loss of generality, we focus on the case with $\alpha - \beta s < \sum h_i$, because otherwise, the external demand can support the fair price of the asset even when every agent in the network defaults. Finally, assume that the fire-sale procedure goes through the iterative procedure discussed in the previous subsection, which follows the methods in Greenwood, Landier, and Thesmar (2015) and Duarte and Eisenbach (2021). The new market clearing condition becomes

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(p)} h_j p = \sum_{i \in N} \phi_i(p) - \alpha + \beta p \qquad \text{if} \quad 0 \leq p < s \tag{12}$$

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(s)} h_j s \geq= \sum_{i \in N} \phi_i(s) - \alpha + \beta s \qquad \text{iff} \quad p = s. \tag{13}$$

All the arguments in Proposition 1 go through under this setup. This extended model also includes the dynamics of fire sales in Greenwood, Landier, and Thesmar (2015), which assume linear changes in net returns for fire sales—that is,

$$F_2 \equiv \frac{p_2 - p_1}{p_1} = L\phi,$$

where p_2 and p_1 are asset prices before and after the sales, ϕ is the fire-sale amount, and L is the fire-sale parameter. The last equation can be rearranged to

$$p_2 = L\phi p_1 + p_1.$$

Now consider the model in this extension as $p_1 = M/(\text{sales})$ and $p_2 = M/(\text{sales} + \beta)$, so that

$$\frac{p_2 - p_1}{p_1} = -\beta \text{sales}$$

Thus, the two models are equivalent in the structure of the fire-sale effect. However, our model incorporates the endogenous fire-sale amount from the debt contagion rather than from a given fixed leverage target. Therefore, an interesting direction to extend the model in this paper would be combining the fire-sale spillovers to multiple markets of multiple assets with internal and external agents for the given collateralized debt network.

5.5. Other Generalizations

There are other directions of generalizing the model. First, there are relatively straightforward directions. As in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), allowing partial liquidation of long-term projects as $\zeta > 0$ is relatively straightforward. All the main results of the baseline model hold with minor differences in conditions. Similarly, allowing for multiple shocks as $\kappa > 1$ is relatively straightforward, as in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and, again, most results hold in this extension.

In contrast, there are more difficult directions of generalizing the baseline model. First, allowing for heterogeneous collateral ratios is challenging because the difference in payments depends on each individual collateral ratio for different price levels. Thus, there would be many different price regions of contagion depending on the shock size and network structure. Fortunately, this complexity is uni-directional as more contagion decreases prices, which lead to even more debt payments to depend on the collateral price and debt payments. Therefore, a model with heterogeneous collateral ratios would be easily solved numerically. Related to the numerical model, generalizing the framework with other forms of counterparty exposures is also possible by setting different c_{ij} depending on the contract structure. Second, allowing for more general shock distribution would be quite challenging because of the exceedingly many dimensions to consider analytically. Again, such a model can be solved numerically for a given distribution of ω.

6. Numerical Analysis

To highlight key attributes of the model, we conduct various numerical tests. First, we evaluate the number of defaulting agents for various network structures under different shock regimes, confirming the main results from Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) when the collateral channel is excluded. When agents hold assets and post portions of them as collateral, we extend these results by including a third dimension for collateral ratio. Lastly, we examine the δ-connected network by defining an aggregate measure of debt exposure between two separable components of a network. We then compare levels of default

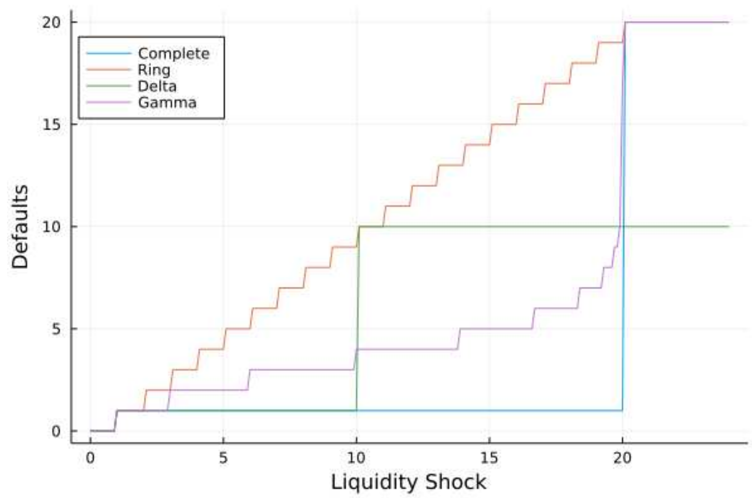


Figure 6: Unsecured debt

and the equilibrium price of assets across changes in inter-component exposure and collateral ratio.

6.1. Network Contagion Patterns without Collateral Channel

As shown by Proposition 2, our model extends Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). Therefore, our models' results should replicate Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) when the asset holdings (h_0) and collateral ratio (c) are 0 for all agents. To test this, we run a numerical simulation that compares the stability and resilience of the complete network, ring network, δ-connected network when $\delta = 0$, and a γ-convex combination of the complete and ring networks when $\gamma = 0.5$ by measuring the number of defaults across varying liquidity shocks (ϵ). The δ-connected network is structured as two discrete components of complete networks, each containing half of the agents in N. We assume homogeneity where all agents have the same total debt (d) obligations and cash endowments (e_0) to ensure that the variation in our results are solely attributed to the variation in network structures. In this setup of $n = 20$ agents with no assets and collateral, e_0 is 1 and d is $2d^* = 38$. Only one agent is hit with the liquidity shock at a given time.

As shown in Figure 6, we reaffirm Proposition 2 when counterparty debt exposure is the only source of contagion. The complete network is the most stable and resilient structure under the small shock regime and becomes as equally fragile as the ring and γ-convex networks under the large shock regime. The δ-connected network is more stable and resilient than the complete network with only half of the agents defaulting under any shock. At most, only half of the agents default because only a single agent receives a liquidity shock which means that the contagion is contained within that one agent's component. Lastly, the ring network is the least stable out of all four networks under any level of liquidity shock.

6.2. Network Contagion Patterns with Collateral Channel

Now, we evaluate how stability, resiliency, and collateral price changes when agents can commit assets as collateral. Using the same setup in the previous section, but c can be greater than 0 and h_0 is equal to 2, we model each network's number of defaults under varying shocks (ϵ) and collateral ratios (c). The collateral ratio ranges from 0 to a value slightly larger than c^*, c^* being 1 and c_* being 0.55. The asset's fair value (s) is 1. The results depicting defaults and collateral prices are presented as 3-dimensional graphs in Figure 7 and Figure 8.

When $\epsilon < \epsilon^* = 20$ or $c \geq c_*$, the complete network is the most stable and resilient network while the ring network is the least stable and resilient with the γ-convex remaining in between the two networks. The price of collateral remains high at its fair value. Both of these results reaffirm Proposition 5. When $\epsilon > \epsilon^*$ and $c < c_*$, all agents default in the complete, ring, and γ-convex networks, making the complete network also the least stable and resilient. The equilibrium price of collateral is 0, which maximizes agents' debt obligations ($d - cdp$). Therefore, contagion from the collateral price channel exacerbates contagion from the debt channel. With δ being sufficiently low, the δ-connected network is more resilient and stable than the complete network, and the collateral price is greater than 0, which is consistent with Proposition 6. When $\epsilon > \epsilon^*$ and $c > c_*$, the price of collateral increases to its fair value s, and contagion is mitigated. An increasing collateral ratio helps reduce the number of defaulting agents for all networks, but most quickly for the complete network and most gradually for the ring network. When $c \geq c^*$ under any liquidity shock, we see collateral fully insulating all networks from debt contagion resulting in only the shocked agent defaulting. Any network is the most stable and resilient for any ϵ as confirmed in Proposition 3.

Overall, these simulations highlight how collateral can be useful in enhancing stability within most networks because it lowers inter-agent liabilities, but only when agents commit a certain threshold of their assets. This threshold can be solved as c_*. When the economy does not meet this threshold, the value of collateral cannot be supported as agents are unable to pay their debt-obligations, which diminishes their collective wealth and renders collateral to be ineffective in promoting network stability. Another important takeaway is the variability in contagion patterns across network structures when measuring defaults and the similarities in contagion patterns when measuring equilibrium price. This shows how collateral price relies on collective wealth, which remains consistent across the different network structures. Having different levels of default despite maintaining the same collateral price shows that the network structure ultimately affects the final distribution and allocation of collective wealth and inefficiency across agents. For example, the ring network has the most defaulting agents, which means collective wealth is concentrated amongst fewer agents relative to the complete network in certain regimes.

6.3. Delta-connected Network

In this experiment, we evaluate a network of 20 agents where again, each agent has 1 unit of cash, 2 assets, and a total liability of $d = 2d^* = 38$. 1 agent out of the 20 receives a large liquidity shock of $\epsilon > \epsilon^* = 20$. The network is segmented into two even components S and S^c where each agent owes a large liability of $(d - 10\delta d)/9$ to every agent within her own component and a small liability of δd to every agent in the opposing component. Across agents i and j, let the inter-component exposure be defined as

$$\Delta = \sum_{i \in S} \sum_{j \in S^c} \delta.$$

We also define Δ^* as the minimal level of inter-component exposure for both components to fully default. $\Delta^* = 0.48$ and $c_* = 0.55$ in this setup. We evaluate changes in inter-component exposure against changes in the collateral ratio and measure the network's number of defaults and the equilibrium asset price as shown in figures 9a and 9b.

When $c < 0.55$ and $\Delta < 0.48$, defaults remain consistently at 10 while the price declines from 0.5. Even when price is close to 0, the number of defaults can still remain at 10. When $\Delta = 0.48$, the number of defaults jumps from 10 to 20 and price becomes 0. Defaults and price remain at 20 and 0 when $\Delta \geq 0.48$. We find that Δ^* is relatively small, meaning that it requires little exposure for the contagion to spread from one component to the other. Furthermore, contagion is well contained within half the agents when $\Delta < 0.48$ but is maximized when $\Delta \geq 0.48$ exhibiting a quick phase transition. When $c \geq 0.55$, the network becomes safe and insulated with maximum price and minimal defaults.

Our results reinforce the notion that unless the collateral ratio is at least a certain threshold c_*, collateral will not improve the network's stability, implying that a network's architecture can be more important than the magnitudes of the liabilities themselves.

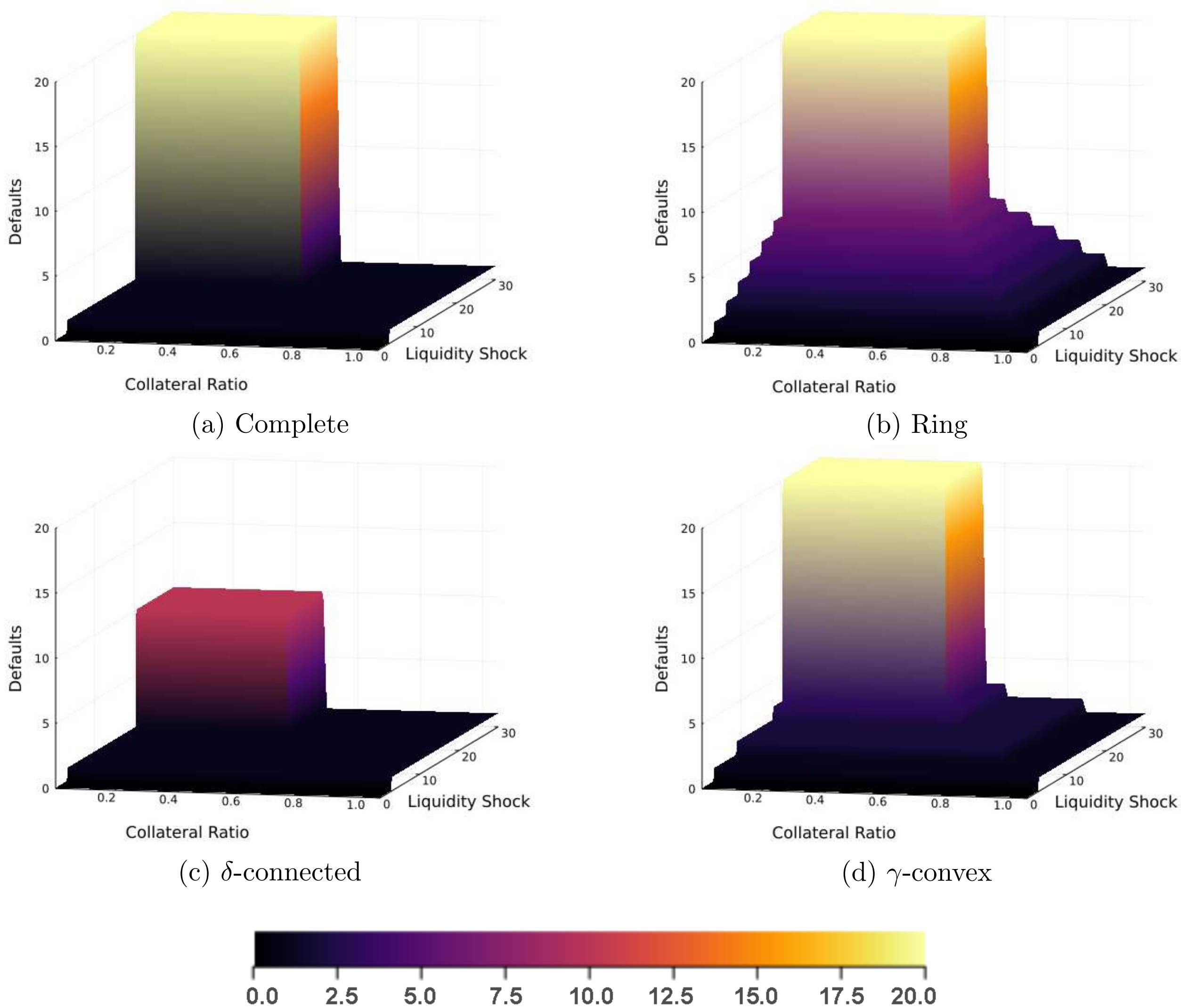
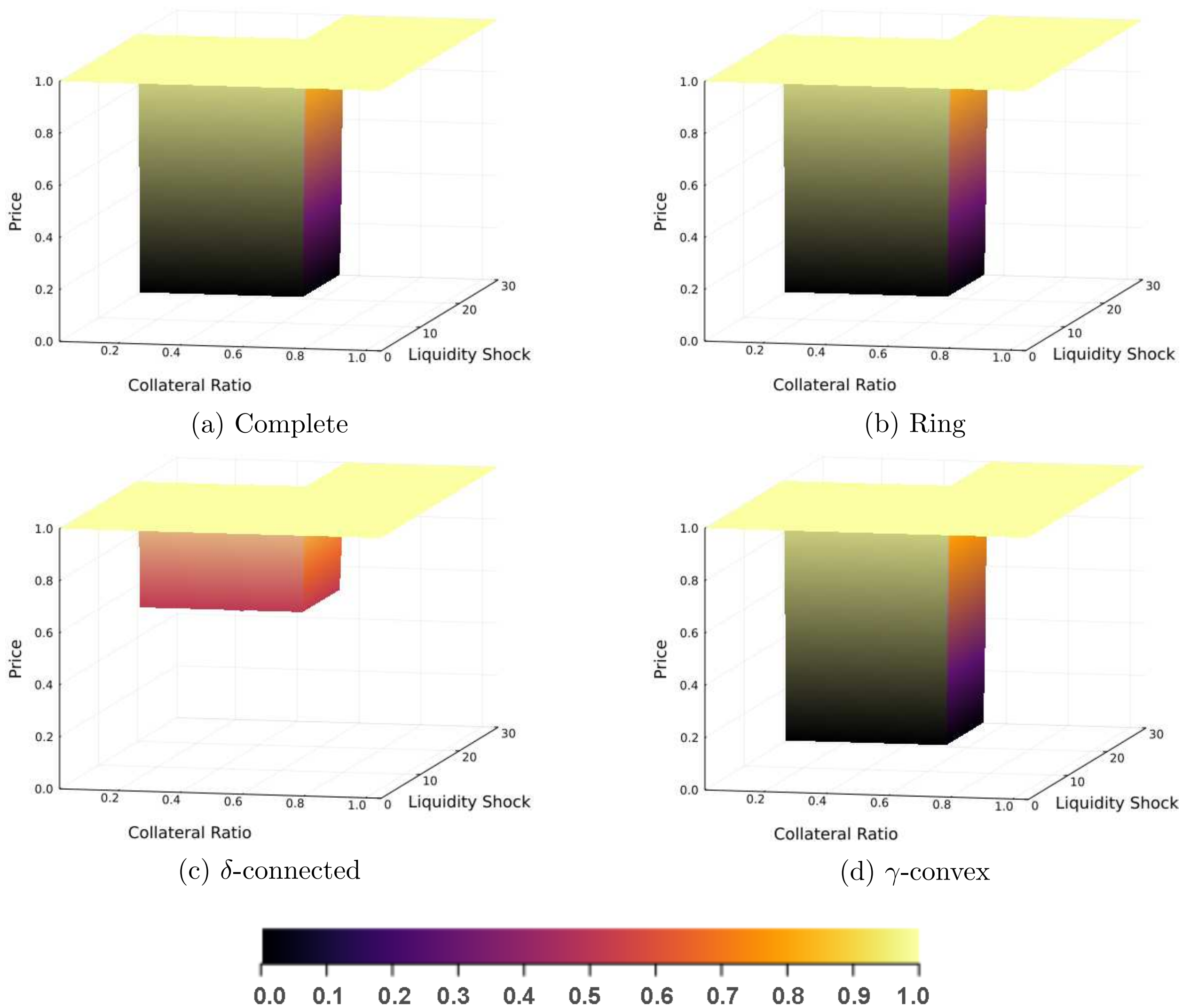


Figure 7: Contagion under collateralized debt



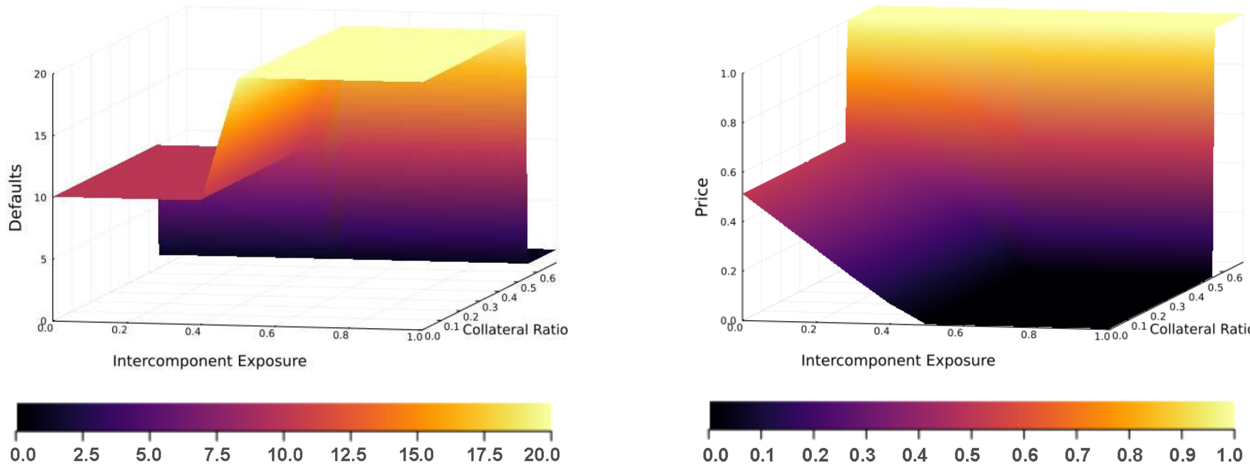
(a) Complete

(b) Ring

(c) δ-connected

(d) γ-convex

Figure 8: Price changes under collateralized debt



(a) Defaults (b) Price

Figure 9: Changes in inter-component exposure, Δ, and collateral ratio, c

7. Conclusion

This paper constructed a model with both debt and collateral market contagion with endogenous fire-sale prices. The collateral can mitigate debt contagion by guaranteeing the payment in case of borrower default. If the collateral ratio is sufficiently large, the economy is fully insulated by the collateral. Therefore, any network with any size of the liquidity shock will generate the same most stable and resilient outcome. If the collateral ratio is somewhat high, then the collateral price is its fundamental value regardless of the network structure and the size of the negative liquidity shock. In this case, the economy is in a robust regime, as the complete network is the most stable and resilient network. However, if the collateral ratio is not so high and the liquidity shock is large, then all agents in the complete network defaults and the asset price goes to zero. Such extremely stark difference in collateral prices depending on the collateral ratio show the fragility of collateral as a buffer for counterparty exposures. Since the payoff of collateral asset is public and fixed, our results also highlight the importance of liquidity flows in bilateral lending relationships during market stress and fire sales. In the last case, a network with two components with very limited interconnections with each other performs better as the shock from one component does not spread to the other component. This result implies that the growing interconnectedness between the (traditional) financial markets and digital assets markets is very concerning for financial stability.

The model also provides insights on macroprudential policy for financial stability. For the same network structure, if the value of the collateral asset decreases, the threshold levels that attain robust and fully-insulated regimes increase. Thus, as the aggregate economy changes, the model can provide the minimum collateral ratio required to attain a robust or fully insulated regime.

Finally, the model is general yet flexible enough to encompass and accommodate many directions of extensions, including the recent literature on fire-sale spillovers. While the literature on financial networks with collateral is limited, the model in this paper can shed light on the framework that can be useful in empirical and numerical analysis for financial stability and systemic risk. Furthermore, this paper provides insights on the roles of collateral in financial markets in general.

0. Appendix: Omitted Proofs

A. Preliminaries

A.1. Collateral Netting

We define a useful way of considering network contagion for a given asset price p, which is *collateral netting*. The role of collateral netting is to pre-calculate any over-collateralized payments as they are guaranteed to be paid in full by the collateral posted. In addition, collateral netting also lumps the collateral of under-collateralized payments into the lender's asset holdings, as the collateral guarantees the market value of collateral even if the borrower pays nothing. Therefore, collateral netting will simplify the derivations while keeping the equilibrium payments the same.

For a given collateralized debt network and environment $(N, C, D, e, h, s, \omega)$ and asset price p, define debt obligations, cash holdings, and asset holdings after collateral netting as the following for any $i, j \in N$:

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij}d_{ij}p]^+ \tag{14}$$

$$\hat{e}_j(p) = e_j + \sum_{\substack{k \in N \\ c_{jk}p > 1}} d_{jk} - \sum_{\substack{i \in N \\ c_{ij}p > 1}} d_{ij} \tag{15}$$

$$\hat{h}_j(p) = h_j + \sum_{\substack{k \in N \\ c_{jk}p \leq 1}} c_{jk}d_{jk} - \sum_{\substack{i \in N \\ c_{ij}p \leq 1}} c_{ij}d_{ij}. \tag{16}$$

This collateral netting derives an interim network after netting out the collateral and payments across agents for a given price p. For example, if a contract d_{ij} is over-collateralized, $c_{ij}p > 1$, then $\hat{d}_{ij}(p)$ is 0, because full payment is guaranteed by the collateral posted. Collateral netting calculates the transfer of full payment d_{ij} to lender i, which is included in i's cash holdings $\hat{e}_i(p)$ and $\hat{e}_j(p)$, and the transfer of collateral $c_{ij}d_{ij}$ to borrower j, which is included in $\hat{h}_j(p)$. If a contract d_{jk} is under-collateralized, $c_{jk}p \leq 1$, then k owes j additional amount of $\hat{d}_{jk}(p) > 0$ on top of the collateral value. Because the market value of collateral does not depend on who owns the collateral, assume that the collateral is kept by lenders for under-collateralized debts, without loss of generality. Then, the collateral will be in j's balance sheet in the asset holdings $\hat{h}_j(p)$. Hence, collateral netting simplifies the cross-agent debt payments by taking care of payments related to collateral and the ownership of collateral, which do not depend on whether a borrower defaults or not.

A.2. Payment Equilibrium under Collateral Netting

We introduce a matrix notation of payment equilibrium for a collateral-netting network that corresponds to the payment equilibrium of the original network. Let $Q(p) \in \mathbb{R}^{n\times n}$ be the matrix with its (i,j) element as q_{ij} defined in equation (7). Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)'$ be the vector of agents' total inter-agent liabilities and $\mathbf{l} = (l_1, l_2, \ldots, l_n)'$ be the vector of agents' liquidation decisions. Define

$$\hat{z}_j(p) \equiv \hat{e}_j + \hat{h}_j p - \omega_j \epsilon,$$

and $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)'$.

Equations (5) and (6) for every agent can be simplified in matrix notation as the system of equations as below:

$$\hat{\mathbf{x}} = \left[\min\left\{Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}, \hat{\mathbf{d}}\right\}\right]^+ \tag{17}$$

$$\hat{\mathbf{l}}(p) = \begin{cases} \left[\min\left\{\frac{1}{\zeta}\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbf{1}\right\}\right]^+ & \text{if } p \geq s\zeta \\ \xi\mathbf{1} & \text{if } p < s\zeta \end{cases}, \tag{18}$$

where $\hat{x}_j = \sum \hat{x}_{ij}$, $\hat{\mathbf{x}}$ is the vector of $\hat{x}_j$'s, $\mathbf{1}$ is a vector of ones for the appropriate dimension. Note that if Q is not defined, then the payments are trivially determined. The function entry p is omitted here and will be omitted from now on unless necessary for exposition. Thus, we define the payment equilibrium of a collateral-netting network with these modified payment and liquidation rules.

Definition 5. *For a fixed price p and a collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium if it satisfies (17) and (18).*

We now show that the payment equilibrium under the *collateral-netting network*, a debt network with no collateral after performing collateral netting on the original network, is equivalent to the payment equilibrium under the original network. In other words, if $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ satisfies the payment rule and the liquidation rule as above, then the corresponding $(\mathbf{x}, \mathbf{l})$ satisfies the payment and liquidation rules for the original network for the given price.

Lemma 5. *The net wealth and payments of agents in the payment equilibrium of the collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$ for a given full equilibrium price p is the same as the net wealth and payments of agents in the payment equilibrium of the original network $(N, C, D, e, h, s, \omega)$ for a given full equilibrium price p.*

Proof. The payment under collateral-netting network is simplified as below for any $i, j \in N$,

$$\begin{aligned}\hat{x}_{ij}(p) &= \min\left\{\hat{d}_{ij}, q_{ij}(p)\left[\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N}\hat{x}_{jk}\right]^+\right\} \\ &= \left[\min\left\{\hat{d}_{ij}, q_{ij}\left(\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N}\hat{x}_{jk}\right)\right\}\right]^+,\end{aligned} \tag{19}$$

where the second equality holds because $\hat{d}_{ij} \geq 0$. If $c_{ij}p \leq 1$ and the net wealth for the second case is the same with $m_j(p)$, then the payments are the same. If $c_{ij}p > 1$, then $\hat{d}_{ij}(p) = 0$, but from $\hat{e}_j(p)$ and $\hat{h}_j(p)$, the payment $d_{ij} - c_{ij}d_{ij}p$ will be subtracted from j's net wealth. Therefore, the payments are equivalent to x_{ij} for any $i, j \in N$ as long as the net wealth is equivalent. The corresponding net wealth is

$$\begin{aligned}\hat{m}_j(p) \equiv &\hat{e}_j + \hat{h}_j(p)p - \omega_j\epsilon + \zeta l_j(p) + \sum_{k\in N}\hat{x}_{jk} - \sum_{i\in N}\hat{x}_{ij} \\ =&e_j + \sum_{\substack{k\in N \\ c_{jk}p>1}} d_{jk} - \sum_{\substack{i\in N \\ c_{ij}p>1}} d_{ij} + h_j p + \sum_{\substack{k\in N \\ c_{jk}p\leq 1}} c_{jk}d_{jk}p - \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} c_{ij}d_{ij}p \\ &+ \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} x_{jk} - \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} (d_{ij} - c_{ij}d_{ij}p) - \omega_j\epsilon + \zeta l_j(p) \\ =&e_j + h_j p - \omega_j\epsilon + \zeta l_j(p) + \sum_{k\in N} c_{jk}d_{jk}p + \sum_{k\in N} x_{jk} - \sum_{i\in N} d_{ij} \\ =&m_j(p),\end{aligned}$$

which implies the net wealth remains the same as in the original network. ∎

Note that the region where the collateral netting is trivial is when the asset price or collateral ratio is high. Therefore, the payment amount and the market value of the asset-holding under collateral-netting network are increasing in p.

The only thing left is to check the market clearing condition, equation (10) and if it does, the given payment equilibrium values consist a full equilibrium. The following lemma, which is a direct application of Lemma B2 in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), further simplifies the computation of payment equilibrium and full equilibrium.

Lemma 6. *Suppose that p is a price from a full equilibrium. Suppose that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is from a collateral-netting network of a full equilibrium $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$. Then, $\hat{\boldsymbol{x}}$ satisfies*

$$\hat{\boldsymbol{x}} = \left[\min\left\{Q\hat{\boldsymbol{x}} + \hat{z} + \zeta\xi\mathbf{1}, \hat{\boldsymbol{d}}\right\}\right]^+. \tag{20}$$

Conversely, if $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ satisfies (20), then there exists $\hat{\boldsymbol{l}} \in [0, \xi]^n$ such that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium and the corresponding $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$ is a full equilibrium.

Proof of Lemma 6. Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is from a full equilibrium and forms a payment equilibrium for the equilibrium price p. First, suppose that $p < s\zeta$. Then, every agent will liquidate its assets regardless of the payments so $\hat{\mathbf{l}} = \xi\mathbf{1}$ and the payment rule satisfies (20). Now suppose that $p \geq s\zeta$. By liquidation rule (18), $\zeta\hat{\mathbf{l}} = \left[\min\left\{\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \zeta\xi\mathbf{1}\right\}\right]^+$, which yields

$$
\begin{aligned}
Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\} \\
\Rightarrow \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\} &= \min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\}\right\} \\
&= \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}.
\end{aligned}
$$

Thus, $\hat{\mathbf{x}} = [\min\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\}]^+ = [\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}]^+$.

Now we consider the other direction. Again, if $p < s\zeta$, then agents will liquidate all of their projects. Therefore, if p is an equilibrium price, then there exists an equilibrium with $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$. Finally, suppose that $p \geq s\zeta$. Then, from (18) and (20) we have $\hat{\mathbf{l}}(p) = \left[\min\left\{1/\zeta\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbf{1}\right\}\right]^+$ satisfied. Plugging this expression into the notation of $\mathbf{X}$ becomes

$$
\begin{aligned}
Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\} \\
\Rightarrow \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\}\right]^+ &= \left[\min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\}\right\}\right]^+ \\
&= \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right]^+ = \hat{\mathbf{x}}.
\end{aligned}
$$

as in the other direction. Therefore, the equilibrium payment rule is also satisfied, and $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is a payment equilibrium for price p. Because p is a full equilibrium price, the corresponding $(\mathbf{x}, \mathbf{l}, \mathbf{m}, p)$ is a full equilibrium. ■

A.3. Useful Lemmas

The following lemma, which is Lemma B1 from Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) is used for the proof of Proposition 1.

Lemma 7. *Suppose that $\beta > 0$. Then,*

$$
|\,[\min\{\alpha, \beta\}]^+ - [\min\{\hat{\alpha}, \beta\}]^+\,| \leq |\alpha - \hat{\alpha}|. \tag{21}
$$

Furthermore, the inequality is tight only if either $\alpha = \hat{\alpha}$ or $\alpha, \hat{\alpha} \in [0, \beta]$.

Proof of Lemma 7. **Case 1.** Suppose that $\beta < \alpha, \hat{\alpha}$. Then, the inequality (21) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

and the inequality is tight only if $\alpha = \hat{\alpha}$.

Case 2. Suppose that $0 \leq \alpha, \hat{\alpha} \leq \beta$. Then, the inequality (21) becomes

$$|\alpha - \hat{\alpha}| = |\alpha - \hat{\alpha}|.$$

Therefore, the inequality is always tight if $\alpha, \hat{\alpha} \in [0, \beta]$.

Case 3. Suppose that either $\alpha < 0 \leq \hat{\alpha}$ or $\hat{\alpha} < 0 \leq \alpha$ holds. Then, the left-hand side of (21) is either $|\hat{\alpha}|$ or $|\alpha|$, which is less than the right-hand side of (21), $|\alpha - \hat{\alpha}|$, and the inequality is tight only if $\alpha = \hat{\alpha} < 0$.

Case 4. Suppose that $\alpha, \hat{\alpha} < 0$. Then, the inequality (21) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

which is tight only if $\alpha = \hat{\alpha}$.

Case 5. Suppose that either $\hat{\alpha} \leq \beta < \alpha$ or $\alpha \leq \beta < \hat{\alpha}$ holds. Then, the left-hand side of (21) is either $|\beta - \hat{\alpha}|$ or $|\alpha - \beta|$, which is less than the right-hand side of (21), $|\alpha - \hat{\alpha}|$, and the inequality can never be tight as $\alpha \neq \hat{\alpha}$. ■

Lemma 8. *For a full equilibrium under the assumptions in Section 4 with $\kappa = 1$, define $\epsilon^* \equiv ne_0$. Then, the following statements are true:*

1. *If $\epsilon < \epsilon^*$, at least one agent does not default.*
2. *If $\epsilon > \epsilon^*$ and $cp < 1$, at least one agent defaults and cannot even pay liquidity shock.*
3. *If $cp \geq 1$, no agent defaults on inter-agent debt.*

Proof of Lemma 8. For the first statement, suppose $\epsilon < \epsilon^*$ and use the collateral-netting network for the given equilibrium price. Suppose all agents default. Then, the only possible equilibrium price is $p = 0$ by (10). Because every agent defaults,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \leq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all $j \in N$ yields

$$ne_0 - \epsilon \leq 0,$$

which is a contradiction to $\epsilon < \epsilon^* = ne_0$.

For the second statement, suppose $\epsilon > \epsilon^*$ and $cp < 1$ and no one defaults. Then,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \geq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all the equations yields

$$n(e_0 + h_0 p) - \epsilon \geq 0, \tag{22}$$

and the only way to satisfy the inequality is that p is large enough. However, because $ne_0 < \epsilon$, there will be no cash in the market to clear the market with $p > 0$, as

$$\begin{aligned} nh_0 p &= n(e_0 + h_0 p) - \epsilon \\ 0 &= ne_0 - \epsilon < 0, \end{aligned}$$

where the last inequality comes from $\epsilon > \epsilon^*$, so p becomes zero, and the above inequality (22) becomes

$$ne_0 - \epsilon \geq 0,$$

which is a contradiction.

For the third statement, recall that the payment under the collateral-netting network is

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ = 0$$

and any payment is fully covered by collateral. ■

B. Characteristics of Full Equilibrium

Proof of Lemma 1. First, for the given asset price p, denote the set of agents defaulting as $\mathcal{D}(p)$ and the complement set as $\mathcal{S}(p) \equiv N \backslash \mathcal{D}(p)$. If $\sum_{j \in N} [m_j(p)]^+ = 0$, then it cannot decrease further and it is trivially increasing in p. Thus, suppose that $\sum_{j \in N} [m_j(p)]^+ > 0$.

Recall that

$$\begin{aligned} m_j(p) =& e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in N} x_{jk}(p) \\ =& e_j + h_j p + \sum_{k \in \mathcal{D}(p)} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in \mathcal{S}(p)} d_{jk} + \sum_{k \in \mathcal{D}(p)} x_{jk}(p). \end{aligned}$$

Summing up the net wealth of non-defaulting agents implies

$$\sum_{j \in \mathcal{S}(p)} m_j(p) = \sum_{j \in \mathcal{S}(p)} \left(e_j + h_j p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in \mathcal{D}(p)} d_{ij} \right) + \sum_{j \in \mathcal{S}(p)} \sum_{k \in \mathcal{D}(p)} \left(c_{jk} d_{jk} p + x_{jk}(p) \right).$$

Note that the coefficients of p are positive. Also note that changes in the liquidation amount $l_j(p)$ does not decrease net wealth. If $p < s\zeta$, then $l_j(p) = \xi$, which is non-decreasing in p. If $p \geq s\zeta$, $l_j(p)$ does not decrease net wealth when p increases, because the liquidation amount should cover the discrepancy $b_j(p) - a_j(p)$, if there is any, and the liquidation amount should be just enough to maintain $m_j(p) = 0$. Hence, an increase in price does not decrease the net wealth of non-defaulting agents through the changes in $l_j(p)$. Therefore, $\sum_{j \in \mathcal{S}} m_j(p)$ is strictly increasing in p if

$$\sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{D}} \left(x_{jk}(p) + c_{jk} d_{jk} p \right)$$

is strictly increasing in p. Recall that

$$\begin{aligned} x_{jk}(p) = \min \Bigg\{ & d_{jk} - c_{jk} d_{jk} p, q_{jk}(p) \Bigg[e_k + h_k p + \sum_{l \in N} c_{kl} d_{kl} p - \sum_{l \in N} c_{lk} d_{lk} p \\ & + \sum_{l \in N} [c_{lk} d_{lk} p - d_{lk}]^+ + \sum_{l \in N} x_{kl}(p) + \zeta \xi - \omega_k \epsilon \Bigg]^+ \Bigg\}, \end{aligned}$$

for any $k \in \mathcal{D}(p)$. We will focus on the case in which $\sum_{l \in N} [c_{lk} d_{lk} p - d_{lk}]^+ = 0$, which is for over-collateralized contracts and increases k's net wealth. If $c_{lk} d_{lk} p - d_{lk} > 0$ for $l \neq j$, then $x_{jk}(p)$ will increase even further with the increase in p and the argument below still holds.

Also, if $c_{jk}d_{jk}p - d_{jk} > 0$, then $x_{jk}(p) + c_{jk}d_{jk}p = d_{jk}$ will be non-decreasing in p. Therefore, it is enough to show that the statement is true in the case where there is no $l \in N$ such that $c_{lk}d_{lk}p - d_{lk} > 0$ for any $k \in \mathcal{D}(p)$.

Given that, we can simplify the expression as

$$\begin{aligned}
&\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[e_k + h_k p + \sum_{l\in N}c_{kl}d_{kl}p - \sum_{l\in N}c_{lk}d_{lk}p + \sum_{l\in N}x_{kl}(p) + \zeta\xi - \omega_k\epsilon\right]^+ + c_{jk}d_{jk}p\right) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[F_k(p) + \sum_{l\in\mathcal{D}(p)}c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)}x_{kl}(p) - \sum_{l\in N}c_{lk}d_{lk}p\right]^+ + c_{jk}d_{jk}p\right),
\end{aligned} \tag{23}$$

where $F_k(p) = e_k + h_k p + \sum_{l\in\mathcal{S}(p)} d_{kl} + \zeta\xi - \omega_k\epsilon$ is strictly increasing in p.

Case 1. Consider the case in which

$$F_k(p) + \sum_{l\in\mathcal{D}(p)}c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)}x_{kl}(p) - \sum_{l\in N}c_{lk}d_{lk}p > 0, \qquad \forall k \in \mathcal{D}(p)$$

so no agents are defaulting on their liquidity shocks (senior debt). The sum of weights should add up to 1 as $\sum_{j\in N} q_{jk} = 1$, therefore the payments of defaulting agents should satisfy

$$\begin{aligned}
x_k(p) \equiv \sum_{l\in N}x_{lk}(p) = &\sum_{l\in\mathcal{S}(p)}x_{lk}(p) + \sum_{l\in\mathcal{D}(p)}x_{lk}(p) \\
=&F_k(p) + \sum_{l\in\mathcal{D}(p)}c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)}x_{kl}(p) - \sum_{l\in N}c_{lk}d_{lk}p > 0
\end{aligned}$$

for any $k \in \mathcal{D}(p)$. Thus, (23) can be rearranged as

$$\begin{aligned}
\sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)}(x_{jk}(p) + c_{jk}d_{jk}p) = &\sum_{k\in\mathcal{D}(p)}F_k(p) + \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) + \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)}x_{kl}(p) \\
&- \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{S}(p)}c_{lk}d_{lk}p - \sum_{l\in\mathcal{D}(p)}\sum_{k\in\mathcal{D}(p)}x_{lk}(p) + \sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)}c_{jk}d_{jk}p \\
= &\sum_{k\in\mathcal{D}(p)}F_k(p),
\end{aligned}$$

which is strictly increasing in p.

Case 2. Now suppose that some agents default on their liquidity shocks. Denote the set of such agents as $\mathcal{B}(p)$, which implies $\forall k \in \mathcal{B}(p)$, $x_{jk}(p) = 0$, for any $j \in N$. We will often omit the argument p for the sets $\mathcal{B}(p), \mathcal{D}(p)$, and $\mathcal{S}(p)$ from now on for notational simplicity. Then, rearranging (23) yields

$$\begin{aligned}\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p)+c_{jk}d_{jk}p) = &\sum_{k\in\mathcal{D}\setminus\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) \\ &- \sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{l\in\mathcal{B}} x_{lk}(p) + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl}p. \end{aligned} \tag{24}$$

Case 2.1. If $x_{lk}(p)$ is zero—that is, $q_{lk}(p) = 0$— for all $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$, then the right-hand side of (24) is trivially increasing in p by applying collateral constraints twice. The actual steps are similar to the steps shown in a more general case, Case 2.2., below.

Case 2.2. Suppose that $q_{lk}(p) > 0$ for some $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$. Recall that

$$x_k(p) = F_k(p) + \sum_{l\in\mathcal{D}} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p + \sum_{l\in\mathcal{D}\setminus\mathcal{B}} x_{kl}(p)$$

for any $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$. Therefore, the matrix notation of the aggregate payments from $\mathcal{D}(p)\setminus\mathcal{B}(p)$ becomes

$$x_{\mathcal{D}\setminus\mathcal{B}} = G_{\mathcal{D}\setminus\mathcal{B}} + Q_{\mathcal{D}\mathcal{D}}x_{\mathcal{D}\setminus\mathcal{B}},$$

where $x_{\mathcal{D}\setminus\mathcal{B}}$ is $|\mathcal{D}(p)\setminus\mathcal{B}(p)|\times 1$ vector of $x_k(p)$ for each $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$, $G_{\mathcal{D}\setminus\mathcal{B}}$ is $|\mathcal{D}(p)\setminus\mathcal{B}(p)|\times 1$ vector of $F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p$ for each $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$, and $Q_{\mathcal{D}\mathcal{D}}$ is a $|\mathcal{D}(p)\setminus\mathcal{B}(p)|\times|\mathcal{D}(p)\setminus\mathcal{B}(p)|$ matrix of weights $q_{ij}(p)$ for $i, j \in \mathcal{D}(p)\setminus\mathcal{B}(p)$. Note that the spectral radius of $Q_{\mathcal{D}\mathcal{D}}$ is less than 1 by assumption and $(I - Q_{\mathcal{D}\mathcal{D}})^{-1}$ exists by the property of Neumann series. Hence,

$$x_{\mathcal{D}\setminus\mathcal{B}} = (I - Q_{\mathcal{D}\mathcal{D}})^{-1}G_{\mathcal{D}\setminus\mathcal{B}},$$

and the sum of payments to agents defaulting on senior debt is represented as

$$\sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{l\in\mathcal{B}} x_{lk}(p) = \mathbb{1}'Q_{\mathcal{B}\mathcal{D}}x_{\mathcal{D}\setminus\mathcal{B}} = \mathbb{1}'Q_{\mathcal{B}\mathcal{D}}(I - Q_{\mathcal{D}\mathcal{D}})^{-1}G_{\mathcal{D}\setminus\mathcal{B}},$$

where $Q_{\mathcal{B}\mathcal{D}}$ is a $|\mathcal{B}(p)|\times|\mathcal{D}(p)\setminus\mathcal{B}(p)|$ matrix of weights $q_{lk}(p)$ for $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\setminus\mathcal{B}(p)$.

Since all entries of $Q_{\mathcal{BD}}$ are also less than 1, there exists $\eta < 1$ such that

$$\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} x_{lk}(p) = \eta\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk}p\right].$$

Thus, (24) implies

$$\begin{aligned}\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) =& (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p)\right] \\ &+ \eta\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk}p + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl}p.\end{aligned}$$

Adding $\sum_{j\in\mathcal{S}} h_j p$ from $\sum_{j\in\mathcal{S}} F_j p$ to the right-hand side makes the coefficient on p as

$$\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \eta\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} + (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl} - c_{lk}d_{lk})\right], \tag{25}$$

which is again positive by applying collateral constraints twice as we show in the following. From the collateral constraints for $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, we have

$$\begin{aligned}&\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{kl}d_{kl} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}} c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj} \\ \geq& \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{lk}d_{lk} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}} c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} \\ \Rightarrow& \sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{kl}d_{kl} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} c_{lk}d_{lk} \geq \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj}.\end{aligned} \tag{26}$$

Similarly, from the collateral constraints for $j \in \mathcal{S}(p)$, we have

$$\begin{aligned}&\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{jk}d_{jk} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}} c_{jj'}d_{jj'} \\ \geq& \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}} c_{j'j}d_{j'j} \\ \Rightarrow& \sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} \geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk}.\end{aligned} \tag{27}$$

Hence, plugging (26) and (27) into (25) implies

$$
\begin{aligned}
&\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \eta\sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} + (1-\eta)\left[\sum_{k\in\mathcal{D}\setminus\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl} - c_{lk}d_{lk})\right] \\
\geq &\sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\setminus\mathcal{B}} c_{kj}d_{kj} - (1-\eta)\sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} \\
&+ (1-\eta)\left[\sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\setminus\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj}\right] \\
= &\sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \eta\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\setminus\mathcal{B}} c_{kj}d_{kj} > 0,
\end{aligned}
$$

thus the coefficient on p is positive, implying that the aggregate positive net wealth is strictly increasing in p.

Finally, we consider the changes in the set of defaulting agents with an increase in p. Because the aggregate net wealth is increasing in p as in Case 1, increase in p will only weakly decrease the number of defaulting agents. Therefore, there are no agents in $j \in \mathcal{S}(p)$, who will default due to an increase in p, and there can be agents $j \in \mathcal{D}(p)$, who will be solvent under higher p and their net wealth would be added to the sum of $\sum\limits_{j\in\mathcal{S}(p)} m_j(p)$, increasing the aggregate positive net wealth further. ■

Proof of Lemma 2. Suppose that $p < s$ makes the market clear. Rearranging the first line of equation (10) yields

$$
\begin{aligned}
\sum_{j\notin\mathcal{D}(p)} [m_j(p) - h_j p]^+ &= \sum_{i\in N} \min\left\{[h_j p - m_j(p)]^+, h_j p\right\} \\
\sum_{j\notin\mathcal{D}(p)} (m_j(p) - h_j p) &= \sum_{i\in\mathcal{D}(p)} h_j p \\
\sum_{j\in N} [m_j(p)]^+ &= \sum_{j\in N} h_j p \\
p &= \frac{\sum_{j\in N}[m_j(p)]^+}{\sum_{j\in N} h_j},
\end{aligned}
$$

which holds because $\phi_i(p) = h_i p$ for any $i \in \mathcal{D}(p)$.

Now suppose that $p = s$ satisfies the second line of equation 10, which implies

$$\begin{aligned}
\sum_{j \notin \mathcal{D}(s)} [m_j(s) - h_j s]^+ &\geq \sum_{i \in N} \min \left\{ [h_j s - m_j(s)]^+ , h_j s \right\} \\
\sum_{j \notin \mathcal{D}(s)} (m_j(s) - h_s p) &\geq \sum_{i \in \mathcal{D}(s)} h_j s \\
\sum_{j \in N} [m_j(s)]^+ &\geq \sum_{j \in N} h_j s \\
s &\leq \frac{\sum_{j \in N} [m_j(s)]^+}{\sum_{j \in N} h_j}.
\end{aligned}$$

Combining the two cases implies that the market price is bounded by s, and otherwise the ratio between the aggregate positive net wealth and the total supply of assets, therefore (11). ■

Proof of Proposition 1.

The first step, which is based on the proof of Proposition 1 in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), is to show that there exists a payment equilibrium that is generically unique for any given p. The second step is to show that there exists an equilibrium price p that satisfies the market clearing condition for the given payment equilibrium payment and liquidation vectors.

Existence of payment equilibrium.

First, fix an asset price p. By Lemma 6, it is sufficient to show that there exists $\mathbf{x}^* \in \mathbb{R}_+^n$ that satisfies $\mathbf{x}^* = \left[\min \left\{ Q\mathbf{x}^* + \hat{z} + \zeta\xi\mathbf{1}, \hat{\mathbf{d}} \right\} \right]^+$. Define the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ as

$$\Phi(\mathbf{x}) = \left[\min \left\{ Q\mathbf{x} + \hat{z} + \zeta\xi\mathbf{1}, \hat{\mathbf{d}} \right\} \right]^+$$

where $\mathcal{X} = \Pi_{i=0}^n [0, d_i]$. This mapping is continuous and its domain, which is the same as its range, is a convex and compact subset of the Euclidean space. Thus, there exists $\mathbf{x}^* \in \mathcal{X}$ such that $\Phi(\mathbf{x}^*) = \mathbf{x}^*$ by the Brouwer fixed-point theorem. The corresponding $\mathbf{l}^*$ can be obtained and the pair $(\mathbf{x}^*, \mathbf{l}^*)$ satisfies the payment and liquidation rules in the original network for any given price p.

Generic uniqueness of payment equilibrium.

Assume that the financial network is connected without loss of generality, as we can apply the proposition for each component of a network that is not connected. Suppose that for the same equilibrium price p, there exist two distinct payment equilibria (X, l) and $(\tilde{X}, \tilde{l})$ such that $X \neq \tilde{X}$. Then, payments from each equilibrium should satisfy (20). Hence, for each agent j,

$$\begin{aligned}|\hat{x}_j - \hat{\tilde{x}}_j| &= \left| \left[\min\left\{(Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi, \hat{d}_j\right\}\right]^+ - \left[\min\left\{\left(Q\hat{\tilde{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi, \hat{d}_j\right\}\right]^+ \right| \\ &\leq \left|(Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j\right|\end{aligned}$$

where the last inequality is coming from the fact that both terms have the same upper bound and by triangle inequality. Taking L^1 norm for the vector representation of both sides of the above inequality becomes

$$\begin{aligned}||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}|| &\leq \left|\left|Q\left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}\right)\right|\right| \\ &\leq ||Q|| \cdot \left|\left|\left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}\right)\right|\right| \\ &= ||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}||,\end{aligned}$$

because Q is column stochastic from the weighting rule (7). Therefore, all the inequalities are binding and

$$|\hat{x}_j - \hat{\tilde{x}}_j| = \left|(Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j\right|$$

holds. Because $\hat{\mathbf{x}} = \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right]^+$ and by Lemma 7, either

$$(Q\hat{\mathbf{x}})_j = \left(Q\hat{\tilde{\mathbf{x}}}\right)_j$$

or

$$\begin{aligned}0 &\leq (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi \leq d_j \\ 0 &\leq \left(Q\hat{\tilde{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi \leq d_j.\end{aligned} \tag{28}$$

Therefore, the set of defaulting agents, $\mathcal{D}(p)$, is the same for the two different payment equilibria. For any agent satisfying (28)—that is, if $j \in \mathcal{D}(p)$,

$$(Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j = \hat{x}_j - \hat{\tilde{x}}_j.$$

For the other case, for all $j \notin \mathcal{D}(p)$, the other equality $(Q\hat{\mathbf{x}})_j = \left(Q\hat{\tilde{\mathbf{x}}}\right)_j$ should hold. Because the collateral-netting network eliminates any idiosyncratic collateral ratio, the payment and weighting matrices are invariant to any permutation. Denote $\underline{Q}$ and $\underline{\hat{x}}$ as the weighting matrix and payment vector for collateral-netting matrix after a permutation of the order of agents by having $j \in \mathcal{D}(p)$ first and then $i \notin \mathcal{D}(p)$ later. Therefore,

$$\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right) = \begin{bmatrix} \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}} \\ 0 \end{bmatrix}$$

where the $\underline{x}_{\mathcal{D}}$ is the subvector of $\underline{x}$ only including the agents in $\mathcal{D}(p)$ and

$$\left|\left|\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right)\right|\right| = ||\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}||.$$

Thus, $\hat{x}_j = \hat{\hat{x}}_j$ for any $j \notin \mathcal{D}(p)$ and

$$\underline{Q}_{\mathcal{D}}(\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}) = \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}} \tag{29}$$

where $\underline{Q}_{\mathcal{D}}$ is the submatrix of $\underline{Q}$ for the agents in $\mathcal{D}(p)$ and $\underline{\hat{\mathbf{x}}}_{\mathcal{D}}$ is a subvector of $\underline{\hat{\mathbf{x}}}$ for the agents in $\mathcal{D}(p)$. If the debt network is connected, then $\underline{Q}$ and $\underline{Q}_{\mathcal{D}}$ are irreducible nonnegative matrices by construction. Then, by Perron-Frobenius theorem, there exists a simple eigenvalue and right eigenvector whose components are all positive (Gaubert and Gunawardena, 2004).

If $\mathcal{D}(p)$ is a proper subset of N, then all of the column sums are less than one, and the spectral radius for Q and $\underline{Q}_{\mathcal{D}}$ are less than one. This result is due to $\lim_{k\to\infty} ||Q^k|| = 0$, which implies $0 = \lim_{k\to\infty} Q^k\mathbf{v} = \lim_{k\to\infty} \lambda^k\mathbf{v} = \mathbf{v}\lim_{k\to\infty} \lambda^k$ that leads to $\lim_{k\to\infty} \lambda^k = 0$, where λ and $\mathbf{v}$ are eigenvalue and eigenvector, respectively. All of the eigenvalues of $\underline{Q}_{\mathcal{D}}$ have absolute value less than one, and $\underline{Q}_{\mathcal{D}}\mathbf{v} = \mathbf{v}$ does not have a non-trivial solution. Hence, (29) cannot hold unless $\mathcal{D}(p) = N$. Then, $\hat{x}_j = (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi$ for all $j \in \mathcal{D}(p) = N$ and

$$\begin{aligned}\sum_{j\in N}\hat{x}_j &= \sum_{j\in N}\sum_{i\in N} q_{ij}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi \\ &= \sum_{i\in N}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi.\end{aligned}$$

Furthermore, the only asset price p under $\mathcal{D}(p) = N$ is $p = 0$ from (10). In other words, there cannot be multiple equilibria if the equilibrium price is $p > 0$. Finally, even if all the agents default and $p = 0$, the last equation implies

$$\sum_{j\in N}\hat{z}_j(p) = \sum_{j\in N}(e_j - \omega_j\epsilon) = -n\zeta\xi,$$

which holds only for a non-generic set of parameters, n, e, ω, ζ, ξ and ϵ, which is a line over a multidimensional Euclidean space. Thus, for the given equilibrium price p, the payment and liquidation pair (X, l) is generically unique.

Existence of full equilibrium.

Now the only condition left for an equilibrium is the market clearing condition for price p. Suppose that there exists a unique (X, l) pair that satisfies the two equilibrium conditions for any given price $p \in [0, s]$.[7] If the resulting payment equilibrium $(\mathbf{x}^*, \mathbf{l}^*)$ generates m that satisfies the second line of equation (10), then $(X^*, \mathbf{l}^*, m^*, \phi^*, s)$ is a full equilibrium.

Now suppose the contrary, and only a price $p < s$ makes the market clear. From Lemma 2, we have

$$p = \frac{\sum_{j \in N} [m_j(p)]^+}{\sum_{j \in N} h_j}, \tag{30}$$

and by Lemma 1, the aggregate positive net wealth is continuously (and strictly) increasing in p (as long as $\sum_{j \in N} [m_j(p)]^+ > 0$). Therefore, the numerator of the right-hand side of (30) is continuously increasing in p. Also, $\sum_{j \in N} m_j(p)$ is increasing in p, as shown in the proof of Lemma 1. Thus, $\mathcal{D}(p) \subset \mathcal{D}(p')$ for any $p > p'$, and an increase in p will increase the price even further by including more agents on the numerator.

Define the mapping $\Psi : [0, s] \to [0, s]$

$$\Psi(p) = \frac{\sum_{j \in N} \left[m_j^*(p)\right]^+}{\sum_{j \in N} h_j},$$

where $m_j^*(p)$ is the corresponding net wealth for agent j under $(X^*, \mathbf{l}^*)$ that are derived from the corresponding payments and liquidation amounts after collateral netting for given price p. Because $\Psi(p)$ is a continuously (and strictly) increasing function of p from $[0, s]$ to $[0, s]$ (in the region $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $0 < \Psi(p) < s$), there exists a fixed point p^*, which is an equilibrium price. Therefore, a full equilibrium $(X^*, \mathbf{l}^*, m^*, \phi^*, p^*)$ exists, and there exists the maximum price $\bar{p}$, which is a full equilibrium price greater than any other full equilibrium prices. ■

[7] This is true for any price $p > 0$ as shown in the proof of generic uniqueness of payment equilibrium.

C. Characteristics of Contagion

Proof of Lemma 3. Suppose that (X, l, m, p) is a full equilibrium. Then any of the cash and asset will generate the given payoff to the whole economy so $n(e_0 + h_0 s)$ should be part of the welfare. However, the total long-term projects $n\xi$ may not remain intact as some or all of the project can be liquidated by l_j amount for each $j \in N$. The total liquidation amount will be $\sum_{i \in N} l_i$ while the cost of early liquidation is $(1 - \zeta)$, as only the ζ proportion will be salvaged. Therefore, the social surplus of the economy is $U = n(e_0 + h_0 s + \xi) - (1 - \zeta) \sum_{i \in N} l_i$. ∎

Proof of Proposition 2. First, compute the collateral-netting network of the original network. The case of fire-sale collapse under $p < s\zeta$ is irrelevant as $nh_0 = 0$. Then, for this collateral-netting network, apply Propositions 4 and 6 of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and the results follow. ∎

Proof of Proposition 3. Without loss of generality, assume that

$$\min \left\{ 1, \frac{(n - \kappa)e_0}{\kappa s h_0} \right\} \geq \zeta,$$

which holds for $\zeta \to 0$. All the payments are covered by the collateral if $cp \geq 1$ by Lemma 8. If all κ number of agents default on their senior debt, collateral covers all the payments, and no non-defaulting agent is liquidating the long-term project, then the total available cash in the economy is $(n - \kappa)e_0$. Also, as only κ number of agents are out of the market, from (11), we have

$$\begin{aligned} nh_0 p &\leq \sum_{j \notin \mathcal{D}} m_j(p) = (n - \kappa)(e_0 + h_0 p) \\ \kappa h_0 p &\leq (n - \kappa)e_0, \end{aligned}$$

which implies the relevant amount of fire sales is κh_0. Therefore, the asset price is either the fundamental value s or the aggregate liquidity divided by total amount of fire sales—that is,

$$p = \min \left\{ s, \frac{(n - \kappa)e_0}{\kappa h_0} \right\}.$$

If $s < \dfrac{(n - \kappa)e_0}{\kappa h_0}$ and $p = s$, then $c \geq 1/s$ will satisfy $cp \geq 1$.
If $s \geq \dfrac{(n - \kappa)e_0}{\kappa h_0}$, then $p = \dfrac{(n - \kappa)e_0}{\kappa h_0}$ and $cp \geq 1$ holds if $c \geq c^\dagger \equiv \dfrac{\kappa h_0}{(n - \kappa)e_0}$. Finally, the

network should satisfy the resource constraints and $c \geq c^\dagger$, thus

$$nh_0 \geq cd \geq \frac{\kappa h_0}{(n-\kappa)e_0} d,$$

which is possible only if $n(n-\kappa)e_0 \geq \kappa d$. Then, for $c^*(s,n) = \dfrac{1}{\min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}}$, any $c \geq c^*$ will satisfy $cp \geq 1$. By uniqueness of the (maximum) full equilibrium, this equilibrium is the only (maximum) equilibrium. In this equilibrium, all of the payments are made in full, and there will be no additional defaults. Thus, any network structure has the most stable and resilient results as the collateral fully insulates any propagation. ∎

Proof of Lemma 4. The lower bound is trivial by the size of the shock causing the agent under the liquidity shock to go default regardless because collateral does not cover the debt obligations by $cp < 1$. Now, consider the upper bound. For the collateral-netting network, recall $\mathcal{D}(p)$ is the set of agents that defaults under price p. Then, for each agent $j \in \mathcal{D}(p)$,

$$\sum_{i \in N} \hat{x}_{ij} = \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \epsilon \right]^+,$$

where the agent can pay a positive amount only if the agent has enough cash inflows to cover the liquidity shock and payments are zero otherwise. Note that the maximum payment amount an agent can receive as a lender is $d - cdp$. Therefore,

$$\sum_{i \in N} \hat{x}_{ij} \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\} \right]^+.$$

Summing over all defaulting agents yields

$$\sum_{j \in \mathcal{D}(p)} \left[(e_0 + h_0 p) + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\} \right]^+ = \sum_{j \in \mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij}.$$

Because agents without liquidity shocks cannot have negative net wealth,

$$\sum_{j \in \mathcal{D}(p)} (e_0 + h_0 p) + \sum_{j \in \mathcal{D}(p)} \sum_{k \in N} \hat{x}_{jk} \leq \sum_{j \in \mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij} + \kappa \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\}, \tag{31}$$

and by canceling out the payments among defaulting agents, we obtain the bound as

$$\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\} - (e_0 + h_0 p)|\mathcal{D}(p)| \geq \sum_{i \notin \mathcal{D}(p)} \sum_{j \in \mathcal{D}(p)} (\hat{d}_{ij} - \hat{x}_{ij}) > 0,$$

where the last inequality comes from the definition of the defaulting agents. Then, rearranging the inequality results in

$$|\mathcal{D}(p)| < \frac{\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\}}{e_0 + h_0 p}, \tag{32}$$

which is decreasing in the asset price p and collateral ratio c. ■

Proof of Proposition 4.

Case 1. $\epsilon < \epsilon^*$. The market clearing condition (11) implies that the aggregate net wealth of non-defaulting agents determines the asset price p. From (11) and (31), the aggregate net wealth of non-defaulting agents is bounded below by

$$\sum_{j \notin \mathcal{D}} (e_0 + h_0 p) - \sum_{i \in N} \sum_{j \in \mathcal{D}} \hat{x}_{ij} + \sum_{j \in \mathcal{D}} (e_0 + h_0 p) + \sum_{j \in \mathcal{D}} \sum_{k \in N} \hat{x}_{jk} - \kappa\epsilon = ne_0 + nh_0 p - \kappa\epsilon,$$

and the market clearing condition becomes

$$p \leq \left[\frac{ne_0 + nh_0 p - \kappa\epsilon}{nh_0}\right]^+.$$

Since $\epsilon < \epsilon^* \equiv \dfrac{ne_0}{\kappa}$, there exists agents with positive net wealth. Then, the market clearing condition implies

$$\underline{p} \leq \underline{p} + \frac{ne_0 - \kappa\epsilon}{nh_0},$$

so the maximum equilibrium would satisfy the market clearing condition with $p = s$.

Case 2. $c \geq c_*$. Suppose $c \geq c_* \equiv \dfrac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$. Also, assume $\epsilon > \epsilon^*$ because the result is trivially true by the first case otherwise. At least one agent under liquidity shock defaults even on the liquidity shock (senior debt) by Lemma 8 and $cp < 1$, implying that all other agents would suffer the total default of the shocked agent in the amount of $\kappa(d - cdp)$ or less. Suppose that the total number of defaulting agents is $k = |\mathcal{D}| < n$. Then, even if all κ agents default on their senior debt, the market clearing condition is

$$
\begin{aligned}
(n-k)(e_0+h_0p)+(k-\kappa)(e_0+h_0p)-\kappa(d-cdp) &\geq nh_0p \qquad (33)\\
(n-\kappa)e_0-\kappa(d-cdp) &\geq \kappa h_0p\\
\kappa(cd-h_0)p &\geq \kappa d-(n-\kappa)e_0,
\end{aligned}
$$

where the left-hand side of the final inequality is increasing in p with $cd > h_0$, which holds when $c \geq c_*$. Therefore, if $c \geq c_*$, $p = s$ holds for the market clearing condition. Also, this implies the left-hand side of (33) is positive, implying that there are agents with positive net wealth—that is, there are solvent agents who can purchase the assets in the market. Therefore, the asset price is $p = s$ in the (maximum) equilibrium regardless of the network structure.

Finally, we show that $c_* \leq c^*$. The value of c^* can be either $1/s$ or $\kappa h_0/(n-\kappa)e_0$.

Case 2.1. Suppose that $\dfrac{(n-\kappa)e_0}{\kappa h_0} \geq s$, therefore, $c^* = 1/s$. Then, $c_* \leq c^*$ holds because

$$
\begin{aligned}
c_* \equiv \frac{d-((n-\kappa)/\kappa)\,e_0+h_0s}{ds} &\leq \frac{1}{s} \equiv c^*\\
d-((n-\kappa)/\kappa)\,e_0+h_0s &\leq d\\
h_0s &\leq ((n-\kappa)/\kappa)\,e_0\\
s &\leq \frac{(n-\kappa)e_0}{\kappa h_0},
\end{aligned}
$$

which holds by the initial assumption.

Case 2.2. Suppose that $\dfrac{(n-\kappa)e_0}{\kappa h_0} < s$, therefore, $c^* = \dfrac{\kappa h_0}{(n-\kappa)e_0}$. Then, $c_* < c^*$ holds because

$$
\begin{aligned}
c_* \equiv \frac{d-((n-\kappa)/\kappa)\,e_0+h_0s}{ds} &< \frac{\kappa h_0}{(n-\kappa)e_0} \equiv c^*\\
d-((n-\kappa)/\kappa)\,e_0+h_0s &< \frac{d\kappa h_0s}{(n-\kappa)e_0}\\
\left(d-\frac{n-\kappa}{\kappa}e_0\right)(n-\kappa)e_0 &< (d-\frac{n-\kappa}{\kappa}e_0)\kappa h_0s\\
\frac{(n-\kappa)e_0}{\kappa h_0} &< s,
\end{aligned}
$$

which holds by the initial assumption. ■

Proof of Proposition 5. Suppose that $\epsilon < \epsilon^*$. Then, the setting becomes similar to that of Proposition 2 except that each agent is endowed with h_0 amount of assets, which is priced as $p = s$ by Proposition 4. Then, the same steps in Proposition 2 apply to this setup with modified endowments for the collateral-netting network.

Now consider the case with $c > c_*$ and $\epsilon > \epsilon^*$. The equilibrium asset price is $p = s$ by Proposition 4. Therefore, each agent's required debt payment is $d - cds$. Because $c > c_*$,

$$d - cds < d - c_* ds = (n-1)e_0 - h_0 s,$$

which is the market clearing condition at $p = s$. Hence, even if the agent under liquidity shock does not pay anything to the remaining agents, the remaining agents combined have sufficient cash amount to pay the debt towards the agent under liquidity shock and purchase the assets on fire sale at $p = s$. Because all agents are symmetric in the complete network, all remaining $n - 1$ agents can pay their debt in full. Hence, the complete network is the most stable and resilient network.

Now consider the ring network. Without loss of generality, suppose that agent 1, who only owes to agent 2, is under liquidity shock. By Lemma 8 and $\epsilon > \epsilon^*$, agent 1 cannot even pay the liquidity shock. Then, agent 1 pays nothing to agent 2, and agent 2 only has the endowment $e_0 + h_0 s$ to pay to agent 3. Agent 3 will reuse this payment from agent 2 and add agent 3's own endowment, so the total payment from agent 3 to agent 4 is $2(e_0 + h_0 s)$. Similarly, agent 4 will pay agent 5 in the amount of $3(e_0 + h_0 s)$, and so forth. Then, agent $k + 1$ would have total available cash of $k(e_0 + h_0 s)$, and agent $k + 1$ is solvent only if

$$k(e_0 + h_0 s) > d - cds,$$

which implies

$$k + 1 > \frac{e_0 + h_0 s + d - cds}{e_0 + h_0 s},$$

which is above the upper bound of the number of defaults in Lemma 4 and (32), implying k reached the upper bound of the number of defaults when $p = s$. Also, note that the lowest k satisfying the above inequality is decreasing in c. Thus, the ring network is the least stable and resilient network, and the number of defaulting agents is decreasing in c.

Given the two results of the ring and complete network, we can use the same method of collateral-netting as in the proof of Proposition 2 and apply Proposition 6 of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) to obtain the last result. ■

Proof of Proposition 6. Suppose that $\epsilon > \epsilon^*$, $c < c_*$, and $d > d^* = (n-1)e_0$. First, note that $cp < 1$ and $p < s$ in any network structure because (33) in the proof of Proposition 4 does not hold. Consider the complete network. The agent under liquidity shock defaults even on the senior debt (liquidity shock) by Lemma 8 and default fully on the debt obligation of $(d - cdp)/(n-1)$ to all other agents. Suppose the contrary that $n-1$ agents do not default. Then, each agent's net wealth equation should satisfy

$$(n-2)\frac{d-cdp}{n-1} + e_0 + h_0 p - (d - cdp) \geq 0. \tag{34}$$

The market clearing condition is

$$(n-1)e_0 - d + cdp \geq h_0 p, \tag{35}$$

where $p > 0$ by the assumption that there are surviving agents. Because $d > (n-1)e_0$, if $cd < h_0$, then there is no price $p > 0$ that can satisfy the market clearing condition, so $p = 0$. Then, (34) implies $(n-1)e_0 \geq d$, which is a contradiction. Now suppose that $cd > h_0$. This inequality implies (35) becomes

$$(cd - h_0)p \geq d - (n-1)e_0, \tag{36}$$

and the left-hand side is maximized when $p = s$, which is the case for the maximum equilibrium we are focusing on. Because $c < c_* \equiv \dfrac{d-(n-1)e_0 + h_0 s}{ds}$, (36) becomes

$$d - (n-1)e_0 \leq (cd - h_0)s < c_* ds - h_0 s = d - (n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$, and again (34) implies $(n-1)e_0 \geq d$, which is a contradiction. Therefore, all agents default and the complete network is the least resilient and least stable network.

Now consider the ring network such that agent 1 borrows from agent 2 who borrows from agent 3, and so on. Without loss of generality, let agent 1 be the agent under negative liquidity shock. Again, by Lemma 8, agent 1 defaults with full amount, $d - cdp$. Agent 2 will pay agent 3 only with the endowment $e_0 + h_0 p$, then agent 3 will reuse this with her own endowment to pay agent 4 in the amount of $2(e_0 + h_0 p)$. Agent 4 will pay agent 5 in the amount of $3(e_0 + h_0 p)$, and so on. In order not to cascade after k length from agent 1,

$$k(e_0 + h_0 p) > d - cdp. \tag{37}$$

If agent $k+1$ pays the debt in full, then all the subsequent agents, $k+2, k+3, \ldots, n$, can pay in full without any sale of their asset holdings. The total market clearing condition is

$$\begin{aligned} (n-k)(e_0+h_0p) - d + cdp + (k-1)(e_0+h_0p) &\geq nh_0p \\ (cd-h_0)p &\geq d-(n-1)e_0, \end{aligned} \tag{38}$$

if $p > 0$, and the left-hand side is maximized at $p = s$. However, because $c < c_*$, (38) evaluated even at the highest $p = s$ implies

$$d-(n-1)e_0 \leq (cd-h_0)s < c_*ds - h_0s = d-(n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$ is the market clearing price. Then, (37) becomes

$$ke_0 > d,$$

which implies $k \geq n$—that is, the first non-defaulting agent should be in the minimum distance of n or higher from agent 1, which exceeds the total number of agents n. Therefore, all agents default and the ring network is the least resilient and least stable network.

Finally, consider a δ-connected network with $\delta < \dfrac{e_0}{(n-1)d}$ and the partition of agents sets are $(\mathcal{S}, \mathcal{S}^c)$ such that $d_{ij} \leq \delta d$ for any $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$. Thus, $\sum_{j \notin S} d_{ij} \leq \delta d|\mathcal{S}^c|$ for any $i \in \mathcal{S}$. Therefore, for any $i \in \mathcal{S}$,

$$\sum_{j \in S}(d_{ij} - d_{ij}cp) \geq d - cdp - \delta(d-cdp)|\mathcal{S}^c| \geq d - cdp - e_0,$$

which implies

$$e_0 + h_0p + \sum_{j \in S}(d_{ij} - d_{ij}cp) \geq e_0 + \sum_{j \in S}(d_{ij} - d_{ij}cp) \geq d - cdp,$$

so agents in $\mathcal{S}$ can fulfill their debt even when all agents in $\mathcal{S}^c$ do not pay any amount to agents in $\mathcal{S}$. Note that the last inequality holds for any given collateral price p. In other words, all agents in $\mathcal{S}$ remain solvent when $\omega_j = 1$ for an agent $j \in \mathcal{S}^c$ regardless of the size of the shock. Therefore, the δ-connected network is more stable and resilient than the complete or ring networks. ■

D. Results from the Extended Models

Proof of Proposition 7. First, we show that the cutoff collateral ratio for full insulation, $c^*(s,n)$, is decreasing in s and n. Recall that $c^*(s,n) = \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\}$. If, $c^*(s,n) = 1/s$, then $c^*(s',n) = 1/s'$, so the cutoff for full insulation is higher at s', that is $c^*(s,n) < c^*(s',n)$. Under this case, $c^*(s,n) \leq c^*(s,n')$, as $c^*(s,n')$ is either $1/s$ or $\frac{\kappa h_0}{(n'-\kappa)e_0}$, which is greater than $1/s$. Otherwise, $c^*(s,n) = c^*(s',n) = \frac{\kappa h_0}{(n-\kappa)e_0}$ and $c^*(s,n) = \frac{\kappa h_0}{(n-\kappa)e_0} < c^*(s,n') = \frac{\kappa h_0}{(n'-\kappa)e_0}$ for $n > n'$.

Now we show that $c_*(s,n)$ is decreasing in s and n. Recall that

$$c_* \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds},$$

which is trivially decreasing in n. The first-order derivative of c_* with respect to s is

$$\begin{aligned}\frac{\partial c_*}{\partial s} &= \frac{h_0 ds - d\left(d - ((n-\kappa)/\kappa)\, e_0 + h_0 s\right)}{(ds)^2} \\ &= \frac{d\left(\frac{n-\kappa}{\kappa}e_0 - d\right)}{(ds)^2} < 0,\end{aligned}$$

where the last inequality comes from $d > (n-1)e_0 > \frac{n-\kappa}{\kappa}e_0$. Therefore, the overall vulnerability region becomes smaller as s and n increases. ■

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