Dynamic optimization; lecture notes

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Dynamic Optimization
Lecture notes

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Chapter 1

Static Optimization

Preliminary definitions $\Rightarrow$ min/max.

Let $x^* \in \mathbb{R}^n$, $n > 1$, $r > 0$, $E \subseteq \mathbb{R}^n$, $d(x,y)$ be a distance on $\mathbb{R}^n$. Two types of distance: (1) **Euclidean distance**$\Rightarrow$ $d(x,y) = \|x-y\|$; (2) **Manhattan distance**$\Rightarrow$ $d(x,y) = \sum_{i=1}^{n}(x_i - y_i)$.
x^* is called an interior point of E if and only if there is $r > Q$ such that $B(x^*, r \subset E)$.

Take an open ball $B(x^*, r)$ centered at x^*, with radius $r \equiv$ the set $\{y|d(x^*, y) < r\}$, x^* is the interior part of E, if $\exists r$ s.t. $(x^*, r) \subseteq E \Leftrightarrow x^* \in into(E)$.

Local/relative maximum $\Rightarrow$ f^* attains a maximum or a minimum at x^* if $\exists$ a neighbourhood V s.t. $\forall x \in V$ dominium of f, $f(x) \leq f(x^*)$.

Global/absolute maximum $\Rightarrow$ with strict inequalities.

1.1 Free Optimization

Let $f : D \subseteq \Re^n \to \Re : x \to f(x)$.

1.1.1 Real functions of one real variable (n=1)

THEOREM A.1.1: First order necessary condition: if f is differentiable at x^*, and x^* is a local maximizer or minimizer of x, them $f'(x^*) = 0$.

PROOF: for a maximizer $\Rightarrow$ f attains a local maximum at x^* $\Rightarrow$ $\exists V$ s.t. $\forall x \in V, f(Vx) \leq f(Vx^*)$. $x = x^*+h, h \leq r \Rightarrow f(x^*+h)-f(x^*) \leq 0, \forall h \leq r$. R.H.S. $lim_{>h\to 0}\frac{f(x^*+h)-f(x^*)}{h} = f(x^*) \leq 0$ (Def. of a derivative).

L.H.S. $lim_{<h \to 0} \frac{f(x^*+h)-f(x^*)}{h} \leq 0 = f'(x^*) \geq 0$. ■

Remark. this is a necessary, not a sufficient condition!

THEOREM A.1.2: second order sufficient condition.
If f is C^2 in a neighborhood of x^*, an interior point of D, and if x^* is a stationary point of f ($\Leftrightarrow \frac{\partial f(x^*)}{\partial x} = 0$), then:
PROOF. $f(x) = x^*$. Taylor expansion $\Rightarrow f$ if C^2 in a neighbourhood of $x^* \Rightarrow \exists \theta \in [0,1], f(x^*+h) = f(x^*) + hf'(x^*) + \frac{1}{2}h^2 f''(x^*+\theta h)$. Recall that a second order Taylor expansion if $f(y) = f(x) + (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(C) \Rightarrow$ with an error term $\Rightarrow C \in [x,y] \to f(x^*+h) - f(x^*) = \frac{1}{2}h^2 f''(x^*+\theta h)$ if h is small enough, because f'' is continuous.
$f''(x) \geq 0$ by assumption $\Rightarrow x^*$ is a minimizer, $\forall h$.■

THEOREM A.1.3. if f is continuously differentiable up to an order $n, (n \geq 2)$ and if for $x^* \int D$, we have $f'(x^*) = 0, f''(x^*) = 0, ..., f^{n-1}(x^*) = 0, f^n(x^*) = 0 \Rightarrow$ if n is even, a maximum occurs at x^* if $f''(x^*) \leq 0$. If n is odd, f has neither a maximum nor a minimum at x^*.

Idea: $f(x^*+h) - f(x^*) = 1, f^n(x^*)h^n > 0$, if n even.
Take some Taylor expansions:
- n even : $\frac{1}{n!}h^n f^n(x^* + \theta * h)$;
- n odd: $\frac{1}{n!}h^n f^n(x^* + \theta)$.

1.1.2 Real functions of n(>1) real variables

THEOREM A.2.1: First order necessary conditions. Let $x^* \int D$. If f is differentiable at x^*, if f attains a relative maximum at x^*, then $\frac{\partial f}{\partial x_i}(x^*) = 0, \forall i = 1, ..., n$ (or $\bigtriangledown f(x^*) = 0$).[1]

PROOF - for a maximum. If a maximum of f occurs at x^*, an interior point of D, then $\exists r > 0$ such that $\forall x \in B(x^*, r)$, $f(x) \leq f(x^*)$.
From there, $\forall i = 1, ..., n$: $f(x_1^*, x_2^*, ..., x_{i-1}^*, x_i, x_{i+1}^*, ..., x_n^*) \leq f(x^*)$, $\forall x_i \in]x_i^* - r; x_i^* + r[$.
The f.o.c. (theorem A.1.1) relative to functions of one variables can be applied yielding

$$\frac{\partial f}{\partial x_i}(x^*) = 0 \forall i = 1, ..., n. \blacksquare \tag{1.1}$$

[1] Differentiability of f is "too strong" as a condition, just $\exists$ of the partial derivative is needed.

THEOREM A.2.2: Second order sufficient conditions. If f^* is C^2 in a neighborhood of x^*, if x^* is a stationary point of f and if:

(i) $Hf(x^*)$ is positive definite $\Rightarrow$ a relative maximum occurs (f is convex);

(ii) $Hf(x^*)$ is negative definite $\Rightarrow$ a relative minimum occurs (f is concave).

$Hf(x^*) \equiv$ the Hessian matrix of f at $x^* \Rightarrow$ cross 2^{nd} partial derivatives.

Recall: a matrix is p.d. if all leading principal minors are > 0, or alls eigenvalues are strictly > 0. Positive semi-definiteness $\Rightarrow$ suppress "strictly". Negative definiteness $\Rightarrow$ if all minors alternate in sign (-) (+), thus are all < 0.

Idea of the proof: Taylor expansion: (1) exact error term (justifies that $Hf(x^* + \theta h)$ stays positive forever); (2) (notes) generic error term goes to zero faster that h does.

Global maxima (minima) $\Rightarrow$ three ways:

1. A function that is continuous on a compact set (in $\mathbb{R}^n$, this is equivalent to a bounded and closed set), attains a global maximum and a global minimum on that set. You can thus find maxima on a case by case basis, ex. $f(x) = x^2$.

2. If $f \in C^2$ in his domain D, that is an open and convex subset of $\mathbb{R}^n$ $\Rightarrow$ if f is strictly concave (convex) on D, and $\bigtriangledown f(x^*) = 0 \Rightarrow f$ has a global maximum (minimum) at x^*.

Reminders. For $\mathbf{n} = \mathbf{1} \Rightarrow f \in C^2(D)$, D open and convex (concave) subset of $\mathbb{R}^n$, then f is concave (convex) if $\forall x \in D$, $f''(x) \leq 0$ (resp ≥ 0). For $\mathbf{n} > \mathbf{1} \Rightarrow f \in C^2(D)$, D open and convex subset of $\mathbb{R}^n$, then f is concave if $\forall x \in D$, $Hf(x)$ is negative semidefinite.

Counterexample. $f(x) = \frac{1}{x}$ is not continuous, closed and bounded. On the contrary $f(x) = e^x$ is continuous, closed and unbounded: hence no max nor min.

1.2 Constrained Optimization

1.2.1 Real function of one variable, "positivity constraints"

Let $f : \mathbb{R} \to \mathbb{R} : x \to f(x)$, where f is differentiable. **Problem**:

$$\max_x f(x) s.t. x \geq 0$$
$$\min_x f(x) s.t. x \geq 0.$$

Two types of maxima are possible:

1. at a point $x^* > 0$. Then, as before, we have $f'(x) = 0$;

2. at a point where $x^* = 0$. Only the right side neighborhood of x^* can be taken into account and f must decrease (or increase) $\Rightarrow$ $f'_{rhs}(x^*) \leq 0$ (≥ 0).

Theorem A.2.1 - FONC. If f is differentiable at x^*, an interior point of D, and if local max (min) of f subject to $x \geq 0$ occurs at x^*, then $x^* f'(x^*) = 0$, $x^* > 0$ and $f'(x^*) \leq 0$ or (> 0).

Proof. As in the unconstrained case. ■

Theorem A.2.2 - SOSC. If f is C^2 in the neighborhood of x^*, interior point of D, if $x^* > 0$ is a stationary point of f, then:

1. if $f''(x^*) > 0$, a local min of f subject to $x \geq 0$ at x^*.

2. if $f''(x^*) < 0$, a local max of f subject to $x \geq 0$ at x^*.

If $x^* = 0$, then:

1. if $f'(x^*) > 0$, a local min of f subject to $x \geq 0$ at x^*;

2. if $f'(x^*) < 0$, a local max of f subject to $x \geq 0$ at x^*.

Example. Optimize $f(x) = x^2 - 5x + 6$ subject to $x \geq 0$.

$$\begin{aligned} x^* f'(x^) &= 0 \Leftrightarrow x^*(2x - 5) = 0 \\ \Leftrightarrow x^* &= 0 \Leftrightarrow x^* = \frac{5}{2}. \end{aligned} \tag{1.2}$$

$f'' = 2 > 0 \Rightarrow \frac{5}{2}$ is a local minimum.

$f'(0) = -5 < 0 \Rightarrow 0$ is a local maximum.

1.2.2 Real functions of $n(> 1)$ real variables - "positivity constraints"

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Problem. $\max_{x \in D} f(x)$ subject to $x \geq 0$, $(\forall i, x_i \geq 0)$. Let $f : D \subset \mathbb{R}^n \to \mathbb{R} : x \to f(x)$.

Theorem A.3.1 - FONC. If f is differentiable at x^*, an interior point of D, and if a max (min) of f subject to $x^{\geq}0$ occurs at x^*, then $\forall i = 1, ..., n$,

$$x_i^* \frac{\partial f}{\partial x}(x^*) = 0, x_i^* \geq 0, \frac{\partial f}{\partial x}(x^*) \leq 0 (\geq 0).$$

Remark. In some situations not all variables need to be restricted as ≥ 0. The positivity constraint holds for all $x_i \geq 0$, $\forall i \in \{1, ..., m\}$ where $m < n$. Then the necessary condition becomes:

$$\begin{aligned}
& xi_i^* \frac{\partial f}{\partial x_i}(x^*) = 0, \forall i = 1, ..., n \\
& \frac{\partial f}{\partial x_i}(x^*) = 0, \forall m \neq 1, ..., n \\
& x_i^* \geq 0, \forall i =, ..., m \\
& \frac{\partial f}{\partial x_i}(x^*) \leq 0
\end{aligned} \tag{1.3}$$

Theorem A.3.2 - SOSC If f is C^2 in the neighborhood of x^*, an interior point of D, then:

1. if $x^* > 0$ is a stationary point, then $Hf(x^*)$ is a positive (negative) definite matrix, f attains a local maximum (minimum) at x^*.

2. If $x^* = 0$ is a stationary point, the if $\forall i = 1, ..., n$, $\frac{\partial f}{\partial x_i}(x^*) > 0$, f attains a local minimum at x^* (resp. max).

Examples. **a.** $f(x_1, x_2) = 3x_1 + 2x_2$ s.t. $x_1, x_2 \geq 0$.
b. $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2$ s.t. $x_1, x_2 \geq 0$.

Solutions. **a.**

$$x_1^*(2(x_1^*-1))=0$$
$$x_2^*(2(x_2^*-2))=0$$

. (0,0)

$$\frac{\partial f}{\partial x1}(0,0)=-1<0$$
$$\frac{\partial f}{\partial x_2}(0,0)=-2<0$$

$\Rightarrow$ local max.
. (0,2)

$$\frac{\partial f}{\partial x_1}(0,2)=-1<0$$
$$Hf=\frac{\partial^2 f}{\partial x_2^2}(0,2)=2>0$$

$\Rightarrow$ saddle point.
. (1,0)

$$\frac{\partial f}{\partial x_1}(1,0)=-4<0$$
$$Hf=\frac{\partial^2 f}{\partial x_2^2}=2>0$$

$\Rightarrow$ saddle point.

1.2.3 Real functions of $n(>1)$ real variables subject to $m<n$ equality constraints

. Let $f : D \subset \mathbb{R}^n \to \mathbb{R} : x \to f(x)$, $g_j : D \subset \mathbb{R}^m \to \mathbb{R} : x \to g_j(x)$, $j=1,...,m$, $c \in \mathbb{R}^m$.

Problem. $\max_{x\in D} f(x)$ subject to $g_j(x)=c_j$, $\forall j=1,...,m$.

Theorem FONC - n=2, m=1 If f and g are C^1 in the neighborhood of x^*, an interior point of D, such that $\bigtriangledown g(x^*)\neq 0$. If a max (min) of f subject to $g(x)=c$ occurs at x^*, then there exists a $\lambda^*\in\mathbb{R}$ such that

$$\bigtriangledown f(x^*)=\lambda^* \bigtriangledown g(x^*). \tag{1.4}$$

In other words, we may set up a Lagrangian equation, containing the objective function and the constraint with attached a Lagrangian multiplier λ, $L(x,\lambda) = f(x) - \lambda(g(x) - c)$ then we have $\lambda^* \in \mathbb{R}$ ... and we have one solution for λ:

$$\lambda^* = \left(\frac{\frac{\partial f}{\partial x_2}(x^*)}{\frac{\partial g}{\partial x_2}(x^*)} \right).$$

It remains to show that this value of λ^* satisfies as well equation $\frac{\partial f}{\partial x_i}(x^*) - \lambda \frac{\partial g}{\partial x_i}(x^*) = 0$. From the implicit function theorem, we know that we have equation $g(x) - c = 0$ verified by x^*, this is equivalent to an equation $x_2 = h(x_i)$ in the neighborhood of x^*.
Furthermore, we know that $h(x_1) - c = 0$ and we have $g(x_1), h(x_1) - c = 0$. Differentiating both sides w.r.t. x_1:

$$\frac{\partial g}{\partial x_1}(x_1, h(x_1)) + \frac{\partial g}{\partial x_2}(x_1, h(x_1))h'(x_1) = 0 \tag{1.5}$$

.

In particular this holds for $x_1 = x_1^*$. So, function of one variables $F(x_1) = f(x_1, h(x_1))$ attains an optimum at x_1^*, which is an interior point of $D \Rightarrow F'(x_1^*) = 0 \Leftrightarrow \frac{\partial g}{\partial x_2}(x_1^*, h(x_1^*)) + \frac{\partial f}{\partial x_2}(x_1^*, h(x_1^*))h'(x_1^*) = 0$. From [1] $\frac{\partial f}{\partial x_!}(x^*) + \lambda^* \frac{\partial g}{\partial x_2}(x^*)h'(x_1^*) = 0$. Multiplying eq. [2] by λ^* and substituting eq. [4] yields

$$\lambda^* \frac{\partial g}{\partial x_1}(x^*) - \frac{\partial f}{\partial x_1}(x^*) = 0. \tag{1.6}$$

Note. We may wish to give some economic interpretation to this Lagrangian multiplier λ. Notice that $\bigtriangledown f(x^*)$ is tangent to $\bigtriangledown g(x^*)$ and λ is just a coefficient multiplying it. If

$$\Lambda = f - \lambda(g - c) = f + \lambda(c - g)$$
$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial \lambda_i^*}{\partial_c} = \sum_{i=1}^{n} \lambda \frac{\partial g}{\partial x} \frac{\partial \lambda_i^*}{\partial c}.$$

The optimal solution is a function that depends on c. We know that we have $g(x^*) = c$, hence $\frac{\partial g(x^*)}{\partial c} = 1$. And so $\frac{\partial f(x^*(c))}{\partial c} = \lambda * 1 = \lambda$, so λ is the impact of f when changing c. In economic terms, this means that λ is the shadow price that you are willing to pay to relax the constraint.

Theorem 2 - SOSC. If f and g are C^2 in the neighborhood of x^*, an interior point of D such that $\bigtriangledown g \neq 0$ and such that the $\Lambda(x,\lambda) = f(x) - \lambda g(x - x)$, then there exists $\lambda^* \in \mathbb{R}$ such that $\frac{\partial \Lambda}{\partial x_1}(x_1^*, \lambda^*) = \frac{\partial \Lambda}{\partial x_2}(x^*, \lambda^*) = 0 = \frac{\partial \Lambda}{\partial \lambda}$.

Then, the bordered Hessian matrix,

$$det \begin{pmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial^2 \Lambda}{\partial x_1^2} & \frac{\partial^2 \Lambda}{\partial x_1 \partial x_2} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial^2 \Lambda}{\partial x_2^2} & \frac{\partial^2 \Lambda}{\partial x_2 \partial x_1} \end{pmatrix}. \tag{1.7}$$

If $\det\tilde{H} > 0$, f attains a local maximum at x^*; if $\det\tilde{H} < 0$, then f reaches a minimum at x^*. **Proof**. Intuition: same as in Theorem 1, just applied to $F''(x_1)$, where $F(x_1) = f(x_1, h(x_1))$. First point exactly the same as thm 1. Assume $\frac{\partial g}{\partial x_2}(x^*) \neq 0$. Then set:

$$\frac{\partial g}{\partial x_1}(x_1^*, h(x_1^*)) + \frac{\partial g}{\partial x_2}(x_2^*, h(x_2^*))h'(x_1^*) = 0. \tag{1.8}$$

$$F'(x_1) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}(x_1^*, h(x_1^*)) + \frac{\partial f}{\partial x_2}(x_1^*, h(x_1^*))h'(x_1^*) = 0. \tag{1.9}$$

Thus, (1.8) + (1.9)$-\lambda^*$, we get:

$$F'(x_1) = \frac{\partial \Lambda}{\partial x_1}(x_1^*, h(x_1^*)) + \frac{\partial \Lambda}{\partial x_2}(x_1^*, h(x_1^*))h'(x_1^*); \tag{1.10}$$

differentiating both terms w.r.t. x, we get:

$$F''(x_1) = \frac{\partial^\Lambda}{\partial x_1^2}(x^*, \lambda^*) + \Lambda \frac{\partial^2 \Lambda}{\partial x_1 \partial x_2}(x^*, \lambda^*)h'(x_1^*) + \frac{\partial^2 \Lambda}{\partial x_2^2}(x^*, \lambda^*)[h'(x_1^*)]^2 + \frac{\partial \Lambda}{\partial x_2}(x^*, \lambda^*)h'(x_1^*). \tag{1.11}$$

From (1.9) we have that $h'(x_1^*) = -\frac{\partial g/\partial x_1(x^*)}{\partial g/\partial x_2(x^*)} \neq 0$. Substituting this in equation (1.11):

$$F''(x_1^) = \frac{1}{\left[\frac{\partial g}{\partial x_2}(x^*)\right]^2} \left[\left[\frac{\partial g}{\partial x_2}(x^*)\right] \frac{\partial^2 \Lambda}{\partial x_1^2}(x^*, \lambda^*) - 2\frac{\partial^2 \Lambda}{\partial x_1 \partial x_2}(x^*, \lambda^*)\frac{\partial g}{\partial x_2}(x^*) \right] \tag{1.12}$$

The expression between brackets is the exact negative of the determinant of the "bordered Hessian" given in the Theorem 2.

1.2.4 n>2 and m<n

FONC. Let $f, g_j : D \subset \mathbb{R}^n \to \mathbb{R} : x \to f(x), g_j(x), \forall j = 1, ..., m$ and $c \in \mathbb{R}^n$. If fa and g are C^1 in a neighborhood of x^*, and an interior point of D such that the rank of the Jacobian matrix I_g (the Jacobian matrix of the G_j's) at x^* is m. If a max or a min of f subject to constraints $g_j(x) = c_j$ occurs at x^* and if

$$\Lambda(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c), \tag{1.13}$$

then

$$\exists \lambda^* \in \mathbb{R}^m : \forall i = 1, ..., n, \frac{\partial \Lambda}{\partial x_i}(x^*, \lambda^*) = 0. \tag{1.14}$$

Note. The Jacobian:

$$I_j = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}. \tag{1.15}$$

Proof. Exactly as in 1.2.3.a (Theorem 1, just until we have more constraints and more variables).

Theorem 2 - SOSC. If f and g are C^2 in a neighborhood of x^*, being an interior point of D, at which the Jacobian is of rank n and s.t. if $\Lambda(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j)$, there exists a $\lambda^* \in \mathbb{R}^m$ s.t. $\forall i = 1, ..., n$, $\frac{\partial \Lambda}{\partial x_i}(x^*, \lambda^*) = 0$ and if $\forall j = 1, ..., m$ $\frac{\partial \Lambda}{\partial \lambda_j}(x^*, \lambda^*) = 0$. Then the bordered Hessian, look like:

$$\tilde{H}(x^*, \lambda^*) = \begin{bmatrix} 0 & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ 0 & 0 & \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} & \frac{\partial^2 \Lambda}{\partial x_i \partial x_j} & \frac{\partial^2 \Lambda}{\partial x_i \partial x_j} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} & \frac{\partial^2 \Lambda}{\partial x_i \partial x_j} & \frac{\partial^2 \Lambda}{\partial x_i \partial x_j} \end{bmatrix} \tag{1.16}$$

If the $n - m$ last principal minors alternate in sign and if $|\tilde{H}_{m+1}(x^*, \lambda^*)|$ has some sign as $(-1)^{m+1}$, then f attains a local maximum subject to the constraint $g_j(x) = c_j$ at x^*.

If those principal minors all have the same sign as $(-1)^{m+1}$, there is a local minimum at x^* subject to the constraint $g_j(x) = c_j$.

Without proof.

1.2.5 Real functions of $n > 1$ variables and $m < n$ equality constraints and positivity constraints.

Teorem 1. Let $f, g : D \subset \mathbb{R}^n \to : x \to f(x), g_j(x)$ with $g_j(x) = c_j$ and $x \geq 0$. If f and g_j are C^2 is the neighborhood of x^*, is a n interior of D, at which the Jacobian is of maximal rank (where the indexes i correspond to those of strictly positive x's). If f subject to $g_j(x) = c_j$ with $j = 1, ..., m$ and $x \geq 0$ attains a local maximum (minimum) at x^*, then:

$$\Lambda(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j) \tag{1.17}$$

, there exists $\lambda^* \in \mathbb{R}^m$ such that:

$$\begin{aligned} & xi_i^* \frac{\partial \Lambda}{\partial x_i}(x^*, \lambda^*) = 0 \\ & x_i^* \geq 0 \\ & \frac{\partial \Lambda}{\partial x_i}(x^*, \lambda^*) \leq 0 \end{aligned}$$

$\forall i = 1, .., n$.

No proof.

Example. Maximize $U(x_1, x_2) = (1 + x_1)(1 + x_2)$ subject to $4x_1 + x_2 = 1$ and $x_1 \geq 0$, $x_2 \geq 0$. Solution: $\Lambda(x, \lambda) = (1 + x_1)(1 + x_2) - \lambda(4x_1 + x_2 - 1)$, which yields the following f.o.c.s $x_1^*(1 + x_2^* - 4\lambda) = 0$, $x_2^*(1 + x_1^* - \lambda) = 0$, and $4x_1^* + x_2^* = 1$. Rewrite this system of equations as: $x_1^*(2 - 4x_1^* - 4\lambda^*) = 0$, $(1 - 4x_1^*)(1 + x_1^* - \lambda) = 0$, and $x_2^* = 1 - 4x_1^*$. $\rightsquigarrow$ **1.** $x_1^* = 0$ from first equation; which implies that $\lambda^* = 1$; which means that $x_2^* = 1 \Rightarrow \frac{\partial \Lambda}{\partial x_1}(0, 1, 1) = -2 < 0$ and $\frac{\partial \Lambda}{\partial x_2}(0, 1, 1) = 0 \leq 0$. Ok, local max. **2.** $x_1^* = 1/4$ from second equation $\Rightarrow \lambda^* = 1/4 \Rightarrow x_2^* = 0 \Rightarrow$ positivity constraint ok. $\frac{\partial \Lambda}{\partial x_1}(1/4, 0, 1/4) \geq 0$, $\frac{\partial \Lambda}{\partial x_2}(1/4, 0, 1/4) = 0 \geq 0 \rightsquigarrow$ local min. **3.** $x_1^{=} 1/2 \Rightarrow (-1 + 4\lambda)(3/2 - 2\lambda) = 0$ $\Rightarrow \lambda = 1/4$, $x_1 = 1/4$, $x_2 = 0$ or $\lambda = 3/4$, $x_1 = -1/4$ not good! **4.** $x_1 = \lambda = 1$, $x_1 = 0$, $x_2 = 1$ or $\lambda = 3/4$, $x_1 = -1/4 < 0$, not good!

1.2.6 Real functions of $n > 1$ variables with inequality constraints (Kuhn and Tucker I).

Let $f, g_j : D \subset \mathbb{R}^n \to \mathbb{R} : x \to f(x), g_j(x)$, where $b \subset \mathbb{R}^n, \forall i = 1, ..., m$. We will look at the problem where $m = 1$. **Problem**: $\max_{x \in D} f(x)$ s.t. $g(x) \leq b$. The essential breakthrough considering this problem came from Kuhn and Tucker, who proposed to look at it as an equality constraint. This was because at that point, the equality constraint case was already known how to be solved. They added a so called "slack" variable z to the problem, which reflects the difference between the left hand side and the right hand side of the inequality constraint. Since, when $g << 0$, it will need to be brought up to zero, and for this we need to add a positive quantity to the left hand side. The result of the transformed problem is $\max_{x\in D} f(x)$s.t.$g(x)+z^2 = b$. (This subsection of the program is different than that of last year). Then you can just set up a Lagrangian, $\Lambda(x,\lambda) = f(x) + \lambda(b - g(x) - z^2)$. Set of variables is $\{x, z, \lambda\}$. The F.O.C. are:

$$\begin{aligned}
&\frac{\partial\Lambda}{\partial x_i} = f_i - \lambda_i = 0\\
;&\frac{\partial\Lambda}{\partial z} = -2\lambda_2 = 0;\\
&\frac{\partial\Lambda}{\partial\lambda} = -(g(x) + z^2) = 0.
\end{aligned}$$

If $\lambda < 0$, we have a problem $\rightsquigarrow \lambda \geq 0$. Since the second F.O.C. is equivalent to $-\lambda z^2 = 0$ (by multiplication of 2/2) we can incorporate the third F.O.C. into this $\rightsquigarrow \lambda g(x) = 0$. Knowing that $z^2 = b - g(x)$, b assumed to be zero $\rightsquigarrow b \geq g(x) \rightsquigarrow$

$$\begin{aligned}
&f_i - \lambda g_i = 0\\
&\lambda g(x) = 0\\
&\lambda \geq 0\\
&b \geq g(x).
\end{aligned}$$

Example. $\max_{x\in D} bx_1x_2$ such that $2x_1 + x_2 \leq 10$. Write the Lagrangian as $\Lambda = 6x_1x_2 + \lambda(10 - 2x_1 - x_2 - z^2) \rightarrow \lambda \geq 0$. Kuhn and Tucker condition implies that $\frac{\partial\Lambda}{\partial x_1} = 0 \Rightarrow 6x_2 - 2\lambda = 0$ and $\frac{\partial\Lambda}{\partial x_2} = 0 \Rightarrow 6x_1 - \lambda = 0$, also $\frac{\partial\Lambda}{\partial z} = 0 \Rightarrow \lambda(10 - 2x_1 - x_2) = 0$. So $2x_1 + x_2 \leq 10$. **Case 1:** $\lambda > 0 \Rightarrow \frac{6x_2}{6x_1} = 2 \Rightarrow x_2 = 2x_1$. $\lambda \neq 0 \Rightarrow 2x_1 + x_2 = 0 \Rightarrow x_1 = 2 - 5$, $\Rightarrow x_2 = 5$. Check the inequality $2 \times 2.5 + 5 \leq 10$, ok, then $f(2.5, 5) > 0$. **Case 2:** $\lambda = 0 \Rightarrow x_1 = x_2 = 0$ and the inequality $2 \times 0 + 0 \leq 10$, ok, $f(0, 0) = 0$.

1.2.7 Real function of $n > 1$ variables with inequality constraints and positivity constraints (Kuhn and Tucker II).

Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R} : x \to f(x), g_j(x), j = 1, ..., m$ and $c \in \mathbb{R}^m$. If f and g are C^1 in the neighborhood of x^*, an interior point of D, such that the Jacobian calculated in x^* has maximum rank. If f subject to the constraints $g_j(x) - c_j \leq 0$ and $x \geq 0$ attains a max (min) at x^* if $\Lambda(x, \lambda) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j)$. Then there exists $\lambda^* \in \mathbb{R}^m$ such that $\frac{\partial \Lambda}{\partial x_1}(x^*, \lambda^*) \leq 0$, $x_i^* \geq 0$ and $x_i^* \frac{\partial \Lambda}{\partial x_1}(x^*, \lambda^*) = 0$. And $\frac{\partial \Lambda}{\partial \lambda_j}(x^*, \lambda^*) = 0$, $\lambda_j \geq 0$ and $\lambda_j^* \frac{\partial \Lambda}{\partial \lambda_j}(x^*, \lambda^*) = 0$.
Example. $\text{Max}_{x,y} f(x, y) = 2x + 2y - x^2 - y^2$ s.t. $x + y \leq 1$, and $x, y \geq 0$. $\Lambda(x, y, \lambda) = 2x + 2y - x^2 - y^2 - \lambda(x + y - 1)$, take the first partial derivatives w.r.t. to the different variables of the model and get $x(2 - 2x - \lambda) = 0$; $y(2 - 2y - \lambda) = 0$; and $\lambda(x + y - 1) = 0$. **Case 1:** $(0, 0, 0)$; **case 2.** $(0, 0, x+y = 1) \rightsquigarrow$ contradiction; **case 3.** $(0, 1, 0)$; **case 4.** $(0, 1, x+y = 1) \rightsquigarrow$ (0,1,0); **case 5.** $2 - 2x - \lambda = 0$, $\lambda = 0$, $y = 0$, $x = 0 \rightsquigarrow$ (1,0,0). **case 6.** $2-2x-\lambda = 0$, $y = 0$, $x+y = 1 \rightsquigarrow$ (1,0,0); **case 7.** $2-2-\lambda = 0$, $2-2y-\lambda = 0$, $\lambda = 0 \rightsquigarrow$ contradiction; **case 8.** $2 - 2x - \lambda = 0$, $2 - 2y - \lambda = 0$, $x + y = 1$, $x = y = \frac{1}{2}$, $(\frac{1}{2}, \frac{1}{2}, 1)$.

1.3 Errata corrige

Inequality constraints (Kuhn -Tucker). Equality $\max_x f(x)$ s.t. $g(x) - z^2 = b$. Lagrangian $\Lambda(x, \lambda) = f(x) + \lambda(b - g(x) - z^2)$. FOC $\Rightarrow \frac{\partial \Lambda}{\partial \lambda} = 0 \Rightarrow (b - g(x) - z^2) = 0 \Rightarrow -[g(x) + z^2 - b] = 0$.

Chapter 2

Envelope Theorems and Integrals

2.1 Integrals

In economics, we mainly use integrals for two purposes:
i. to compute the primitive of a given function, e.g. we have the marginal profit and we want to reconstruct profit function to retrieve the state variables $y(t)$. The fundamental theorem of calculus says that $\int_a^x f(t)dt = F(x)$ and $F'(x) = f(x)$. We can go back and forth, using derivatives and integrals. The only information that probably we lose is the additive constant, allied constant of integration.
Note: *i.* to deal with the constant of integration you need boundary condition; *ii.* it is needed that f is continuous to obtain differentiability of F.
Example: $f : x \rightarrow= 1 \text{if} 0 \leq x < 1; 2 \text{if} 1 \leq x \leq 2$.
We can compute the integral, given the property $\int_a^b f = \int_a^c f + \int_c^b f$. For example, if $\int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt = [t]_0^1 + [2t]_1^2$. So, at each separate integral, we obtain a differentiable function. However, gluing the two together gives us a non-differentiable function. And this is only at the point 1. $F : t \rightarrow t$ if $0 \leq t < 1$ and $2t$ if $1 \leq t \leq 2$. If f is not continuous, cut it in parts and remember that you can only obtain non differentiability in points where you cut.
ii. The second application is to compute "continuous sums". In dynamic optimization we often have to sum the $y(t)$ for any $t \in 0, T[$. This can be seen as an infinite, continuous sum and to compute it we use integrals $\int_0^T y(t)dt$. So, for example consider, $\int_a^b f(t)dt$ = surface between a and b and f and the x-axis. Rienmann integral tells you that you have

to partition the integral into infinitesimal parts and then sum their areas up in order to approach more and more the function and have less and less empty spaces between the rectangles and the graph of the function. Lower sums $s(f,P) = \sum_{i=1}^{n}(x_i - x_{i-1})\inf[x_{i-1},x_i]$ or too big $\rightsquigarrow$ upper sum $S(f,P) = \sum_{i=1}^{n}(x_i - x_{i-1})\sup[x_{i-1},x_i]$. A function is then said to be Rienmann-integrable if and only if $\lim_P s(f,P) = \lim_P S(f,P)$, meaning that approximations should tend to the same number.
Remarks: *i.* this method only works if and only if f is bounded. Otherwise, the sum could be equal to ∞; *ii.* a strange way to reformulate the surface is saying that you "add up the lines".
Some properties: $\int_a^b = \int_a^c f + \int_c^b f$; $\int_a^b kf = k\int_a^b f$; $\int_a^b fg \neq \int_a^b f \int_a^b g$[1].

2.1.1 Main tricks for integral calculation.

i. Substitution. This is simply reformulating the integral such that you recognize the primitive. E.g. we all know that $\int e^t dt = e^t + te$. But first sight, $\int(3t^2+2t)e^{t^3+t^2}dt$ can seem complicated. We may simply introduce a new variable $y = t^3 + t^2$, then $dy = (3t^2+2t)dt$ and so we get $\int(3t^2+2t)e^{t^3+t^2} = \int e^y dy = e^y + cte = e^{t^3+t^2} + cte$.
Remarks: **i.** it is not allowed to use y and t at the same time; **ii.** normally, you end the computation by going back to your original variable; **iii.** inspiration for a good substitution comes from known integrals.
Exercises: 1. $\int_0^1 \frac{t}{t^2+1}dt$; 2. $\int_0^1 \frac{t^3}{t^2+1}dt$.
1. Let $\dot{y} = t^2 + 1 \rightsquigarrow dy = 2tdt$. $t = 0 \rightsquigarrow y = 1$, $t = 1 \rightsquigarrow y = 2$. $\int_0^1 \frac{t}{t^2+1} = \frac{1}{2}\int_1^2 \frac{1}{y}dy = \frac{1}{2}[lny]_1^2 = \frac{ln2}{2}$.
ii. Partial Integration.[2] This trick stems from of insight of taking differentials: $\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} \rightsquigarrow \int\frac{d}{dx}(f(x)g(x)) = \int\frac{df(x)}{dx}g(x) + \int f(x)\frac{dg(x)}{dx}$, then by the fundamental theorem of calculus we have that $f(x)g(x) = \int\frac{df(x)}{dx}g(x) + \int f(x)\frac{dg(x)}{dx}$ and thus $\int\frac{df(x)}{dx}g(x)dx = f(x)g(x) - \int f(x)\frac{dg(x)}{dx}dx$. So if your integral can be written as something like the l.h.s., we can replace it by the r.h.s.
Example: $\int te^t dt = \int\frac{de^t}{dt}tdt = e^t t - \int e^t 1dt = e^t - e^t + c$.
Note: this trick is often used is one has some exponential functions. Inspiration come from the fact that you want to eliminate g by taking derivatives.
Example: $\int_0^1 sin(2\pi t)e^t dt \rightsquigarrow df = sin(2\pi t) \rightsquigarrow f = \frac{-1}{2\pi}cos2\pi t$, thus $g = e^t$ $\rightsquigarrow dg = e^t$, the original expression is equal to $\left[\frac{1}{2\pi}cos(\pi t)e^t\right]_0^1 + \int\frac{1}{2\pi}cos2\pi te^t dt$,

[1] By chain rule, for composite functions you have $\rightsquigarrow (fg)' = f'g + fg'$.
[2] Also known as integration by parts.

recall that $df = cos2\pi t \rightsquigarrow f = \frac{1}{2\pi}sin2\pi t$, then we get: $\big[-\frac{1}{2\pi}cos2\pi te^t\big]_0^1 + \frac{1}{2\pi}\Big[\big[\frac{1}{2\pi}sin2\pi te^t\big] - \int_0^1 \frac{1}{2\pi}2\pi te^t dt\Big] = \big[\big(-\frac{1}{2\pi}cos2\pi t + \frac{1}{4\pi^2}sin2\pi t\big)e^t\big]_0^1 - \frac{1}{4\pi^2}\int_0^1 \frac{1}{2\pi}sin2\pi te^t dt$
$\Rightarrow \int_0^1 sin(2\pi t)e^t dt = -\frac{1}{1+\frac{1}{4\pi^2}}\big(\frac{1}{2\pi}e^1 - \frac{1}{2\pi}\big)$.

2.1.2 Extensions

Improper Integrals

As a standard definition we use bounded functions on bounded intervals. As such it is obvious that we will obtain a finite surface. But what if our time horizon goes to infinity or our function is unbounded. E.g. $\int_0^1 \frac{1}{t}dt$, $\int_1^\infty \frac{1}{t^2}dt$, $\int_0^\infty \frac{1}{t}dt$. We can still compute integrals, we may simply may the functions and integrals bounded by cutting off the interval and in a second step taking limits. **Example:** $\int_0^1 \frac{1}{t}dt = \lim_{a\to 0}\int_a^1 \frac{1}{t}dt = \lim_{a\to 0}[\ln t]_a^1 = \lim_{a\to 0} - \ln a$.
$\int_1^\infty \frac{1}{t^2}dt = \lim_{a\to +\infty}\int_1^a \frac{1}{t^2}dt = \lim_{a\to +\infty}\big[\frac{1}{t}\big]_1^a = \lim_{a\to +\infty}\big(-\frac{1}{a}+1\big) = 1$.
Remark: so it can be an infinite or a finite number.
Exercise: $\int_0^\infty \frac{1}{x^n}dx$ with $n \neq 1$.
Solution: $\int_0^\infty \frac{1}{x^n}dx = \int_0^1 \frac{1}{x^n}dx + \int_1^\infty \frac{1}{x^n}dx \rightsquigarrow \int_0^1 \frac{1}{x^n}dx = \lim_{a\to 0}\frac{1}{1-n} + \frac{a^{1-n}}{n-1} = \frac{1}{1-n}$ if $n < 1$ and $= \infty$ if $n > 1$. On the other side, $\int_1^\infty \frac{1}{x^n}dx = \lim_{a\to +\infty}\Big[\frac{x^{-n+1}}{-n+1}\Big] = \lim_{a\to +\infty}\frac{a^{-n+1}}{-n+1} + \frac{1}{n-1} = \frac{1}{n-1}$ if $n > 1$ and $= +\infty$ if $n < 1$.

Integrals with multiple variables

In a similar way as far as for simple variable functions you define lower and upper sums, but you now have to partition a rectangle instead of an interval. E.g. $n = 2 \rightsquigarrow [a_1, b_1]\times[a_2, b_2]$. Otherwise $\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)dx_n, ..., dx_1$.
Example: $\int_a^1\int_3^4 (x^2ty^2)dydx = \int_a^1 [x^2y + \frac{y^3}{3}]_3^4 dx = \int_0^1 (4x^2 + \frac{64}{3} - 3x^2 - 9)dx = \int_0^1 (x^2 + \frac{37}{3}x)dx = \big[\frac{x^3}{3} + \frac{37}{3}x\big]_0^1 = \frac{38}{3}$.
There is however a complication. Before it was natural to start from a given interval, but now we start from several integration areas $\int\int_D f(x,y)dydx$. As long as D is a bounded set, then it can be enclosed in a rectangle and we can switch the integrals if we know that the function is continuous ($\Rightarrow$ Fubini's Thm.). D can be sometimes called *regular*, which means that its boundaries are functions. For example, having two functions $y = 2x$ and $y = x^2$, you want to integrate over the domain lying between the two functions. You get $x = \frac{y}{2}$ and $x = \sqrt{y}$. $D = \{(x,y)|0 \leq x \leq 1, x^2 \leq y \leq 2x\} = \{(x,y)|0 \leq y \leq 1, \frac{y}{2} \leq x \leq \sqrt{y}\}$ v $\{(x,y)|1 \leq y \leq 2, \frac{y}{2} \leq x \leq 1\}$.

Note: i. The order is important! A volume is a number so the function in the boundaries should be included in the inner integrals. ii. You have to compute several times a simple integral. So intuitively the properties for the simple integral hold here. iii. One exception, which is substitution. This method becomes more complicated because you have to use the Jacobian matrix.

Example: 1. $\int_{-1}^{1}\int_{0}^{\sqrt{1+x^2}} e^{(x^2+y^2)}dydx = x = r\cos\theta$ and $y = r\sin\theta \rightsquigarrow y = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ and $dydx = |J|drd\theta = r(\cos^2\theta + \sin^2\theta)drd\theta = rdrd\theta$.

Then $(*)(*) = \int_0^1\int_0^\pi re^{r^2}d\theta dr = \int_0^1 \pi re^{r^2}dr = \frac{\pi}{2}\left[e^{r^2}\right]_0^1 = \frac{\pi}{2}(e^1 - 1)$.

Compute $\int\int_D \frac{x}{y}e^y dxdx \rightsquigarrow \int_0^1\int_y^{\sqrt{y}} \frac{x}{y}e^y dxdx = \int_0^1\left[\frac{x^2e^y}{2y}\right]_y^{\sqrt{y}}dy = \int_0^1\left[\frac{e^y}{2} + \frac{ye^y}{2}\right]dy = \left[e^y\frac{1+y}{2}\right]_0^1 - \int_0^1\frac{e^y}{2}dy = e - \frac{1}{2} - \left[\frac{e^y}{2}\right]_0^1 = \frac{e}{2} - 1$. Second part is $\int_0^1\int_{x^2}^{x}\frac{xe^y}{y}dydx = \int_0^1 x(\int_{x^2}^{x}\frac{e^y}{y}dy)dx$.

Exercises:

1. $\frac{d}{da}\int_a^b 3e^{t^2}dt$;
2. $\frac{d}{dx}\int_{x^2}^{e^2}\sin txdt$;
3. $\frac{d}{da}\int_5^{\ln x}(t+x)^2dt$;
4. $\int_3^7\frac{2t^2+1}{4t^3+6t+5}dt$;
5. $\int_1^4(2t+5)e^{t^2+3}dt$;
6. $\int_1^4 t\ln tdt$;
7. $\int_0^1 t^2e^{3t}dt$;
8. $\int_0^\infty e^{-ax}dx$;
9. $\int_0^1\int_0^1 e^{2x+3y}dydx$;
10. $\int\int_D(xty)dxdy$ with $y = 3x$ and $y = x$.

Resolutions:

1. $\frac{d}{da}\int_a^b 3e^{t^2}dt = \frac{3d}{da}\int_a^b e^tdt = \frac{d}{da}3\left[e^t\right]_a^b = \frac{3d}{da}(e^b - e^a) = -3e^{a^2}$, because $\frac{d}{da}\int_a^b 3e^tdt = \frac{d}{da}3\int_a^b e^{t^2}dt = -3e^{a^2}$.

2. $\frac{d}{dx}\int_{x^2}^{e^2}\sin txdt$ solve it by substitution assuming that $u = tx$, then $\frac{d}{dx}\int_{x^2}^{e^{2x}}\sin\ udu = \frac{d}{dx}\left[-\cos u\right]_{x^2}^{e^{2x}} = -\left[\frac{u}{t}\right]^2$, or otherwise:
$\int_{x^2}^{e^{2x}}\sin(tx)dt = \int\sin udu = \frac{1}{x}\left[-cos(tx)\right]_{x^2}^{e^{2x}} = \frac{1}{x}\left[-\cos(e^{2x}x) + \cos(x^3)\right]$.

Formula for $F : \mathbb{R}^+ \to \mathbb{R} : x \to \int_0^\infty\frac{e^{-xt}-e^{-t}}{t}dt$ (Hint: compute first a formula for $\frac{dF}{dt}$, i.e. derive under the sign of integral!).

$\frac{\partial f(x,t)}{\partial x} = \left[\frac{e^{-xt}-e^{-t}}{t}\right]' = \left[\frac{e^{-xt}}{t} - \frac{e^{-t}}{t}\right]' = \frac{1}{t}\left[e^{-xt} - e^{-t}\right]' = \frac{1}{t}\left[\frac{1}{e^{xt}} - \frac{1}{e^t}\right]' = \frac{1}{t}\left[\frac{1}{te^{xt}} - \frac{1}{e^t}\right] = \frac{1}{t^2e^{xt}} - \frac{1}{te^t} = \frac{1}{te^t}\left[\frac{1}{te^x} - 1\right] = \frac{1}{te^t}\left[\frac{1-te^x}{te^x}\right] = ... = t\frac{1}{te^{xt}} = \int_0^\infty\frac{1}{e^{xt}}dt =$

$\left[+\frac{e^{xt}}{x}\right]_0^\infty = \frac{0}{x}$. $\frac{dF}{dx}F(x) = -\frac{1}{x} \Rightarrow F(x) = \ln x + c$. $F(1) = c \Rightarrow c = 0$. Thus, $\int_0^\infty \frac{e^{-1t}-e^{-t}}{t}dt$.

2.1.3 Leibniz's rules

Namely, differentiation under the sign of integral.
Fundamental Theorem of Calculus. Let $f : [a,b] \to \mathbb{R} : x \to f(x)$ be a continuous function on $[a,b]$. Then, $F(x) = \int_a^x f(t)dt$ is a primitive of $f(x)$, i.e. $F'(x) = f(x), \forall x \in [a,b]$. To prove the theorem, we need the following lemma (mean-value theorem). **Lemma.** Let f be continuous on $[a,b]$, then $\exists$ a $\xi \in [a,b]$ such that $\int_a^b f(t)dt = f(\xi)(a-b)$. **Proof.** Assume $m = \inf_{[a,b]} f$, $M = \sup_{[a,b]} f$, and $m \leq f \leq M$, $\forall x \in [a,b] \Rightarrow$ it holds $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a) \Leftrightarrow m \leq \frac{1}{b-a}\int_a^b f(t)dt \leq M$. As f is continuous, it takes all the values between m and $M \Rightarrow \exists$ a $\xi \in [a,b]$ s.t. $f(\xi) = \frac{1}{b-a}\int_a^b f(t)dt \Leftrightarrow (b-a)f(\xi) = \int_a^b f(t)dt\square$.
Proof of the theorem. Assume $x \in]a,b[$, let Δx such that $x + \Delta x \in]a,b[$, then $F(x+\Delta x) = \int_a^{x+\Delta x} f(t)dt \Rightarrow F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_a^x f(t)dt + \int_a^{\Delta x} f(t)dt - \int_a^x f(t)dt = \int_a^{\Delta x} f(t)dt$. From the lemma, $x = f(\xi)(x+\Delta x - x)$ for some $\xi \in [x, x+\Delta x]$. Then, $F(x+\Delta x) - F(x) = \Delta x f(\xi) \Rightarrow f(\xi) = \frac{F(x+\Delta x)}{\Delta x}$, taking limits, $\lim_{\Delta x \to 0} \frac{F(x+\Delta x)}{\Delta x} = f(x)$, for $\xi \to x$, since f is continuous. Same argument holds for $\Delta x < 0$. $F'(x) = f(x) \to \forall x \in]a,b[\to F'(x+\Delta x)$. For $x = a \to F'_r(x)$ and for $x = b \to F'_l(x)$.

Theorem 2. Let $f : [a,b] \Rightarrow \mathbb{R} : t \Rightarrow f(t)$ be continuous on its domain. Let $u(x)$ and $v(x)$ be differentiable functions on $[c,d] \Leftrightarrow u, v : [c,d] \longrightarrow [a,b] : x \longrightarrow u(x), v(x)$ and $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f'(v(x))v'(x) - f'(u(x))u'(x)$, $\forall x \in [c,d]$. **Remark.** This comes generalizing the former theorem: take $u(x) = a$ and $v(x) = x$, $\forall x \Rightarrow \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(x)x' - f(a)a' = f(x)$.
Proof. Let $F(x)$ be a primitive for f. Then, $h(x) = F(v(x)) - F(u(x))$. Also $h'(x) = F'(v(x))v'(x) - F'(u(x))u'(x)\square$.
Example. $\frac{d}{dt}\int_{2x+3}^{x^2} 3t^2 dt$. Two ways to solve this. (1) $\frac{d}{dx}\left[[x^2]^3 - (2x+3)^3\right] = 6x^5 - 6(2x+3)^2$. (2) Using theorem 2, we may proceed as follows $(3x^2)^2 2x - 3(2x+3)^2 2 = 6x^5 - 6(2x+3)^2$.
Remark. Sometimes way (1) doesn't work. For example $\frac{d}{dx}\int_0^{x^2} e^{-t^2}dt$ doesn't have a closed form solution.

Leibniz rule #1. Let $f : [c,d] \times [a,b] \longrightarrow \mathbb{R} : (x,t) \longrightarrow f(x,t)$ be C^1 on

$[c,d]\times[a,b]$. Then, $\int_a^b f(x,t)dt$ is differentiable on $[c,d]$ and $\frac{\partial}{\partial x}\int_a^b f(x,t)dt = \int_a^b \frac{\partial}{\partial x} f(x,t)dt = \int_a^b f_x(x,t)dt$.
Notes. (1) Actually, we only need f being C^1 w.r.t. x and continuous w.r.t. t. (2) The proof uses the mean-value theorem of Lagrange.
Lemma. Let $g : [a,b]$ be continuous on $[a,b]$, differentiable on $]a,b[$ such that $g(b)-g(a) = g'(\xi)(b-a) \Leftrightarrow \frac{g(b)-g(a)}{b-a} = g'(\xi)$.
Proof. Let $I(x) = \int_{a(x)}^{b(x)} f(x,t)dt \Rightarrow I'(x) = \lim_{\Delta x\to 0}\frac{I(x+\Delta x)-I(x)}{\Delta x}$. Then, $I(x+\Delta x) - I(x) = \int_{a(x+\Delta x)}^{b(x+\Delta x)} f(x+\Delta x,t)dt - \int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)}[f(x+\Delta x,t) - f(x,t)]dt + \int_{a(x)}^{a(x+\Delta x)} f(x+\Delta x,t)dt - \int_{b(x)}^{b(x+\Delta x)} f(x+\Delta x,t)dt$. Now let's analyse the single components of the latter equality. The first block $= \int_{a(x)}^{b(x)} f(\xi,t)(x+\Delta x - x)dt$ for some ξ s.t. $|\xi - x| \leq \Delta x$ (M. value). The second block $= f(x+\Delta x\xi)[b(x_1+\Delta x)-b(x)]$ for some $\xi \in [b(x)+b(x+\Delta x)]$. Finally, the third block is transformed analogously than the second one.
$\lim_{\Delta x\to 0}\frac{I(x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(1)}{\Delta x}\frac{(2)}{\Delta x}\frac{(3)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(1)}{\Delta x} + ...$
$\lim_{\Delta x\to 0}\frac{(1)}{\Delta x} = \lim_{\Delta x}\int_{a(x)}^{b(x)} f(\xi,t)dt = \int_{a(x)}^{b(x)} f_x(x,t)dt$, $\xi \in [x, x+\Delta x]$.
$\lim_{\Delta x\to 0}\frac{(2)}{\Delta x} = \lim_{\Delta x\to 0} f(x+\Delta x\xi)\frac{b(x+\Delta x)-b(x)}{\Delta x} = f(x,b(x))b'(x)$.
$\lim_{\Delta x\to 0}\frac{(3)}{\Delta x} =$ similar.

Leibniz rule #2. If $f(x,t)$ is C^1 on $[c,d]\times[a,b]$, if $a(x)$ and $b(x)$are C^1 on $[c,d]$ with values in $[a,b]$, the $\int_{a(x)}^{b(x)} f(x,t)dt$ is differentiable on $[c,d]$ and $\forall x \in [c,d] : a(x) \Rightarrow \frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} f_x(x,t)dt + f(x,b(x))b'(x) - f(x,a(x))a'(x)$. This theorem is a fusion between the previous two.

Leibniz rule #3. Let $f(x,t) \in C^0$ on $[c,d]\times[a,b]$, $\alpha,\beta \in [a,b]$. The $\in_\alpha^\beta\ f(x,t)dt$ is differentiable with respect to α and β on $[a,b] \Rightarrow \alpha$ and $\frac{\partial}{\partial\beta}\int_\alpha^\beta f(x,t)dt = f(\alpha,\beta)$. $\frac{\partial}{\partial\alpha}\int_\alpha^\beta f(x,t)dt - f(x,\alpha)$.
Proof. example $\to$ intuition.
Example. $\frac{d}{dp}\int_0^{p^2}\frac{\partial}{\partial p}[t^2(p^2-t)]dt = (p^3)^2(p^2-p^3)3p^2 = \int_0^{p^3} pt^2dt + 3p(1-p) \to$ no!

2.2 Envelope Theorems

In any optimization problem (constrained or not), if f involves some parameters, one could wonder how the optimal value of the function changes as a function of the changes in the parameters. Let $f(x,\alpha)$ be C^1 for $x \in D$, an open subset of $\Re^n$, $\alpha \in \Re^s$, $x = (x_1,...,x_n)$, $\alpha = \alpha_1,...,\alpha_s$. For each α, consider the problem $\{\max\ f'(x,\alpha)\}$. Assume that $x^*(\alpha)$ is a solution to the problem of maximization, $x^*(\alpha)$ is $C^1(\alpha)$.

We have $\frac{d}{d\alpha_s} = [f(x^*(\alpha),\alpha)] = \frac{\partial f}{\partial \alpha}(x^*(\alpha),\alpha)$, for $\forall s = 1,...,S$. Recall that the notation $\frac{\partial .}{\partial .}$ differs from $\frac{d.}{d.}$ because it is suitable for partial derivatives, i.e. derivatives of functions of more than one variable.

Remark: only the direct effect of α_s on f is important! Not the influence it has on $x^*(\alpha)$.

Proof: $V(\alpha) = f(x^*(\alpha),\alpha) \rightarrow V$ is C^1 in α.

$$f(x^*(\alpha), x_n^*(\alpha),\alpha) \Rightarrow \frac{\partial V}{\partial \alpha_s} = \sum_{i=1}^{n} \frac{\partial f(x^*(\alpha),\alpha)}{\partial x_1}\frac{\partial x_i^*(\alpha)}{\partial \alpha_s} + \frac{\partial f(x^*(\alpha),\alpha)}{\partial \alpha_s} = \frac{\partial f}{\partial x_s}(x^*(\alpha),\alpha). \tag{2.1}$$

Let $f, g_j \in C^1$ on D, open subset of $\Re^n$, $c \in \Re^n$. Assume $x^*(\alpha)$ is a solution of the problem max $f(x,\alpha)$ subject to $g(x,\alpha) = c_j$. Assume $x^*(\alpha), \lambda_j^*(\alpha)$ (Lagrangian multipliers) are C^1 on $\alpha \Rightarrow$ Jacobian matrix of the constraints $\frac{\partial g_i}{\partial x_i}(x^*(\alpha),\alpha)$ is of full column rank $\forall \alpha$. Then

$$\frac{\partial}{\partial \alpha_s}\left[f^*(\alpha),\alpha\right] = \frac{\partial L}{\partial \alpha_s}\left[x^*(\alpha),\alpha\right], s = 1,...,S. \tag{2.2}$$

The Lagrangian equation is then:

$$L(x,s,\alpha) = f(x,\alpha) = f(x,\alpha) - \sum_{i=1}^{n} \lambda_j \left[(g_i)(x,\alpha) - c_j\right]. \tag{2.3}$$

Remark: Hotelling's Lemma is a version of the general Envelope Theorem.

Application: An efficient firm minimizes costs for a given output level. The problem is $Min_{x_1,x_2}(p_1x_1 + p_2x_2) = C(x_1,x_2)$ such that $f(x_1,x_2) = y$ $\Rightarrow \frac{\partial C(x_1,x_2)}{\partial y} = \frac{\partial L}{\partial y} = \frac{\partial(p_1x_1+p_2x_2-\lambda(f(x_1,x_2))-y)}{\partial y} = \lambda$.

$\Rightarrow \lambda$ is the marginal cost of producing an additional unit of output or the willingness to pay - *shadow price* - for an extra output.

1 variable, 1 parameter $\Rightarrow x^*(\alpha) \to x^* = \emptyset(\alpha)$.

$x^* \equiv$ stationary point $\to \frac{\partial f}{\partial x}(x^*(\alpha), \alpha) = 0 \to \frac{\partial^2 f}{\partial^2 x}(x^*(\alpha), \alpha)\frac{dx}{d\alpha} + \frac{\partial^2 f(x^*,\alpha)}{\partial x \partial \alpha} f(x^*, \alpha)$

$\to \frac{d\phi}{d\alpha} = \frac{dx^*}{d\alpha} = -\frac{\frac{\partial^2 f}{\partial x \partial \alpha}(x^*,\alpha)}{\frac{\partial^2 f}{\partial x^2 (x^*,\alpha)}}$.

Chapter 3

Differential Equations

3.1 Introduction

Definition. A differential equation as being an equation of the type

$$F(t, y(t)y'(t), y''(t), ..., y'^{n}(t))$$

The order of a differential equation is the highest order of differentiation which appears in the equation. For example in

$$y'' + 3y' + 2 = 0$$

is of the second order. A solution to a differential equation is a function $y(t)$ verifying the equation. The differential equation is linear if

$$F(y, y, y', ...) = y'^{n}, f_n(t) + y'^{n-1} f_{n-1}(t) + ... + y' f_1(t) + f_0(t) = 0$$

The degree of a differential equation is of the highest power of the highest derivative, e.g.

$$\begin{aligned} y'' + 4y' + y = 4 &\rightsquigarrow \text{degree1} \qquad (3.1)\\ y''^{2} + 4y'^{2} + y = 4 &\rightsquigarrow \text{degree2} \end{aligned}$$

If F is linear, then it is of degree 1.
Issues. (i) We want to find all the solutions that satisfy the equation. (ii) Solve a *Cauchy Problem*: find a solution $y(t)$ meeting a set of initial conditions $y(t_0) = y_0, y'(t_0) = y'_0, ..., y'^{n-1}(t_0) = y'^{n-1}$, where $y_0, y'_0, ...y_0'^{n-1}$ are given numbers. (iii) Solve a *limit problem*: find a solution $y(t)$ such that $y(t_0) = y_0, y(t_F) = y_F$, where y_0, y_F are given numbers.

3.2 First order differential equations

3.2.1 Separable variables' differential equations

Type $\rightsquigarrow y'(t) = f(t)g(y)$, with $g(y) \neq 0$, $f, g \in C^0$ given. Method $\rightsquigarrow \frac{y'(t)}{g(t)} = f(t)$. If $y(t)$ is the solution in C^1, then $\frac{y'(t)}{g(t)} = f(t) \Leftrightarrow \int_{t_0}^{t} \frac{y'(t)}{g(t)} = \int_{t_0}^{t} f(t)$. Define y as $y(t) \rightsquigarrow dy = y'(t)dt \rightsquigarrow y'(t) = \frac{dy}{dt}$ and $y(t_0) = t_0 \Leftrightarrow \int_{y_0}^{y} \frac{dy}{dt} = \int_{t_0}^{t} f(t)dt$. If we assumed that $\int \frac{dy}{dt} = G(y) + c$ and $\int f(t)dt = F(t) + c \Leftrightarrow G(y) - G(y_0) = F(t) - F(t_0)$. Example:

$$\begin{aligned}
&y' = \frac{t}{3y^2}, y \neq 0 &(3.2)\\
&3y^2y' = t \Leftrightarrow 3(y(t))^2 y'(t) = t\\
&3\int_{t_0}^{t} y^2 y' dt = \int_{t_0}^{t} tdt\\
&\int_{y_0}^{y} y^2 dy = \int_{t_0}^{t} tdt\\
&[y^3]_{y_0}^{y} = [\frac{1}{2}]_{t_0}^{t}\\
&y^3 - y_0^3 = \frac{t^2}{2} - \frac{t_0^2}{2}\\
&y = [\frac{t^2}{2} - \frac{t_0^2}{1} + y_0^2]^{1/3}.
\end{aligned}$$

From there the solution of the differential equation which corresponds to the Cauchy problem where we have the condition $y(0) = 3 \rightsquigarrow t = 0 \rightsquigarrow y(t) = [\frac{t^2}{2} + 27]^{1/3}$. Example: $y' = -\frac{y}{2t} \rightsquigarrow \frac{dy}{dt} = -\frac{y}{2t} \rightsquigarrow \frac{dy}{y} = -\frac{2t}{dt} \rightsquigarrow \int \frac{dy}{y} = -\frac{1}{2}\int \frac{1}{t}dt + c$. If $t_0 > 0$, $y_0 > 0 \rightsquigarrow \ln y - \ln y_0 = -\frac{1}{2}(\ln t - \ln t_0) \rightsquigarrow \ln \frac{y}{y_0} = -\frac{1}{2}\ln\frac{t}{t_0}$. If $t_0 > 0$, and $y_0 < 0$, $\rightsquigarrow \ln(-y) - \ln(-y_0) = -\frac{1}{2}(\ln t - \ln t_0) \rightsquigarrow \ln\frac{y}{y_0} = -\frac{1}{2}\ln\frac{t}{t_0} \rightsquigarrow y = y_0\sqrt{\frac{t}{t_0}}$.If $t_0 < 0$, $y_0 < 0$ or $t_0 < 0$ and $y_0 > 0 \longrightarrow$ always obtain $y = y_0\sqrt{\frac{t}{t_0}}$ but in each quadrant separately! Other example: $y' = t^3 y$, for $y \neq 0 \Leftrightarrow \frac{y'}{y} = t^3 \Leftrightarrow \int_{t_0}^{t}\frac{y'}{y} = \int_{t_0}^{t} t^3 dt \Leftrightarrow \int_{y_0}^{y}\frac{dy}{y} = \int_{t_0}^{t} t^3 dt \Leftrightarrow \ln y - \ln y_0 = \frac{t^4}{4} - \frac{t_0^4}{4} \Leftrightarrow y = y_0 e^{\frac{t^4}{4} - \frac{t_0^4}{4}}$, if $y, y_0 > 0$. If $y, y_0 < 0$, you get exactly the same solution. And so the general solution is given by $y = y_0 e^{\frac{t^4}{4} - \frac{t_0^4}{4}} = y_0 e^{-\frac{t_0^4}{4}} e^{\frac{t^4}{4}} = Le^{\frac{t^4}{4}}$.

3.2.2 Homogeneous differential equations

Type $y' = F\left[\frac{y}{t}\right]$ with $F \in C^0$, $t \neq 0 \longrightarrow F(t,y,y') = y' - g(\frac{y}{t}) = 0$. Method: substitution $u = \frac{y}{t} \rightsquigarrow ut = y \rightsquigarrow y' = u't + u \rightsquigarrow u't + u = F(u) \rightsquigarrow u' = \frac{1}{t}[F(u) - u]$. And we are back to the case of separable variables. Example $y' = -\frac{t^2+y^2}{2ty}$, $t, y \neq 0 \rightsquigarrow y' = -\frac{1+\frac{y}{t}}{2\frac{y}{t}} \rightsquigarrow$ homogeneous differential equation. Let $u = \frac{y}{t} \rightsquigarrow ut = y \rightsquigarrow y' = u't + t \rightsquigarrow u't + u = -\frac{1+u^2}{2u} \rightsquigarrow u' = \frac{1}{t}[\frac{-1+u^2}{2u} - u] \rightsquigarrow u' = -\frac{1}{t}[\frac{1+3u^2}{2u}] \rightsquigarrow \frac{2u}{1+3u^2}u' = -\frac{1}{t} \rightsquigarrow$ separable differential equation $\rightsquigarrow \int_{t_0}^{t} \frac{2u}{1+3u^2}u'dt = \int_{t_0}^{t} -\frac{1}{t} \rightsquigarrow \int_{u_0}^{u} \frac{2u}{1+3u^2}du = -\int_{t_0}^{t} \frac{1}{t} \rightsquigarrow \frac{1}{3}\ln\frac{1+3u^2}{1+3u_0^2} = -\ln t + \ln t_0 \rightsquigarrow \left(\frac{1+3u^2}{1+3u_0^2}\right)^{\frac{1}{3}} = \frac{t_0}{t} \rightsquigarrow u^2 = \frac{1}{3}[(\frac{t_0}{t})^3(1+3u_0^2) - 1] \rightsquigarrow y^2 = \frac{t^2}{3}[(\frac{t_0}{t})^3(1+3(\frac{y_0}{y})^2) - 1]$. Example $y' = \frac{t}{y} + \frac{y}{t}$, with $t, y \neq 0$ and $t, y >$ $\rightsquigarrow y' = \frac{1}{\frac{y}{t}} + \frac{y}{t} \rightsquigarrow$ homogeneous differential equation. Let $u = \frac{y}{t} \rightsquigarrow ut = y \rightsquigarrow y' = u't + u \rightsquigarrow u't + u = \frac{1}{u} + u \rightsquigarrow u'u = \frac{1}{t} \rightsquigarrow \int_{t_0}^{t} u'udt = \int_{t_0}^{t} \frac{1}{t}dt \rightsquigarrow \int_{u_o}^{u} udu = \int_{t_0}^{t} \frac{1}{t}dt \rightsquigarrow \frac{1}{2}(u^2 - u_0^2) = \ln\frac{t}{t_0} \rightsquigarrow u^2 = \ln t^2 - \ln t_0^2 + u_0^2 = \ln t^2 + c \rightsquigarrow y^2 = t^2(\ln t^2 + c)$, where $c = \ln t_0^2 + u_0^2$.

3.2.3 Linear differential equation

The unknown function and its first derivative are of degree one (lagged to the power of one). Type: $y' + u(t)y(t) = w(t)$, where $u(t), w(t) \in C^0$ and we denote this equation by [1]. Method: to find the general solution: general solution of $y' + u(t)y(t) = 0[2]$, the associated homogeneous equation + plus a particular solution of the whole initial problem [1]. Let $y_1(t)$ and $y_2(t)$ be two potential solutions of [1]. We have that $y_1'(t) + u(t)y_1 = w(t)$. Taking the difference between those two equations yields $[y_1'(t) - y_2'(t) + u(t][y_1(t) + y_2(t)] = 0$. From there, if we let $y_3(t) = y_1(t) - y_2(t)$ be a solution of our homogeneous equation [2]. We have a general solution. Method: (i) solving [2] $y'(t) + u(t)y(t) = 0 \Leftrightarrow y' = -u(t)y \Leftrightarrow \frac{y'}{y} = -u(t)$. If $u(t) \neq 0$, we get $\int \frac{dy}{y} = -\int u(t)dt + c$. Then $y > 0$ or $y < 0 \Rightarrow \ln(+/-y) = -\int u(t)dt + c \Rightarrow +/-y = e^{-\int u(t)dt+c} \Rightarrow y = ke^{-\int u(t)dt+c}$, which is the general solution of [2] with $k > 0, k < 0$. (ii) Finding a particular solution of [1], we have to methods: (1) particular solution method; (2) variations of constants' method.

Particular solution method. We assume that $u(t) = a(\in \mathbb{R}) \Rightarrow$ this makes the linear differential equation with constant coefficient. Thus we have $y' + ay = w(t)$, $a \neq 0$. If $w(t) = P_n(t)$ (polynomial of degree n) then $y(t) = Q_n(t)$. If $w(t) = P_n(t)e^{ct}$ then $y(t) = Q_n(t)e^{ct}$ if $a \neq -c$ or

$y(t) = Q_n(t)e^{ct}$ if $a = -c$. If $w(t) = P_n(t)e^{ct[A\cos\alpha t + B\sin\alpha t]}$ then $y(t) = Q_n(t)e^{ct}[k_1\cos\alpha t + k_2\sin\alpha t]$.
Variation of constants. The general solution of [2] is of the following form $y(t) = k\phi(t)$ where $\phi(t)$ is a solution to $y'(t) + tu(t)y(t) = 0$. One assumes that a particular solution of [1] is $y = k(t)\phi(t)$. And so by differentiating and substituting it in [1], a function $k(t)$ is determined and hence a particular solution sod [1] is found. We have $y = k(t)\phi(t) \rightsquigarrow y' = k'(t)\phi(t) + k(t)\phi'(t)$. Substituting in $y' + u(t)y = w(t)$ yields $k'(t)\phi(t) + k(t)\phi'(t) + u(t)k(t)\phi(t) = w(t)$, the last two terms on the l.h.s. are equal to zero because $\phi(t)$ is a solution of [2] $\Rightarrow k'(t) = \frac{w(t)}{\phi(t)} \Rightarrow k(t) = \int \frac{w(t)}{\phi(t)} \Rightarrow y(t) = \left[\int \frac{w(t)}{\phi(t)}dt\right]\phi(t)$ which is a particular solution of [1]. The general solution is therefore given by $y = ke^{-\int u(t)dt} + [\int w(t)e^{\int u(t)dt}]e^{-\int u(t)dt}$.
Examples. (1) $y' + 3y = 1$ we denote it by [1]. 3 is the constant coefficient. First step: set up a homogeneous D.E. $y' + 3y = 0$ which is our [2], for $y \neq 0$ $\Leftrightarrow y' = -3y \Leftrightarrow \frac{y'}{y} = -3 \Leftrightarrow \ln\frac{y}{y_0} = -3t + 43t_0 \Leftrightarrow y = y_0e^{-3t+3t_0} \Leftrightarrow y = Le^{-3t}$. Second step: finding a particular solution $\rightsquigarrow$ **particular solution method:** $w(t) = 1 \rightsquigarrow y(t) = \alpha(\in \mathbb{R})$. If $y = \alpha \rightsquigarrow y' = 0$. Substituting in [1] $\rightsquigarrow y' + 3y = 1 \rightsquigarrow 0 + 3\alpha = 1 \rightsquigarrow \alpha = \frac{1}{3}$. From this, we have the following general solution $y(t) = Le^{-3t} + \frac{1}{3}$. **Variation of constants:** From solving [2], we have that $y(t) = Le^{-3t}$. So we assume for the particular solution of [1] that $y = L(t)e^{-3t} \rightsquigarrow y' = L'(t)e^{-3t} - 3L(t)e^{-3t}$. Substitution in [1] $\rightsquigarrow$ $L'(t)e^{-3t} - 3L(t)e^{-3t} + 3L(t)e^{-3t} = 0$, the second two terms of the l.h.s. are equal to zero by definition, thus $L'(t)e^{-3t} = 1 \rightsquigarrow L'(t) = e^{3t} \rightsquigarrow L(t) = \frac{1}{3}e^{3t}$. And so the general solution for [1] is $y(t) = Le^{-3t} + \frac{1}{3}e^{3t}e^{-3t} = Le^{-3t} + \frac{1}{3}$.
(2) $y' + 2ty = t$ which is our equation [1]. We solve first for $y' + 2ty = 0 \rightsquigarrow \frac{y'}{y} = -2t \rightsquigarrow \ln\frac{y}{y_0} = -t^2 + t_0^2 \rightsquigarrow y = y_oe^{-t^2+t_0^2} \rightsquigarrow y = ke^{-t^2}$. Here, differently that in the previous example, we may use only the **variation of constants method**. Let us assume that a particular solution of [1] is given by $y(t) = k(t)e^{-t^2}$. From there we have that $k'(t)e^{-t^2} = t$, which implies that $k'(t) = te^{t^2} \rightsquigarrow k(t) = e^{t^2}\frac{1}{2}$, by integrating without putting any constants neither boundaries, just because we are interested in only **one** solution. A particular solution to [1] is $y = \frac{1}{2}e^{t^2}e^{-t^2} \Leftrightarrow y = \frac{1}{2}$. And so the general solution is given by $y(t) = ke^{-t^2} + \frac{1}{2}$.
(3) $y' - \frac{1}{t}y = t$ is our [1] with $y(1) = 2$. Homogeneous D.E. is $y' - \frac{1}{t}y = 0$ which is our [2]. We solve [2] $\rightsquigarrow \frac{y'}{y} = \frac{1}{t} \rightsquigarrow \ln\frac{y}{y_0} = \ln\frac{t}{t_0} \rightsquigarrow y = \frac{y}{y_0}t = kt$. **Variation of constants:** $y = k(t)t$ is a particular solution of [1]. $y'(t) = k'(t)t + k(t) \rightsquigarrow$ substitute this in [1]: $k'(t)t + k(t) - \frac{1}{t}k(t)t \rightsquigarrow k'(t)t = t \rightsquigarrow k'(t) = 1 \rightsquigarrow k(t) = t$ and so $y_p(t) = tt = t^2$. Finally, the general solution is $y(t) = kt + t^2$. Solving the Cauchy problem yields $y(1) = 2 = k + 1 \rightsquigarrow k = 1$,

the unique solution of all the problem verifying the initial condition.

3.2.4 Differential equations reducible to linear equations (Bernoulli)

Type $tu(t)y = w(t)y^m$ where $m \in \mathbb{R}$ except $\{0,1\}$, $u(t), w(t) \in C^0$. Note that if $m = 0$, then we are in the setting of separable D.E.; if $m = 1$, then we are in the case of linear D.E. Method: obviously $y(t) = 0 \forall t$ is a solution. $y(t) \neq 0$, we divide our equation be $y^m \rightsquigarrow \frac{\dot{y}}{y^m} + \frac{u(t)}{y^{n-1}} = w(t) \Leftrightarrow \dot{y}y^{-m} + u(t)y^{1-m} = w(t)$. Substitute $y^{1-m} = z \Rightarrow \dot{z}(t) = (1-m)y^{-m}(t)\dot{y}(t) \Rightarrow \frac{\dot{z}(t)}{(1-m)} + u(t)z(t) = w(t) \Leftrightarrow \dot{z}(t) + (1-m)u(t)z(t) = (1-m)u(t)$ if $m \neq 1$. And this is a linear differential equation (see previous subsection).

Example. $\dot{y} + ty = 3ty^2$

- $y(t) = 0 \forall t$ is a solution
- $y(t) \neq 0$, divide by y^2 and so we have $\frac{\dot{y}}{y^2} + \frac{t}{y} = 3t$.

Let $z = y^{-1} \Rightarrow \dot{z} = -y^{-2}\dot{y} \Rightarrow -\dot{z} + tz = 3t \Leftrightarrow \dot{z} - tz = -3t \Leftrightarrow \dot{z} = t(z-3) \Leftrightarrow \frac{\dot{z}}{z-3} = t$, for $z \neq 3$. If so, then is $z > 3 \Rightarrow \ln(z-3) = \frac{t^2}{2} + c$, with $c \in \mathbb{R}$, $Rightarrowz - 3 = e^{\frac{t^2}{2}+c} \Rightarrow z = ke^{\frac{t^2}{2}} + 3$, for $k > 0$. If $z < 3 \Rightarrow \ln(3-z) = \frac{t^2}{2} + c \Leftrightarrow z = ke^{\frac{t^2}{2}} + 3$, with $k < 0$. General solution $\dot{z} = t(z-3)$ is $z = ke^{\frac{t^2}{2}} + 3$. Therefore $y(t) = \frac{1}{ke^{\frac{t^2}{2}}+3}$, with the conditions that $k \neq -3e^{\frac{t^2}{2}}$.

Example. $\dot{y} + \frac{1}{t}y = y^3$.

- $y(t) = 0 \forall t$ is a solution
- $y(t) \neq 0 \Rightarrow \frac{\dot{y}}{y^3} + \frac{1}{t}\frac{1}{y^2} = 1$.

Take $z = \frac{1}{y^2} \Rightarrow \dot{z} = \frac{-2}{y^3}\dot{y} \Rightarrow -\frac{1}{2}\dot{z} + \frac{1}{t}z = 1 \Rightarrow \dot{z} - \frac{2}{t}z = -2$, where we set $-\frac{2}{t} = u(t)$ and $-2 = w(t)$. Homogeneous differential equation: $\dot{z} - \frac{2}{t}z = 0 \Rightarrow \frac{\dot{z}}{z} = \frac{2}{t} \Rightarrow \ln\frac{z}{z_0} = \ln(\frac{t}{t_0})^2$. Variation of constants for the particular solution: $z_p = k(t)t^2 \Rightarrow \dot{z}_p = \dot{k}t^2 + 2tk \Rightarrow \dot{k}(t)t^2 + k(t)2t - 2k(t)t = -2 \Rightarrow \dot{k}(t) = -\frac{2}{t^2} \Rightarrow \dot{k}(t) = -\frac{-2}{t^2} \Rightarrow k(t) = \frac{2}{t} \to z_p = 2t$. So $y^2 = \frac{1}{z} = \frac{1}{2t+kt^2}$ where $(2t + kt^2) \neq 0$.

3.2.5 Exact differential equations

In general, any first-order differential equation may be written as

$$M(t,y)dt + N(t,y)dy = 0$$

where $M, N \in C^1$. The differential equation $M(t,y)dt + N(t,y)dy = 0$ is called exact, meaning that there exists a function $u(t,y)$ with differential $du(t,y)$,

$$du(t,y) = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial y}dy.$$

And so $\frac{\partial u}{\partial t} = M$ and $\frac{\partial u}{\partial y} = N$. This implies that $u(t,y) \equiv$ constant so that $du(t,y) = 0$. The quantity $du(t,y)$ is called "exact", "perfect" or "total"differential. Test for exactness: a necessary and sufficient condition for differential eq. $M(t,y)dt + N(t,y)dy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

Method: how can you find a solution for $u(t,y)$?
(1) Let $u(t,y) = \int M(t,y)dt + \phi(y) \rightarrow (\frac{\partial u}{\partial t} = M(t,y))$.
(2) $\frac{\partial u}{\partial y} = N(t,y) = \frac{\partial}{\partial y}[\int M(t,y)dt]\dot{\phi}(y)$.
(3) Simplify and solve for $\phi(y)$.
(4) Substitute $\phi(y)$ in the expression of the first step (1) and set it equal to 0.
$\Rightarrow$ this is the solution!
Remarks. You can as well do it using in (1) $\frac{\partial u}{\partial y} = N(t,y) \rightarrow u = \int N(t,y)dt + \phi(t)$ and then in step (2) $\frac{\partial u}{\partial t} = M(t,y)$ and then you solve. Separable differential equation is exact, i.e. $\dot{y} - f(t)g(y) = 0 \Leftrightarrow \frac{1}{g(y)}dy - f(t)dt = 0$. So $\frac{\partial M}{\partial y} = \frac{\partial -f(t)}{\partial y} = 0 = \frac{\frac{\partial}{\partial g(y)}}{\partial t} = \frac{\partial N}{\partial t} \Rightarrow$ test ok! So we checked that a separable variable equation is nothing but a specific type of exact first order differential equation.
Example. $(2t^2+3y)dt+(3t+y-1)dy$, where the first polynomial is $M(t,y)$ and the second trinomial is our $N(t,y)$. Test: $\frac{\partial M}{\partial y} = 3 = \frac{\partial N(t,y)}{\partial t} \Rightarrow$ exact. We can rewrite the eq. in the following way: $du(t,y) = (2t^3 + 3y)dt + (3t + y - 1)dy$. Then we have $u(t,y) = \int(2t^3 + 3y)dt + \phi(t) \rightsquigarrow M(t,y) = \frac{\partial u}{\partial t}$. Then, $du(t,y) = \frac{t^4}{2} + 3yt + \phi(y)$. Instead, $N(t,y) = \frac{\partial u(t,y)}{\partial y} = 3t + \dot{\phi}(y) \Leftrightarrow \dot{\phi}(y) = y - 1 \Rightarrow \phi(y) = \frac{1}{2}y^2 - y + c$, with $c \in \mathbb{R}$. From there, $u(t,y) = \frac{t^4}{2} + 3yt + \frac{1}{2}y^2 - y + c \Rightarrow t^4 + 6yt + y^2 - 2y = K$, with $k \in \mathbb{R}$.

If the differential equation $M(t,y)dt + N(t,y)dy = 0$ is not exact it still might be possible to find a solution by finding a corresponding integrating factor $\rightsquigarrow$ there exists some function $\zeta(y)$ such that

$$\zeta(t,y)[Mdt + Ndy] = du(t,y)$$

and$\zeta(t,y)$ is called the integrating factor. And then the test for exactness is

$$\frac{\partial}{\partial y} = \zeta(t,y)M(t,y) = \frac{\partial}{\partial t}\zeta(t,y)N(t,y).$$

A few examples.
A. If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{N} = f(t)$ then $e^{\int f(t)dt}$ is an integrating factor, and if $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{M} = -g(y)$, then $e^{\int g(y)dy}$ is an integrating factor. $\frac{\partial e^{\int f(t)dt}M}{\partial y} = \frac{\partial e^{\int f(t)dt}N}{\partial t} \Leftrightarrow e^{\int f(t)dt}\frac{\partial M}{\partial y} = e^{\int f(t)dt}f(t)N + e^{\int f(t)dt}\frac{\partial N}{\partial t} \Leftrightarrow \frac{\partial M}{\partial t} - \frac{\partial N}{\partial t} = f(t)N$. **B.** If $M(t,y)dt + N(t,y)dy = 0$ is a homogeneous differential equation and $Mt + Ny \neq 0$, then $\frac{1}{Mt+Ny}$ is an integrating factor. **C.** If $M(t,y)dt+N(t,y)dy = 0$ can be written as $yf(t,y)dt + tg(t,y)dy = 0$, with f$(t,y) \neq g(t,y)$, then $\frac{1}{ty[f(t,y)-g(t,y)]} = \frac{1}{Mt+Ny}$ i an integrating factor.
Numerical examples.
1. $(t^2 + y^2 + t)dt + tidy = 0$, first therm is $M(t,y)$, second term equal to $N(t,y)$. Exact? $\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial t} = y \rightsquigarrow$ not exact! However, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{N} = \frac{y}{ty} = \frac{1}{t} = f(t) \rightsquigarrow e^{\int f(t)dt}$ as being the integrating factor. But $e^{\int \frac{1}{t}dt} = e^{\ln t} = t$ if $t > 0$ or $e^{\ln -t} = -t$ is $t < 0$. Therefore $-/+t[(t^2 + y^2 + t)dt + tydy] = 0$. For $t > 0$: $(t^3 + y^2t + t^2)dt + t^2ydy = 0$, first term being $\tilde{M}$ and second term being $\tilde{N}$. Exact ? $\frac{\partial\tilde{M}}{\partial y} = 2yt = \frac{\partial N}{\partial t} \rightsquigarrow$ exact! We have $\frac{\partial u}{\partial t} = \tilde{M}(t,y) = t^3 + y^2t + t^2 \rightsquigarrow u(t,y) = \int(t^3 + ty^2 + t^2)dt + \phi(y) = \frac{t^4}{4} + \frac{y^2}{2}t^2 + \frac{t^3}{3} + \phi(y)$. $\tilde{N}(t,y) = \frac{\partial u}{\partial y} = yt^2 + \dot{\phi}(y)$, last term being equal to zero since $\dot{\phi}(y) = c$, with $c \in \mathbb{R}$. General solution is $3t^4 + 6y^2t^2 + 4t^3 = k$, with $k \in \mathbb{R}$. For $t < 0$, is the same just multiplying by $-1 \rightsquigarrow$ same result as $k \in \mathbb{R}$. Switch roles, $\tilde{N}(t,y) = \frac{\partial u}{\partial y} = t^2y \Rightarrow u(t,y) = \frac{t^2y^2}{2} + \phi(t)$. $\tilde{M}(t,y) = \frac{\partial u}{\partial t} = ty^2 + \dot{\phi}(t) \Rightarrow \dot{\phi}(t) = t^3 + t^2 \Rightarrow \phi(t) = \frac{t^4}{4} + \frac{t^3}{3} + c \Rightarrow u(t,y) = \frac{t^2y^2}{2} + \frac{t^4}{4} + \frac{t^3}{3} + c$, and so the solution is $6t^2y^2 + 3t^4 + 4t^3 = k$, $k \in \mathbb{R}$.
2. $(2t^4 + 3y)\dot{y} + 4t^3y = 0 \Leftrightarrow (2t^4 + 3y)dy + 4t^3ydt = 0$, where the first term is equal to $N(t,y)$ and the second one to $M(t,y)$. Exact? $\frac{\partial M}{\partial y} = 4t^3 \neq \frac{\partial N}{\partial t} = 8t^3$, not exact! However $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}}{M} = -\frac{4t^3}{4t^3y} = -\frac{1}{y} = -g(y)$. So $e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = y$ if $y > 0$ and $-y$ if $y < 0$, this is the integrating factor. For $y > 0$: $(2t^4y + 3y^2)dy + 4t^3y^2dt = 0$, where the first term is $\tilde{N}$ and the second one is $\tilde{M}$. Exact? $\frac{\partial\tilde{M}}{\partial y} = 8t^3y = \frac{\partial\tilde{N}}{\partial t} \rightsquigarrow \frac{\partial u}{\partial t} = 4t^3y^2 = \tilde{(}t,y) \Rightarrow u(t,y) = \int 4t^3y^2dt + \phi(y) = t^4y^2 + \phi(y)$; and $\frac{\partial u}{\partial y} = \tilde{N}(t,y) = 2t^4y + \dot{\phi}(y) \Rightarrow \phi(y) = y^3 + c$, with $c \in \mathbb{R} \Rightarrow u(t,y) = t^4y^2 + y^3 + c \Rightarrow$ general solution is $t^4y^2 + y^3 = k$, with $k \in \mathbb{R}$.

3.2.6 Nearly exact differential equations

$F(t,y,\dot{y})=0$, assume the equation is of degree one, it can be rewritten as $M(t,y)\dot{y}+N(t,y)=0$. $M(t,y)\frac{\partial y}{\partial t}+N(t,y)=0 \Leftrightarrow M(t,y)dy+N(t,y)dt=0$, i.e. $d\mu(t,y)=M(t,y)dy+N(t,y)dt \Leftrightarrow$ an implicit solution for it is $\mu(t,y)=k$, for $k\in\mathbb{R}$. $d\mu(t,y)=\frac{\partial\mu}{\partial t}(t,y)dt+\frac{\partial\mu}{\partial y}(t,y)dy$. Consider $t^2+y^2=\cos(t) \Rightarrow y=+/-\sqrt{\cos t-t^2}$ is an implicit solution. When $\mu(y,t)\exists \rightsquigarrow$ the differential equation is said to be "exact". A necessary and sufficient condition for the equation to be exact is that $\frac{\partial M}{\partial t}=\frac{\partial N}{\partial y}$. $M(t,y)=\frac{\partial\mu}{\partial y}(t,y) \Rightarrow \frac{\partial M}{\partial t}=\frac{\partial^2\mu}{\partial y\partial t}(t,y)$ and $N(t,y)=\frac{\partial\mu}{\partial t}(y,t) \Rightarrow \frac{\partial N}{\partial y}=\frac{\partial^2 N}{\partial t\partial y}(t,y)$, with $\mu\in C^1$ and $N,M\in C^1$. $\frac{\partial\mu}{\partial y}=M(t,y) \Rightarrow \mu(t,y)=\tilde{\int}M(t,y)dy+\phi(t)$, where $\phi(t)$ is a constant, and $\frac{\partial\mu}{\partial t}=N(t,y) \Rightarrow \mu(t,y)=\frac{\partial}{\partial t}\left(\tilde{\int}M(t,y)dy\right)+\phi'(t)$, the goal is to find $\phi(t)$. **Example** $2(t^2+3y)dt+(3t-y-1)dy=0$, rewrite the equation as $\underbrace{2t^2+3y}_{M(t,y)dt}+\underbrace{3t-y-1}_{N(t,y)dy}$; $\frac{\partial M}{\partial y}=3=\frac{\partial N}{\partial t}$. Then proceed as follows $\rightsquigarrow$ $\frac{\partial\mu}{\partial y}(t,y)=3t-y-1 \Rightarrow \mu(t,y)=(3ty-\frac{y^2}{2}-y)+\phi(t)$. Then consider a two equations' system $\frac{\partial\mu}{\partial y}(t,y)=3y+\phi'(t)$ and $\frac{\partial\mu}{\partial y}(t,y)=M(t,y)=2t^2+3y \rightsquigarrow \phi'(t)=2t^3 \Leftrightarrow \phi(t)=\frac{t^4}{2}+c$ with $c\in\mathbb{R} \rightsquigarrow \mu(t,y)=3ty-\frac{y^2}{2}-y+\frac{t^4}{4}+c$, with $C\in\mathbb{R}$, an implicit solution for this differential equation.
Remark. (i) If easier, switch the role of M and $N \Rightarrow \frac{\partial\mu}{\partial t}=N(t,y) \Rightarrow \mu(t,y)=\tilde{\int}N(t,y)dt+\phi(y)$ then determine $\phi(y)$ and $\frac{\partial\mu}{\partial y}=M(t,y)$. (ii) In the example, any Cauchy problem will have exactly one solution, i.e. $y(t_0)=y_0$. (iii) Separable equations are exact! $\dot{y}=f(t)g(y) \Leftrightarrow \frac{d}{dt}-f(t)g(t)=0 \Leftrightarrow dy-f(t)g(y)dt=0 \Rightarrow M(t,y)=\frac{1}{g(y)}$; $N(t,y)=-f(t) \rightsquigarrow \frac{dy}{g(y)}-f(t)g(t)=0$; $\frac{\partial M}{\partial t}=0=\frac{\partial N}{\partial y}$.
If the equation is exact, it is always possible to find an "integrating factor" $\xi(t,y)$ such that $\xi(t,y)M(t,y)dy+\xi(t,y)N(t,y)dt=0$ is exact.
$\mu'(t,y)=f(t)dt+g(y)dy=0 \rightsquigarrow$ exact equation. $\mu(t,y)=\int f(t)dt+\int g(y)dy$, just integrating here, $f_1(t)g_1(y)dt+f_2(t)g_2(y)dy=0$, where $\frac{1}{g_1(y)}$ and $\frac{1}{f_2(t)}=\frac{1}{g_1(y)f_2(t)}$ thus we have $\frac{f_1(t)}{g_1(t)}dt+\frac{g_1(y)}{g_2(y)}=0$. $\mu(t,y)=\int\frac{f_1(t)}{f_2(t)}dt+\int\frac{g_1(y)}{g_2(y)}dy$. If $\frac{\frac{\partial M}{\partial t}-\frac{\partial N}{\partial y}}{N}=f(y) \Rightarrow e^{\int f(y)dy}$ is an integrating factor. **Proof.** $e^{\int f(y)dy}M(t,y)dy+e^{\int f(y)dy}N(t,y)dt=0$ is exact if $\frac{\partial}{\partial t}\left[e^{\int f(y)dy}M(t,y)\right]=\frac{\partial}{\partial y}\left[e^{\int f(y)dy}N(t,y)\right] \Rightarrow e^{\int f(y)dy}f(y)M(t,y)+e^{\int f(y)dy}\frac{\partial M}{\partial y}(t,y)=e^{\int f(y)dy} \rightsquigarrow \left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right]/M=-f(y)$.$\square$ If the equation is homogeneous, and $My+Ny\neq 0$, then $(Nt+My)^{-1}$ is an integrating factor. If the equation is of the form:

$yf(t,y)dt + tg(t,y)dy = 0$, with $f(t,y)\neg(t,y) \Rightarrow ty(f(t,y) - g(t,y))^{-1} = (My + Nt)^{-1}$ is an integrating factor. **Example.** $tydy + (t^2 +^2 +t)dt$ where $\frac{\partial M}{\partial t} = y \neq \frac{\partial N}{\partial y} = 2y$. But $\frac{\frac{\partial M}{\partial t} - \frac{\partial N}{\partial y}}{M} = \frac{-y}{ty} = -\frac{1}{t} \rightsquigarrow e^{\int \frac{1}{t}dt}$ is an integrating factor!! $|t| = t$ is $f > 0$ and $|t| = -t$ is $f < 0 \rightsquigarrow$ equation becomes $\underbrace{t^2ydy}_{M(t,y)} + \underbrace{(t^3 + y^2ty + t^2)dt}_{N(t,y)} = 0$ and therefore $\frac{\partial \mu}{\partial y}M(t,y) \rightarrow \mu(t,y) = \tilde{\int} M(t,y)dy + \phi(t) = \frac{t^2y^2}{2} + \phi(t)$; $\frac{\partial \mu}{\partial t}N(t,y) = ty^2 + \phi'(t) \rightsquigarrow N(t,y) = t^3 + y^2t + t^2 \rightsquigarrow \phi'(t) = t^3 + t^2 \rightsquigarrow \phi(t) = \frac{t^4}{4} + \frac{t^3}{3} + c$, with $c \in \mathbb{R} \rightsquigarrow \mu(t,y) = \frac{t^2y^2}{2} + \frac{t^4}{4} + \frac{t^3}{3} + c$, with $c \in \mathbb{R} \rightsquigarrow$ implicit solution is $y(t) = \frac{t^2y^2}{1} + \frac{t^4}{4} + \frac{t^3}{3} + k$, with $k \in \mathbb{R}$.

3.3 Equations of order higher than the first

Type: $y^{(n)}(t) + a_1y^{(n-1}(t) + ... + a_{n-1}y'(t) + a_ny(t) = \pi(t_1)$, with $a_n \in \mathbb{R}, n > 1, \phi \in C^1$. Recall that (n) stands for the highest order of differentiation of a variable $y(t)$ within the equation w.r.t. t. Therefore, $y^{(1)}(t) = \dot{y} = \frac{dy(t)}{dt}$; $y^{(2)}(t) = y'' = \frac{d^2y(t)}{dt^2}$; ...with a linear differential equation $\Rightarrow Y = Y_H + Y_P$, where $Y_H \equiv$ general solution of the associated homogeneous equation (characteristic equation) $\Rightarrow$ replace $\phi(t)$ by 0 and $y = 1$, $y' = x$, $y'' = x^2$, and so on $\Rightarrow$ solve for x.

3.3.1 Second order differential equations

Type: $y''(t) + a_1y'(t) + a_2y(t) = \phi(t)$ where $a_1, a_2 \in \mathbb{R}$ and $\phi \in C^1$. Solution $\Rightarrow$ a particular solution of the whole equation + a general solution of the associated homogeneous equation. **E.g.** $y'' - 5y' + 6y = \underbrace{t^2 + 2}_{\phi(t)}$. **1.** $Y_H \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow \Delta = b^2 - 4c = 25 - 24 = 1 \Rightarrow x_{1,2} = \frac{5+/-\sqrt{1}}{2} = 3 \wedge 2$ (where the general form for finding the two solutions of the characteristic equation is just $ax^2 + bx + c = 0$, a second degree polynomial). $Y_H = \underbrace{A_1e^{3t}}_{Y_{H1}} + \underbrace{A_2e^{2t}}_{Y_{H2}}$. If $\Delta = 0$ or $\Delta < 0 \Rightarrow$ we anyway have two solutions. $\mathbf{2}.\mathbf{Y_P} = (\mathbf{at^2} + \mathbf{bt} + \mathbf{c})\mathbf{t^k}$.

a. Variation of constants;

b. Guess the solution.

b. Look at $\phi(t) = Pn(t)e^{ct}(\lambda \cos \beta t + \mu \sin \beta t) \Rightarrow y_p = \underbrace{Pn(t)}_{\text{polynomial of order } n} e^{ct}(A \cos \beta +$

$B\sin\beta t)t^k$, $k=?$, if $\phi(t)=t^2+2 \Rightarrow \underbrace{P_2(t)}_{\text{polynomial of order 2}}$. $y_p = P_2(t)t^k = (at^2+bt+c)t^k$, here $b=0$. If you have a polynomial of order 2 $\Rightarrow$ in $\phi(t) \Rightarrow$ then, also the particular solution will be a polynomial of the same order.
Goal $\rightarrow$ find a,b,c,k.
Now, we replace Y_P in the original equation:
$\underbrace{y_p''}_{2a} - \underbrace{5y_p'}_{5(2at+bt+c)} + \underbrace{6y_p}_{6(at^2+bt+c)} = t^2+2$. $k=2$ is the algebraic multiplicity of 0 in the characteristic equation $\Rightarrow$ # of times you observe zero in the solutions $\Rightarrow$ zero times in our case. $2a-10at-5b+6at^2+6bt+c=t^2+2 \Rightarrow$ find $a,b,c \Rightarrow a=\frac{1}{6} \Leftrightarrow \frac{1}{3}-\frac{5}{3}t-5b+t^2+bt+c=\alpha t^2+\beta t+\gamma = a^t2+bt+c$; $b=\frac{5}{18}$ and $c=\frac{55}{108}$.
3. $Y=Y_H+Y_P=Ae^{3t}+Be^{2t}+\frac{1}{6}t^2+\frac{5}{18}t+\frac{55}{118}$. $6at^2+6bt-10at+2a-5b+6c = t^2+2$ resolve it in such a way: $\begin{bmatrix} 6at^2=t^2 \Leftrightarrow a=\frac{1}{6} \\ 6bt-10at=0 \Leftrightarrow 6bt-\frac{10}{6}t=0 \Leftrightarrow (6b-\frac{5}{3})t=0 \\ 2a-5b+6c=2 \Leftrightarrow \frac{1-5-6}{3}=6c \Leftrightarrow -\frac{10}{3}\frac{1}{6}=c \end{bmatrix}$.

Ex. $y'''-y''+y'-y=2e^t \Rightarrow \phi(t)=2e^t$.
1. $Y_h=A_1e^t+B\sin(t)+A_3\cos(t) \Rightarrow$ general solution.
$x^3-x^2+x-1=0 \Leftrightarrow x^2(x-1)+x-1=0 \Leftrightarrow x^2(x-1)=-x+1 \Leftrightarrow (x-1)(x^2+1)=0 \Leftrightarrow x_4=1 \wedge x_3=-1 \wedge x_{1,2}=+/-i \Rightarrow Y_{H1}=A_1e^t; Y_{H2}=A_2e^{0t}\sin(t); Y_{H3}=A_3e^{0t}+\cos(t)$.
2. $Y_p \Rightarrow \phi(t)=2e^tt^k$. k is the multiplicity of c in k characteristic equation. $k=1$. **Goal:** find $a \Rightarrow c$ is the coefficient of $e^t \Rightarrow$ you see how many times it occurs ($c=1$) in the solution of the characteristic equation.
$y_p=ae^tt$
$y_p'=ae^t+ae^tt \Rightarrow$ derivative of a composite function
$y_p''=ae^t+ae^t+ae^tt=2ae^t+ae^tt$
$y_p'''=ae^t+ae^t+ae^t+ae^tt=3ae^t+ae^t$
Replace them in the original equation to find a.
$3ae^t+ae^tt-2ae^t-ae^tt+ae^t+ae^tt=2e^t$
$3a-2a-a+at-at+a+a=2$
$2a=2$
$a=1$.

Remark. $y'''-y''+y+-y=2et^t$
$(Y_H'''+Y_P)-(Y_H''+Y_P)+(Y_H'+Y_P)-(Y_H+Y_P)=2e^t$
$\underbrace{Y_H'''}_{0}+Y_P'''-\underbrace{Y_H''}_{0}-Y_P''+\underbrace{Y_H'}_{0}+Y_P'-\underbrace{Y_H}_{0}-Y_P=2e^t$.
Solution $\Rightarrow Y=Y_P+Y_H$.
1. $Y_H \Rightarrow$ ch. equation $\phi(t)\rightarrow 0$; $y\rightarrow 1$; $y'\rightarrow x$; $y''=x^2$;... Solve for x and

reach one of the three cases:
$\Delta > 0, \forall i : \sum_i Y_{Hi} = A_i e^{x_i t} \Rightarrow Y_H = \sum_i Y_{Hi}$.
$Delta = 0, x_1, x_1 \Rightarrow Y_{H1} = A_1 e^{x_1 t}; Y_{H2} = A_2 e^{x_1 t} t; Y_{H3} = A_3 e^{x_1 t} t^2$
$\Delta < 0$; different complex solutions $\alpha+\beta i, \alpha-\beta i \Rightarrow Y_{H1} = A_1 e^{\alpha t} \sin(\beta t); Y_{H2} = A_2 e^{\alpha t} \cos(\beta t)$.
Some complex solution.
E.g.: $x_{1,2,3} = 1, 1, 2 \longrightarrow Y_{H1} = A_1 e^t; Y_{H2} = A_2 e^t t; Y_{H3} = A_3 e^t t^2$.
$x_{1,2,3} = 2-i; 2+i; 2-i; 2+i \longrightarrow Y_{H1} = A_1 e^{2t} \cos(t); Y_{H2} = A_2 e^{2t} \sin(t); Y_{H3} = A_3 e^{2t} \cos(t) t; Y_{H4} = A_4 e^{2t} \sin(t) t$, you multiply by t if you have multiple solutions.
2. $Y_P \Rightarrow \phi(t) = Pn(t) e^{Ct}(C_1 \sin \beta t + C_2 \cos \beta t)$. $Y_p = Q_n(t) e^{Ct}(k_1 \sin \beta t + k2 \cos \beta t) t^k$. $k =$ multiplicity of 0 in the solution of the characteristic equation (if $\Delta > 0$). $k =$ multiplicity of c with $Delta = 0$ in the roots of the characteristic equation. $k =$ multiplicity of $c + \beta i; c - \beta i$ is $\Delta < 0$ in the roots of the characteristic equation. In the first two cases, you have to find a, b, and c, whereas in the latter one you have to find k_1 and k_2.
$y'' - 4y' + 13y = 10 \cos 2t + 25 \sin t \rightsquigarrow$ characteristic equation $x^2 - 4x + 13 = 0 \Rightarrow x_{1,2} = \frac{4+/-\sqrt{16-25}}{2} = x_1 = \frac{4-6i}{2} = 2 - 3i \wedge x_2 = \frac{4+6i}{2} = 2 + 3i$. Then $Y_H = A_1 e^{2t} \sin 3t + A_2 e^{2t} \cos 3t$; $\phi(t) = P_0(t) e^{0t}(10 \cos 2t + 25 \sin 2t) t^k$; $Y_p = e^{0t}(\alpha k_1 \cos 2t + \alpha k_2 \sin 2t) t^k$, $k =$ mult. of $c + / - \beta i$ and $0 + / - 2i \Rightarrow k = 0$. $Y_p = (\beta_1 \cos 2t + \beta_2 \sin 2t) t^0 = (\beta_1 \cos 2t + \beta_2 \sin 2t)$; $Y_p' = -2\beta_1 \sin 2t + 2\beta_2 \sin 2t$; $Y_p'' = -4\beta_1 \cos 2t - 4\beta_2 \sin 2t$ plug the partial derivatives in the main general equation to find the values of the constants: $-4\beta_1 \cos 2t - 4\beta_2 \sin 2t + 8\beta_1 \sin 2t - 8\beta_2 \cos 2t + 13\beta_1 \cos 2t + 13\beta_2 \sin 2t = 10 \cos 2t + 25 \sin 2t$ resolving this algebraic expression should yield the following result: $\beta_1(9 \cos 2t + 8 \sin 2t) = 10 \cos 2t \wedge \beta_2(-8 \cos 2t + 9 \sin 2t) = 25 \sin 2t$.

Chapter 4

Dynamic Optimization

4.1 Introduction

STATIC OPTIMIZATION $\equiv$ at some point in time find the value of one or several variables, subject or not to constraints which maximizes or minimizes a given function.

E.g. $\max_{C,L} U(\underbrace{C}_{\text{consumpt}}, \underbrace{L}_{\text{labour}})$ s.t. $C + \underbrace{w}_{\text{wage rate}} L = w + \underbrace{B}_{\text{bonds}}$

DYNAMIC OPTIMIZATION $\equiv$ find a path of one or several variables, eventually subject to some constraints, in such a way to maximize a given functional (variable $\equiv$ function of time).

E.g. $\max_{\{C_t,L_t\}_{t=0}^{\infty}} \sum_{t=1}^{T} \underbrace{U_t(C_t, L_t)}_{\text{utility funct.}}$ s.t. $\underbrace{A_t}_{\text{assets}} = (1 + r_t)A_{t-1} + B_t + W_t$

E.g which is the optimal path between two points A and B? We may either consider a *continuous* problem with confidence bands or a *discrete* problem with intermediate states.

Whether a variable is continuous or discrete, four things:

- an initial and terminal state, i.e. $[0, T]$;
- a set of admissible paths;
- a set of values associated with each admissible path;
- an objective functional to be optimized.

The initial point/state will be denoted by $y(0) \equiv y(0, A)$ and the terminal point/state will be denoted by $y(T) \equiv y(T, Z)$ as far as the problem is considered on a interval $[0, T]$. Letting $(0, A)$ being given, three scenarios may occur:

1. T is fixed, but $y(T)$ is free $\Rightarrow$ FREE TERMINAL STATE;

2. T is free, but $y(T) = B$ is fixed $\Rightarrow$ FINITE HORIZON TERMINAL PROBLEM;

3. T is free, and $y(T) = \phi(T)$ $\Rightarrow$ FREE TERMINAL CURVE.

4.2 Calculus of Variations

$V[y(t)] = \int_0^T F[t, y(t), \dot{y}(t)]dt \Rightarrow$ functional equation.

Calculus of Variations $\Rightarrow$ $\max / \min V[y(t)] = \int_0^T F[t, y(t), \dot{y}(t)]dt$ s.t. $y(0) = A$ and $y(T) = Z$, with A, T, Z given; $y(t) \in C^1_{[0,T]}$, $F \in C^2$.

$V[\epsilon] = \int_0^T F[t, y^*(t) + \epsilon\eta(t), \dot{y}^*(t) + \epsilon\eta(t)]dt$, apply Leibniz's rule and chain rule, to get the F.O.C.: $\frac{dV}{d\epsilon}[\epsilon] = \frac{d}{d\epsilon}\int_0^T F[t, y^* + \epsilon\eta(t), \dot{y}^*(t) + \epsilon\eta(t)]dt = \int_0^T \left[\frac{\partial F}{\partial y}[t, y^*(t) + \epsilon\eta(t), \dot{y}^*(t) + \epsilon\eta(t)]\eta(t) + \frac{\partial F}{\partial \dot{y}}[t, y^*(t) + \epsilon\eta(t), \dot{y}^*(t) + \epsilon\eta(t)]\dot{\eta}(t)\right]dt$

$\frac{dV}{d\epsilon}[0] = 0 \Leftrightarrow \int_0^T \left[\frac{\partial F}{\partial y}[t, y^*(t), \dot{y}^*(t)]\eta(t) + \frac{\partial F}{\partial \dot{y}}\underbrace{[t, y^*(t), \dot{y}^*(t)]}_{\dot{u} = \frac{d}{dt}\frac{\partial F}{\partial \dot{y}}} \underbrace{\dot{\eta}(t)}_{v = \eta(t)}\right]dt$

apply partial integration on the second element of the integral $\Leftrightarrow$ do integration by parts on $\int u\dot{v}dt = uv + \int \dot{u}vdt \Leftrightarrow \int_0^T \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}}\right]\eta(t)dt + \left[\eta(t)\frac{\partial F}{\partial \dot{y}}\right]_0^T = 0$ where the second term is equal to zero, due to terminal and initial conditions. $\Leftrightarrow \int_0^T \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}}\right]\eta(t)dt = 0$.

Exercises:

1. $V[y(t)] = \int_0^2 (12ty + \dot{y}^2)dt$ with $y(0) = 0$ and $y(2) = 8$.

Euler - Lagrange equation $\rightarrow$

$$\begin{aligned} 12t - \frac{d}{dt}[2\dot{y}] &= 0 \\ 12t &= 2\ddot{y} \\ 6t &= \ddot{y} \\ A + 3t^2 &= \dot{y} \\ B + At + t^3 &= y, A, B \in \mathbb{R} \end{aligned}$$

$y(0) = 0$ and $y(2) = 8 \Leftrightarrow B + 0 + 0 = 0 \Leftrightarrow B = 0$ and $2A + 8 = 8 \Leftrightarrow A = 0 \Rightarrow y(t) = t^3$ is the unique extremal.

2. $V[y(t)] = \int_1^5 [3t + \dot{y}^{1/2}]dt$ with $y(1) = 3$ and $y(5) = 7$.

E.L.:

$$\begin{aligned} 0 - \frac{d}{dt}\big[\frac{1}{2}\dot{y}^{-1/2}\big] &= 0 \\ -\frac{1}{4}\dot{y}^{-1/2}\ddot{y} &= 0, \ddot{y} \neq 0 \\ 4\dot{y}^{1/2}\big(-\frac{1}{4\dot{y}^{1/2}}\ddot{y}\big) &= 0 \times 4\dot{y}^{1/2} \\ \ddot{y} &= 0 \\ \dot{y} &= A, A \in \mathbb{R} \\ y &= At + B, B \in \mathbb{R} \end{aligned}$$

$$\Leftrightarrow = \begin{cases} A = 3 - B \\ 7 = 15 - 5B + B \end{cases}$$

$A = 1$, $4B = 8$, $B = 2 \Rightarrow y^*(t) = t + 2$.

3. $V[y(t)] = \int_0^T [t + y^2 + 3\dot{y}]dt$ s.t. $y(0) = 0$ and $y(5) = 3$.

E.L. eq.: $2y - \frac{d}{dt}[3] = 0 \Leftrightarrow 2y(t) = 0 \Leftrightarrow y(t) = 0$,

$$\Leftrightarrow = \begin{cases} y(0) = 0 \\ y(5) = 3 \end{cases}$$

true, $\forall t$. No extremal because no solution to the Euler-Lagrange equation $\Leftrightarrow$ impossible to verify the equality.

4. $V[y(t)] = \int_0^T \dot{y}(t)dt$ with $y(0) = \alpha$ and $y(T) = \beta$.

EL: $0 - \frac{d}{dt}[1] \Leftrightarrow 0 = 0$ latter equation satisfied by any admissible path, in fact $\int_0^T \dot{y}(t)dt = [y(T) - y(0)] = \beta - \alpha$ depends solely on the initial and terminal conditions and not on the path in between $\Rightarrow$ infinite # of extremals.

SPECIAL CASES:

1. $F = F[t, \dot{y}] \longrightarrow F_y = 0$

$\Rightarrow$ E.L. $\frac{d}{dt}F_{\dot{y}} = 0 \Rightarrow F_{\dot{y}} = c$

2. $F = F[y, \dot{y}]$, E.L. $\Rightarrow F - \dot{y}F_{\dot{y}} = c \Rightarrow F$ does not explicitly depend on t (autonomous problem)[1]

3. $F = F[t, y]$, E.L. $\Rightarrow F_y = 0$

4. $F = F[\dot{y}] \Rightarrow$ E.L. $\Rightarrow F_y - \frac{d}{dt}[F_{\dot{y}}] = 0$

$-F_y + F_{t\dot{y}} + F_{y\dot{y}}\dot{y} + F_{\dot{y}\dot{y}}\ddot{y} = 0$

$\Leftrightarrow F_{\dot{y}\dot{y}}\ddot{y} = 0 \Rightarrow \ddot{y} = 0$ and $F_{\dot{y}\dot{y}} = 0$

[1] $\frac{d}{dt}\big[F - \dot{y}F_{\dot{y}}\big] = F_t + F_y\dot{y} + F_{\dot{y}}\ddot{y} - \ddot{y}F_{\dot{y}} - \dot{y}F_{\dot{y}t} - (\dot{y})^2F_{\dot{y}y} - \ddot{y}\dot{y}F_{\dot{y}} = \dot{y}\big[F_y - \frac{d}{dt}[F_{\dot{y}}]\big] = 0 \Leftrightarrow \dot{y}\big[F_y - \frac{d}{dt}[F_{\dot{y}}]\big] = 0 \Leftrightarrow F - \dot{y}F_{\dot{y}} = c$

4.2.1 Generalizations: several variables

$V[t, y_1(t), y_2(t), ..., y_n(t)]dt = \int_0^T F[t, y_1(t), y_2(t), ..., y_n(t), \dot{y}_1(t), \dot{y}_2(t), ..., \dot{y}_n(t)]dt$. With initital and terminal conditions for each $y_i(t), i = 1, ..., n$. It can be easily shown that the E-L equation, the FONC, becomes for this problem, a system of n-equations to be solved: $F_{y_i} - \frac{d}{dt}F_{\dot{y}_i} = 0, \forall i = 1, ..., n \Leftrightarrow$

$$\begin{cases} F_{y_1} - \frac{d}{dt}F_{\dot{y}_1} = 0 \\ \vdots \\ F_{y_n} - \frac{d}{dt}F_{\dot{y}_n} = 0 \end{cases}$$

$$\Leftrightarrow \underbrace{\mathbf{F}_y}_{n \times 1} - \frac{d}{dt}\underbrace{\mathbf{F}_{\dot{y}}}_{n \times 1} = \underbrace{\mathbf{0}}_{n \times 1}$$

Proof: similar to that for the Euler - Lagrange equation. You have to show that for y_j, keeping the other y_i's constant, and saying you do this for all $j = 1, ..., n$.

4.2.2 Presence of derivatives of higher order

cases such as: $V[y(t)] = \int_0^T F[t, y(t), \dot{y}(t), \ddot{y}(t), ..., \dot{y}(t)^{(n)}]dt$, with initial and terminal conditions for $y, \dot{y}, \ddot{y}, ..., \dot{y}^{(n)}$. This can be done as previously by applying a substitution of the type $z = \dot{y}$, $\dot{z} = \ddot{y}$, $u = \ddot{y}$, $\dot{u} = \dddot{y}$,.... ". Euler - Poisson equation:

$$F_y - \frac{d}{dt}[F_{\dot{y}}] + \frac{d}{dt}[F_{\ddot{y}}] - ... + (-1)^n \frac{d^n}{dt^n}[F_{\dot{y}^{(n)}}] = 0, \forall t \in [0, T],$$

a differential equation of the n^{th} order.

4.2.3 Transversality conditions

Solving the E.L. equation involves solving a second order differential equation. Solving it leads to a solution with two degrees of freedom. Conditions $y(0) = A$ and $y(T) = Z$ allow to find extremals, by solving a Cauchy problem. If part of the initial conditions are "missing", i.e. the initial or terminal point aren't fixed $\Rightarrow$ use the "tranversality conditions" allowing to replace the "missing" conditions.

$V[y(t)] = \int_0^T F[t, y, \dot{y}]dt$ by letting $y(t) = y^*(t) + \epsilon\eta(t)$, T free. But, now, $\eta(0) = 0$, where $\eta(t) \in C^1_{[0,T]}$, once continuously differentiable on the domain of interest with no conditions on $\eta(0) = 0$ and pick an $\epsilon > 0$ sufficiently small. Let $T = T^* + \Delta T\epsilon$. ΔT is fixed and therefore $y(T) = y^*(t) + \varepsilon\eta(t)$.

$$\begin{cases} y(T) = y^*(T) + \varepsilon\eta(T) \\ y(T) = y^*(T^*) + \varepsilon\Delta y_T \end{cases}$$

$\Rightarrow$ rewrite the optimal stopping time in such a fashion $\Rightarrow y^*(T)+\varepsilon\eta(T) = y^*(T^*) + \varepsilon\Delta y_T$ following from before.

$\Rightarrow \eta(T) = \frac{-[y^*(T)-y^*(T^*)]+\varepsilon\Delta y_T}{\varepsilon}$

$\Rightarrow \eta(T^* + \varepsilon\Delta T) = \frac{-[y^*(T^*+\varepsilon\Delta T)-y^*(T^*)]}{\varepsilon\Delta T}\Delta T + \Delta y_T$

$\Rightarrow \lim_{\varepsilon\to 0}\eta(T^*) = \dot{y}^*(T^*)\Delta T + \Delta T$.

We have $V[\varepsilon] = \int_0^{T^*+\varepsilon\Delta T} F[t, y(t), \dot{y}(t)]dt$.

Let $T(\varepsilon) = T^* + \varepsilon\Delta T$, change the extremals of integration.

$\Rightarrow V[\varepsilon] = \int_0^{T(\varepsilon)} F[t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}(t)]dt$

$\frac{d}{dt}[\varepsilon] \underbrace{=}_{\text{Leibniz}} \int_0^{T(\varepsilon)} \left[F_y[...]\eta(t) + F_{\dot{y}}[...]\dot{\eta}(t)\right]dt + [F]_{T(\epsilon)}\frac{dT(\varepsilon)}{d\varepsilon}$

Integration by parts (as in the E.L. equation).

$\Rightarrow \int_0^{T(\varepsilon)} \left[F_y\eta(t) - \frac{d}{dt}F_{\dot{y}}\eta(t)\right]dt + \left[F_{\dot{y}}\eta(t)\right]_0^{T(\varepsilon)} + \left[F\right]_{T(\varepsilon)}\Delta T =$

$= \int_0^{T(\varepsilon)} \left[F_y - \frac{d}{dt}F_{\dot{y}}\right]\eta(t)dt + \left[F_{\dot{y}}\right]_{T(\varepsilon)}\eta(T(\varepsilon)) + [F]_{T(\varepsilon)}\Delta T =$

$\Rightarrow \frac{dV[0]}{d\varepsilon} = 0 \Leftrightarrow \in_0^{T^*} \left[F_y - \frac{d}{dt}F_{\dot{y}}\right]\eta(t)dt + \left[F_{\dot{y}}\right]_{T^*}\eta(T^*) + \left[F\right]_T\Delta T$.

F.O.C. $\int_0^{T^*} \left[F_y - \frac{d}{dt}F_{\dot{y}}\right]\eta(t)dt + \left[F_{\dot{y}}\right]_{T^*}\Delta y(t) + \left[F - \dot{y}F_{\dot{y}}\right]_{T^*}\Delta T = 0$, depending on which situation of TVC you are.

(A) Vertical terminal line $\Rightarrow T^*$ is fixed $\Rightarrow \Delta T = 0 \Rightarrow$ necessarily $[F_{\dot{y}}]_{T^*} = 0$.

(B) Horizontal terminal line $\Rightarrow y(T^*)$ is given $\Rightarrow \Delta y_T = 0 \Rightarrow$ necessarily $[F - \dot{y}F_{\dot{y}}] = 0$.

(C) It can be that $y(t^*) = \phi(T^*) \longrightarrow$ terminal curve $\Rightarrow [F_y + (\dot{\phi}-\dot{y})F_{\dot{y}}] = 0$ because the idea is that $\Delta y_t \approx \dot{\phi}(T^*)\Delta T$.

Ex. Find the extremals.

1. $V[y(t)] = \int_0^1 (y + y\dot{y} + \dot{y} + \frac{\dot{y}^2}{2})dt$ with $y(0) = 2$ and $y(1) = 5$.

Euler Lagrange equation: $F_y - \frac{d}{dt}F_{\dot{y}} = 0$.

$F[t, y(t), \dot{y}(t)] = y(t) + y(t)\dot{y}(t) + \dot{y}(t) + \frac{\dot{y}(t)^2}{2}$

$\frac{\partial F}{\partial y} = 1 + \dot{y}(t)$; $\frac{\partial F}{\partial \dot{y}} = y(t) + 1 + \dot{y}(t)$; $\frac{\partial F_{\dot{y}}}{\partial t} = \dot{y}(t) + \ddot{y}(t)$.

$\Rightarrow 1 + \dot{y}(t) - \dot{y}(t) - \ddot{y}(t) = 0$

$\ddot{y}(t) = 1$

$\dot{y}(t) = t + A, A \in \mathbb{R}$

$y(t) = \frac{t^2}{2} + At + B, B \in \mathbb{R}$

$y(0) = 2$ and $y(1) = 5 \Leftrightarrow B = 2$ and $\frac{1}{2} + A + 2 = 5 \Leftrightarrow A = 3 - \frac{1}{2} = \frac{6-1}{2} = \frac{5}{2} \Rightarrow y^*(t) = \frac{t^2}{2} + \frac{5}{2}t + 2$.

2. $V[y(t)] = \int_0^{\pi/2}[y^2 - \dot{y}^2]dt$ with $y(0) = 0$ and $y(\pi/2) = 1$.

$F[t, y(t), \dot{y}(t)] = y^2(t) - \dot{y}^2(t)$, the functional of interest
$\frac{\partial F}{\partial y} = 2y(t)$; $\frac{\partial F}{\partial \dot{y}} = -2\dot{y}(t)$; $\frac{\partial F_{\dot{y}}}{\partial t} = -2\ddot{y}(t)$.
$2y(t) + 2\ddot{y}(t) = 0$
$\ddot{y}(t) + y(t) = 0$
$\lambda^2 + 1 = 0$
$\lambda_{1,2} = \pm\sqrt{1}$
$\lambda_{1,2} = \pm i$

linear homogenous differential equation, whose characteristic equation has two complex solutions, of the type $\alpha \pm i\beta$, $\alpha = 0$ and $\beta = 1$. Two independent solutions of the form:

$$y_1(t) = e^{\alpha t}\cos\beta t = \cos t$$
$$y_2(t) = e^{\alpha t}\sin\beta t = \sin t$$

linear combination $\Rightarrow$

$$\begin{aligned} y(t) &= \lambda y_1(t) + \mu y_2(t) \\ &= \lambda\cos\beta t + \mu\sin\beta t, \lambda, \mu \in \mathbb{R} \end{aligned}$$

$$\begin{cases} y(0) = 0 & \Leftrightarrow \lambda\cos(0) + \mu\sin(0) = 0 & \Leftrightarrow \lambda = 0 \\ y(\pi/2) = 1 & \Leftrightarrow \lambda\cos(\pi/2) + \mu\sin(\pi/2) = 1 & \Leftrightarrow \mu = 1 \end{cases}$$

$$\Rightarrow y^*(t) = \sin(t)$$

3. $V[t, y(t), z(t)] = \int_0^{\pi/2}(2yz + \dot{y}^2 + \dot{z}^2)dt$ with $y(0) = z(0) = 0$ and $y(\pi/2) = 0 = z(\pi/2)$. $F[t, y(t), z(t), \dot{y}(t), \dot{z}(t)] = 2y(t)z(t) + \dot{y}(t)^2 + \dot{z}(t)^2$.
$\frac{\partial F}{\partial y} = 2z(t)$; $\frac{\partial F}{\partial \dot{y}} = 2\dot{y}(t)$; $\frac{\partial F_{\dot{y}}}{\partial t} = 2\ddot{y}(t)$.
E.L.: $2z(t) = 2\ddot{y}(t)$
$z(t) = \ddot{y}(t)$
$\frac{\partial F}{\partial z} = 2y(t)$; $\frac{\partial F}{\partial \dot{z}} = 2\dot{z}(t)$; $\frac{\partial F_{\dot{z}}}{\partial t} = 2\ddot{z}(t)$.
E.L.: $2y(t) = 2\ddot{z}(t)$
$y(t) = \ddot{z}(t)$

$$\begin{cases} z(t) = \ddot{y}(t) & (1) \\ y(t) = \ddot{z}(t) & (2) \end{cases}$$

plug (1) into (2) and get:
$y(t) = \ddddot{y}(t) \Rightarrow \ddddot{y}(t) - y(t) = 0 \Rightarrow \lambda^4 - 1 = 0 \Rightarrow (1-\lambda^2)(1+\lambda^2) row \underbrace{\lambda_{1,2} = \pm 1}_{\text{(A) } \Delta > 0}$ and $\underbrace{\lambda_{3,4} = \pm i}_{\text{(B) } \Delta < 0}$.

(A) Former two solutions are:

$$y_1(t) = e^{\lambda_1 t} = e^t$$
$$y_2(t) = e^{\lambda_2 t} = e^{-t}$$
$$y_A(t) = \lambda e^t{+}\mu e^{-t}, \lambda, \mu \in \mathbb{R}$$

(B) Latter two solutions:

$$y_3(t) = e^{0t}\cos\beta t = \cos t$$
$$y_4(t) = e^{0t}\sin\beta t = \sin t$$
$$y_b(t) = \nu\cos t + \eta\sin t, \nu, \eta \in \mathbb{R}$$

(A) + (B)

$$\Rightarrow y(t) = \lambda e^t + \mu e^{-t} + \nu\cos t + \eta\sin t$$

now, plug back in (2) and $z(t) = \ddot{y}(t)$.
$\dot{y}(t) = \lambda e^t - \mu e^{-t} - \nu\sin t + \eta\cos t$
$\ddot{y}(t) = \lambda e^t + \mu e^{-t} - \nu\cos t + -\eta\sin t = z(t)$

$$\begin{cases} y(t) = \lambda e^t + \mu e^{-t} + \nu\cos t + \eta\sin t & (*) \\ z(t) = \lambda e^t + \mu e^{-t} - \nu\cos t - \eta\sin t & (**) \end{cases}$$

$\lambda, \nu, \nu, \eta \in \mathbb{R}$, find them!

$$\begin{cases} y(0) = 0 & z(0) = 0 \\ y(\pi/2) = 1 & z(\pi/2) = 1 \end{cases}$$

$$\begin{cases} \lambda e^0 + \mu e^0 + \nu\cos(0) + \eta\sin(0) = 0 & (1') \\ \lambda e^0 + \mu e^0 - \nu\cos(0) - \eta\sin(0) = 0 & (2') \\ \lambda e^{\pi/2} + \mu e^{-\pi/2} + \nu\cos(\pi/2) + \eta\sin(\pi/2) = 1 & (3') \\ \lambda e^{\pi/2} + \mu e^{-\pi/2} - \nu\cos(\pi/2) - \eta\sin(\pi/2) = 1 & (4') \end{cases}$$

$$\begin{cases} \lambda + \mu + \nu = 0 \\ \lambda + \mu - \nu = 0 \\ \lambda e^{\pi/2} + \mu e^{-\pi 2/} + \eta = 1 \\ \lambda e^{\pi/2} + \mu e^{-\pi/2} - \eta = 1 \end{cases}$$

$(1') - (2') = \nu + \nu = 0 \Leftrightarrow 2\nu = 0 \Leftrightarrow \nu = 0.$
$(3') - (4') = 0 + 0 + 2\eta = 1 \Rightarrow \eta = \frac{1}{2}$
$\Rightarrow \lambda = -\mu \Rightarrow e^{pi/2} - \lambda e^{-\pi/2} = \frac{1}{2} \Leftrightarrow \lambda[e^{-\pi/2} + e^{\pi/}] = \frac{1}{2}$
$\lambda = \frac{1}{2[e^{\pi/2}+e^{-\pi/2}]}$ and $\mu = \frac{1}{2[e^{\pi/2}+e^{-\pi/2}]}$.
The solution to the variational problem is:

$$\begin{cases} y^*(t) = \frac{e^{\pi/2}}{2(e^{\pi/2}+e^{-\pi/2})} - \frac{e^{-\pi/2}}{2(e^{\pi/2}+e^{-\pi/2})} + \frac{1}{2}\sin t \\ z^*(t) = \frac{e^{\pi/2}}{2(e^{\pi/2}+e^{-\pi/2})} - \frac{e^{-\pi/2}}{2(e^{\pi/2}+e^{-\pi/2})} - \frac{1}{2}\sin t \end{cases}$$

4.3 Constrained problems

- EQUALITY CONSTRAINTS $\Rightarrow$ Lagrangian + associated E.L. eqution + satisfy equality constraints.

- CONSTRAINTS TAKING THE FORM OF DIFFERENTIAL EQ. $\Rightarrow$ this problem is similar to the first one but the equality constraints are such that:

$$\begin{cases} g_1(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) = c_1 \\ \vdots \\ g_m(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) = c_m \end{cases}$$

- INEQUALITY CONSTRAINTS:

$$\max / \min \int_0^T F[t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n] dt$$

$$\begin{cases} g_1(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) \leq c_1 \\ \vdots \\ g_m(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) \leq c_m \end{cases}$$

the size of the system is $m \times n$, subject to # of state variables $n > m$ is possible $\Rightarrow$ if you have less variables than constraints. Agin you write the Lagrangian equation:

$$L = F - \sum_{j=1}^{m} \lambda_j(t)[g_j(.) - c_j]$$

you have the following necessary conditions:

$$L_{y_i} - \frac{d}{dt}[L_{\dot{y}_i}] = 0, \forall i = 1, ..., n$$

system of $m-$ equations to solve:

$$\lambda_j(t)[g_j(.) - c_j] = 0, \forall j = 1, ..., n, \forall t \in [0, T] +$$

+ boundary conditions; + sign of λ_j (≥ 0: max; ≤ 0: min).

- ISOPERIMETRIC PROBLEMS $\Rightarrow \exists$ constraints involving integrals $\Rightarrow \int_0^T G[t,y,\dot{y}]dt = k,\ k \in \mathbb{R}$ with some boundary conditions $\Rightarrow$ in general the problem is written as follows:

$$\max / \min \int_0^T F[t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n]dt$$

such that:

$$\begin{cases} \int_0^T G_1(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) = k_1 \\ \vdots \\ \int_0^T G_M(t, y_1, ..., y_n, \dot{y}_1, ..., \dot{y}_n) = k_m \end{cases}$$

with $k_1, ..., k_m \in \mathbb{R}$ + boundary conditions.

Let us consider the following:

$$\max \int_0^T F[t,y,\dot{y}]dt s.t. \int_0^T G[t,y,\dot{y}]dt = k$$

We redefine $(t) = \int_0^T G[t,y,\dot{y}]dt$. We have the condition that $\Gamma(0) = 0$ and $\Gamma(T) = k$; so $\Gamma(.)$ measures in fact the accumulation of $G(.)$ from 0 to T, $\dot{\Gamma}(t) = G[T,y,\dot{y}]$ from the Fundamental Theorem of Calculus.

$G[t,y,\dot{y}] - \dot{\Gamma}(t) = 0$, which can be seen as the constraint $g(.) - c = 0$ where $G - \dot{\Gamma} = g$ and $c = 0$ so that the integrand constraint can be transformed into a new constraint taking the form of differential equation (2)

$\longrightarrow \bar{L} = F - \lambda(t)[G - \dot{\Gamma}]$. From Euler - Lagrange equation:

$\bar{L}_y - \frac{d}{dt}[\bar{L}_{\dot{y}}] = 0$ and $\bar{L}_\lambda - \frac{d}{dt}[\bar{L}_{\dot{\lambda}}] = 0 \longrightarrow -\frac{d}{dt}[L_{\dot{\lambda}}] = 0 \Rightarrow \frac{d}{dt}[\lambda(t)] = 0 \Rightarrow \lambda(t) = const. \Rightarrow$ it is simply λ depending on t not any more.

$L = F - \sum_{j=1}^m \lambda_j G_j$ + solving the corresponding Euler - Lagrange equation + satisfying boundary conditions. Ex:

1. Extremals of $\int_0^T \dot{y}^2 dt$ s.t. $\int_0^T y dt = k$ and $y(0) = 0$, $y(T) = T$.
2. $y(t)$ and $z(t)$ opt. $\int_0^T \dot{y}^2 + \dot{z}^2 dt$ s.t. $y - \dot{z} = 0$.

1. $L = \dot{y}^2 - \lambda[y] \Rightarrow L_y - \frac{d}{dt}[L_{\dot{y}}] = 0 = -\lambda - \frac{d}{dt}[2\dot{y}]$

$\lambda + 2\ddot{y} = 0 \Rightarrow \int(\lambda + 2\ddot{y})dt = 0 \Rightarrow \lambda t + 2\dot{y} + C = 0$

$\frac{\lambda t^2}{2} + 2y + Ct + D = 0 \Leftrightarrow y(0) = 0$ and $y(T) = T$

$0 + 0 + 0 + D = 0$; $\frac{\lambda T^2}{2} + 2T + CT = 0 \Leftrightarrow$

$\frac{\lambda T^2}{2} + (2 + C)T = 0$

$T[\lambda T + 4 + 2C] = 0$

$T = 0$ and $T = \frac{-2C-4}{\lambda}$

$y(t) = -\frac{\lambda}{4}t^2 + (-2 - \frac{\lambda T}{2})t$

2. $F = \dot{y}^2 + \dot{z}^2$, $G = y - \dot{z}$
$L = \dot{y}^2 + \dot{z}^2 - \lambda(t)(y - \dot{z})$

$$\begin{cases} L_y - \frac{d}{dt}[L_{\dot{y}}] = 0 & \Leftrightarrow -\lambda(t) - \frac{d}{dt}[2\dot{y}] = 0 \\ L_z - \frac{d}{dt}[L_{\dot{z}}] = 0 & \Leftrightarrow 0 = \frac{d}{dt}[2\dot{z} + \lambda] \end{cases}$$

$$\begin{cases} \lambda = -2\ddot{y} & \text{eliminate the multiplier} \\ -2\ddot{z} = \dot{\lambda} & 2\ddot{y} = 2\ddot{z} \end{cases}$$

$\Rightarrow \int \ddot{z}dt = \int \ddot{y}dt \Leftrightarrow \dot{z} + C = \dot{y} + D \Leftrightarrow z + Ct + E = y + Dt + H.$

We also have to solve the differential equation $2\ddot{y} - 2\dot{y} = 0$, throughout the method of the variation of constants or with the guess one.

$$\begin{aligned} \ddot{y} - \dot{y} &= 0 \\ \int \ddot{y}dt &= \int \dot{y}dt \\ \dot{y} - y + C &= 0 \\ y - yt + Ct + D &= 0 \end{aligned}$$

In other problems, λ will depend on t.

$\ddot{y} - \dot{y} = 0 \Leftrightarrow x^2 - x = 0 \Leftrightarrow x(x-1) = 0 \Leftrightarrow x_1 = 0$ and $x_2 = 1$.

$y_1(t) = e^{0t} = 1$ and $y_2(t) = 1e^t = e^t$.

$y(t) = \sum_{i=1}^{n} A_i(t)y_i(t)$, where $y_i(t)$ are the particular solutions of the homogenous associated equation. Isomorphism between $\mathbb{R}^2$ and the set of solutions...determine two linearly independent solutions of (**).

$\lambda(t) = -2\ddot{y} = -2\ddot{z} \Rightarrow 2\ddot{z} - 2\ddot{y} = 0$

$2x^2 - 2x^4 \Leftrightarrow x^2(1 - x^2) = 0$ from linear differential equation, $x_1 = 0$ and $x_{2,3} = \pm 1 \Rightarrow z(t) = A + Bt + Ce^t + De^{-t}$ and so $y(t) = B + Ce^t - De^{-t}$ and we can also find backwardly $\lambda(t) = -2(Ce^t - De^{-t}) = 2\ddot{y}(t)\blacktriangle$.

4.4 Optimal Control Theory

The calculus of variations is the classical method to solve dynamic optimization problems.But it allows only for interior solutions. Optimal control theory can have corner solutions and can take into account functions that are not everywhere differentiable. In calculus of variations the goal is to find the optimal path of the state variable $y(t)$, denoted by $y^*(t)$. In optimal control the aim is to find the optimal control variable $u(t)$, denoted by $u^*(t)$, which will help to determine $y^*(t)$ because the control variable $u(t)$ drives the state variable $y(t)$.

4.4.1 The simplest problem of optimal control

One *control* variable $u(t)$ and one *state* variable $y(t)$. $u(t)$ can be seen as a policy instrument which influences or drives $y(t)$. To each $u(t)$ corresponds one and only one $y(t)$.

Special features

(a) $u(t)$ does not necessarily have to be continuous in order to be admissible. It just has to be piece-wise continuous. (b) In optimal control, the control variable $u(t)$ can have some constraints on it, such as $u(t) \in U$, a set, $\forall t \in [0, T] \Rightarrow$ set closed and bounded (*compact*) $\rightarrow$ "bang-bang" solution. (c) The simplest problem in optimal control is to consider that y_T is free. The problem is stated as follows: $\max V[u(t), y(t)] = \int_0^T F[t, y, u]dt$ subject to the law of motion of the state variable $\dot{y} = f(t, y, u)$, $y(0) = A$, $y(T) = free$, T, A are given and $u(t) \in U, (t \in [0, T])$.

Note: (i) we focus on max. Clearly, min $= -$max. (ii) Motion equation $\Rightarrow$ given a time t and state y, it tells us in which direction doe y move if we choose the policy $u \Rightarrow$ this equation describes the pattern as a function of the policy u. (iii) If $\dot{y} = u$, then the problem boils down to $\max \int_0^T F[y, y, \dot{y}]dt$ with $y(0) = A$, $y(T) = free$ (A, T given) which is a problem of C.o.V. with a vertical terminal line $\Leftrightarrow$ use some transversality condition.

4.4.2 Maximum Principle (Pontryagin)

Let us define the Hamiltonian equation:

$$H[t, y, u, \lambda] = F[t, y, u] + \lambda(t)f[t, y, u]$$

λ is called the co-state variable. The Hamiltonian is similar to a Lagrangian equation.

Theorem: the necessary conditions for $u^*(t)$ and $y^*(t)$ to be optimal are that $\exists$ a $\lambda(t)$ such that $\forall t \in [0, T]$:

$H[t, y^*, u^*, \lambda^*] = \max_u H[t, y^*, u, \lambda^*]$;

$\dot{y}^* = \frac{\partial H}{\partial \lambda}() \Rightarrow$ equation of motion of y;

$\dot{\lambda}^* = -\frac{\partial H}{\partial y}() \Rightarrow$ equation of motion of λ;

$\lambda^*(t) = 0 \Rightarrow$ TVC for a free terminal point.

(1) $\max_u \int_0^2 (y-u^2)dt$ s.t. $\dot{y} = u$ with $y(0) = 0$, $y(2) = free$ and u is free. $H[t, y, u, \lambda] = F[t, y, u] + \lambda(t) f(t, y, u) = (y - u^2) + \lambda(t)u$.

(a) $\frac{\partial H}{\partial u} = 0 \Leftrightarrow 2u = \frac{\lambda(t)}{2}$

(c) $\dot{\lambda} = -\frac{\partial H}{\partial y} \Leftrightarrow \dot{\lambda} = 1 \Leftrightarrow \lambda = -t + A$

(d) $\lambda(T) = \lambda(2) = 0 \to$ in C

$\Rightarrow -2 + A = 0$

$$\begin{cases} \lambda^* = -t + 2 \\ u^* = \frac{-t+2}{2} = -\frac{t}{2} + 2 \end{cases}$$

(b) $\dot{y}^* = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{y} = u \Rightarrow y = -\frac{t^2}{4} + t + B$, $B = 0$ and $y(0) = 0 \Rightarrow y^* = -\frac{t^2}{4} + t$.

Note: we could have used the E.L. eq. to get the same result $\Rightarrow F_y - \frac{d}{dt}[F_{\dot{y}}] = 0 \Leftrightarrow 2\ddot{y} = -1 \Leftrightarrow \ddot{y} = -\frac{1}{2}t + A \Rightarrow y = -\frac{1}{2}t^2 + At + B$, $y(0) = 0 \Rightarrow B = 0$, vertical terminal line (TVC), $[F_{\dot{y}}]_{t=T} = 0 \Leftrightarrow 2 = 2A \Leftrightarrow A = 1 \Rightarrow y = -\frac{t^2}{4} + t + B = -\frac{t^2}{4} + t$.

4.4.3 Justification of the Maximum Principle

Just an intuition of the proof...skip it!!

4.4.4 Alternative transversality conditions

1. Fixed terminal point $(T, y_T fixed) \Rightarrow$ maximum principle remains unchanged, except condition (d) $\to$ (d') $y(T) = y_T$.

2. Horizontal terminal line (y_T given, T free) $\to$ transversality condition $[H]_{t=T} = 0$. No restrictions on $\lambda(t)$.

3. Terminal curve, some function $\phi(T)$, $(y_T = \phi(T), \phi$ given$) \Rightarrow$ TVC $[H - \lambda\dot{\phi}]_{t=T} = 0 \to$ calculus of variations $[F + (\dot{\phi} - \dot{y}F_{\dot{y}})]_{t=T} = 0$.

4. Truncated vertical terminal line $\Rightarrow T$ is given ($y_T > y_{min}$, given) $\Rightarrow$ either $y_T^* > y_{min}$ or $y_T^* = y_{min}$ reduced the problem in consideration to be a problem of fixed terminal point, back to (a).

5. Truncated horizontal line $\Rightarrow y_T$ given, $T < T_{max}$ (given), $[H]_{t=T} \geq 0, T \leq T_{max}$, $[T - T_{max}] \times [H]_{t=T} = 0 \Leftrightarrow$ TVC.

Ex. $\max V[u(t), y(t)] = \int_0^1 (-u^2)dt$ such that $\dot{y} = y + u$ with $y(0) = 1$ and $y(1) = 0$. Hamiltonian: $H[t, u, y, \lambda] = -u^2 + \lambda(t)[y + u]$,

(a) $\frac{\partial H}{\partial u} = 0 \Leftrightarrow -2u(t) + \lambda(t) = 0 \Leftrightarrow \lambda(t) = 2u(t) \Leftrightarrow u = \frac{\lambda}{2}$

(c) $\dot{\lambda} = -\frac{\partial H}{\partial y} \Leftrightarrow \dot{\lambda} = -\lambda \Leftrightarrow \dot{\lambda} + \lambda = 0 \Leftrightarrow x + 1 = 0 \Leftrightarrow x = -1 \Rightarrow \lambda(t) = Ce^{-t}$, $C \in \mathbb{R}$

(d) $\lambda(T) = \lambda(1) = 0$, $0 = e^1$, $C = 1/e$

(b) $\dot{y} = \frac{\partial H}{\partial \lambda} \Leftrightarrow \dot{y} = y + u \Leftrightarrow \dot{y} - y - u = 0$

4.4.5 The "Current Hamiltonian"

The integrand $F[t, y, u]$ is of the form[2] $G[t, y, u]e^{-\rho t}$ and so $\max V[y, u] = \int_0^T G[t, u, y]e^{-\rho t}dt$ such that $\dot{y} = f(t, y, u)$ with initial and terminal conditions. The Hamiltonian function is

$$H[t, y, u, \lambda] = G[t, y, u]e^{-\rho t} + \lambda(t)f(t, y, u) \tag{4.1}$$

and this is called the current Hamiltonian:

$$H_c = He^{\rho t} = G[t, y, u] + m(t)f(t, y, u) \tag{4.2}$$

with $m(t) = \lambda(t)e^{\rho t}$.

What does the maximum principle become?

(a) $\max H_c$ w.r.t. $u(t)$

(b) $\dot{y} = \frac{\partial H_c}{\partial m} = f(t, y, u)$, the eq. of motion

(c) $\dot{m} = -\frac{\partial H_c}{\partial y} + \rho m$ and $\lambda = me^{-\rho t} \Leftrightarrow \dot{\lambda} = \dot{m}e^{-\rho t} - \rho m e^{-\rho t}$

$\frac{\partial H}{\partial y} = -\frac{\partial H_c}{\partial y}e^{-\rho t}$

$\dot{m} - \rho m = -\frac{\partial H_c}{\partial y}$

$\dot{m} = -\frac{\partial H_c}{\partial y} + \rho m$

(d) $m(T)e^{-\rho t} = 0$, TVC, since $\lambda(T) = 0$.

[2] It has a discount factor attached to it.

4.4.6 Sufficient conditions

1. Mangasarian $\Rightarrow$ **Theorem**: for the standard problem $V[y,u] = \int_0^T F[t,y,u]dt$ subject to $\dot{y} = f(t,y,u)$ with $y(0) = 0 = y(T)$. T given, $y(T)$ is free, the necessary conditions of the MP are sufficient for a global maximum of V if (a) F and f are differentiable and concave in (y,u); (b) $\lambda^*(t) \geq 0$, $\forall t \in [0,T]$ if f is non-linear in y or u.
   Proof: follows more or less the same lines as the proofs in CoV.

2. Arrow $\Rightarrow$ this condition, weaker than the previous one, needs a new Hamiltonian $H^0[t,y,u,\lambda] = F[t,y,u^*] + \lambda(t)f(t,y,u^*)$, where $u^*(t,y,\lambda)$ is such that $H[t,y,u^*,\lambda] = \max_u H[t,y,u,\lambda]$.
   Remark: $H^0 \neq H[t,y^*,u^*,\lambda^*]$.
   Theorem $\Rightarrow$ the conditions obtained in the maximum principle are sufficient for the maximization of V if $H^=$ is concave in y, $\forall t \in [0,T]$, λ fixed. Here, you just check the convexity or concavity of the Hamiltonian itself, not the multiplier that is assumed to be fixed. Equivalent to the Legendre's condition.

4.4.7 Several state or control variables

$\max V[y(t)] = \int_0^T F[t,y,u]dt$ subject to $\dot{y} = f(t,u,y)$ with $y(0) = y_0$ and $y(T) = y_T$ and $u(t) \in U$, where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix}$$

and

$$f(t,u,y) = \begin{pmatrix} f_1(t,y,u) \\ \vdots \\ f_n(t,y,u) \end{pmatrix}.$$

$y(0), y_0, y(T), y_T \in \mathbb{R}^{n\times 1}$, $U = (u1, \cdots, u_m)'$, m and n can take any values. The Maximum Principle: define

$$H[t,y,u,\lambda] = F[t,y,u] + \sum_{j=1}^{m} \lambda_j f_j(t,y,u)$$

or, in matrix notation

$$H[t, y, u, \lambda] = F[t, y, u, \lambda] + \lambda' f(t, y, u)$$

where $\lambda = (\lambda_1 \cdots \lambda_n)'$.

(a) $\max H$, $u \in \mathbb{R}^{m\times 1}$;

(b) $\dot{y}_j = \frac{\partial H}{\partial \lambda_j}$, $\forall j = 1, ..., m$;

(c) $\dot{\lambda}_j = -\frac{\partial H}{\partial y_j}$, $\forall j = 1, ..., m$;

(d) $y(T) = y_T$.

The TVC as well remains, just be careful, now you are dealing with vectors.

4.4.8 Infinite horizon problems

From $\int_0^T$ to $\int_0^\infty \Rightarrow$ discussion on the convergence, see calculus of variations. Transversality conditions $\Rightarrow$ simply put a $\lim_{T\to\infty}$ before the conditions. The sufficient conditions stay true. And if extra conditions are needed, then put a $\lim_{T\to\infty}$ before the condition.

4.4.9 The problem of constrained optimal control: constraints involving the control variable

Equality constraints

For example, $\max \int_0^T F[t, y, u_1, u_2]dt$ such that $\dot{y} = f(t, y, u_1, u_2)$ and $g(t, y, u_1, u_2) = c$ + some boundary conditions $\longrightarrow H[t, y, u_1, u_2, \lambda] = F[t, y, u_1, u_2] + \lambda(t) f(t, y, u_1, u_2) \Rightarrow$ take the Lagrangian:

$$L(t, y, u_1, u_2, \lambda, \theta) = H - \theta(t)[g(t, y, u_1, u_2) - c]$$

(MP) $\Rightarrow$ a. $\frac{\partial L}{\partial u_1} = 0$ and $\frac{\partial L}{\partial u_2} = 0$, $\forall t \in [0, T]$;

b. $\dot{y} = \frac{\partial L}{\partial \lambda} = \frac{\partial H}{\partial \lambda}$;

c. $\dot{\lambda} = -[\frac{\partial H}{\partial y} + \theta(t)\frac{\partial g}{\partial y}]$

+ TVC.

Inequality constraints

$\max \int_0^T F[t, y, u_1, u_2]dt$ subject to $\dot{y} = f(t, y, u_1, u_2)$, $g_1(t, y, u_1, u_2) \leq c_1$, $g_2(t, y, u_1, u_2) \leq c_2$ and some limit conditions.

We have $L(t, y, u_1, u_2, \lambda, \theta_1, \theta_2) = H - \theta_1[g_1(t, y, u_1, u_2) - c_1] - \theta_2[g_2(t, y, u_1, u_2) - c_2]$.

(MP) $\Rightarrow \frac{\partial L}{\partial u_1} = 0$; $\frac{\partial L}{\partial u_2} = 0$

$\frac{\partial L}{\partial \theta_1}\theta_1 = 0$; $\frac{\partial L}{\partial \theta_1} = 0$;

$\frac{\partial L}{\partial \theta_2}\theta_2 = 0$; $\frac{\partial L}{\partial \theta_2} = 0$[3].

b. $\dot{y} = \frac{\partial L}{\partial \lambda}$; c. $\dot{\lambda} = -\frac{\partial L}{\partial y}$ + d. TVC.

Isoperimetric problems

$\max_{y,u} \int_0^T F[t, y, u]dt$ such that $\dot{y} = f(t, y, u)$, $\int_0^T G[t, y, u]dt = k$, ($k$ given), $y(0) = 0$, $y(T) \equiv$ free for T given. New state variable $z(t)$ is introduced in order to replace the integral constraint. Let $z(t) = -\int_0^T G[t, y, u]dt \Rightarrow \dot{z} = -G[t, y, u]$ with $z(0) = 0$ and $z(T) = -k$ (given). The problem boils down to:

$\max \int_0^T F[t, y, u]dt$ s.t. $\dot{y} = f(t, y, u)$, $\dot{z} = -G[t, y, u]$ with $y(0) = 0$, y_T free, T given. $z(0) = 0$, $z(T) = -k_T$, $k \in \mathbb{R}$. This problem can be seen as an unconstrained optimal control problem with two state variables.

$$H[t, y, z, u, \lambda, \mu] = F[t, y, u] + \lambda(t)f(t, y, u) + \mu(t)G[t, y, u]$$

still to be maximized with respect to $u(t)$:

(M.P.) a. $\max_u H$, $\forall t \in [0, T]$;

b. $\dot{y} = \frac{\partial H}{\partial \lambda}$, $\dot{z} = \frac{\partial H}{\partial \mu}$;

c. $\dot{\lambda} = -\frac{\partial H}{\partial y}$, $\dot{\mu} = -\frac{\partial H}{\partial z}$;

d. $\lambda(T) = 0$.

You transformed an integral constrained into another state variable...

[3] From Khun - Tucker conditions.

Inequality integral constraints

$\max \int_0^T F[t,y,u]dt$ such that $\dot{y} = f(t,y,u)$ with $y(0) = 0$, y_T free with T given. $\int_0^T G[t,y,u]dt$. As before, we let $z(t) = -\int_0^T G[t,y,u]dt \Rightarrow \dot{z}(t) = -G[t,y,u]$ with $z(0) = 0$ and $z(T) \geq -k$. Thus, the problem can be written as:

$\max \int_0^T F[t,y,u]dt$ s.t. $\dot{y} = f(t,y,u)$ and
$dotz = -G[t,y,u]$ with $y(0) = 0$, y_T free, T given, $z(0) = 0$, $z(T) \geq -k$
$\Rightarrow H = F + +\mu(-G)$.

(MP) a. $\max_u H$, $\forall t \in [0,T]$;

b. $\dot{y} = \frac{\partial H}{\partial \lambda}$, $\dot{z} = \frac{\partial H}{\partial \mu}$;

c. $\dot{\lambda} = -\frac{\partial H}{\partial y}$, $\dot{\mu} = -\frac{\partial H}{\partial z}$;[4]

d. $\lambda(T) = \lambda_T$, $z(T) \geq -k$.

Exercises

1. $\max V[y(t),u(t)] = \int_0^T -(1-u^2)^{1/2}dt$ such that $\dot{y} = u$ with $y(0) = A$, $y(T) \equiv$ free, $(A,T$ given).

2. $\max \int_0^4 3ydt$ s.t. $\dot{y} = y + u$, $y(0) = 5$, $y(4)$ free and $u(t) \in [0,2]$.

3. $\max \int_0^1 (-u^2)dt$ such that $\dot{y} = 2u$ with $y(0) = 1$ and $\mathrm{d}y(1) \geq 3$.

4. $\max \int_0^T (-1)dt$ s.t. $\dot{y} = y + u$ with $y(0) = 5$, $y(T) = 11$, T free and $u(t) \in [-1,1]$.

Solutions

1. $H = F[t,y,u] + \lambda(t) f(t,y,u) = -(1+u^2)^{1/2} + \lambda(t)[u]$

(i) $\frac{\partial H}{\partial u} = 0 \Leftrightarrow -\frac{1}{2}(1+u^2)^{-1/2} 2u + \lambda(t)$

$\lambda(t) = \frac{1}{2}\frac{2u}{(2+u^2)^{1/2}}$

$\lambda^2(t) = \frac{1}{4}\frac{4u^2}{1+u^2}$

$(1+u^2)\lambda^2(t) = u^2$

$\lambda^2(t) = u^2(1-\lambda(t)^2)$

$\frac{\lambda^2(t)}{1-\lambda^2(t)} = u^2(t)$

[4] $\dot{\mu} = 0 \Rightarrow \mu = const.$ as in the isoperimetric problem in the CoV.

$\frac{\lambda(t)}{(1+\lambda^2(t))^{1/2}} = u(t)$

(ii) $\dot{\lambda} = -\frac{\partial H}{\partial y} \Leftrightarrow \dot{y} = 0 \Leftrightarrow \lambda = C$, $C \in \mathbb{R}$.

(iii) $\dot{y} = \frac{\partial H}{\partial \lambda} \Leftrightarrow \dot{y} = u = \frac{\lambda(t)}{(1-\lambda^2(t))^{1/2}} = \frac{C}{1-C^2}^{1/2} = 0$ check for the value of C using the boundary conditions, $y(0) = A$, $y(T)$ free.

2. $H = F[t, y, u] + \lambda(t) f(t, y, u) = 3y + \lambda(t)(y + u)$

(i) $\frac{\partial H}{\partial u} = 0 \Leftrightarrow \lambda(t) = 0$ slope of u is $\lambda \Rightarrow u^*(t) = \begin{cases} 0 & \text{if} \lambda < 0 \\ 1 & \text{if} \lambda > 0; \end{cases}$

(ii) $\dot{\lambda} = -\frac{\partial H}{\partial y} \Leftrightarrow \dot{\lambda} = -3 - \lambda(t) \Leftrightarrow \lambda = 3t$ because $\lambda(t) = 0$;

(iii) $\dot{y} = \frac{\partial H}{\partial \lambda} \Leftrightarrow \dot{y} = y + u \Rightarrow u(t) = 0$or $u(t) = 2$...get corner solutions...$\dot{y} = y$ or $\dot{y} = y + 2$

$\dot{y} - y = 0$

$x - 1 = 0$

$x = 1$

$y = Ce^{1t}$, $C \in \mathbb{R}$

$\dot{\lambda} = -3 - 2\lambda$

$\dot{\lambda} + \lambda + 3 =$

$x + 1 = 0$

$x = -1$

$\lambda = Ke^{-1t}$, $K \in \mathbb{R}$

$\dot{y} - y - 2 = 0$

$x - 1 = 0$

$x = 1$

$y = Ce^{1t} + 2$, $C \in \mathbb{R}$.

4.5 Dynamic Programming

[...] This is a backwards reasoning procedure to find the optimal values of x_0, x_1,..., x_T and u_0, u_1,...,u_T. This sequence will lead us in the end to a sequence of optimal value functions.

Example: $\max_u \sum_{t=0}^{3}(1+x_t-u_t^2)$ s.t. $x_{t+1} = x_t+u_t$, $\forall t = 0,1,2$. Note $T = 3$, $F[t,x,u] = 1+x-u^2$ and $x_0 = 0$, $u_t \in \mathbb{R}$, $f(t,x,u) = x+u$, $s = T = 3$, $\max 1+x-u^2$,

FOC $\frac{d\cdot}{du} = 0 \Rightarrow -2u = 0 \Rightarrow u = 0$

This is a maximum as u is concave. Then, $u_3 = 0$, whatever the value of x_3 is.

The value function is $I_3 = 1+x+0$.

$s = 2$, $\max_u \underbrace{1+x-u^2}_{F} + \underbrace{1+u+x}_{I_3(f(t,x_t,u_t))}$

FOC: $2u+1 = 0 \Rightarrow u = \frac{1}{2} \Rightarrow u_2 = \frac{1}{2}$ and $I_2 = 1+x-\frac{1}{4}+1+\frac{1}{2}+x = 2x+\frac{9}{4}$.

$s = 1$, $\max_u 1+x-u^2+2(x+u)+\frac{9}{4}$

FOC $-2u+3u = 0 \Rightarrow u = \frac{3}{2}$.

Thus, $u_0 = \frac{3}{2}$ and $I_0 = 1+x-\frac{9}{4}+3x+\frac{9}{2}+\frac{17}{4} = 4x+\frac{15}{2}$.

Given the initial value $x_0 = 0$, we can compute values for the state variable, $x_1 = x_0+u_0 = \frac{3}{2}$, $x_2 = x_1+u_1 = \frac{5}{2}$, $x_3 = x_2+u_2 = 3$.

4.5.1 Euler Equation

It's the cornerstone to add stochastic uncertainty to the problem. Clearly, this is something that you encounter in macroeconomics to derive Euler Equations, we will state the problem in the following terms:

$\max_{x_0,x_1,...,x_T} \sum_{t=0}^{T} F[t,x_t,x_{t+1}]$ with x_0 given, $x_1,...,x_T \in \mathbb{R}$.

There is no control variable, but, contrarily to before, when we focused on $\max F[t,x_t,u_t]$ s.t. $x_{t+1} = f(t,x_t,u_t)$, we now suppose that we can "invert" the function to obtain $u_t = \phi(t,x_t,x_{t+1})$. This is because of the theorem of invertibility $\Leftrightarrow$ there exists a unique u_t making $x_{t+1} = f(t,x_t,u_t)$.

Then, we can of course eliminate u_t,

$$\max \sum F[t, x_t, \phi(t, x_t, x_{t+1})] = \max \sum \tilde{F}(t, x_t, x_{t+1})$$

To derive the Euler equation, we focus on the first order conditions:

$$\max \sum_{t=0}^{T} F[t, x_t, x_{t+1}] = \max[F(0, x_0, x_1) + F(1, x_1, x_2) + ...]$$

$\Rightarrow x_0$ is given, $\nexists$ any F.O.C. to calculate $\Rightarrow \forall t = 1, ..., T$, $\frac{\partial \sum F(\cdot)}{\partial x_t} = F_3(t-1, x_{t-1}, x_t) + F_2(t, x_2, x_1) \Rightarrow T+1 : F_3(T, x_T, x_{T+1}) = 0$.

Remark: in the continuous world we find that: E.-L. eq.: $F_y - \frac{d}{dt} F_{\dot{y}} = 0 \Leftrightarrow F_2 - \frac{d}{dt} F_3$.

Example: $\max \sum_{t=0}^{T-1} \ln u_t + \ln x_T$ s.t. $x_{t+1} = \frac{1}{a}(x_t - u_t), \forall t = 0, ..., T \Rightarrow x_{t+1} = \frac{1}{a}(x_t - u_t) \Rightarrow u_t = x_t - a x_{t+1} \Rightarrow \max \sum_{t=0}^{T-1} \ln(x_t - a x_{t+1}) + \ln x_T$, disappears u_t.

t = 1: $\ln(x_0 - a x_1) + 2 \ln x_T + \ln(x_1 - a x_2)$

FOC: $\frac{\partial}{\partial x_1} = \frac{1}{x_0 - a x_1}(-a) + \frac{1}{x_1 - a x_2} = 0$

$\forall t = 1, ..., T-1$: $\frac{1}{x_{t-1} - a x_t}(-a) + \frac{1}{x_t - a x_{t+1}} = 0$

t = T: $\frac{-a}{x_{T-1} - a x_T} - \frac{1}{x_T} = 0 \Leftrightarrow x_{T-1} = 2 a x_T$

Let's do the backwards substitution ($t = T - 1$):

$$\begin{cases} x_{T-1} = 2 a x_T \\ \frac{-a}{x_{T-2} - a x_{T-1}} + \frac{1}{x_{T-1} + a x_T} = 0 \Leftrightarrow \end{cases}$$

$$\begin{cases} x_{T-1} = 2 a x_T \\ \frac{-a}{x_{T-2} - 2a^2 x_T} + \frac{1}{a x_T} = 0 \Leftrightarrow \end{cases}$$

$$\begin{cases} x_{T-1} = 2 a x_T \\ a^2 x_T = x_{T-2} - 2a^2 x_T \Leftrightarrow \end{cases}$$

$$\begin{cases} x_{T-1} = 2 a x_T \\ x_{T-2} = 3a^2 x_T \Leftrightarrow \end{cases}$$

By repeating the procedure, we obtain for:

$x_{t-k} = (k+1) a^k x_T \Rightarrow x_T = \frac{1}{T+1} \frac{1}{a^T}, x_0, x_1, ...$

4.5.2 Bellman Equation

This is the setting where the time horizon in $+\infty$. This is relevant if the horizon is reached or, in order to avoid that one needs an extra exogenous variable, because, e.g., T could be endogenous. We will thus focus on the following problem: $\max_{u_0,u_1} \sum_{t=0}^{\infty} \beta^t F(x_t, u_t)$ s.t. $x_{t+1} = f(x_t, u_t), \forall t = 0, ..., \infty$, x_0 given, $u_t \in U$.

Note: (i) F and f do not directly depend on t. (ii) $\tilde{F}(t, x_t, u_t) = \beta^t F(x_t, u_t)$. (iii) $\beta \in]0,1[$ since it's a discount factor, otherwise, explosive behaviour.

The optimal value function can be written as:

$$I(x) = \max_{u \in U} F(x, u) + \beta I(f(x, u)).$$

As a comparison, to $I_S = F(\cdot) + I_{S+1}(x)$, the impact of the times s and $s+1$ disappears due to the infinite horizon and we discount the future that is why β^1 disappears.

The Bellman equation is called a "functional equation" because we are interested in I and not in the variables.

Example: $\max \sum_{t=0}^{\infty} \beta^t (u_t, x_t)^{1-\gamma}$ s.t. $x_{t+1} = a(1 - u_t)$, $\forall t$, x_0 given, $a > 0$ given, $u_t \in (0, 1)$, $\gamma \in (0, 1)$, $\beta \in (0, 1)$.

In economics, $x_t \equiv$ wealth, $(1 - u_t) \equiv$ path to be solved, $(u_t + x_t)^{1-\gamma}$ is utility. Guess a solution and verify it $\Rightarrow$ and guess for $I(x)$ is $Cx^{1-\gamma}$.

The Bellman equation is $I(x) = \max_{u \in (0,1)} (ux)^{1-\gamma} + \beta I(a(1 - u)x)$

$Cx^{1-\gamma} = \max (ux)^{1-\gamma} + \beta C(a(1 - u)x)^{1-\gamma}$

$C = \max u^{1-\gamma} + \beta C a^{1-\gamma} (1 - u)^{1-\gamma}$

FOC $1 -^{-\gamma} + (1 + \gamma) D(1 - u)^{-\gamma} = 0$

$\Leftrightarrow u = \frac{1}{1+D^{1/\gamma}}$

This implies that $C = (1 + d^{1/\gamma})^{\gamma}$.

In summary: $u = 1 - (\beta a^{1-\gamma})^{1/\gamma}$

$I(x) = \frac{x^{1-\gamma}}{(1-(\beta a^{1-\gamma})^{1/\gamma})^{\gamma}}$

$x_1 = a(\beta a^{1-\gamma})^{1/\gamma} x_0 = E x_0$

$x_2 = a(\beta a^{1-\gamma})^{1/\gamma} x_1 = E^2 x_0$

$\vdots$

$x_k = E^k x_0$

REMARKS: Dynamic programming is mostly used in discrete setting, whereas optimal control theory is mostly used in continuous time. In continuous time one has to solve the partial differential equation of Hamilton-Jacobi-Bellman.