Sensitivity Analysis among Commodities and Coupons during Utility Maximization

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Abstract

In mathematical economics utility is the vital concept. In the society utility is considered as the tendency of an object or action that increases or decreases overall happiness. This study tries to discuss sensitivity analysis during utility maximization investigation. Actually the method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus. In this paper mathematical formulation of Lagrange multipliers is derived through the optimal devise. This article has taken attempts to find relations among commodities and coupons when the prices of different commodities increase or decrease.

Keywords: Coupon constraints, Lagrange multipliers, sensitivity analysis, utility maximization

1. Introduction

In economics, new problems are emerging continuously in every moment. In the 21st century, global economy has become challenging and use of mathematics becomes inevitably essential in economics (Chaturvedi & Singh, 2020; Zheng & Liu, 2022). Mathematical modeling in economics is the application of mathematics in economics to represent theories and analyze problems, where algebra, geometry, set theory, calculus, etc. are used to explain economic behavior of optimization (Samuelson, 1947). It deals with the prices, production, employment, saving, investment, etc. to analyze their logical implications. It is the primary tool to explain or predict about economic issues and problems (Glaister, 1991; Diewert et al., 1993). It is a tool for describing, reasoning, and calculating the relationships between things and the laws arising from changes in their nature by using certain quantitative and quantifiable objects (Lin & Peng, 2019; Zheng & Liu, 2022).

Utility is the property in any object that produces benefit, pleasure, happiness, advantage, or good (positive utility), i.e., to prevent the happening of unhappiness, pain, mischief, or evil (negative utility) of a particular individual or of a society (Bentham, 1780; Stigler, 1950). In economics, utility is a term used to determine the worth or value, i.e., the total satisfaction or benefit gain from consuming of a good or service (Castro & Araujo, 2019). The idea of utility was developed in the late 18^{th} century by the English moral philosopher, jurist, and social reformer Jeremy Bentham (1748-1832) and English philosopher, political economist, Member of Parliament (MP) and civil servant John Stuart Mill (1806-1873) (Bentham, 1780; Marshall, 1920). Utility indicates that individuals seek to obtain the highest level of satisfaction from their purchasing goods (Kirsh, 2017; Lin & Peng, 2019). If utility is represented without mathematical procedures, individuals fail to achieve utility maximization. Therefore, mathematical demonstration of utility maximization provides better result (Gauthier, 1975).

In this study we want to develop sensitivity analysis among commodities and coupons during the utility maximization problem; considering two constraints: budget constraint and coupon constraint (Mohajan, 2021a; Mohajan & Mohajan, 2022c,d). We have included four commodity variables to analyze the optimization methods of multivariate calculus. We have used the determinant of 6×6 Hessian matrix, and later we have used 6×10 Jacobian during the sensitivity analysis (Moolio et al., 2009; Islam et al., 2010; Mohajan, 2021c, 2022c; Mohajan & Mohajan, 2022a,b).

2. Literature Review

US mathematician John V. Baxley and economist John C. Moorhouse have worked on the utility maximization method through the mathematical formulation, where they have introduced an explicit example of optimization (Baxley & Moorhouse, 1984). Renowned mathematician Jamal Nazrul Islam and his coauthors have discussed utility maximization and other optimization problems, where they have considered reasonable interpretation of the Lagrange multipliers (Islam et al., 2009a,b, 2010, 2011).

Lia Roy and her coauthors have considered cost minimization of a running industry by a Cobb-Douglas production function considering three variables capital, labor, and other inputs. They have applied necessary and sufficient conditions to prepare the economic model for the cost minimization problem of an industry for its sustainability (Roy et al., 2021). Jannatul Ferdous and Haradhan Kumar Mohajan have tried to calculate a maximum profit from sale items of an industry. They stress that for the survival of an industry, profit maximization policy is essential (Ferdous & Mohajan, 2022). Devajit Mohajan and Haradhan Kumar Mohajan have studied Cobb-Douglas production function with detail mathematical analysis. They have tried to discuss sensitivity analysis during profit maximization of an industry. They have taken attempts to give economic predictions of future production through the comparative statics (Mohajan & Mohajan, 2022a, b).

Haradhan Kumar Mohajan has considered the maximization of utility problem of consumers of Bangladesh subject to two constraints: budget constraint and coupon constraint (Mohajan, 2017a, 2021a). In two studies he has explored interpretation of Lagrange multiplier to predict the cost minimization policy using Cobb-Douglas production function. He tried to show the production of garments in minimum cost by using statistical analysis (Mohajan, 2021c, 2022c). In a published book, he and his coauthors have investigated optimization problems for the social welfare (Mohajan et al., 2013). Pahlaj Moolio and his coauthors have used a Lagrange multiplier to develop and solve optimization problems (Moolio et al., 2009).

3. Methodology of the Study

Research is a hard-working search, scholarly inquiry, and investigation aimed at the discovery of new facts and findings (Adams et al., 2007). Methodology is a system of explicit rules and procedures in which research is based, and against which claims of knowledge are evaluated (Ojo, 2003). Research methodology is the systematic procedure adopted by researchers to solve a research problem (Kothari, 2008). It indicates the logic of development of the process used to generate theory that is procedural framework within which the research is conducted (Remenyi et al., 1998). It helps to identify research areas and projects within these areas (Blessing et al., 1998; Mohajan, 2018).

At the starting we have tried to present a mathematical economic model, where we have used four commodity variables: X_1, X_2, X_3, and X_4; to operate bordered Hessian. During the sensitivity analysis we have used 6×10 Jacobian. In the study we have worked using total 16 variables, such as

γ_1, γ_2, a_1, a_2, a_3, a_4 p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and C. We have provided economic predictions with the mathematical evidences (Mohajan & Mohajan, 2022a).

In the study we have tried our best to maintain the reliability and validity. We have also tried to cite references properly both in the text and reference list (Mohajan, 2017b, 2020). We have used secondary data sources to enrich this article. To collect secondary data we have used both published and unpublished data sources (Mohajan, 2011, 2022a,b). To prepare this paper we consulted books of famous authors, websites, national and international journals, e-journals, handbooks, theses, etc. (Mohajan, 2014, 2017c).

4. Objective of the Study

The main objective of this study is to discuss the sensitivity analysis of the commodities with respect to coupons during the investigation of utility maximization. The other related objectives are as follows:

- to analyze the bordered Hessian and Jacobian,
- to interpret the mathematical results precisely, and
- to show mathematical calculations in some details.

5. A Mathematical Economic Model

Let us consider an economic world where there are only four commodities that are X_1, X_2, X_3, and X_4 (Moolio et al., 2009; Roy et al., 2021). Let a consumer wants to buy only a_1, a_2, a_3, and a_4 amounts from these four commodities X_1, X_2, X_3, and X_4, respectively. The utility function on these four commodities is given by (Mohajan, 2022c; Mohajan & Mohajan, 2022b),

$$u(a_1, a_2, a_3, a_4) = a_1 a_2 a_3 a_4 . \qquad (1)$$

The budget constraint of the consumer can be represented as,

$$B(a_1, a_2, a_3, a_4) = p_1 a_1 + p_2 a_2 + p_3 a_3 + p_4 a_4 \qquad (2)$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities a_1, a_2, a_3, and a_4, respectively. Now the coupon constraint is given by,

$$C(a_1, a_2, a_3, a_4) = k_1 a_1 + k_2 a_2 + k_3 a_3 + k_4 a_4 \qquad (3)$$

where k_1, k_2, k_3, and k_4 are the coupons necessary to buy a unit of commodity of a_1, a_2, a_3, and a_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $u(a_1, a_2, a_3, a_4, \gamma_1, \gamma_2)$ as (Mohajan, 2017a; Ferdous & Mohajan, 2022),

$$u(a_1, a_2, a_3, a_4, \gamma_1, \gamma_2) = a_1 a_2 a_3 a_4 + \gamma_1 (B - p_1 a_1 - p_2 a_2 - p_3 a_3 - p_4 a_4)$$
$$+ \gamma_2 (C - k_1 a_1 - k_2 a_2 - k_3 a_3 - k_4 a_4). \quad (4)$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where γ_1 and γ_2 are two Lagrange multipliers that are used as devices of mathematical procedures. Now we consider the bordered Hessian (Roy et al., 2021; Mohajan & Mohajan, 2022a),

$$|H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -C_1 & -C_2 & -C_3 & -C_4 \\ -B_1 & -C_1 & u_{11} & u_{12} & u_{13} & u_{14} \\ -B_2 & -C_2 & u_{21} & u_{22} & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{31} & u_{32} & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{41} & u_{42} & u_{43} & u_{44} \end{vmatrix}. \quad (5)$$

Now taking first and second order and cross-partial derivatives in (4) we obtain (Mohajan & Mohajan, 2022c);

$$B_1 = p_1, \quad B_2 = p_2, \quad B_3 = p_3, \quad B_4 = p_4.$$
$$C_1 = k_1, \quad C_2 = k_2, \quad C_3 = k_3, \quad C_4 = k_4. \quad (6)$$
$$u_{11} = 0, \quad u_{12} = u_{21} = a_3 a_4, \quad u_{13} = u_{31} = a_2 a_4,$$
$$u_{14} = u_{41} = a_2 a_3, \quad u_{22} = 0, \quad u_{23} = u_{32} = a_1 a_4,$$
$$u_{24} = u_{42} = a_1 a_3, \quad u_{33} = 0, \quad u_{34} = u_{43} = a_1 a_2, \quad u_{44} = 0. \quad (7)$$

We use $p_1 = p_3$ and $p_2 = p_4$, i.e., a pair of prices are same, and $k_1 = k_3$ and $k_2 = k_4$, i.e., a pair of coupon numbers are same. Now we consider that every term contains $p_1 p_2 k_1 k_2$, then (5) becomes (Mohajan & Mohajan, 2022b);

$$|H| = -2 p_1 p_2 k_1 k_2 < 0. \quad (8)$$

We can determine Lagrange multiplier $\gamma_2 > 0$ as,

$$\gamma_2 = a_3 a_4 \frac{a_2 p_2 - a_1 p_1}{c_1 p_2 - c_2 p_1}. \quad (9)$$

For a_1, a_2, a_3, a_4, γ_1, and γ_2 in terms of p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and C we can calculate the sixty partial derivatives, such as $\frac{\partial \gamma_1}{\partial p_1}$, $\frac{\partial \gamma_2}{\partial p_1}$, …, $\frac{\partial \gamma_1}{\partial k_1}$, $\frac{\partial \gamma_2}{\partial k_1}$, …, $\frac{\partial a_1}{\partial p_1}$, …, $\frac{\partial a_1}{\partial k_1}$, …,

$\frac{\partial \gamma_1}{\partial B}$, …, $\frac{\partial \gamma_1}{\partial C}$, etc. (Islam et al., 2010). Now we consider 6×6 Jacobian matrix (Mohajan & Mohajan, 2022a);

$$J = H = \begin{bmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -C_1 & -C_2 & -C_3 & -C_4 \\ -B_1 & -C_1 & u_{11} & u_{12} & u_{13} & u_{14} \\ -B_2 & -C_2 & u_{21} & u_{22} & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{31} & u_{32} & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \quad (10)$$

which is non-singular at the optimum point $\left(a_1^*, a_2^*, a_3^*, a_4^*, \gamma_1^*, \gamma_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (10) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem. We have total 16 variables in our study, such as γ_1, γ_2, a_1, a_2, a_3, a_4 p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and C. By the implicit function theorem, we can write (Moolio et al., 2009; Islam et al., 2011),

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, C\right). \quad (11)$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by (Mohajan, 2021a),

$$J_G = \begin{bmatrix} \frac{\partial \gamma_1}{\partial p_1} & \frac{\partial \gamma_1}{\partial p_2} & \frac{\partial \gamma_1}{\partial p_3} & \frac{\partial \gamma_1}{\partial p_4} & \frac{\partial \gamma_1}{\partial k_1} & \frac{\partial \gamma_1}{\partial k_2} & \frac{\partial \gamma_1}{\partial k_3} & \frac{\partial \gamma_1}{\partial k_4} & \frac{\partial \gamma_1}{\partial B} & \frac{\partial \gamma_1}{\partial C} \\ \frac{\partial \gamma_2}{\partial p_1} & \frac{\partial \gamma_2}{\partial p_2} & \frac{\partial \gamma_2}{\partial p_3} & \frac{\partial \gamma_2}{\partial p_4} & \frac{\partial \gamma_2}{\partial k_1} & \frac{\partial \gamma_2}{\partial k_2} & \frac{\partial \gamma_2}{\partial k_3} & \frac{\partial \gamma_2}{\partial k_4} & \frac{\partial \gamma_2}{\partial B} & \frac{\partial \gamma_2}{\partial C} \\ \frac{\partial a_1}{\partial p_1} & \frac{\partial a_1}{\partial p_2} & \frac{\partial a_1}{\partial p_3} & \frac{\partial a_1}{\partial p_4} & \frac{\partial a_1}{\partial k_1} & \frac{\partial a_1}{\partial k_2} & \frac{\partial a_1}{\partial k_3} & \frac{\partial a_1}{\partial k_4} & \frac{\partial a_1}{\partial B} & \frac{\partial a_1}{\partial C} \\ \frac{\partial a_2}{\partial p_1} & \frac{\partial a_2}{\partial p_2} & \frac{\partial a_2}{\partial p_3} & \frac{\partial a_2}{\partial p_4} & \frac{\partial a_2}{\partial k_1} & \frac{\partial a_2}{\partial k_2} & \frac{\partial a_2}{\partial k_3} & \frac{\partial a_2}{\partial k_4} & \frac{\partial a_2}{\partial B} & \frac{\partial a_2}{\partial C} \\ \frac{\partial a_3}{\partial p_1} & \frac{\partial a_3}{\partial p_2} & \frac{\partial a_3}{\partial p_3} & \frac{\partial a_3}{\partial p_4} & \frac{\partial a_3}{\partial k_1} & \frac{\partial a_3}{\partial k_2} & \frac{\partial a_3}{\partial k_3} & \frac{\partial a_3}{\partial k_4} & \frac{\partial a_3}{\partial B} & \frac{\partial a_3}{\partial C} \\ \frac{\partial a_4}{\partial p_1} & \frac{\partial a_4}{\partial p_2} & \frac{\partial a_4}{\partial p_3} & \frac{\partial a_4}{\partial p_4} & \frac{\partial a_4}{\partial k_1} & \frac{\partial a_4}{\partial k_2} & \frac{\partial a_4}{\partial k_3} & \frac{\partial a_4}{\partial k_4} & \frac{\partial a_4}{\partial B} & \frac{\partial a_4}{\partial C} \end{bmatrix}. \quad (12)$$

$$=-J^{-1}\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & 0 & 1 \\ -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 \end{bmatrix}. \tag{13}$$

The inverse of Jacobian is, $J^{-1}=\frac{1}{|J|}C^T$, where $C=\left(C_{ij}\right)$, the matrix of cofactors of J, and T indicates transpose, then (13) becomes (Mohajan, 2017a; Islam et al., 2011),

$$J_G=-\frac{1}{|J|}C^T\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & 0 & 1 \\ -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_1 & 0 & 0 & 0 & -\gamma_2 & 0 & 0 \end{bmatrix}. \tag{14}$$

Now 6×6 transpose matrix C^T can be represented by,

$$C^T=\begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \tag{15}$$

Using (15) we can write (12) as 6×10 Jacobian matrix;

$$J_G=-\frac{1}{|J|}\begin{bmatrix} -a_1C_{11}-\gamma_1C_{31} & -a_2C_{11}-\gamma_1C_{41} & -a_3C_{11}-\gamma_1C_{51} & -a_4C_{11}-\gamma_1C_{61} & -a_1C_{21}-\gamma_2C_{31} \\ -a_1C_{12}-\gamma_1C_{32} & -a_2C_{12}-\gamma_1C_{42} & -a_3C_{12}-\gamma_1C_{52} & -a_4C_{12}-\gamma_1C_{62} & -a_1C_{22}-\gamma_2C_{32} \\ -a_1C_{13}-\gamma_1C_{33} & -a_2C_{13}-\gamma_1C_{43} & -a_3C_{13}-\gamma_1C_{53} & -a_4C_{13}-\gamma_1C_{63} & -a_1C_{23}-\gamma_2C_{33} \\ -a_1C_{14}-\gamma_1C_{34} & -a_2C_{14}-\gamma_1C_{44} & -a_3C_{14}-\gamma_1C_{54} & -a_4C_{14}-\gamma_1C_{64} & -a_1C_{24}-\gamma_2C_{34} \\ -a_1C_{15}-\gamma_1C_{35} & -a_2C_{15}-\gamma_1C_{45} & -a_3C_{15}-\gamma_1C_{55} & -a_4C_{15}-\gamma_1C_{65} & -a_1C_{25}-\gamma_2C_{35} \\ -a_1C_{16}-\gamma_1C_{36} & -a_2C_{16}-\gamma_1C_{46} & -a_3C_{16}-\gamma_1C_{56} & -a_4C_{16}-\gamma_1C_{66} & -a_1C_{26}-\gamma_2C_{36} \end{bmatrix.}$$

$$\begin{matrix} -a_2C_{21}-\gamma_2C_{41} & -a_3C_{21}-\gamma_2C_{51} & -a_4C_{21}-\gamma_2C_{61} & C_{11} & C_{21} \\ -a_2C_{22}-\gamma_2C_{42} & -a_3C_{22}-\gamma_2C_{52} & -a_4C_{22}-\gamma_2C_{62} & C_{12} & C_{22} \\ -a_2C_{23}-\gamma_2C_{43} & -a_3C_{23}-\gamma_2C_{53} & -a_4C_{23}-\gamma_2C_{63} & C_{13} & C_{23} \\ -a_2C_{24}-\gamma_2C_{44} & -a_3C_{24}-\gamma_2C_{54} & -a_4C_{24}-\gamma_2C_{64} & C_{14} & C_{24} \\ -a_2C_{25}-\gamma_2C_{45} & -a_3C_{25}-\gamma_2C_{55} & -a_4C_{25}-\gamma_2C_{65} & C_{15} & C_{25} \\ -a_2C_{26}-\gamma_2C_{46} & -a_3C_{26}-\gamma_2C_{56} & -a_4C_{26}-\gamma_2C_{66} & C_{16} & C_{26} \end{matrix}\Bigg]. \tag{16}$$

Now we analyze the nature of consumption of commodity a_1 when its price increases. Taking T_{35}, (i.e., term of 3rd row and 5th column) from both sides of (16) we get (Islam et al., 2011; Mohajan & Mohajan, 2022c),

$$\frac{\partial a_1}{\partial k_1} = -\frac{1}{|J|}\left[-a_1 C_{23} - \gamma_2 C_{33}\right]$$

$$= \frac{1}{|J|}\left[a_1 C_{23} + \gamma_2 C_{33}\right]$$

$$= \frac{a_1}{|J|}\left[C_{23}\right] + \frac{\gamma_2}{|J|}\left[C_{33}\right]$$

$$= \frac{a_1}{|J|}\text{Cofactor of } C_{23} + \frac{\gamma_2}{|J|}\text{Cofactor of } C_{33}$$

$$= -\frac{a_1}{|J|}\begin{vmatrix} 0 & 0 & -B_2 & -B_3 & -B_4 \\ -B_1 & -C_1 & u_{12} & u_{13} & u_{14} \\ -B_2 & -C_2 & u_{22} & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{32} & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{42} & u_{43} & u_{44} \end{vmatrix} + \frac{\gamma_2}{|J|}\begin{vmatrix} 0 & 0 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -C_2 & -C_3 & -C_4 \\ -B_2 & -C_2 & u_{22} & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{32} & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{42} & u_{43} & u_{44} \end{vmatrix}$$

$$= -\frac{a_1}{|J|}\left\{-B_2\begin{vmatrix} -B_1 & -C_1 & u_{13} & u_{14} \\ -B_2 & -C_2 & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{43} & u_{44} \end{vmatrix} + B_3\begin{vmatrix} -B_1 & -C_1 & u_{12} & u_{14} \\ -B_2 & -C_2 & u_{22} & u_{24} \\ -B_3 & -C_3 & u_{32} & u_{34} \\ -B_4 & -C_4 & u_{42} & u_{44} \end{vmatrix} - B_4\begin{vmatrix} -B_1 & -C_1 & u_{12} & u_{13} \\ -B_2 & -C_2 & u_{22} & u_{23} \\ -B_3 & -C_3 & u_{32} & u_{33} \\ -B_4 & -C_4 & u_{42} & u_{43} \end{vmatrix}\right\}$$

$$+ \frac{\gamma_2}{|J|}\left\{-B_2\begin{vmatrix} 0 & 0 & -C_3 & -C_4 \\ -B_2 & -C_2 & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{43} & u_{44} \end{vmatrix} + B_3\begin{vmatrix} 0 & 0 & -C_2 & -C_4 \\ -B_2 & -C_2 & u_{22} & u_{24} \\ -B_3 & -C_3 & u_{32} & u_{34} \\ -B_4 & -C_4 & u_{42} & u_{44} \end{vmatrix} - B_4\begin{vmatrix} 0 & 0 & -C_2 & -C_3 \\ -B_2 & -C_2 & u_{22} & u_{23} \\ -B_3 & -C_3 & u_{32} & u_{33} \\ -B_4 & -C_4 & u_{42} & u_{43} \end{vmatrix}\right\}$$

$$= -\frac{a_1}{|J|}\left[-B_2\left\{-B_1\begin{vmatrix} -C_2 & u_{23} & u_{24} \\ -C_3 & u_{33} & u_{34} \\ -C_4 & u_{43} & u_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & u_{23} & u_{24} \\ -B_3 & u_{33} & u_{34} \\ -B_4 & u_{43} & u_{44} \end{vmatrix} + u_{13}\begin{vmatrix} -B_2 & -C_2 & u_{24} \\ -B_3 & -C_3 & u_{34} \\ -B_4 & -C_4 & u_{44} \end{vmatrix} - u_{14}\begin{vmatrix} -B_2 & -C_2 & u_{23} \\ -B_3 & -C_3 & u_{33} \\ -B_4 & -C_4 & u_{43} \end{vmatrix}\right\}\right.$$

$$+ B_3\left\{-B_1\begin{vmatrix} -C_2 & u_{22} & u_{24} \\ -C_3 & u_{32} & u_{34} \\ -C_4 & u_{42} & u_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & u_{22} & u_{24} \\ -B_3 & u_{32} & u_{34} \\ -B_4 & u_{42} & u_{44} \end{vmatrix} + u_{12}\begin{vmatrix} -B_2 & -C_2 & u_{24} \\ -B_3 & -C_3 & u_{34} \\ -B_4 & -C_4 & u_{44} \end{vmatrix} - u_{14}\begin{vmatrix} -B_2 & -C_2 & u_{22} \\ -B_3 & -C_3 & u_{32} \\ -B_4 & -C_4 & u_{42} \end{vmatrix}\right\}$$

$$\left. - B_4\left\{-B_1\begin{vmatrix} -C_2 & u_{22} & u_{23} \\ -C_3 & u_{32} & u_{33} \\ -C_4 & u_{42} & u_{43} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & u_{22} & u_{23} \\ -B_3 & u_{32} & u_{33} \\ -B_4 & u_{42} & u_{43} \end{vmatrix} + u_{12}\begin{vmatrix} -B_2 & -C_2 & u_{23} \\ -B_3 & -C_3 & u_{33} \\ -B_4 & -C_4 & u_{43} \end{vmatrix} - u_{13}\begin{vmatrix} -B_2 & -C_2 & u_{22} \\ -B_3 & -C_3 & u_{32} \\ -B_4 & -C_4 & u_{42} \end{vmatrix}\right\}\right]$$

$$+\frac{\gamma_2}{|J|}\left[-B_2\left\{-C_3\begin{vmatrix}-B_2 & -C_2 & u_{24}\\ -B_3 & -C_3 & u_{34}\\ -B_4 & -C_4 & u_{44}\end{vmatrix}+C_4\begin{vmatrix}-B_2 & -C_2 & u_{23}\\ -B_3 & -C_3 & u_{33}\\ -B_4 & -C_4 & u_{43}\end{vmatrix}\right\}+B_3\left\{-C_2\begin{vmatrix}-B_2 & -C_2 & u_{24}\\ -B_3 & -C_3 & u_{34}\\ -B_4 & -C_4 & u_{44}\end{vmatrix}+C_4\begin{vmatrix}-B_2 & -C_2 & u_{22}\\ -B_3 & -C_3 & u_{32}\\ -B_4 & -C_4 & u_{42}\end{vmatrix}\right\}\right.$$

$$\left.-B_4\left\{-C_2\begin{vmatrix}-B_2 & -C_2 & u_{23}\\ -B_3 & -C_3 & u_{33}\\ -B_4 & -C_4 & u_{43}\end{vmatrix}+C_3\begin{vmatrix}-B_2 & -C_2 & u_{22}\\ -B_3 & -C_3 & u_{32}\\ -B_4 & -C_4 & u_{42}\end{vmatrix}\right\}\right]$$

$$=-\frac{a_1}{|J|}\left[B_1B_2C_2u_{34}^2 - B_1B_2C_4u_{23}u_{34} - B_1B_2C_3u_{24}u_{34} - B_2^2C_1u_{34}^2 + B_2B_4C_1u_{23}u_{34} + B_2B_3C_1u_{24}u_{34}\right.$$
$$+B_2^2C_4u_{13}u_{34} - B_2B_4C_2u_{13}u_{34} - B_2B_3C_4u_{13}u_{24} + B_2B_4C_3u_{13}u_{24} - B_2^2C_3u_{14}u_{34} - B_2B_3C_2u_{14}u_{34}$$
$$+B_2B_3C_4u_{14}u_{23} - B_2B_4C_3u_{14}u_{23} - B_1B_3C_2u_{24}u_{34} + B_1B_3C_3u_{24}^2 - B_1B_3C_4u_{23}u_{24} + B_2B_3C_1u_{24}u_{34} - B_3^2C_1u_{24}^2$$
$$+B_3B_4C_1u_{23}u_{24} - B_2B_3C_4u_{12}u_{34} - B_3B_4C_3u_{12}u_{24} + B_3^2C_4u_{12}u_{24} - B_3B_4C_3u_{12}u_{24} - B_2B_3C_3u_{14}u_{24}$$
$$+B_2B_3C_4u_{14}u_{23} + B_3^2C_2u_{14}u_{24} - B_3B_4C_2u_{14}u_{23} - B_1B_4C_2u_{23}u_{34} - B_1B_4C_3u_{23}u_{24} + B_1B_4C_4u_{23}^2$$
$$+B_2B_4C_1u_{23}u_{34} + B_4^2C_1u_{23}^2 - B_2B_4C_3u_{12}u_{34} - B_3B_4C_2u_{12}u_{34} - B_3B_4C_4u_{12}u_{23} + B_4^2C_3u_{12}u_{23} + B_2B_4C_3u_{13}u_{24}$$
$$\left.-B_2B_4C_4u_{13}u_{23} - B_3B_4C_2u_{13}u_{24} + B_4^2C_2u_{13}u_{23}\right] + \frac{\gamma_2}{|J|}\left[-B_2^2C_3C_4u_{34} + B_2B_4C_2C_3u_{34} + B_2B_3C_3C_4u_{24}\right.$$
$$-B_2B_4C_3^2u_{24} - B_2^2C_3C_4u_{34} - B_2B_3C_2C_4u_{34} - B_2B_3C_4^2u_{23} + B_2B_4C_3C_4u_{23} + B_2B_3C_2C_4u_{34} - B_3B_4C_2^2u_{34}$$
$$+B_3^2C_2C_4u_{24} - B_3B_4C_2C_3u_{24} - B_2^2C_3C_4u_{24} + B_2^2C_4^2u_{23} + B_2B_3C_2C_4u_{24} - B_2B_4C_2C_4u_{23} + B_2B_4C_2C_3u_{34}$$
$$\left.-B_3B_4C_2^2u_{34} + B_3B_4C_2C_4u_{23} - B_4^2C_2C_3u_{23} - B_2^2C_3^2u_{24} + B_2^2C_3C_4u_{23} + B_2B_3C_2C_3u_{24} - B_2B_4C_2C_3u_{23}\right]$$

$$=-\frac{a_1}{|J|}\left[p_1p_2k_2a_1^2a_2^2 - p_1p_2k_4a_1^2a_2a_4 - p_1p_2k_3a_1^2a_2a_3 - p_2^2k_1a_1^2a_2^2 + p_2p_4k_1a_1^2a_2a_4 + p_2p_3k_1a_1^2a_2a_3\right.$$
$$+p_2^2k_4a_1a_2^2a_4 - p_2p_4k_2a_1a_2^2a_4 - p_2p_3k_4a_1a_2a_3a_4 + p_2p_4k_3a_1a_2a_3a_4 - p_2^2k_3a_1a_2^2a_3 - p_2p_4k_2a_1a_2^2a_3$$
$$+p_2p_3k_4a_1a_2a_3a_4 - p_2p_4k_3a_1a_2a_3a_4 - p_1p_3k_3a_1^2a_2a_3 + p_1p_3k_3a_1^2a_3^2 - p_1p_3k_4a_1^2a_3a_4 + p_2p_3k_1a_1^2a_2a_3$$
$$-p_3^2k_1a_1^2a_3^2 + p_3p_4k_1a_1^2a_3a_4 - p_2p_3k_4a_1a_2a_3a_4 - p_3p_4k_3a_1a_3^2a_4 + p_3^2k_4a_1a_3^2a_4 - p_3p_4k_3a_1a_3^2a_4$$
$$-p_2p_3k_3a_1a_2a_3^2 + p_2p_3k_4a_1a_2a_3a_4 + p_3^2k_2a_1a_2a_3^2 - p_3p_4k_2a_1a_2a_3a_4 - p_1p_4k_2a_1^2a_2a_4 - p_1p_4k_3a_1^2a_3a_4$$
$$+p_1p_4k_3a_1^2a_4^2 - p_2p_4k_1a_1^2a_2a_4 + p_3p_4k_1a_1^2a_3a_4 + p_4^2k_1a_1^2a_4^2 - p_2p_4k_3a_1a_2a_3a_4 - p_3p_4k_2a_1a_2a_3a_4$$
$$\left.-p_3p_4k_4a_1a_3a_4^2 + p_4^2k_3a_1a_3a_4^2 + p_2p_4k_3a_1a_2a_3a_4 - p_2p_4k_4a_1a_2a_4^2 - p_3p_4k_2a_1a_2a_3a_4 + p_4^2k_2a_1a_2a_4^2\right]$$
$$+\frac{\gamma_2}{|J|}\left[-p_2^2k_3k_4a_1a_2 + p_2p_4k_2k_3a_1a_2 + p_2p_3k_3k_4a_1a_3 - p_2p_3k_3^2a_1a_3 - p_2^2k_3k_4a_1a_2 - p_2p_3k_2k_4a_1a_2\right.$$
$$-p_2p_3k_3^2a_1a_4 + p_2p_4k_3k_4a_1a_4 + p_2p_3k_2k_4a_1a_2 - p_3p_4k_2^2a_1a_2 + p_3^2k_2k_4a_1a_3 - p_3p_4k_2k_3a_1a_3 - p_2^2k_3k_4a_1a_3$$
$$+p_2^2k_4^2a_1a_4 + p_2p_3k_2k_4a_1a_3 - p_2p_4k_2k_4a_1a_4 + p_2p_4k_2k_3a_1a_2 - p_3p_4k_2^2a_1a_2 + p_3p_4k_2k_4a_1a_4 - p_4^2k_2k_3a_1a_4$$
$$\left.-p_2^2k_3^2a_1a_3 + p_2^2k_3k_4a_1a_4 + p_2p_3k_2k_3a_1a_3 - p_2p_4k_2k_3a_1a_4\right]$$

$$=-\frac{a_1}{|J|}\left[\left(p_1p_2k_2 - p_2^2k_1\right)a_1^2a_2^2 + \left(p_1p_3k_3 - p_3^2k_1\right)a_1^2a_3^2 + \left(p_1p_4k_3 + p_4^2k_1\right)a_1^2a_4^2 - 3p_3p_4k_2a_1a_2a_3a_4\right.$$
$$+\left(-p_1p_2k_4 - p_1p_4k_2\right)a_1^2a_2a_4 + \left(p_2p_3k_1 - p_1p_2k_3 - p_1p_3k_3 + p_2p_3k_1\right)a_1^2a_2a_3 + \left(p_2^2k_4 - p_2p_4k_2\right)a_1a_2^2a_4$$
$$+\left(-p_2^2k_3 - p_2p_4k_2\right)a_1a_2^2a_3 + \left(p_3^2k_2 - p_2p_3k_3\right)a_1a_2a_3^2 + \left(p_3^2k_4 - p_3p_4k_3 - p_3p_4k_3\right)a_1a_3^2a_4$$
$$\left.+\left(2p_3p_4k_1 - p_1p_3k_4 - p_1p_4k_3\right)a_1^2a_3a_4 + \left(p_4^2k_2 - p_2p_4k_4\right)a_1a_2a_4^2\right]$$

$$+\frac{\gamma_2}{|J|}\left[\left(2p_2p_4k_2k_3-2p_2^2k_3k_4-2p_3p_4k_2^2\right)a_1a_2\right.$$

$$+\left(p_2p_3k_3k_4-p_2p_3k_3^2+p_3^2k_2k_4-p_3p_4k_2k_3-p_2^2k_3k_4+p_2p_3k_2k_4-p_2^2k_3^2+p_2p_3k_2k_3\right)a_1a_3$$

$$\left.+\left(p_2p_4k_3k_4-p_2p_3k_3^2+p_2^2k_4^2-p_2p_4k_2k_4+p_3p_4k_2k_4-p_4^2k_2k_3+p_2^2k_3k_4-p_2p_4k_2k_3\right)a_1a_4\right]. \quad (17)$$

Now using $a_1 = a_2 = a_3 = a_4 = 1$ in (17) we get,

$$\frac{\partial a_1}{\partial k_1} = -\frac{1}{|J|}\left[\left(p_1p_2k_2+p_4^2k_1+2p_2p_3k_1+p_2^2k_4+p_3^2k_2+2p_3p_4k_1+p_4^2k_2\right)-\left(p_3^2k_1+p_2^2k_1+3p_3p_4k_2\right.\right.$$

$$\left.\left.+p_1p_2k_4+p_1p_4k_2+p_1p_2k_3+2p_2p_4k_2+p_2^2k_3+p_2p_3k_3+p_3^2k_4+2p_3p_4k_3+p_1p_3k_4+p_2p_4k_4\right)\right]$$

$$+\frac{\gamma_2}{|J|}\left[\left(p_2p_4k_2k_3+p_2p_3k_3k_4+p_3^2k_2k_4+p_2p_3k_2k_4+p_2p_3k_2k_3+p_2p_4k_3k_4+p_2^2k_4^2+p_3p_4k_2k_4\right)\right.$$

$$\left.-\left(2p_2^2k_3k_4+2p_3p_4k_2^2+2p_2p_3k_3^2+p_3p_4k_2k_3+p_2^2k_3^2+p_2p_4k_2k_4+p_4^2k_2k_3\right)\right]. \quad (18)$$

We consider; $\gamma_2 = \frac{1}{k}$, $p_1 = p_2 = p_3 = p_4 = p$, and $k_1 = k_2 = k_3 = k_4 = k$, then $|J| = -2p^2k^2$; and (18) gives,

$$\frac{\partial a_1}{\partial k_1} = \frac{5}{k} > 0. \quad (19)$$

Inequality (19) indicates that if the quantity of surrendering coupon for the purchase of the commodity a_1 increases, the level of consumption of a_1 will also increase. It seems that commodity a_1 is a superior good and it has no supplementary goods (Islam et al., 2010; Mohajan, 2021b; Mohajan & Mohajan, 2022d).

We consider $p_3 = p_1$ and $p_4 = p_2$; and $k_3 = k_1$, and $k_4 = k_2$, then $|J| = |H| = -2p_1p_2k_1k_2$, and from (9) using $a_1 = a_2 = a_3 = a_4 = 1$ we get,

$$\gamma_2 = \frac{p_2 - p_1}{k_1p_2 - k_2p_1}. \quad (20)$$

Using the value of γ_2 from (20) in (18) we can write,

$$\frac{\partial a_1}{\partial k_1} = \frac{p_1^2k_1+p_2^2k_1+4p_1p_2k_2+p_1^2k_2+p_2^2k_2}{-2p_1p_2k_1k_2} + \frac{(p_2-p_1)\left(p_1p_2k_1k_2+p_1^2k_2^2-p_2^2k_1k_2-2p_1p_2k_1^2-p_2^2k_1^2\right)}{-2p_1p_2k_1k_2(k_1p_2-k_2p_1)}.$$

$$\frac{\partial a_1}{\partial k_1} = \frac{p_1^2p_2k_1^2+3p_1p_2^2k_1k_2+p_2^3k_1^2+p_1^2p_2k_1k_2+p_2^3k_1k_2-4p_1^2p_2k_2^2-p_1^3k_1k_2-p_1^3k_2^2-p_1p_2^2k_2^2}{-2p_1p_2k_1k_2(k_1p_2-k_2p_1)}$$

$$+\frac{2p_1p_2^2k_1k_2+p_1^2p_2^2k_2^2+p_1p_2^2k_1^2-p_2^3k_1k_2-p_2^3k_1^2-p_2^3k_2^2-p_1^2p_2k_1k_2}{-2p_1p_2k_1k_2(k_1p_2-k_2p_1)} \quad (21)$$

where $k_1p_2 \neq k_2p_1$. Now let $k_1 = k_2 = k$, then from (21) we get,

$$\frac{\partial a_1}{\partial k_1}=\frac{3p_1^2p_2+p_2^3+2p_1^3-5p_1p_2^2-p_1^2p_2^2}{p_1p_2(p_2-p_1)}$$

$$\frac{\partial a_1}{\partial k_1}=\frac{\left(2p_1^2-p_2^2\right)}{p_1p_2}-\frac{(p_2-5)p_1^2p_2+4p_1p_2^2}{p_1p_2(p_2-p_1)}, \tag{22}$$

where $p_1 \neq p_2$.

If $p_1 > p_2 > 5$ then from (22) we can write,

$$\frac{\partial a_1}{\partial k_1}>0. \tag{23}$$

Inequality (23) bears the same properties as of the relation (19).

If $p_2 > p_1$ and $p_2 > 5$ then (22) gives,

$$\frac{\partial a_1}{\partial k_1}<0. \tag{24}$$

Relation (24) indicates that if the price of commodity a_1 increases, the level of consumption of a_1 will decrease. This situation seems reasonable result in the sense that commodity a_1 has many substitutes; and hence consumers switch to substitutes when the price of commodity a_1 goes up.

Now we analyze the nature of consumption of commodity a_2 when the price of a_1 increases, where relations are connected to coupons. Taking T_{41}, (i.e., term of 4th row and 1st column) from both sides of (14) we get (Islam et al., 2010; Mohajan & Mohajan, 2022c),

$$\frac{\partial a_2}{\partial k_1}=-\frac{1}{|J|}\left[-a_1C_{24}-\gamma_2C_{34}\right]$$

$$=\frac{1}{|J|}\left[a_1C_{24}+\gamma_2C_{34}\right]$$

$$=\frac{a_1}{|J|}\left[C_{24}\right]+\frac{\gamma_2}{|J|}\left[C_{34}\right]$$

$$=\frac{a_1}{|J|}\text{Cofactor of } C_{24}+\frac{\gamma_2}{|J|}\text{Cofactor of } C_{34}$$

$$=\frac{a_1}{|J|}\begin{vmatrix} 0 & 0 & -B_1 & -B_3 & -B_4 \\ -B_1 & -C_1 & u_{11} & u_{13} & u_{14} \\ -B_2 & -C_2 & u_{21} & u_{23} & u_{24} \\ -B_3 & -C_3 & u_{31} & u_{33} & u_{34} \\ -B_4 & -C_4 & u_{41} & u_{43} & u_{44} \end{vmatrix}-\frac{\gamma_2}{|J|}\begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_4 \\ 0 & 0 & -C_1 & -C_2 & -C_4 \\ -B_2 & -C_2 & u_{21} & u_{22} & u_{24} \\ -B_3 & -C_3 & u_{31} & u_{32} & u_{34} \\ -B_4 & -C_4 & u_{41} & u_{42} & u_{44} \end{vmatrix}$$

$$
=\frac{a_1}{|J|}\left\{-B_1\begin{vmatrix}-B_1 & -C_1 & u_{13} & u_{14}\\ -B_2 & -C_2 & u_{23} & u_{24}\\ -B_3 & -C_3 & u_{33} & u_{34}\\ -B_4 & -C_4 & u_{43} & u_{44}\end{vmatrix}+B_3\begin{vmatrix}-B_1 & -C_1 & u_{11} & u_{14}\\ -B_2 & -C_2 & u_{21} & u_{24}\\ -B_3 & -C_3 & u_{31} & u_{34}\\ -B_4 & -C_4 & u_{41} & u_{44}\end{vmatrix}-B_4\begin{vmatrix}-B_1 & -C_1 & u_{11} & u_{13}\\ -B_2 & -C_2 & u_{21} & u_{23}\\ -B_3 & -C_3 & u_{31} & u_{33}\\ -B_4 & -C_4 & u_{41} & u_{43}\end{vmatrix}\right\}
$$

$$
-\frac{\gamma_2}{|J|}\left\{-B_1\begin{vmatrix}0 & 0 & -C_2 & -C_4\\ -B_2 & -C_2 & u_{22} & u_{24}\\ -B_3 & -C_3 & u_{32} & u_{34}\\ -B_4 & -C_4 & u_{42} & u_{44}\end{vmatrix}+B_2\begin{vmatrix}0 & 0 & -C_1 & -C_4\\ -B_2 & -C_2 & u_{21} & u_{24}\\ -B_3 & -C_3 & u_{31} & u_{34}\\ -B_4 & -C_4 & u_{41} & u_{44}\end{vmatrix}-B_4\begin{vmatrix}0 & 0 & -C_1 & -C_2\\ -B_2 & -C_2 & u_{21} & u_{22}\\ -B_3 & -C_3 & u_{31} & u_{32}\\ -B_4 & -C_4 & u_{41} & u_{42}\end{vmatrix}\right\}
$$

$$
=\frac{a_1}{|J|}\left[-B_1\left\{-B_1\begin{vmatrix}-C_2 & u_{23} & u_{24}\\ -C_3 & u_{33} & u_{34}\\ -C_4 & u_{43} & u_{44}\end{vmatrix}+C_1\begin{vmatrix}-B_2 & u_{23} & u_{24}\\ -B_3 & u_{33} & u_{34}\\ -B_4 & u_{43} & u_{44}\end{vmatrix}+u_{13}\begin{vmatrix}-B_2 & -C_2 & u_{24}\\ -B_3 & -C_3 & u_{34}\\ -B_4 & -C_4 & u_{44}\end{vmatrix}-u_{14}\begin{vmatrix}-B_2 & -C_2 & u_{23}\\ -B_3 & -C_3 & u_{33}\\ -B_4 & -C_4 & u_{43}\end{vmatrix}\right\}\right.
$$

$$
+B_3\left\{-B_1\begin{vmatrix}-C_2 & u_{21} & u_{24}\\ -C_3 & u_{31} & u_{34}\\ -C_4 & u_{41} & u_{44}\end{vmatrix}+C_1\begin{vmatrix}-B_2 & u_{21} & u_{24}\\ -B_3 & u_{31} & u_{34}\\ -B_4 & u_{41} & u_{44}\end{vmatrix}-u_{14}\begin{vmatrix}-B_2 & -C_2 & u_{21}\\ -B_3 & -C_3 & u_{31}\\ -B_4 & -C_4 & u_{41}\end{vmatrix}\right\}-B_4\left\{-B_1\begin{vmatrix}-C_2 & u_{21} & u_{23}\\ -C_3 & u_{31} & u_{33}\\ -C_4 & u_{41} & u_{43}\end{vmatrix}\right.
$$

$$
\left.+C_1\begin{vmatrix}-B_2 & u_{21} & u_{23}\\ -B_3 & u_{31} & u_{33}\\ -B_4 & u_{41} & u_{43}\end{vmatrix}-u_{13}\begin{vmatrix}-B_2 & -C_2 & u_{21}\\ -B_3 & -C_3 & u_{31}\\ -B_4 & -C_4 & u_{41}\end{vmatrix}\right\}-\frac{\gamma_2}{|J|}\left[-B_1\left\{-C_2\begin{vmatrix}-B_2 & -C_2 & u_{24}\\ -B_3 & -C_3 & u_{34}\\ -B_4 & -C_4 & u_{44}\end{vmatrix}+C_4\begin{vmatrix}-B_2 & -C_2 & u_{22}\\ -B_3 & -C_3 & u_{32}\\ -B_4 & -C_4 & u_{42}\end{vmatrix}\right\}\right.
$$

$$
\left.\left.+B_2\left\{-C_1\begin{vmatrix}-B_2 & -C_2 & u_{24}\\ -B_3 & -C_3 & u_{34}\\ -B_4 & -C_4 & u_{44}\end{vmatrix}+C_4\begin{vmatrix}-B_2 & -C_2 & u_{21}\\ -B_3 & -C_3 & u_{31}\\ -B_4 & -C_4 & u_{41}\end{vmatrix}\right\}-B_4\left\{-C_1\begin{vmatrix}-B_2 & -C_2 & u_{22}\\ -B_3 & -C_3 & u_{32}\\ -B_4 & -C_4 & u_{42}\end{vmatrix}+C_2\begin{vmatrix}-B_2 & -C_2 & u_{21}\\ -B_3 & -C_3 & u_{31}\\ -B_4 & -C_4 & u_{41}\end{vmatrix}\right\}\right]\right]
$$

$$
=\frac{a_1}{|J|}\Big[B_1^2C_2u_{34}^2 -B_1^2C_4u_{23}u_{34} -B_1^2C_3u_{24}u_{34} -B_1B_2C_1u_{34}^2 +B_1B_4C_1u_{23}u_{34} +B_1B_3C_1u_{23}u_{34} -B_1B_2C_4u_{13}u_{34}
$$

$$
-B_1B_4C_2u_{13}u_{34} +B_1B_3C_4u_{13}u_{24} -B_1B_4C_3u_{13}u_{24} +B_1B_2C_3u_{14}u_{34} -B_1B_3C_2u_{14}u_{34} +B_1B_3C_4u_{14}u_{23}
$$

$$
+B_1B_4C_3u_{14}u_{23} -B_1B_3C_2u_{14}u_{34} +B_1B_3C_4u_{12}u_{34} +B_1B_3C_3u_{14}u_{24} -B_1B_3C_4u_{13}u_{24} +B_2B_3C_1u_{14}u_{34}
$$

$$
-B_3B_4C_1u_{12}u_{34} -B_3^2C_1u_{14}u_{24} -B_1B_4C_1u_{13}u_{24} -B_2B_3C_3u_{14}^2 +B_2B_3C_4u_{13}u_{14} +B_3^2C_2u_{14}^2 -B_3B_4C_2u_{13}u_{14}
$$

$$
-B_1B_4C_2u_{13}u_{34} +B_1B_4C_3u_{12}u_{34} -B_1B_4C_3u_{14}u_{23} +B_1B_4C_4u_{13}u_{23} +B_2B_4C_1u_{13}u_{34} -B_3B_4C_1u_{12}u_{34}
$$

$$
+B_3B_4C_1u_{14}u_{23} -B_4^2C_1u_{13}u_{23} +B_2B_4C_3u_{13}u_{14} -B_2B_4C_4u_{13}^2 -B_3B_4C_2u_{13}u_{14} +B_4^2C_2u_{13}^2 +B_3B_4C_4u_{12}u_{13}
$$

$$
-B_4^2C_3u_{12}u_{13}\Big] -\frac{\gamma_2}{|J|}\Big[-B_1B_2C_2C_4u_{34} +B_1B_4C_2^2u_{34} +B_2B_3C_2C_4u_{24} -B_2B_4C_2C_3u_{24} -B_1B_2C_3C_4u_{24}
$$

$$
+B_1B_2C_4^2u_{23} +B_1B_3C_2C_4u_{24} -B_1B_4C_2C_4u_{23} +B_2^2C_1C_4u_{34} -B_2B_4C_1C_3u_{34} -B_2B_3C_1C_4u_{24} +B_2B_4C_1C_3u_{24}
$$

$$
+B_1B_2C_3C_4u_{14} -B_1B_2C_4^2u_{13} +B_1B_3C_2C_4u_{14} -B_1B_4C_2C_4u_{13} -B_1B_3C_4^2u_{12} +B_1B_4C_3C_4u_{12} +B_2B_4C_1C_3u_{24}
$$

$$
-B_2B_4C_1C_4u_{23} -B_3B_4C_1C_2u_{24} +B_4^2C_1C_2u_{23} -B_2B_4C_2C_3u_{14} +B_2B_4C_2C_4u_{13} -B_3B_4C_2^2u_{14} +B_4^2C_2^2u_{13}
$$

$$
-B_3B_4C_2C_4u_{12} +B_4^2C_2C_3u_{12}\Big]
$$

$$
=\frac{a_1}{|J|}\Big[p_1^2k_2a_1^2a_2^2 -p_1^2k_4a_1^2a_2a_4 -p_1p_2k_1a_1^2a_2^2 +p_1p_4k_1a_1^2a_2a_4 +p_1p_3k_1a_1^2a_2a_4 -p_1p_2k_4a_1a_2^2a_4
$$

$$
-p_1p_4k_2a_1a_2^2a_4 +p_1p_3k_4a_1^2a_2a_3 -p_1p_4k_3a_1a_2a_3a_4 +p_1p_2k_3a_1a_2^2a_3 +p_1p_3k_2a_1a_2^2a_3 +p_1p_3k_4a_1a_2a_3a_4
$$

$$+p_1p_4k_3a_1a_2a_3a_4 - p_1p_3k_2a_1a_2^2a_3 + p_1p_3k_4a_1a_2a_3a_4 + p_1p_3k_3a_1a_2a_3^2 - p_1p_3k_4a_1a_2a_3a_4 + p_2p_3k_1a_1a_2^2a_3$$
$$-p_3p_4k_1a_1a_2a_3a_4 - p_3^2k_1a_1a_2a_3^2 - p_1p_4k_1a_1a_2a_3a_4 - p_2p_3k_3a_2^2a_3^2 + p_2p_3k_4a_2^2a_3a_4 + p_3^2k_2a_2^2a_3^2$$
$$-p_3p_4k_2a_2^2a_3a_4 - p_1p_4k_2a_1a_2a_3a_4 + p_1p_4k_3a_1a_2a_3a_4 - p_1p_4k_3a_1a_2a_3a_4 + p_1p_4k_4a_1a_2a_4^2 + p_2p_4k_1a_1a_2^2a_4$$
$$-p_3p_4k_1a_1a_2a_3a_4 + p_3p_4k_1a_1a_2a_3a_4 - p_4^2k_1a_1a_2a_4^2 + p_2p_4k_3a_2^2a_3a_4 - p_2p_4k_4a_2^2a_4^2 - p_3p_4k_2a_2^2a_3a_4$$
$$+p_4^2k_2a_2^2a_4^2 + p_3p_4k_4a_2a_3a_4^2 - p_4^2k_3a_2a_3a_4^2\Big] - \frac{\gamma_2}{|J|}\Big[-p_1p_2k_2k_4a_1a_2 + p_1p_4k_2^2a_1a_2 + p_2p_3k_2k_4a_1a_3$$
$$-p_2p_4k_2k_3a_1a_3 - p_1p_2k_3k_4a_1a_3 + p_1p_2k_4^2a_1a_4 + p_1p_3k_2k_4a_1a_3 - p_1p_4k_2k_4a_1a_4 + p_2^2k_1k_4a_1a_2$$
$$-p_2p_4k_1k_3a_1a_2 - p_2p_3k_1k_3a_1a_3 + p_2p_4k_1k_3a_1a_3 + p_1p_2k_3k_4a_2a_3 - p_1p_2k_4^2a_2a_4 + p_1p_3k_2k_4a_2a_3$$
$$-p_1p_4k_2k_4a_2a_4 - p_1p_3k_4^2a_3a_4 + p_1p_4k_3k_4a_3a_4 + p_2p_4k_1k_3a_1a_3 - p_2p_4k_1k_4a_1a_4 - p_2p_4k_1k_2a_1a_3$$
$$+p_4^2k_1k_2a_1a_4 - p_2p_4k_2k_3a_2a_3 + p_2p_4k_2k_4a_2a_4 - p_3p_4k_2^2a_2a_3 + p_4^2k_2^2a_2a_4 - p_3p_4k_2k_4a_3a_4 + p_4^2k_2k_3a_3a_4\Big]$$

$$= \frac{a_1}{|J|}\Big[\left(p_1^2k_2 - p_1p_2k_1\right)a_1^2a_2^2 + \left(p_4^2k_2 - p_2p_4k_4\right)a_2^2a_4^2 + \left(p_3^2k_2 - p_2p_3k_3\right)a_2^2a_3^2 + \left(p_1p_3k_1 + p_1p_4k_1\right)a_1^2a_2a_4$$
$$+\left(p_1p_3k_4 - p_3p_4k_1 - p_1p_4k_2 - p_1p_4k_1\right)a_1a_2a_3a_4 + \left(p_1p_3k_4 - p_1^2k_3 - p_1^2k_4\right)a_1^2a_2a_3$$
$$+\left(p_2p_4k_1 - p_1p_4k_2 - p_1p_2k_4\right)a_1a_2^2a_4 + \left(p_1p_2k_3 + p_2p_3k_1\right)a_1a_2^2a_3 + \left(p_1p_3k_3 - p_3^2k_1\right)a_1a_2a_3^2$$
$$+\left(p_2p_3k_4 - p_3p_4k_2\right)a_2^2a_3a_4 + \left(p_1p_4k_4 - p_4^2k_1\right)a_1a_2a_4^2 + \left(p_2p_4k_3 - p_3p_4k_2\right)a_2^2a_3a_4$$
$$+\left(p_3p_4k_4 - p_4^2k_3\right)a_2a_3a_4^2\Big] - \frac{\gamma_2}{|J|}\Big[\left(p_1p_4k_2^2 + p_2^2k_1k_4 - p_1p_2k_2k_4 - p_2p_4k_1k_3\right)a_1a_2 + \left(p_2p_4k_2k_4 - p_1p_2k_4^2 - p_1p_4k_2k_4\right)a_2a_4$$
$$+\left(p_1p_2k_4^2 + p_4^2k_1k_2 - p_1p_4k_2k_4 - p_2p_4k_1k_4\right)a_1a_4 + \left(p_1p_2k_3k_4 + p_1p_3k_2k_4 - p_2p_4k_2k_3 - p_3p_4k_2^2\right)a_2a_3$$
$$+\left(p_2p_3k_2k_4 + p_1p_3k_2k_4 + 2p_2p_4k_1k_3 - p_2p_4k_2k_3 - p_1p_2k_3k_4 - p_2p_3k_1k_3 - p_2p_4k_1k_2\right)a_1a_3$$
$$+\left(p_1p_4k_3k_4 + p_4^2k_2^2 + p_4^2k_2k_3 - p_1p_3k_4^2 - p_3p_4k_2k_4\right)a_3a_4\Big]. \quad (25)$$

Now we use $p_1 = p_3$ and $p_2 = p_4$ where pair of prices are same, and $k_1 = k_3$ and $k_2 = k_4$, i.e., two types of coupon numbers are same. We put $a_1 = a_2 = a_3 = a_4 = 1$ then (25) becomes (Mohajan & Mohajan, 2022b),

$$\frac{\partial a_2}{\partial k_1} = \frac{1}{|J|}\left(3p_1^2k_2 - p_1p_2k_1 - 2p_1p_2k_2\right) - \frac{\gamma_2}{|J|}\left(p_1^2k_1k_2 - p_1p_2k_1^2 + p_2^2k_1^2 - 3p_1p_2k_2^2 + 2p_2^2k_2^2\right). \quad (26)$$

Now we use $\gamma_2 = \frac{1}{k}$, $k_1 = k_2 = k$, and $|J| = -2p_1p_2k^2$ in (26), and then we get,

$$\frac{\partial a_2}{\partial k_1} = \frac{(2p_1 + 3p_2)(p_2 - p_1)}{2p_1p_2k}. \quad (27)$$

Now if $p_2 > p_1$ in (27) we get,

$$\frac{\partial a_2}{\partial k_1} > 0. \quad (28)$$

Inequality (28) indicates that if the number of surrendering coupon to purchase the commodity a_1 increases, the level of consumption of a_2 will increase. This supports that goods a_1 and a_2 are supplementary goods; that is, when price of a_1 goes up people switch to its supplementary commodity a_2 (Islam et al., 2010; Mohajan & Mohajan, 2022c).

Now if $p_1 > p_2$ in (27) we get,

$$\frac{\partial a_2}{\partial k_1} < 0. \tag{29}$$

Inequality (29) indicates that if the number of surrendering coupon to purchase the commodity a_1 increases, the level of consumption of a_2 will decrease. This situation shows that goods a_1 and a_2 are complementary goods; that is, when price of a_1 goes up people buy less of it, consequently level of consumption of a_2 also decreases.

Now if $p_2 = p_1$ in (27) we get,

$$\frac{\partial a_2}{\partial k_1} = 0. \tag{30}$$

Inequality (30) indicates that if the number of surrendering coupon to purchase the commodity a_1 increases, there seems no effect on the level of the consumption of goods a_1. This is reasonable result in the sense that commodities a_1 and a_2 are unrelated goods, for example, mango and mathematics textbook.

Using $\gamma_2 = \frac{p_2 - p_1}{k_1 p_2 - k_2 p_1}$ and $|J| = -2p_1 p_2 k_1 k_2$ in relation (26) we get,

$$\frac{\partial a_2}{\partial k_1} = \frac{1}{-2p_1 p_2 k_1 k_2}\left(3p_1^2 k_2 - p_1 p_2 k_1 - 2p_1 p_2 k_2\right)$$

$$+\frac{1}{-2p_1 p_2 k_1 k_2}\frac{p_2 - p_1}{k_1 p_2 - k_2 p_1}\left(p_1^2 k_1 k_2 - p_1 p_2 k_1^2 + p_2^2 k_1^2 - 3p_1 p_2 k_2^2 + 2p_2^2 k_2^2\right)$$

$$\frac{\partial a_2}{\partial k_1} = -\frac{4p_1^2 p_2 k_1 k_2 + 2p_1^2 p_2 k_2^2 - p_1 p_2^2 k_1^2 - 2p_1 p_2^2 k_1 k_2 - 3p_1^3 k_2^2}{2p_1 p_2 k_1 k_2 (k_2 p_1 - k_1 p_2)}$$

$$-\frac{p_1^2 p_2 k_1 k_2 + p_2^3 k_1^2 + 2p_2^3 k_2^2 + 3p_1^2 p_2 k_2^2 - p_1 p_2^2 k_1^2 - 5p_1 p_2^2 k_2^2 - p_1^3 k_1 k_2}{2p_1 p_2 k_1 k_2 (k_2 p_1 - k_1 p_2)} \tag{31}$$

where $k_2 p_1 \neq k_1 p_2$. Now let, $k_1 = k_2 = k$, and then from (31) we get,

$$\frac{\partial a_2}{\partial k_1} = \frac{3(p_1 + p_2)^2 + 6p_1 p_2}{p_1 p_2 k} > 0. \tag{32}$$

Equation (32) is true for all $p_1, p_2, k \in \mathbf{R}^+$; which has same explanations as in relation (28).

6. Conclusions

In this study we have discussed utility maximization policy subject to two constraints: budget constraint and coupon constraint. We have analyzed the sensitivity analysis and have tried to find relationships among commodities and coupons. We have used four commodity variables to operate the mathematical formulation efficiently. We have applied the technique of Lagrange multipliers to investigate the optimization problems. During the sensitivity analysis we have used 6×6 bordered Hessian matrix and 6×10 Jacobian matrix. In this study we have faced total sixteen variables. When we face difficulties in mathematical procedures we consider two commodities equal to unity. At one stage we have considered all commodities are of unit amount, and prices of two commodities are same, and also two types of coupon numbers are same. In this study we have tried to show mathematical calculations in some details.

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