Stable and metastable contract networks

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Abstract

We consider a hypergraph (I, C), with possible multiple (hyper)edges and loops, in which the vertices $i \in I$ are interpreted as *agents*, and the edges $c \in C$ as *contracts* that can be concluded between agents. The preferences of each agent i concerning the contracts where i takes part are given by use of a *choice function* f_i possessing the so-called *path independent* property. In this general setup we introduce the notion of stable network of contracts.

The paper contains two main results. The first one is that a general problem on stable systems of contracts for (I, C, f) is reduced to a set of special ones in which preferences of agents are described by use of so-called *weak orders*, or utility functions. However, for a special case of this sort, the stability may not exist. Trying to overcome this trouble when dealing with such special cases, we introduce a weaker notion of *metastability* for systems of contracts. Our second result is that a metastable system always exists.

Keywords: Plott choice functions, Aizerman-Malishevski theorem, stable marriage, roommate problem, Scarf lemma

JEL classification: C71, C78, D74

1 Introduction

In their lives, people often have to make joint actions and organize groups in order to achieve some goals. We call such cooperations *contracts*. Examples are: house exchange, purchase or sale, marriage, loan or deposit of money, hiring, co-financing society, cartel, military or economic union of countries. Contracts can involve not only individuals, but also larger entities; for convenience, we call the parties of a contract *agents* or *participants*. Some contracts include only two agents (we call such contracts *binary*), but many other ones can include a larger number of agents. Note also that agents are allowed to enter several different contracts at once.

Contracts bring some benefits to the participants, but also require them to expend money, time or other costs. It is important for participants to know more precisely what

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they can count on. For this purpose, the agreements should be as detailed and formalized as possible, although not everything can be taken into account. For example, a marriage contract may include how much time the spouses can spend in the family and how much 'outside', how to share household efforts, how many children to have, etc. The more all this is worked out in detail, the better the participants represent the pros and cons and can compare different contracts. We further assume that each contract can be unambiguously evaluated by each of its participants.

The information about the 'preference' of contracts primarily affects the choice of contracts that will actually be concluded (signed). Here we are based on the premise of voluntariness of contracts. No one can force an agent to sign a contract if the agent does not like to do this. On the other hand, no one can forbid a group to sign a contract if its all members agree. These two requirements lead to the concept of a *stable* system (or network) of contracts, which will be the main topic of our work. The concept of stability originally appeared in the work of Gale and Shapley [9] and gradually has become the subject of extensive researches both theoretical and practical. Gale and Shapley showed that in case of marriages, a stable system always exists. They assumed the marriages to be monogamous and heterosexual (which is shared not always nowadays). Without these conditions, stable systems may not exist. For example, so is in the famous problem of stable 'roommates', or 'division into pairs'. Even to a greater extent, this concerns non-binary contracts.

The main task that we are going to deal with in this paper is to find out under which conditions stable systems do exist.

One remark is needed to us here. When agents are allowed to conclude many contracts, they should be able to compare not only individual contracts, but also arbitrary subsets of contracts. Therefore, it is not enough to attribute a value to each contract only. Instead, we prefer to use the so-called *choice functions* (CFs, for brevity), which tell us which contracts from the available list are 'the best' ones to be chosen for signing. This approach was initiated in [13], where the authors revealed importance of the condition of "substitutability" for the existence of stable allocations. Subsequently, a number of researches have shown that this condition is applicable to all problems with 'bilateral' contracts (see [3, 7, 10, 15]). We show that in a more general setting, this condition on agents' CFs is also adequate.

The paper contains two main results. The first one is that a general problem on stable systems of contracts is reduced to a more special situation in which preferences of agents are described by use of *weak orders*, or utility functions. Roughly speaking, agents conclude contracts having the maximum utility, abandoning the rest. But even in such situations the stability need can not exist. To overcome this trouble, we propose (when dealing with a weakly ordered setup) a weaker notion of *metastable* systems of contracts. Our second result is that such a metastable system always exists.

2 Basic definitions and settings

A general setup can be stated as follows. There are a finite set of *agents* I and a finite set C of *contracts* which the agents can, in principle, conclude. Each agent can conclude several contracts. Each contract $c \in C$ is shared by a nonempty set of participants $P(c) \subseteq I$. If $P(c)$ is a singleton $\{i\}$, the contract c is called *autarkic*; this can be thought of not as a contract in reality, but rather as an 'activity' available to i alone. Thus, the object that we deal with can be interpreted as a hypergraph, with possible parallel hyperedges. So, when needed, we may use the language of (hyper)graphs, understanding agents as vertices and contracts as (hyper)edges. Equivalently, the situation is described by a bipartite graph with the parties I and C.

For a set $S \subseteq C$, let $S(i)$ denote the set of contracts $s \in S$ such that i is a participant of s.

As mentioned above, the contracts have some kind of 'utility', which brings benefits to their participants. Guided by these utilities, agents conclude some contracts and refuse others. So an evaluation of contracts via utilities is the most important part of the problem. The simplest way to define a utility is to express it numerically, by assigning a (real or integer) number $u_i(c)$ to each contract c of $C(i)$. However, this is not the most general way to set 'preferences' of agents. Since an agent can conclude several contracts, it is often important for it to know not only the utilities of individual contracts, but also the ones of their collections.

Sufficiently large flexibility and generality are obtained by use of describing preferences via choice functions. A *choice function* f on an (abstract) set of 'alternatives' X indicates a subset $f(A) \subseteq A$ for any set ('menu') $A \subseteq X$. In our case, the choice of agent i is taken within the set of $C(i)$ contracts.[1]

It light of this, the second important ingredient of the problem is to assign, for each agent $i \in I$, an appropriate choice function f_i on the set $C(i)$. We call such a preference assignment by an *equipment* of the hypergraph $G = (I, C)$. Using this, we now can talk about the stability of a contract system (or network) $S \subseteq C$. Roughly speaking, this is a system S such that nobody wants to change it, either by renouncing some contract, or by concluding a new contract, perhaps by breaking some existing ones in S. At the same time, it is assumed that contracts are concluded voluntarily. This means that any agent can refuse to conclude any contract, and that any contract can be concluded only with the unanimous consent of all its participants. The formal definition is as follows.

Definition. A network S is called *stable* if the following two conditions hold:

S0. $f_i(S(i)) = S(i)$ for any $i \in I$;

S*. If a contract b does not belong to S, then $b \notin f_i(S(i) \cup b)$ for some $i \in P(b)$.

[1]Here we default assume that an agent does not care of what contracts are concluded without its participation. For example, in the situation of hiring workers by firms, one assumes that it is important to the worker in which firms he will work, but it does not matter to him who else works in these firms. On the other hand, it is important to a firm who will work in it, but it does not matter where else the employee works. In some situations, such an assumption looks not realistic, but it can be accepted as a first approximation.

(Hereinafter, for a set S and a singleton s, we may write $S \cup s$ for $S \cup \{s\}$.)

The first condition expresses the possibility of renouncing any contract. The second one says that if a contract b is interesting to its all participants, then it should be concluded. And its absence in S indicates incompleteness of the process of building a system of contracts. Sometimes one says that such a contract b *blocks* the system S.

The main issue that we will be dealing with concerns the existence of stable networks. The answer depends on both the structure ('geometry') of the original network C and (to a greater extent) the 'preferences' of agents. For example, if (I, C) is a bipartite graph, then a stable network $S \subseteq C$ exists under rather weak conditions on preferences. And if the agents behave indifferently, then the original network C is stable. On the other hand, even in case of binary contracts with the best individual preferences, the stability may not take place.

We will focus on preferences without imposing a priori restrictions on the original network C. As is mentioned earlier, preferences are given by choice functions. But there are a lot of CFs, and most of them do not correspond to an intuitive concept of the 'best choice'. To somehow clarify this situation, let us demonstrate a couple of examples of CFs that are regarded as 'rational'.

Example 1. Let $\leq$ be a preorder on a set X (that is, a reflexive and transitive binary relation). And let $f(A)$ consist of all maximal elements in $A \subseteq X$ relative to $\leq$ (one often writes $f = \max_{\leq}$). A CF of this kind is considered as rational, since a rational reason for including one or another alternative in the choice is clearly seen, namely, the lack of a better alternative. Note that this choice is nonempty (when a menu A is such).

Two special cases of this construction deserve to be mentioned. The first one is when $\leq$ is a *weak order*, which means a complete preorder. The second one is when $\leq$ is a *linear* (full) order, or ranking. In the latter case, the choice $f(A)$ consists of a single element (if $A \neq \emptyset$), and we are talking about *linear preferences*, or linear equipment.

Example 2. Let $\leq$ be a linear order, but the choice includes b best items from menu. The number b could be understood as a quota. Such a choice rule is viewed as rational as well, and it has been considered in many works on stable b-matchings. See e.g. [5, 7, 8, 12] where generalizations (of type 'many-to-many') of stable marriages and roommates are studied.

Both examples are special cases of the so-called *path-independent* choice functions. Such functions are given by the following functional equation:

$$f(A \cup B) = f(f(A) \cup B).$$

This equation says that the answer does not change if to replace a part of the menu by its best elements. In other words, the choice can be performed step by step, in several stages, and the answer does not depend on the order ('path'). This condition was introduced by Plott [14], and we call such CFs *Plottian* or *Plott functions*. It turned out that this condition is quite suitable for studying stability, and later on we will assume everywhere that the CFs in question are just Plottian. Such CFs have been investigated extensively in the literature (especially Aizerman and Malishevski's paper [2] should be distinguished); the facts about Plottian CFs needed for us are given in Appendix C.

When the equipment is given by Plott functions, any stable network is Pareto optimal. However, one can address the question: how to understand the optimality if preferences are given by CFs? We can do this in the following way. Let f be a Plott function on a set X. One can associate with f the following hyper-relation (a relation on 2^X) $\preceq = \preceq_f$ (introduced by Blair [4]), which is given by the expression:

$$A \preceq B \quad \text{if } f(A \cup B) \subseteq B.$$

This hyper-relation is transitive and reflexive. If CF f is given by the weak order $\leq$, then $A \preceq B$ implies $max(A) \leq \max(B)$.

Proposition 2.1 *Suppose that all CFs f_i are Plottian and that $\preceq_i$ are the corresponding hyper-relations. Let S be a stable system, and T an arbitrary system in C. If for any i, the inequality $S(i) \preceq_i T(i)$ takes place, then $S(i) = f_i(T(i))$ holds for any i.*

In other words, if the system T is not worse than S for all agents, then S and T are in fact equivalent; so T is not better than S.

Proof The condition $S(i) \preceq_i T(i)$ means that

$$f_i(S(i) \cup T(i)) \subseteq T(i).$$

Note that $f_i(S(i) \cup T(i)) = f_i(S(i) \cup T(i) \cup T(i)) = f_i(f_i(S(i) \cup T(i)) \cup T(i)) = f_i(T(i))$ (since $f_i(S(i) \cup T(i)) \subseteq T(i)$). Replacing $T(i)$ by $f_i(T(i))$, we may assume that $f_i(S(i) \cup T(i)) = T(i)$.

If the opposite inequality $T(i) \preceq_i S(i)$ holds for all i, then $f_i(S(i) \cup T(i)) = f_i(S(i)) = S(i)$, and therefore $S(i) = f_i(T(i))$. So we may assume that for some agent (denote it as 0) the set $T(0) = f_0(S(0) \cup T(0))$ is not contained in $S(0)$. Then there is a contract t belonging to $T(0)$ but not to $S(0)$. Now let j be an arbitrary participant of the contract t; so $t \in T(j) = f_j(S(j) \cup T(j))$. Since t is taken (by the CF f_j) from a larger set $S(j) \cup T(j)$, t is also taken in the smaller set $S(j) \cup t$. So that $t \in f_j(S(j) \cup t)$. And this is true for any element j of $P(t)$. Since t does not belong to S, this all contradicts the stability condition **M***. □

The first result of our work is that the problem with a general Plott functions can be reduced to a problem in which the preferences of agents are given by weak orders. This allows us to focus on such 'weakly ordinal' situations.

3 The reduction theorem

Theorem 3.1 *Suppose that an equipment of a hypergraph (I, C) is given by Plott CFs. Then there exist a hypergraph (I', C') equipped with weak orders and a mapping $\pi : (I', C') \to (I, C)$ such that*

a) for any stable system S' in C', its image $\pi(S')$ is stable in C;

b) for any stable system S in C, there is a stable system S' in C' such that $\pi(S') = S$.

This assertion is based on a theorem in [2] saying that any Plott function is representable as the union of several linear CFs. We construct the desired hypergraph (I', C') by splitting each agent $i \in I$ into a set of its 'subagents' $\widetilde{i}$, which already have weak orders as preferences on the contracts available to them. To make the construction more transparent, we describe in detail the 'splitting' of only one agent, which is denoted as 0.

Let us assume that the CF f_0 of this agent is represented as the union of several 'simpler' Plott functions $f_1, \dots, f_\ell$ (for example, linearly or weakly ordered ones).[2] We even may assume that $\ell = 2$, and accordingly 'split' the agent 0 into two new agents 0_1 and 0_2, which are denoted simply as 1 and 2. Each of them has the same set $C(0)$ of contracts, but their preferences differ and are given by CFs f_1 and f_2, respectively. More formally, the new set of contracts $\widetilde{C}$ is arranged as follows:

$$\widetilde{C} := (C - C(0)) \sqcup \widetilde{C}(1) \sqcup \widetilde{C}(2),$$

where $\widetilde{C}(1) = C(0) \times \{1\}$, $\widetilde{C}(2) = C(0) \times \{2\}$ are two copies of $C(0)$. When it is not confusing, we will identify each of $\widetilde{C}(1)$ and $\widetilde{C}(2)$ with $C(0)$. In other words, each contract c involving agent 0 is duplicated, turning into two contracts, c_1 and c_2, concluded by the same agents except for 0 which is replaced by 1 and 2, respectively. The mapping π sends agents 1 and 2 to 0, as well as c_1 and c_2 to c.

We have already described the preferences of agents 1 and 2; namely, they are given by CFs f_1 and f_2. For the other agents (which will be usually denoted as j, j', etc.), the contracts c_1 and c_2 are equivalent (they perceive them as contracts with agent 0). More formally, agent j (considered as an element of the set $\widetilde{I} = (I - \{0\}) \cup \{1, 2\}$) chooses $\widetilde{c}$ from a menu $A \subseteq \widetilde{C}(j)$ if and only if $c = \pi(\widetilde{c})$ is chosen from $\pi(A)$:

$$\widetilde{f_j}(A) = A \cap \pi^*(f_j(\pi A)).$$

In Appendix C, we show that $\widetilde{f_j}$ is a Plott CF as well.

Note that even if the old CF f_j was linear, the new CF $\widetilde{f_j}$ is given in general by a weak order, because the twins c_1 and c_2 for the agent j are equivalent (indifferent). This is the reason why we are able to reduce the problem not to the linear case, but merely to the weakly ordered one.

So, we have described the new system $\widetilde{C}$ of contracts, and now we can formulate the first assertion; it will be proved in Appendix A.

Proposition 3.2 *For C and $\widetilde{C}$ as above, if $\widetilde{S}$ is a stable system in $\widetilde{C}$, then the system $S = \pi(\widetilde{S})$ is stable in C.*

Let $\mathbf{St}(G)$ be the set of stable networks for the equipped hypergraph G, and $\mathbf{St}(\widetilde{G})$ a similar set for $\widetilde{G}$. Then Proposition 3.2 determines the mapping

$$\pi : \mathbf{St}(\widetilde{G}) \to \mathbf{St}(G)$$

[2]Recall that the union $f_1 \cup \dots \cup f_\ell$ is given by the formula $(f_1 \cup \dots \cup f_\ell)(A) = f_1(A) \cup \dots \cup f_\ell(A)$.

that translates any stable system $\widetilde{S}$ into $\pi(\widetilde{S})$. We claim that this mapping is surjective, that is, for any stable system S in G, there exists a stable system $\widetilde{S}$ in $\widetilde{G}$ such that $\pi(\widetilde{S}) = S$. Moreover, we will build such a system $\widetilde{S}$ canonically.

Construction of $\widetilde{S}$. If a contract s belongs to S and does not contain agent 0, then s is lifted in $\widetilde{C}$ in a natural way and included in $\widetilde{S}$. Therefore, we only have to explain how to form $\widetilde{S}(1)$ and $\widetilde{S}(2)$. We put $\widetilde{S}(1) := f_1(S(0))$ and $\widetilde{S}(2) := f_2(S(0))$ (identifying $\widetilde{C}(1)$ and $\widetilde{C}(2)$ with $C(0)$).

Proposition 3.3 *The system* $\widetilde{S}$ *constructed as above is stable.*

The proof is given in Appendix A.

Now let us return to Theorem 3.1. A required mapping $\pi : G' \to G$ is constructed by iterating the above splitting construction. The induction is proceeded by the number of vertices j in (the current hypergraph) G for which CF f_j is not weakly ordered. If there are no such vertices, we are done. So let 0 be a vertex of G for which CF f_0 is not weakly ordered. Discarding unnecessary edges, we may assume that f_0 is not empty-valued. By Aizerman–Malishevski's theorem (see Appendix C), f_0 is representable as $f_0 = f_1 \cup \cdots \cup f_\ell$ with all $f_1, \dots, f_\ell$ implemented by weak orders (and even by linear ones). Let $\widetilde{G}$ be the hypergraph constructed as above, but with splitting 0 not into two vertices, but into ℓ ones. The proofs given above can be extended in an obvious way to this case as well. The CFs in the new vertices $1, \dots, \ell$ are already weakly ordered. But what about the other vertices j? When f_j was weakly ordered, the new $\widetilde{f}_j$ is again weakly ordered. Indeed (see Appendix C), if f_j was generated by a weak order $\leq_j$ on $C(j)$, then $\widetilde{f}_j$ is generated by the weak order $\pi^*(\leq_j)$ on $\widetilde{C}(j)$, where π is the projection of $\widetilde{C}(j)$ on $C(j)$. $\square$

Thus, we obtain a reduction of a general case (with Plottian CFs) to the special case where the preferences of agents are given by weak orders. And now it is reasonable to analyze this special case.

4 Metastable networks of contracts

From now on, we assume that the choice functions f_i are always given by weak orders $\leq_i$. We reformulate the stability conditions in terms of these orders. The first condition **S0** means that all elements of the set $S(i)$ are equivalent to each other. The second condition **S*** says that if a contract c is not worse than those in $S(i)$ for all participants i of this contract, then $c \in S$.

In many stability researches one usually assumes that the preferences are given by linear orders, and many known results are obtained in the framework of this assumption. It may seem that the situation when preferences admit so-called 'ties' would not complicate the problem too much. But this is a rather controversial issue.

To begin with, note that the previous construction (with vertex splitting) does not allow us to reduce weak orders to linear ones. Consider the graph G as in Fig. 1.

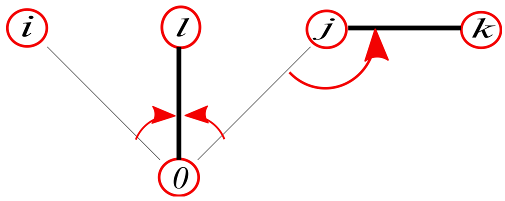


Figure 1: The orders on edges are represented by the arrows. The neighbors i and j of agent 0 are equivalent and both are worse than ℓ. The edges of a stable network are drawn in bold.

In this graph G, split vertex 0 into vertices 1 and 2; agent 1 considers i slightly better than j, agent 2 conversely. The edge $d = 0\ell$ is split into d_1 and d_2, and agent ℓ must distinguish between these edges. Let us assume that d_1 is slightly better for him than d_2.

Below Figure 2 shows a stable network $\widetilde{S}$ of the graph $\widetilde{G}$ for which $\pi(\widetilde{S})$ is unstable. And Figure 1 shows a stable network S which is not lifted to a stable one in $\widetilde{G}$.

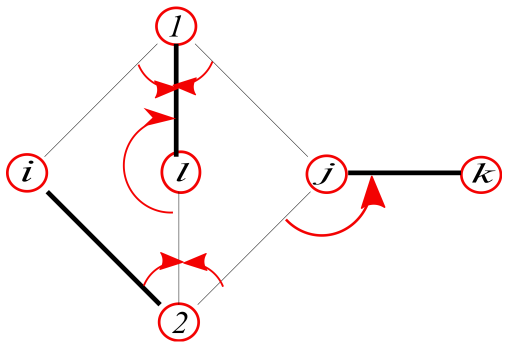


Figure 2: Lifted graph $\widetilde{G}$.

Another technique deals with splitting ties. We slightly perturb preferences without violating strict inequalities and breeding equivalences. To understand what one can expect here, look at simple examples illustrated in Fig. 3.

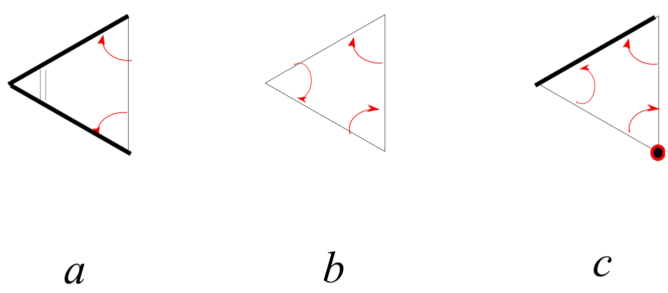


Figure 3: Examples with a triangular graph. Fragment a shows a tie. In fragment b, there is no stable network. In fragment c, a stable network consists of one edge and one lone agent.

We can see that the original problem and the perturbed one behave differently. Stable networks in a perturbed situation may have no relation to stable networks in the original one.

Stable networks themselves may or may not exist. Classics in the field dealt with such situations in the case of graphs (viz. binary contracts) with linear orders of agents. Irving [11] developed an efficient algorithm (for arbitrary graphs) which either finds a stable matching or declares that none exists. Tan [16] introduced the notion of stable partition, which always exists and, moreover, enables us to conclude whether a stable matching does exist or not. A significant clarification of the latter concept was given in [1] by using the so-called Scarf lemma. Below we propose a close concept of a metastable network, which also always exists.

In order to get an analogue of stability which 'always exists', we need to weaken (preferably not too much) the previous requirements for stability. There are two of those, namely, **S0** and **S***. They act in opposite directions, in a sense. If a system S satisfied **S0**, then any of its subsystems satisfies **S0** as well (this is a property of Plottian CFs). Conversely, if S satisfies **S***, then any of its super-systems satisfies **S*** as well. So one may say that "the stability is a thin balance between two opposing forces". One of them tends to reject 'bad' contracts. The other one strives to concluding 'profitable' contracts. Now, in our new setup, the agents deal with weak orders and evaluate groups of contracts on the basis of these orders.

The concept of metastability proposed below preserves condition **S*** but slightly weakens condition **S0**, restricting rejections of unprofitable contracts. The refusal of one participant i from some contract c may cause that another participant j will have to turn to a new contract c' (which he previously ignored, as having a more favorable contract c). Other participants of c' can support the desire of j and conclude c', perhaps rejecting other contracts. In other words, one can start an unpredictable series (a cascade) of renegotiations of contracts, and it is difficult to say how this process could finish (and whether this finishes at all). The metastability permits to break a contract only if this action does not threaten to cause a collapse.[3]

Next we come to the exact definition of *metastability*. Here it is more convenient to use utility functions $u_i : C(i) \to \mathbb{R}$ rather than weak orders. This does not affect the meaning of the concept in essence, since we do not compare the utilities of different agents, nor add up the utilities of one of them.

Let $S \subseteq C$ be some system of contracts. For each $i \in I$, we define

$$u_i(S) = \min(u_i(s),\ s \in S(i)).$$

To make this well-defined, we impose the first condition of the metastability:

MS0. $S(i) \neq \emptyset$ for any $i \in I$.

One can understand the number $u_i(S)$ as the guaranteed utility that the system S gives to agent i. Agent i takes part in several contracts (namely, in $S(i)$), and many of them may give him more utility than $u_i(S)$. The contracts in $S(i)$ that give exactly $u_i(S)$ are called *marginal* for agent i. There may be several of these if the order $\leq_i$ is not linear.

[3] One may happen that the conclusion of a new agreement could cause a collapse, too. We think that the proposed method works in a suitable way as a rule, yet cannot be well-motivated in all situations.

The second condition of the metastability is the following:

MS*. For any contract $c \in C$, there is a participant i of c such that $u_i(c) \le u_i(S)$.

In other words, no contract c can give to each of its participants strictly more than the system S guarantees. That is, if there would be such a contract c, this should be a serious reason for the participants of c to conclude it, abandoning their marginal contracts, which give them strictly less.

Definition. A system S is called *metastable* if both properties **MS0** and **MS*** hold.

From condition **MS*** it is immediately clear that *any contract in a metastable network S is marginal for some of its participants.*

An interest to the concept of metastability is explained by the facts that, first, there is a close connection with the notion of stability, and second, metastable systems always exist. Both facts require, however, that the original contract system C satisfies a rather weak assumption of 'autarky'. Namely, that *each participant $i \in I$ has an autarkic contract.* In what follows, we assume that this requirement is imposed.

Proposition 4.1 *Any stable contract system S is metastable.*

Proof Let us check **MS0**. Suppose, for a contradiction, that $S(i)$ is empty for some agent i. Let c be the autarkic contract of i. Since c is not worse than the set $S(i) \cup c = \{c\}$ for the unique participant of c, the second stability condition implies $c \in S(i)$, contrary to the emptiness of $S(i)$.

Now we check **MS*** for a stable network S. To do this, note that for any agent i, any contract from $S(i)$ (where all members are equivalent for i) is marginal and has the utility $u_i(S)$. Now let c be an arbitrary contract; one has to show that some $i \in P(c)$ satisfies $u_i(c) \le u_i(S)$. This is true if $c \in S$ (since then c is marginal and its utility is equal to $u_i(S)$). And if $c \notin S$, then by the stability condition **S*** there is $s \in S(i)$ (for some $i \in P(c)$) such that $u_i(c) < u_i(s) = u_i(S)$. $\square$

Also it is obvious that if all contracts in a metastable network are marginal for each of its participants, then this network is stable.

An importance of the metastability is expressed in the following theorem whose proof will be given in Appendix B.

Theorem 4.2 *Under the condition of autarky, for any contract system C with weakly ordered preferences of all agents, there always exists a metastable network.*

5 Minimal metastable networks

The concept of metastable networks is still loose, not rigid enough. Let us imagine that s is an autarkic contract in a metastable system S. Then it is obviously marginal for his unique participant i. Assume that $S(i)$ contains a contract c different from s. Then removing s from S, we obtain a system which is again metastable. Moreover, it is even preferable for i, since the guaranteed gain of i can only increase (and the guarantees of the others agents do not change).

This observation is applicable not only to autarkic contracts. Let S be a metastable system, and $T \subseteq S$ some subsystem in it. Then T satisfies condition **MS***. In its turn, condition **MS0** may be violated for T. But if it is still valid, then the system T is also metastable.[4] This justifies the introduction of the following notion.

Definition. *A minimal metastable system* is a metastable system minimal by inclusion.

There is a simple criterion of the minimality. Let S be some system of contracts. Let us say that agent i is *monogamous*[5] if $S(i)$ consists of a single contract. This situation is typical in classical marriage or roommate problems.

Proposition 5.1 *A metastable system S is minimal if and only if any contract s of S contains a monogamous participant.*

Indeed, if s is a contract without monogamous participants, then the system $S - \{s\}$ is also metastable. $\square$

Similar reasonings show that any metastable contract is a union of minimal metastable ones. Therefore, in principle, we can restrict ourselves by studying minimal metastable networks. Especially since such systems provide the largest guaranteed utility.

Linearization. Let $\leq_i$ be weak orders of participants i on sets $C(i)$, and let $\preceq_i$ be linear orders extending $\leq_i$ on the same sets. (Then $c <_i c'$ implies $c \prec_i c'$, that is, strict preferences preserve while equivalences are eliminated. In terms of utility functions, the original functions are slightly perturbed.). So, without changing the hypergraph (I, C), we will strengthen the initial preferences of agents to get linear orders $\preceq$.

Proposition 5.2 *1) If S is a metastable network with respect to linear orders $\preceq$, then S is metastable for the original weak orders $\leq$.*

2) Conversely, if a network S is minimal metastable for $\leq$, then there is a linear extension $\preceq$ of $\leq$ such that S is metastable with respect to $\preceq$.

Proof Assertion 1) is quite simple. We only need to check that condition **MS*** for $\leq$ is valid. Suppose it is violated, that is, there is a contract c such that $u_i(c) > u_i(S)$ for any participant i of c. But then $\tilde{u}_i(c) > \tilde{u}_i(S)$ for all $i \in P(c)$, which contradicts the metastability of S with respect to $\preceq$ (where $\tilde{u}$ stands for the utilities for $\preceq$).

To see 2), let a system S be metastable for weak orders $\leq_i$. The 'splitting' of ties of non-marginal contracts is not important, so we can focus on marginal ties. Fix a participant i and denote by $M(i)$ the set of contracts $c \in C(i)$ such that $u_i(c) = u_i(S)$. Such contracts are divided into two groups. The group belonging to S is denoted by $M_+(i)$, and the rest by $M_-(i)$. The first group is certainly nonempty. Choose some

[4]Note that stable networks do not allow this possibility: if S is stable, then its any proper subsystem is unstable.

[5]The term monogamous is appropriate if '$\gamma\alpha\mu o\zeta$' (marriage) is understood as a contract.

contract s_i in it and remain its utility unchanged; the utilities $\tilde{u}_i$ of the other members of $M_+(i)$ are slightly increased. The utilities of contracts in $M_-(i)$ are slightly decreased.

Doing such operations for all agents, we eventually get a system with linear orders $\preceq_i$ extending the original weak orders $\leq_i$. We assert that this system is metastable.

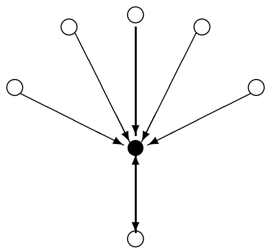
Indeed, let c be an arbitrary contract, and suppose that for its all participants i, there holds $\tilde{u}_i(c) > \tilde{u}_i(S)$. Then $u_i(c) \geq u_i(S)$. Such inequalities cannot be strict for all i, in view of the metastability of S. So for some i, the equality $u_i(c) = u_i(S)$ is fulfilled; then $c \in M(i)$. If $c \in M_-(i)$, its perturbed utility $\tilde{u}_i$ is slightly less than the utility of marginal contract s_i, contrary to the supposition $\tilde{u}_i(c) > \tilde{u}_i(S)$. Hence $c \in S$. Now since $\tilde{u}_i(c) > \tilde{u}_i(S)$, c is not a unique contract in $S(i)$ for all $i \in P(c)$. This contradicts the minimality of S. $\square$

We obtain that when dealing with minimal metastable networks, one may assume, w.l.o.g., that the preferences of all participants are given by linear orders. In other words, all contracts for any agent i are comparable and non-equivalent. In this situation, we have a closer relationship between the stability and minimal metastability, as follows.

Proposition 5.3 *Suppose that the preferences of all agents are given by linear orders. If S is a stable network of contracts, then S is a minimal metastable one.*

Proof Metastability has already been established. It remains to check that any contract $s \in S$ is unique for some of its participants i. (Equivalently, any contract $s \in S$ has a monogamous participant.) This follows from the fact that (in the case of linear preferences) each set $S(i)$ consists of a single contract. $\square$

Note that when the preferences are linear and the original network C is binary (viz. (I, C) is a graph), the minimal metastable network splits into connected components that are either isolated vertices or have the form of 'dandelions':



Here there is a central agent (the dark vertex), and several monogamous agents associated with it. The marginal contract of the central agent is represented by a two-sided arrow. In a particular case, this structure degenerates into a single binary contract.

Appendixes

A Proofs of Propositions 3.2 and 3.3

To prove these propositions, we use the following lemma, where $S(1) = \pi(\widetilde{S}(1))$, $S(2) = \pi(\widetilde{S}(2))$, and $S(j) = \pi(\widetilde{S}(j))$ for $j \neq 0, 1, 2$.

Lemma A.1 (a) $f_j(S(j)) = S(j)$ *for each agent* j *different from 0.*

(b) $f_1(S(1)) = f_1(S(0))$*, and similarly* $f_2(S(2)) = f_2(S(0))$.

Proof (a) Let $s \in S(j)$. If all participants of a contract $s \in S$ are different from 0, then $s = \pi(\widetilde{s})$ for a single (actually, equal to s) contract $\widetilde{s}$ from $\widetilde{S}$. By condition **S0**, $\widetilde{s} \in \widetilde{f}_j(\widetilde{S}(j))$, that is, $s \in f_j(S(j))$. Therefore, we may assume that 0 is one of the participants of s. The contract s is a projection of some $\widetilde{s}$ from $\widetilde{S}$, and 1 or 2 is a participant of $\widetilde{s}$. Let for definiteness $\widetilde{s} = s_1 \in \widetilde{S}(1)$; then $s_1 \in \widetilde{S}(j)$. By condition **S0**, for $\widetilde{S}$ we have the equality $\widetilde{f}_j(\widetilde{S}(j)) = \widetilde{S}(j)$. Thus, $s_1 \in \widetilde{f}_j(\widetilde{S}(j))$. By definition of $\widetilde{f}_j$, this means that $s = \pi(s_1)$ belongs to $f_j(\pi(S(j))) = f_j(S(j))$.

(b) Recall that $S(0) = S(1) \cup S(2)$. It is enough to show that $f_1(S(0)) \subseteq S(1)$, since then the Outcast property (defined in Appendix C) gives the desired equality.

Suppose, for a contradiction, that some contract s of $f_1(S(0))$ does not belong to $S(1)$. Since s is selected from the larger set $S(0)$, it is also selected from the smaller set $S(1) \cup s$, yielding $s \in f_1(S(1) \cup s)$. Then $s_1 \in \widetilde{f}_1(\widetilde{S}(1) \cup s_1)$. This shows that the contract s_1 is not autarkic. Because in this case $s_1 \in \widetilde{S}(1)$ and $s \in S(1)$ contrary to the supposition.

Now let j be another participant of the contract s (or s_1). We assert that $s_1 \in \widetilde{f}_j(\widetilde{S}(j) \cup s_1)$. To show this (see the definition of $\widetilde{f}_j$), we have to make sure that $\pi(s_1) = s$ belongs to $f_j(\pi(\widetilde{S}(j) \cup s_1)) = f_j(S(j) \cup s)$. Note that $s_2 \in \widetilde{S}(2)$; then $s_2 \in \widetilde{S}(j)$ and $s \in S(j)$. So $S(j) \cup s = S(j)$, and $f_j(S(j) \cup s)$ is equal to $S(j)$ and contains s.

Thus, s_1 belongs to both $\widetilde{f}_1(\widetilde{S}(1) \cup s_1)$ and $\widetilde{f}_j(\widetilde{S}(j) \cup s_1)$ for any participant $j \neq 1$ of s_1. By condition **S***, we obtain $s_1 \in \widetilde{S}(1)$ and $s \in S(1)$, yielding a contradiction. □

Proof of Proposition 3.2. One has to verify properties **S0** and **S*** for the system S.

We first check **S0** for agent 0, that is, $f_0(S(0)) = S(0)$. Let $s \in S(0)$; one may assume that $s \in S(1)$. Since $S(1) = f_1(S(1))$ (according to **S0** for $\widetilde{S}$ at the vertex 1), s belongs to $f_1(S(1))$, which is equal to $f_1(S(0))$ (by Lemma A.1(b)), and therefore s belongs to $f_0(S(0))$. For other agents j, the needed equality is established in Lemma A.1(a).

Next we check **S***. Suppose that there is a blocking contract b for S. That is, $b \notin S$, but $b \in f_i(S(i) \cup b)$ for any $i \in P(b)$. If $0 \notin P(b)$, then b also blocks $\widetilde{S}$. Therefore, we may assume that $0 \in P(b)$.

In this case, $b \in f_0(S(0) \cup b)$. This means that b lies either in $f_1(S(0) \cup b)$ or in $f_2(S(0) \cup b)$. Let $b \in f_1(S(0) \cup b)$. Due to the Heredity property of CF f_1, we have $b \in f_1(S(1) \cup b)$. Then $b_1 \in \widetilde{f}_1(\widetilde{S}(1) \cup b)$, where $b_1 = (b, 1)$.

Now let us examine the inclusion $b \in f_j(S(j) \cup b)$, where $j \in P(b) - \{0\}$. By the definition of $\widetilde{f}_j$, we have $b_1 \in \widetilde{f}_j(\widetilde{S}(j) \cup b_1)$ since $\pi(\widetilde{S}(j) \cup b_1) = S(j) \cup b$.

Finally, b_1 does not belong to $\widetilde{S}$ since $b \notin S$. Then the contract b_1 is blocking for $\widetilde{S}$, contrary to the stability of $\widetilde{S}$. □□

Proof of Proposition 3.3. Recall how the 'covering' $\widetilde{S}$ is arranged. If s belongs to S and does not contain 0 as a participant, then s is lifted in $\widetilde{C}$ in a natural way and is included in $\widetilde{S}$. As for $\widetilde{S}(1)$ and $\widetilde{S}(2)$, they are defined as $\widetilde{S}(1) := f_1(S(0))$ and $\widetilde{S}(2) := f_2(S(0))$ (where we identify $\widetilde{C}(1)$ and $\widetilde{C}(2)$ with $C(0)$). Since $f_1(S(0)) \cup f_2(S(0)) = f_0(S(0)) = S(0)$ (by condition **S0**), we have

$$\pi(\widetilde{S}(1) \cup \widetilde{S}(2)) = S(0).$$

Similar equalities hold for vertices j different from 0. We need two additional lemmas.

Lemma A.2 *Let a vertex j be different from 0. Then $\pi(\widetilde{S}(j)) = S(j)$.*

Proof Let $s \in S(j)$. We have to show that s appears from $\widetilde{S}(j)$. This is immediate from the construction if agent 0 does not participate in the contract s. So assume that $0 \in P(s)$. Then $s \in S(0)$, and by condition **S0**, the contract s is selected from $S(0)$ either by CF f_1 or by CF f_2. Assume that $s \in f_1(S(0))$; then $s_1 \in \widetilde{S}(1)$. Therefore, $s = \pi(s_1)$ belongs to $\pi(\widetilde{S})$, whence $s \in \pi(\widetilde{S}(j))$.

Conversely, let $s = \pi(\widetilde{s})$ for $\widetilde{s} \in \widetilde{S}(j)$, and let for definiteness $1 \in P(\widetilde{s})$. Since $\widetilde{s} \in \widetilde{S}(1)$, we have $s \in f_1(S(0)) \subseteq S(0)$, which means that s belongs to $S(j)$. $\square$

Corollary A.3 $\pi(\widetilde{S}) = S$.

Lemma A.4 *Each $c \in C(0)$ satisfies $\widetilde{f}_1(\widetilde{S}(1) \cup c_1) = f_1(S(0) \cup c)$.*

Proof $\widetilde{S}(1) = f_1(S(0))$. Therefore $\widetilde{f}_1(\widetilde{S}(1) \cup c_1) = f_1(f_1(S(0) \cup c) = f_1(S(0) \cup c)$. $\square$

Now we are ready to finish the proof of Proposition 3.3. One has to verify properties **S0** and **S*** for $\widetilde{S}$.

Verification of **S0.** We have to show that $\widetilde{f}_i(\widetilde{S}(i)) = \widetilde{S}(i)$ for any vertex i of the hypergraph $\widetilde{G}$.

For vertex 1, we have $\widetilde{S}(1) = f_1(S(0)) = f_1(f_1(S(0))) = f_1(S(1)) = \widetilde{f}_1(\widetilde{S}(1))$. Similarly for vertex 2.

Now let j be different from 1 and 2. Let $\widetilde{s} \in \widetilde{S}(j)$; we have to show that $\widetilde{s}$ is selected (by CF $\widetilde{f}_j$) from $\widetilde{S}(j)$. This is obvious if there are no agents 1 or 2 among the participants of $\widetilde{s}$. Assume that agent 1 participates in $\widetilde{s}$, that is, $\widetilde{s} = s_1$ for some edge s of $f_1(S(0)) \subseteq S(0)$. But then $s \in S(j) = f_j(S(j))$, by **S***. And since $s = \pi(s_1)$ is selected (by CF f_j) from $S(j) = \pi(\widetilde{S}(j))$ (see Lemma A.2), we have $s_1 \in \widetilde{f}_j(\widetilde{S}(j))$ (by the definition of $\widetilde{f}_j$).

Verification of **S***. We show that there are no blocking contracts for $\widetilde{S}$. Suppose, for a contradiction, that such a contract $\widetilde{b}$ exists. If there are no vertices 1 and 2 among the participants of $\widetilde{b}$, then its projection $b = \pi(\widetilde{b})$ blocks S, which contradicts the stability of S. Therefore, we may assume that agent 1, say, participates in $\widetilde{b}$.

Assume that 1 is the unique participant of $\widetilde{b}$ (that is, $\widetilde{b}$ is autarkic). Then $\widetilde{b}$ does not belong to $\widetilde{S}(1)$, and $b = \pi(\widetilde{b})$ does not belongs to $f_1(S(0))$. If $b \in S(0)$, then $b \notin f_1(S(0) \cup b) = f_1(S(0))$. But due to Lemma A.4, $f_1(S(0) \cup b) = \widetilde{f}_1(\widetilde{S}(1) \cup b_1)$. So $\widetilde{b} = b_1$ is not blocking, contrary to the supposition. Therefore, we may assume that $b \notin S(0)$. Since S is stable, $b \notin f_0(S(0) \cup b)$ and, moreover, b does not belong to $f_1(S(0) \cup b)$. This contradicts the stability of S.

Now consider a participant j in $P(b_1)$ different from 1. Since b_1 blocks $\widetilde{S}$, we have

1) $b_1 \notin \widetilde{S}(1)$;

2) $b_1 \in \widetilde{f}_1(\widetilde{S}(1) \cup b_1)$, and

3) $b_1 \in \widetilde{f}_j(\widetilde{S}(j) \cup b_1)$.

The first relation 1) can be rewritten as $b \notin f_1(S(0))$.

The second relation can be rewritten as $b \in f_1(S(0) \cup b)$ (in view of Lemma A.4). This together with the first relation implies that $b \notin S(0)$.

The third relation (with the definition of $\widetilde{f}_j$) gives $b \in f_j(\pi(\widetilde{S}(j) \cup b)) = f_j(S(j) \cup b)$ (by Lemma A.2).

As a consequence, we obtain that b blocks S, contrary to the stability of S. This completes the proof of Proposition 2. $\Box\Box$

B Proof of Theorem 4.2

We will think of each contract c as a partially defined real function on the set of agents I. The definition domain $Dom(c)$ of this function coincides with $P(c)$, the set of participants of c, and the value at $i \in Dom(c)$ is $u_i(c)$.

Definition. A function $x : I \to \mathbb{R}$ is called *compromise* for C if the following two properties are fulfilled:

1) no function c from C can be strictly greater than x within its domain; in other words, for any $c \in C$, there exists $i \in Dom(c)$ such that $c(i) \leq x(i)$;

2) for any participant $i \in I$, there exists $c \in C$ such that $i \in Dom(c)$ and $x \leq c$ (within the domain of c).

The first property is something like a coalition rationality: the coalition $Dom(c)$ refuses the 'distribution' x if c gives strictly more than x to every participant of the coalition. In particular, x is no worse than autarkic contracts. The second condition says that x cannot be too large: the 'payment' to any agent i must be 'justified' by its participation in some 'good' contract (which gives all participants of the contract at least x).

The second condition can also be expressed as follows. Let $S = S(x)$ be the set of 'good' (compared with x) contracts, that is, the contracts c with $x \leq c$. Then the condition says that S covers I. So S is a metastable system. Thus, any compromise function x gives the metastable system $S(x)$ of contracts (those that are not worse than x). On the other hand, if S is a metastable system of contracts, then its 'lower envelope' is a compromise function. Note that the first condition pulls x 'upward', while the second one pulls 'downward'; so x cab be viewed a compromise between these two 'forces'.

Theorem 4.2 is equivalent to the following

Theorem B.1 *Under the hypotheses of Theorem 4.2, there exists a compromise function.*

To show this, one could appeal to Scarf lemma, yet we prefer to give a direct and concise proof inspired by [6]. We will construct a correspondence F ($x \mapsto F(x)$) whose fixed points coincide with the compromises. The existence of fixed points follows from Kakutani's theorem, or, ultimately, from Brauer's theorem.

Construction of the correspondence F. Take a 'big cube' $X = [-N, N]^I$ in the space $\mathbb{R}^I$ (where N is large compared with the maximal value of functions from C), and construct a convex-valued correspondence $F : X \Rightarrow X$. To do this, one need to define the 'image' $F(x)$ of any point $x \in X$. This set $F(x)$ is constructed as a parallelepiped of the form $\times_i F_i(x)$, where $F_i(x)$ is a closed segment in $[-N, N]$. Moreover, $F_i(x)$ is either the whole segment $[-N, N]$ or one of its ends.

To do this, fix i and consider the set $C(i) = \{c \in C\colon i \in Dom(c)\}$. There are three possible situations:

Right: there is a function $c \in C(i)$ such that $c > x$.

Left: each function c from $C(i)$ is less than x at some point (that is, there exists $j \in Dom(c)$ such that $c(j) < x(j)$).

Intermediate: not left and not right.

In Right case, the interval $F_i(x)$ consists of the right point $\{N\}$. In Left case, it consists of the left point $\{-N\}$. And in Intermediate case, $F_i(x)$ is the interval $[-N, N]$.

Lemma B.2 *The fixed points of F are compromises for C, and vice versa.*

Proof Let us check property 1 from the above definition. Suppose this is not valid, namely, there is a function c strictly greater than x. Then for some i, we have $x(i) = N$. But then we obtain Left situation, and $x(i) = -N$. A contradiction.

Now check property 2. Suppose, for a contrary, that for some agent i, every of its contract c is worse than x for some participant of c. Then we have Left situation. So $x(i) = -N$. But in this case, any autarkic function c_i is strictly greater than x. So, the situation is Right, and therefore $x(i) = N$. A contradiction again. □

Note that Left and Right situations are open relative to x: if we slightly disturb x, the situation remains unchanged. It follows that the interval $F_i(x)$ depends on x in an upper semi-continuous way. Since this property holds for any i, we obtain that $F(x)$ *depends on x in an upper semi-continuous way.* Now Kakutani's theorem can be applied to the correspondence F, yielding that there exists $x \in F(x)$.

The converse assertion is trivial. □□

C Plott choice functions

In this section, X is a finite (for simplicity) set. Recall that a CF on X is a mapping $f : 2^X \to 2^X$ such that $f(A) \subseteq A$ for any 'menu' $A \subseteq X$. Such a CF is called a *Plott function* if the following equality holds for any menus A and B:

$$f(A \cup B) = f(f(A) \cup B).$$

This immediately gives $f(A \cup B) = f(f(A) \cup f(B))$, as well as $f(f(A)) = f(A)$.

Let us fix some Plott CF f. A subset $N \subseteq X$ is called *null* (or insignificant) if $f(N) = \emptyset$. It can be seen from earlier reasonings that there is the largest null subset N^*; this is the union of all null sets. Adding any null set to a menu does not change the choice. So, by removing N^* by X, we may assume that $\emptyset$ is the only null set, that is, assume that f is a 'non-empty-valued' CF.

Plott functions have two characteristic properties.

Heredity (or substitutability): if $A \subseteq B$, then $f(B) \cap A \subseteq f(A)$. In other words, if $a \in A$ is chosen in a larger set B, then a is chosen in A as well.

Corollary. *If $A \subseteq f(B)$, then $f(A) = A$.*

Outcast (or independence from rejected alternatives, IRA): if $f(A) \subseteq B \subseteq A$, then $f(A) = f(B)$.

Conversely, it can be shown that holding Heredity and Outcast properties implies that the CF is Plottian.

One may ask: how to build Plott functions?

Let $\leq$ be a preorder on X and let $f(A) = \max_{\leq}(A)$ consist of all maximal elements in A. Such a CF is obviously Plottian. In particular, if $\leq$ is a weak or a linear order.

Another interesting example. Let $\leq$ be a linear order on X, and assign to each element $x \in X$ its 'cost' $c(x)$. Also let a 'budget constraint' b be given. The selection from A consists in a successive choice of the best elements until their total cost exceeds b.

It is easy to check that the union of (two or more) Plott functions is again Plottian. Aizerman and Malishevski [2] showed that any (non-empty-valued) Plott function can be represented as the union of several linear CFs.

Another construction has been encountered earlier. Let $\pi : X \to Y$ be a mapping of sets, and g a CF on Y. Define the CF $f = \pi^*(g)$ on X by the following formula:

$$f(A) = A \cap \pi^{-1}(g(\pi(A))) \quad \text{for } A \subseteq X.$$

In other words, a is chosen from A if $\pi(a)$ is chosen from $\pi(A)$. In particular, $\pi(f(A)) = g(\pi(A))$.

Proposition C.1 *The CF f is Plottian if g is Plottian.*

Proof Let us check that f satisfies Heredity and Outcast.

To see Heredity, let $A \subseteq B$, $a \in A$ and $a \in f(B)$. The second means that $\pi(a) \in g(\pi B)$. Since $\pi A \subseteq \pi B$, by Heredity for g, we get $\pi(a) \in g(\pi A)$, that is, $a \in f(A)$.

To see Outcast, it suffices to show that if $f(B) \subseteq A \subseteq B$, then $f(A) \subseteq f(B)$ (the converse inclusion follows from Heredity). Let $a \in f(A)$; then $\pi(a) \in g(\pi A)$. Applying π to the inclusions $f(B) \subseteq A \subseteq B$, we get $g(\pi B) = \pi(f(B)) \subseteq \pi A \subseteq \pi B$. From Outcast for g, we get $g(\pi A) \subseteq g(\pi B)$. Therefore, $\pi(a) \in g(\pi)$, implying $a \in f(B)$. $\square$

Obviously, for $\pi : X \to Y$ as above, if CF g is given by a weak order $\leq_Y$, then $f = \pi^*(g)$ is given by the weak order $\leq_X = \pi^*(\leq_Y)$.

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