microeconomic theory: general equilibrium

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Andreu Mas-Colell, Michael D. Winston and Jerry Green[1]
"MICRO - ECONOMIC THEORY" - Oxford University Press, 1996

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1.1 PREFERENCE RELATIONS
$\succsim \longrightarrow$ binary relation on the set of alternative $X \rightarrow$
$\rightarrow$ allowing the comparison between pairs of alternatives $x, y \in X$.

$Sales_t := \alpha + \beta_1 \times C_t + \beta_2 \times LabourCost_t + \beta_3 \times no.Trademarks_t + \beta_4 \times AdvertExp_t + \epsilon_t$

$\rightarrow$ reduced form

Δ amount of sales $\leftrightarrow$ trademarks.

trademark value: ROE, ROA, ROS;
cash flow methods;
accrual methods;
cash flow from assets;

(i) strict preference relation $\rightarrow$
$\rightarrow \succ \rightarrow x \succ y \leftrightarrow x \succsim y$ but not $y \succsim x$.

(ii) indifference relation $\sim \rightarrow x \succsim y \wedge y \succsim x$;

DEF. rationality of $\succsim$:
(i) completeness $\rightarrow$
$\forall x, y \in X \rightarrow x \succsim y \vee y \succsim x$;
(ii) transitivity $\rightarrow \forall x, y, z \in X$,
if $x \succsim y \wedge y \succsim z \rightarrow x \succsim z$.

i has to be well defined
preferences $\rightarrow$ btwn $x \wedge y \in X$;
PROP. if $\succsim$ is rational $\rightarrow$

(i) $\succ$ is both irreflexive
($x \succ x$ never holds) and transitive
(if $x \succ y$ and $y \succ z \rightarrow x \succ z$);

(ii) $\sim$ is irreflexive ($x \sim x, \forall x \in X$),
transitive (if $x \sim y$ and $y \sim z \rightarrow x \sim z$);
and symmetric[2] (il $x \sim y \rightarrow y \sim x$);

(iii) if $x \succ y \succsim z \rightarrow x \succ z$.

just perceptible difference
choice $\leftrightarrow$ framing
a utility function $u(x)$
assigns a numerical value
to each element in $X \rightarrow$ ranking;

DEF. a function $u : x \rightarrow \mathbb{R}$ is a utility
function representing pref. relation
$\succsim$ if, $\forall x, y \in X$:

$x \succsim y \leftrightarrow u(x) \geq u(y)$.
ordinal concept.

***

PROP. a preference relation can be
represented by a $u(x)$ iff[3]
it's rational.

***

CHOICE RULES $\rightarrow (\mathcal{B}, C(.)) \rightarrow$
(i) $\beta(.)$ is a set of $\emptyset$ subsets
of $X \rightarrow \mathcal{B} \in \beta \rightarrow$ budget
set;
(ii) $C(.)$ is a choice rule $\rightarrow$
assigns a non-empty set of chosen
elements of $C(B) \subset \mathcal{B}$ for $\forall \mathcal{B} \in \beta$.

$\rightarrow$ elements of $C(B)$ are the acc=
eptable alternatives of $\mathcal{B}$.

WARP $\rightarrow$ consistency $\rightarrow$
samuelson (1947) $\rightarrow$ is satisfied $\rightarrow$ by the choice
structure $[\mathcal{B}, C(.)]$ if the foll. hold:
if, for some $\mathcal{B} \in \beta$, with $x, y \in \mathcal{B}$

[2]or commutative
[3]if and only if

we've $x \in C(\mathcal{B}) \to$ for $\forall \mathcal{B}' \in \beta$
with $x, y \in \mathcal{B} \to x \in C(\mathcal{B}')$.

DEF. $\to$ given a choice structure $[\mathcal{B}, C(.)] \to$ the revealed preference relation $\succsim^*$ is defined as;

$x \succsim^* y \leftrightarrow$ some $B \in \beta$:
$x, y \in \mathcal{B} \wedge x \in C(\mathcal{B})$.
$x \succsim^* y \leftrightarrow \exists x$ is at least as good as y.

RELATIONSHIPS BETWEEN PREF. RELATIONS AND CHOICE RULES

PROP. suppose $\succsim$ is a rational preference relation : $\to$ the choice structure generated by $\succsim$, $[\mathcal{B}; C^*(.), \succsim]$ satisfies the weak axiom.

PROOF $\to$

***

MAS-COLELL, '94 - OXFORD UNIVERSITY PRESS
CHAPTER 17 - THE POSITIVE THEORY OF EQUILIBRIUM

17.A intro $\to$ theoretical predictive power of the walrasian equilibrium model.

model of a private ownership economy
$\to$ aggregate excess demand function;
$\to$ WALRASIAN EQUILIBRIA := solutions to a system of aggregate excess demand equation.

$\exists$ of a walrasian equilibrium;

$\to$ identification of a set of conditions, ensuring existence $\to$ CONVEXITY of the decision problem $\to$
$\to$ of individual economic agents.

$\to$ finite no. of equilibria $\to$ locally isolated equilibria; $\to$ this no. is odd.

$\to$ two categories of equilibria.

$\to$ sufficient conditions for the uniqueness of equilibria $\to$ comparative statics + stability.

weak axiom of revealed preferences: in the aggregate $\to$ property of gross substitution.

***

17.B equilibrium $\to$ definitions and basic equations.

$\exists$ I consumers and J firms;
every consumer $\to i \to X_i \subset \mathbb{R}^L \to$ has her own consumption set;
a preference relation $\succsim_i$ on X_i,

an initial endowment vector
$\omega_i \in \mathbb{R}^L$ and an ownership
share $\theta_{ij} \geq 0$ of a firm
$\forall$ firms $j = 1, \ldots, J$
and the shares are such
that $\sum_{i=1}^{J} \theta_{ij} = 1$;

each firm $j = 1, \ldots, J$ is characterized
by a production set $Y_j \subset \mathbb{R}^L$;

an **ALLOCATION** for **such** an **economy**
is a **collection** of **consumption** and **production vectors**:

$(x, y) = (x_1, \ldots, x_I; y_1, \ldots, y_J) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J$

$\rightarrow$ **walrasian equilibrium**; for a system
of markets where consumers and
firms are **PRICE TAKERS**;
and the wealth of consumers
derives from their endowments

and π[4] shares;

DEF. 17.B.1: given a private
ownership economy specified by
$\left(\{X_i, \succsim_i\}_{i=1}^{I}; \{Y_j\}_{j=1}^{J}; \{\omega_i, \theta_{i1}, \ldots, \theta_{iJ}\}_{i=1}^{I}\right)$

$\rightarrow$ an allocation $(x^*; y^*)$ and a price
vector $p = (p_1, \ldots, p_L) \rightarrow$ is a **WAL=
RASIAN EQUILIBRIUM**[5] if:

(a) for $\forall j, y_j^* \in Y_j \max \pi_j$ in $Y_j \leftrightarrow$
$\leftrightarrow p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$;

(b) for $\forall i, x_i^* \in X_i$ is the maximal for $\succsim_i$
in the budget set $\longrightarrow \longrightarrow \longrightarrow$

$\left\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*\right\}$

$\rightarrow$ consumer $i = 1, \ldots, I$ is an utility
maximizer over her preferences
over the set B;

(c) $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$

$\rightarrow$ the equilibria are the solutions
of such a system of equations.

***

exchange economies and demand
$\longrightarrow$ pure exchange economies
$\rightarrow$ only possible production activities
are those of free disposal;

$J = 1$ and $Y_1 = -\mathbb{R}_+^L$.

$X_i = \mathbb{R}_+^L \rightarrow$ outset that each consumer's
preferences are **CONTINUOUS, STRICTLY
CONVEX** and **LOCALLY NON-SATIATED**.

[4] profits
[5] or market or competitive or price taking

$\sum_{i=1}^{I} \omega_i >> 0$;

***

PROP. 17.B.1: in a pure exchange economy in which consumer preferences are con= tinuous, strictly convex and locally non - satiated, $p > 0$ is a Walrasian equilibrium price vector if and only if

$$\sum_{i=1}^{I} \big[\underbrace{x_i(p, p \cdot \omega_i) - \omega_i}_{z_i(p)}\big] \leq 0$$

DEF.: the excess demand function of consumer i is $z_i(p) = \underbrace{x_i(p, \cdot\omega_i)}\, - \omega_i$

where $x_i \rightarrow$ consumer i's walra= sian demand function;

the aggregate demand function of the economy $\rightarrow z(p) = \sum_{i=1}^{I} z_i(p)$.

the domain of this function is a set of non-negative price vectors that includes all strictly > 0 price vectors

$\ll p \in \mathbb{R}_+^L$ is an equilibrium price vector if and only if $z(p) \leq 0 \gg$.

***

if p is a walrasian equilibrium price vector in a pure exchange economy w/locally non - satiated preferences $\rightarrow$ $p \geq 0$, $z(0) \leq 0$ and $p \cdot z(p) = \sum_{i=1}^{I} p \cdot z_i(p) =$ $= \sum_{i=1}^{I} \big[p \cdot x_i(p, p \cdot \omega_i) - \omega_i\big] = 0 \rightarrow$ $\forall l$, we only have $z_l(p) =$, but also $z_l(p) = 0$ if $p_l > 0 \rightarrow$ at an equilibrium $\rightarrow$ a good l can be in excess supply, $z_l(p) < 0$, but only if it is free.

$\rightarrow$ consumer's preferences are strictly monotone.

we let $\rightarrow x_i = \mathbb{R}_+^L$ for all i and assume that all preference relations $\succsim_i$ are continuous, strictly convex and strongly monotone $\rightarrow$ any walrasian equilibrium must involve a strictly > 0 price vector $p >> 0$; otherwise consumers would de= mand an unboundedly large amount of all the free goods. $\rightarrow$ w/strong monotonicity property $\rightarrow$ of prefs. $\rightarrow$ a price vector $p = (p_1, \ldots, p_L)$ is a walrasian equilibrium price

vector if and only if it ≪clears
all markets≫↔ iff it solves the
system of L equations in L unknowns.

$z_l(p) = 0$ for every $l = 1, \ldots, L$.

***

PROP. 17.B.2: for every consumer i
$X_i = \mathbb{R}^L_*$ and $\succsim_i$ is continuous,
strictly convex and strongly monotone;

$\sum_{i=1}^I \omega_i >> 0$; → the aggregate
excess demand function$z(p)$, defined
for all price vectors $p >> 0$ → has the foll.:

a. $z(.)$ is continuous;

b. $z(.)$ is homogeneous of degree zero;

c. $p \cdot z(p) = 0$ for all p (Walras' law);

d. $\exists$ and $s > 0 : z_l(p) > -s$ for $\forall l, \forall p$;

e. if $p^n \to p$, where $p \neq 0$ and $p_l = 0$
for some $l \to \max\left\{z_1(p^n), \ldots, z_L(p^n)\right\} \to \infty$.

production economies →
assume production sets are closed,
strictly convex and bonded above
→ for any price vector $p >> 0$,
we can let $\pi_j(p)$ and $y_j(p)$
be the maximum profits.
and the profit maximizing
production vector for firm j;

$\bar{z}(p) = \sum_{i=1}^I x_i(p, p \cdot \omega_i + \sum_{j=1}^J \theta_{ij}\pi_j(p)) -$

$- \sum_{i=1}^I \omega_i - \sum_{j=1}^J y_j(p)$.

→ production inclusive exceeds demand
function → p is a walrasian equi=
librium price vector if and only if
it solves the system
of equations $\bar{z}(p) = 0$.

convex technologies →
→ not much loss of generality
in assuming the production
sector of the economy was composed
by a single firm endowed w/a constant
returns production technology.

→ no need to keep the identity of firms
separate in order to compute the wealth
of consumers → production decision

of the firms...

with constant returns → production sets
are neither strictly convex nor bounded
above → we could view the equilibria as
the zeros of a ≪production inclusive excess

demand correspondence$\gg\to$ that doesn't make good equational systems $\to$ they can't be diff= ferentiated. extended system of equations involving the production and the consumpt. sides of the economy.

a single constant returns Y can be sees as a description of a long run state of know= ledge of the economy $\to$ for the purpose of setting up firms $\to$ or simply for the house= hold production $\to$ firms and profit maximization condition $\to$ two - stage process, where consumers chose a vector $v_i \in \mathbb{R}^L$ subject to the budget constraint

$p \cdot v_i \leq p \cdot \omega_i \to$ equilibrium market clearing cond. $\to \sum_i p \cdot v_i = \sum_i p \cdot \omega_i \leftrightarrow \sum_{i=1}^{I} v_i = \sum_{i=1}^{I} \omega_i$.

***

$\exists$ OF WALRASIAN EQUILIBRIUM

$\to$ existence problem $\to$ logical test of consistency $\to$ mathematical model is well suited to its purposes.

exchange economies $\to$ excess demand functions $\to$ with $\sum_{i=1}^{I} \omega_i >> 0$;

$\to$ continuous, strictly convex and strongly monotone preferences $\to$ satisfying these five conditions $\to$ admits a solution. i.e. a price vector p s.t. $z(p) = 0$.

a walrasian equilibrium $\exists$.

production economy $\to$ only exchange takes place.

behavioural assumption of price taking + institutional assumption of competitive markets.

***

$\to$ of $z(.) \to$
normalize $p_2 = 1$ and look for equilibrium price vectors $\to (p_1, 1)$.
due to walras' law $\to$ an equilibrium can be obtained $\to z_1(p_1, 1) = \to$ one variables problem $\to z_1(p_1', 1) > 0$;
identify the commodity with positive excess demand as the one whose relative price is very low.

$p_1^* \in \left[p_1', p_1''\right] \to$ intermediate price with $z_1\left(p_1^*, 1\right) = 0 \to$ an equilibrium price vector must $\exists$.

PROOF $\to$ Kakutani's fixed - point theorem.

suppose $A \subset \mathbb{R}^N$ is a non - empty , compact, convex set, and $f : A \to A$ is an upper hemi - continuous correspondence from A into itself with the property that the set $f(x) \subset A \to f(.)$ has a fixed point $\to \exists$ an $x \in A : x \in f(x)$.■

***

PROP. 17.c.1: $z(p)$ is a function defined for all strictly > 0 price vectors $p \in \mathbb{R}^*_+$ and satisfying conditions a.,. . . ,e.
a walrasian equilibrium $\exists$ in a pure exchange economy where $\sum_{i=1}^{I} \omega_i >> 0$ and every consumer has continuous , strictly convex and strongly mono= tone preferences.

PROOF: (sketch) $\to$

1. **construction of the fixed - point correspondence for** $p \in$ **interior** $\Delta \to$ $f(p) = \{q \in \Delta : z(p) \cdot q \geq z(p) \cdot q \text{ for all } q' \in \Delta\}$;
2. **construction of the fixed - point corres= pondence for** $p \in$ **boundary** Δ;
3. **a fixed - point of** $f(.)$ **is an equilibrium**;
4. **fixed point correspondece is convex valued and upper hemi - continuous**;

a fixed point $\exists$. ■

***

PROP. 17.C.2: if $z(p)$ is a function defined for all non-zero, non-negative price vectors $p \in \mathbb{R}^L_+$ and satisfying conditions a., b., c. of prop. 17.B.2

$\to$ CONTINUITY;
$\to$ HOMOGENEITY OF DEGREE ZERO;
$\to$ WALRAS' LAW;
$\to \exists$ a price vector $p^* : z(p^*) \leq 0$

***

17.D LOCAL UNIQUENESS AND THE INDEX THEOREM.

walrasian model of competitive equilibrium is very parsimonious

$\to$ a complete theoretical account of an economy
$\to$ list of commodities
$\to$ state of technology

→ preferences and endowments[6] of consumers.

EDGEWORTH BOX → local uniqueness
locally isolated → local uniqueness
property.

history → locally determined equi=
librium → every consumer i
is specified by $\left(\succsim_i, \omega_i\right)$ →

$\succsim_i$ is a continuous, strictly convex
and strongly monotone preference
relation on $\mathbb{R}^*_+$ and $\omega_i >> 0$.

$z(.)$ satisfies $a., \ldots, e.$ of prop. 17.B.2.
continuous and differentiable;
relative prices → $p_L = 1$,
$\hat{z}(p) = \left(z_1(p), \ldots, z_{L-1}(p)\right)$

vector of excess demand for the first
$L-1$ goods. a normalized price
vector $p = (p_1, \ldots, p_{L-1})$ is a
walrasian equilibrium if and only if
it solves a system of $L-1$ equations
in $L-1$ unknowns.

regular economies → **PROP. 17.D.1**→
an equilibrium price vector $p = (p_1, \ldots, p_{L-1})$
is regular if the $(L-1) \times (L-1)$ matrix of
price effects $D\hat{z}(p)$ is non-singular,
it has rank $L-1$. if every normalized
equilibrium price vector is regular →
→ the economy is regular.

***

PROP. 17.D.2: any regular (normalized)
equilibrium price vector $p = (p_1, \ldots, p_{L-1}, 1)$
is locally isolated (or locally unique)
↔ ∃ an $\epsilon > 0$: if $p' \neq p$, $p'_L = p_L = 1$
and $||p' - p < \epsilon||$ → $z(p') \neq 0$. if the
economy is regular → the #
of normalized equilibrium
price vectors is finite.

DEF. 17.D.1: suppose that $p = (p_1, \ldots, p_{L-1}, 1)$
is a regular equilibrium of the economy.

→ index $p = (-1)^{L-1}$, sign $|D\hat{z}(p)|$.

where $|D\hat{z}(p)|$ is the determinant
of the $(L-1) \times (L-1)$ matrix $D\hat{z}(p)$.

if $L = 2$ → $|D\hat{z}(p)|$ is just the slope of $z_1(p)$.

***

PROP. 17.D.2: THE INDEX THEOREM
for any regular economy, we've:

[6] dote inziale di beni - monetari o non.

$\sum_{\{p:z(p)=0,p_L=1\}}$ index $p = +1$.
$\rightarrow$ the no. of equilibria
of a regular economy
is odd.

uniqueness and multiplicity of the equilibria.

genericity analysis $\rightarrow$ methodology.
system of M equation in N unknowns.

$f_1(v_1, \ldots, v_N = 0)$

...

...

...

$f_M(v_1, \ldots, v_N)$

$\leftrightarrow f(\underbrace{\mathbf{v}}_{M \times N}) = 0$

$N - M$ degrees of freedom available
for describing the solutions' set.

***

valutazione degli *intangibles*[7]:

il valore attribuibile è in funzione della sopportabilità degli oneri conseguenti:

1. gli ammortamenti *eventualmente* derivanti dall'iscrizione degli *intangibles* $\rightsquigarrow$ ove l'eventualità ha solo il significato di escludere quei beni immateriali che non perdono valore nel tempo;
2. l'attribuzione convenzionale di un'adeguata retribuzione ai capitali investiti.

in assenza del contemporaneo problema di rivalutazione di beni materiali soggetti ad ammortamento, le formule conclusive sono del tutto simili $T_e = a_{ni'} \cdot (R + A - i''C')$, dove C' è il capitale netto diminuito del valore contabile degli immobilizzi tecnici; R è il reddito medio atteso; i'' è il tasso unitario annuo medio-normale di remunerazione del capitale investito; A sono gli ammortamenti dei quali si è tenuto conto nella stima di R; T_e è il valore economicamente accettabile di T ovvero il valore tecnico degli immobilizzi.

i *valori economicamente accettabili* sono perciò:

$$BI_e = (R + A - i'' \cdot C')a_{ni}$$

$$BI_e = \sum_{i=1}^{n}(R_s + A_s - i'' \cdot C')v^s$$

nel caso di presenza contemporanea di plus valenze iscrivibili sia sui beni materiali ammortizzabili sia sugli immateriali, la verifica non poò che avvenire congiuntamente. il valore che ne risulta è la somma $T_e + BI_e$, senza possibilità di un'obiettiva ripartizione di tale unico valore su le due categorie di plus valenze. la distribuzione può avvenire solo discrezionalmente. problema analogo a quello dell'imputazione del *badwill* a $\searrow$ delle varie categorie di valori nelle tipiche applicazioni del metodo misto.

[7]Luigi Guatri, TRATTATO SULLA VALUTAZIONE DELLE AZIENDE, Egea, Milano, 1998.

MICRO ECONOMETRIC MODEL
TO VALUATE FRB'S TRADEMARK

$$\underbrace{y_t}_{\text{ricavi}} = \alpha_t + \beta_t \underbrace{D_t}_{\text{marchio}} + \gamma_t \underbrace{X_t}_{\text{altro}} + \epsilon_t$$

altro := e.g. immobilizzazioni, patrimonio netto, brevetti, costi;

marchio $\rightarrow$ è il punto di cutoff, ovvero il momento in cui è stato registrato il marchio - a partire da cui l'impresa ha iniziato a beneficiare di un surplus di ricavi dovuto al bene immateriale $\rightarrow$ shock locle endogeno;

metodo di stima $\rightarrow$ regression discontinuity design — diff-in-diff;

$\hat{\beta} = (y_{after} - y_{before})$

$D \in \{0; 1\}$

$D = 1$ if $X \geq c \rightarrow *$;

$D = 0$ if $X < c \rightarrow \blacktriangle$;

$\mathbb{E}[y_t(1)|X] \neq \mathbb{E}[y_t(0)|X]$

* marchio;

▲ no marchio;

$\rightarrow$ esperimento randomizzato locale;

$\hat{\beta}$:= weighted average local treatment effect.

durata del marchio $\rightarrow$ logit;

$\rightarrow$ **luoghi geografici di resgistrazione** del marchio $\rightarrow$ mappa;

$\rightarrow$ survival model;

event study analysis.

***

RD designs in STATA[8]

Description

`rd` implements a set of regression-discontinuity estimation methods that are thought to have very good internal validity, for estimating the causal effect of some explanatory variable (called the treatment variable) for a particular subpopulation, under some often plausible assumptions. In this sense, it is much like an experimental design, except that levels of the treatment variable are not assigned randomly by the researcher. Instead, there is a jump in the conditional mean of the treatment variable at a known cutoff in another variable, called the assignment variable, which is perfectly observed, and this allows us to estimate the effect of treatment as if it were randomly assigned in the neighborhood of the known cutoff.

`rd` is an alternative to various regression techniques that purport to allow causal inference (e.g. panel methods such as `xtreg`), instrumental variables (IV) and other IV-type methods (see the `ivreg2` help file and references therein), and matching estimators (see the `psmatch2` and `nnmatch` help files and references therein). The `rd` approach is in fact an IV model with one exogenous variable excluded from the regression (excluded instrument), an indicator for the assignment variable above the cutoff, and one endogenous regressor (the treatment variable).

`rd` estimates local linear or kernel regression models on both sides of the cutoff, using a triangle kernel. Estimates are sensitive to the choice of bandwidth, so by default several estimates are constructed using different bandwidths. In practice, rd uses kernel-weighted `suest` (or `ivreg` if `suest` fails) to estimate the local linear regressions and reports analytic SE based on the regressions.

[8]From the STATA 13 help files.

Further discussion of rd appears in Nichols (2007).

Examples

In the simplest case, assignment to treatment depends on a variable Z being above a cutoff Z_0. Frequently, Z is defined so that $Z_0 = 0$. In this case, treatment is 1 for $Z \geq 0$ and 0 for $Z < 0$, and we estimate local linear regressions on both sides of the cutoff to obtain estimates of the outcome at $Z = 0$. The difference between the two estimates (for the samples where $Z \geq 0$ and where $Z < 0$) is the estimated effect of treatment.

For example, having a Democratic representative in the US Congress may be considered a treatment applied to a Congressional district, and the assignment variable Z is the vote share garnered by the Democratic candidate. At $Z = 50\%$, the probability of treatment=1 jumps from zero to one. Suppose we are interested in the effect a Democratic representative has on the federal spending within a Congressional district. rd estimates local linear regressions on both sides of the cutoff like so:

```
ssc inst rd, replace
net get rd
use votex
rd lne d, gr mbw(100)
rd lne d, gr mbw(100) line(`"xla(-.2 "Repub" 0 .3 "Democ", noticks)"')
rd lne d, gr ddens
rd lne d, mbw(25(25)300) bdep ox
rd lne d, x(pop-vet)
rd lne d, mbw(100) bin binvar(bins) scopt(mcol(black))
```

In a fuzzy RD design, the conditional mean of treatment jumps at the cutoff, and that jump forms the denominator of a Local Wald Estimator. The numerator is the jump in the outcome, and both are reported along with their ratio. The sharp RD design is a special case of the fuzzy RD design, since the denominator in the sharp case is just one.

```
g byte ranwin=cond(uniform()<.1,1-win,win)
rd lne ranwin d, mbw(100)
```

The default bandwidth from Imbens and Kalyanaraman (2009) is designed to minimize MSE, or squared bias plus variance, in a sharp RD design. Note that a smaller bandwidth tends to produce lower bias and higher variance. The optimal bandwidth will tend to be larger for a fuzzy design due to the additional variance arising from the estimation of the jump in the conditional mean of treatment. Unfortunately, a larger bandwidth also leads to additional bias, which will be greater if the curvature of the response function is greater (meaning that a linear regression over a larger range is a poorer approximation). The increase in squared bias due to dividing by the estimated jump in the conditional mean of treatment (using observations away from the discontinuity) can easily dominate the increase in variance and lead to the optimal bandwidth in a fuzzy design to be smaller than in the sharp design. No clear guidance is offered; conducting simulations using plausible generating functions for your specific application are highly recommended. The `rd` option `bdep` facilitates visualizing the dependence of the estimate on bandwidth.

Detailed Syntax and Options

There should be two or three variables specified after the `rd` command; if two are specified, a sharp RD design is assumed, where the treatment variable jumps from zero to one at the cutoff. If no variables are specified after the `rd` command, the estimates table is displayed.

```
rd outcomevar [treatmentvar] assignmentvar [if] [in] [weight] [, options]
```

`mbw(numlist)` specifies a list of multiples for bandwidths, in percentage terms. The default is ”100 50 200” (i.e. half and twice the requested bandwidth) and 100 is always included in the list, regardless of whether it is specified.

`z0(real)` specifies the cutoff Z0 in assignmentvar Z.

`strineq` specifies that mean treatment differs at Z0 from all $Z > Z_0$ (e.g. treatment is 1 for $Z > 0$ and 0 for $Z \leq 0$); the default assumption is that mean treatment differs at Z_0 from all $Z < Z_0$ (e.g. treatment is 1 for $Z \geq 0$ and 0 for $Z < 0$).

`x(varlist)` requests estimates of jumps in control variables varlist.

`ddens` requests a computation of a discontinuity in the density of Z. This is computed in a relatively ad hoc way, and should be redone using McCrary's test described at `http://www.econ.berkeley.edu/~jmccrary/DCdensity/`.

`s(stubname)` requests that estimates be saved as new variables beginning with `stubname`.

`graph` requests that local linear regression graphs for each bandwidth be produced.

`noscatter` suppresses the scatterplot on those graphs.

`cluster(varlist)` requests standard errors robust to clustering on distinct combinations of varlist (e.g. stratum psu).

`scopt(string)` supplies an option list to the scatter plot.

`lineopt(string)` supplies an option list to the overlaid line plots.

`n(real)` specifies the number of points at which to calculate local linear regressions. The default is to calculate the regressions at 50 points above the cutoff, with equal steps in the grid, and to use equal steps below the cutoff, with the number of points determined by the step size.

`bwidth(real)` allows specification of a bandwidth for local linear regressions. The default is to use the estimated optimal bandwidth for a "sharp" design as given by Imbens and Kalyanaraman (2009). The optimal bandwidth minimizes MSE, or squared bias plus variance, where a smaller bandwidth tends to produce lower bias and higher variance. Note that the optimal bandwidth will often tend to be larger for a fuzzy design, due to the additional variance that arises from the estimation of the jump in the conditional mean of treatment.

`bdep` requests a graph of estimates versus bendwidths.

`bingraph` requests a graph of binned means instead of a scatterplot, in bins defined by binvar.

`binvar(varname)` specifies the variable across which binned means should be calculated.

`oxline` adds a vertical line at the default bandwidth.

`kernel(rectangle)` requests the use of a rectangle (uniform) kernel. The default is a triangle (edge) kernel.

`covar(varlist)` adds covariates to Local Wald Estimation, which is generally a Very Bad Idea. It is possible that covariates could reduce residual variance and improve efficiency, but estimation error in their coefficients could also reduce efficiency, and any violations of the assumptions that such covariates are exogenous and have a linear impact on mean treatment and outcomes could greatly increase bias.

Example

```
rd deaths support if inrange(support,-28.82896,28.82896), graph
```

References

Nichols, Austin. 2007. Causal Inference with Observational Data. Stata Journal 7(4): 507-541.

MULTINOMIAL LOGIT MODELS[9]

$\longrightarrow$ similar structure of the multinomial probit
but w/ $\neq$ error structure $\rightarrow$ **logistic distribution**.
it's possible to code $J+1$ alternatives;
the **process** underlying the observations
into specific alternatives is driven by a set
of **latent variables** $\rightarrow y_j^*$, $j = 1, \ldots, J$.
$\rightarrow y_j^* \rightarrow$ utility obtainable by alternative j.

$$\underbrace{y_j^*}_{\text{not observable}} = \underbrace{X_j}_{\text{a vect. of expl. vars.}} \beta_j + u_j \ (*)$$

$y = j$ if alternative j is chosen; ▲
$u_j \rightarrow$ distributed ass iid random variables;
$\searrow$ type I - extreme value distribution;

$$u_j \sim \begin{cases} f(u) = e^{-u-e^{-u}} & \blacksquare \\ F(u) = e^{-e^{-u}} \end{cases}$$

probability of a generic alt. j being chose

$$Pr(u_j > u_h, \forall h \neq j = 1, \ldots, J|X) = Pr(y_j|x) = \frac{e^{X_j\beta_j}}{\sum_{h=0}^J e^{X_h\beta_h}} \bullet$$

$\rightarrow$ **errors** are assumed to be **uncorrelated** across alternatives
$\rightarrow$ replace observable X_{ih}, $\forall h = 0, \ldots, J$
and $\forall i = 1, \ldots, N$ in $\bullet$ $\longrightarrow$ likelihood
contribution of each single observation

$$P_i(\theta) = Pr(y_i = j|X_i) = \frac{e^{X_{ij}\beta_j}}{\sum_{h=0}^J X_{ih}\beta_h} \bigstar$$

$\theta = (\beta_0, \ldots, \beta_j)' \rightarrow$ vector of para=
meters of the model
$L(\theta) = \sum_{i=1}^N \ln P_i(\theta)$ ✠ $\longrightarrow$ **log - likelihood**
MC FADDEN — mixed multinomial logit
$\searrow \exists \neq$ types of explanatory variables.

1. vary only across i and not across j;
2. vary only across j and not across i;
3. vary both across i and j.

$$Pr(y_i = j|X_i) = \frac{e^{z_{ij}\delta_j + m_i\gamma_j}}{\sum_{h=0}^J e^{z_{ih}\delta_h + m_i\gamma_h}} \blacklozenge$$

$\rightarrow \gamma \rightarrow$ not identifiable, γ^* is the true γ.
$\rightarrow$ alternative set of parameters $\rightarrow$
$\zeta = \gamma^* + q \rightarrow q :=$ constant.

$$\begin{aligned} Pr(y_i = j|X_i) &= \frac{e^{z_{ij} + m_i(\gamma_j^* + q)}}{\sum_{h=0}^J e^{z_{ih}\delta_h + m_i(\gamma_h^* + q)}} \\ &= \frac{e^{qm_i} \cdot e^{z_{ij}\delta_j + m_i\gamma_j^*}}{\sum_{h=0}^J e^{qm_i} \cdot e^{z_{ih}\delta_h + m_i\gamma_h^*}} \\ &= \frac{e^{z_{ij}\delta_j + m_i\gamma_j^*}}{\sum_{h=0}^J e^{z_{ih}\delta_h + m_i\gamma_h^*}} \blacktriangledown \end{aligned}$$

classical multinomial logit

$$Pr(y_i = j|X_i) = \frac{e^{X_i\beta_j}}{\sum_{h=0}^J e^{X_i\beta_h}} \odot$$

$\beta_0 = 0 \rightarrow$ first alternative.

[9]From **Michele Pellizzari**'s notes on **micro - econometrics** within the **corso di laurea magistrale in discipline economiche e sociali** of **Università Bocconi**, a.y. **2010/'11**.

$P_i(\theta) = Pr(y_i = j \neq 0|X_i) = \frac{e^{X_i\beta_j}}{1+\sum_{h=0}^{J} e^{X_i\beta_h}}$ ⋈
$Pr(y_i = 0|X_i) = \frac{1}{1+\sum_{h=0}^{J} e^{X_i\beta_h}}$
$\frac{Pr(y_i=j|X_i)}{Pr(y_i=j|X_i)} = e^{X_i\beta_j}$ **
<u>coeff.</u> := % variation in the probabilities
of alternative j relative to the reference alternative 0
due to a marginal Δ in each regressor.

$$\beta_{jk} = \frac{\partial\left[Pr(y_i=j|X_i)/Pr(y_i=0|X_i)\right]}{\partial X_k}$$

* $\frac{Pr(y_i=j|X_i)}{Pr(y_i=j|X_i)} = \frac{e^{X_{ij}\beta_j}}{e^{X_{ik}\beta_k}}$ → iia[10].
→ variance - covariance matrix →
perfectly diagonal elements
→ strong assmpt.
* relative probability of choosing one
over another alternative.

***

P. Haan and A. Uhlendorff, ESTIMATION OF MULTINOMIAL LOGIT MODELS WITH UNOBSERVED HETEROGENEITY USING MAXIMUM SIMULATED LIKELIHOOD, the stata journal, 2006

$$Pr(j|X_{it}, \alpha_i) = \frac{\exp\{X_{it}\beta_j + \alpha_{ij}\}}{\sum_{k=1}^{J} \exp\{X_{it}\beta_k + \alpha_{ik}\}}$$

→ integrating for unobserved heterogeneity

$$\to L = \prod_{i=1}^{N} \int_{-\infty}^{\infty} \prod_{t=1}^{T} \prod_{j=1}^{J} \left\{ \frac{\exp\{X_{it}\beta_j + \alpha_j\}^{d_{ijt}}}{\sum_{k=1}^{J} \exp\{X_{it}\beta_k + \alpha_k\}} \right\} \cdot f(\alpha) d\alpha$$

$\mathbf{W}$→ covariance matrix; $\alpha \sim f(a, \mathbf{W}) \to iid$ across $i = 1, \dots, N$;
→ multivariate normal distribution

$$\alpha \sim f\left\{ \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} var_2 & cov_{23} \\ cov_{32} & var_3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} e_{11} & 0 \\ e_{21} & e_{22} \end{pmatrix} \times \begin{pmatrix} \epsilon_2 \\ \epsilon_3 \end{pmatrix}.$$

***

Gregory S. Carpenter and Donald R. Lehmann, A MODEL OF MARKETING MIX, BRAND SWITCHING AND COMPETITION, Journal of Marketing, 1985

$$\log(\pi_{ij}) = \mu_i + \delta_{1,i}ADV_{ij} + \delta_{2,i}PRICE_{ij} + \delta_3 BRAND_{ij} +$$
$$+\delta_{4,i}FORM_{ij} + \delta_{5,i}(ADV_{ij} \times BRAND_{ij}) + \delta_{6,i}(ADV_{ij} \times FORM_{ij}) +$$
$$+\delta_{7,i}(PRICE_{ij} \times BRAND_{ij}) + \delta_{8,i}(PRICE_{ij} \times FORM_{ij}) +$$
$$+\epsilon_i,\ i, j \in M;$$

where → $ADV_{ij} = \frac{ADV_i}{ADV_j}$;

[10]→ independence of irrelevant alternatives → houthakker's axiom.

$PRICE_{ij} = \frac{PRICE_i}{PRICE_j}$;

$$BRAND_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are the same brand and name} \\ 0 & \text{otherwise} \end{cases}$$

$$FORM_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are the same form} \\ 0 & \text{otherwise} \end{cases}$$

$\epsilon_i \to$ residual; $\mu_i \to$ constant.

utility decomposition into various parts

$$\underbrace{u_{ij}}_{\text{utility}} = \underbrace{v_{ij}}_{\text{deterministic}} + \underbrace{e_{ij}}_{\text{stochastic}}$$

$$v_{ij} = \sum_{k \in C} \alpha_{ik} X_{ijk} + \sum_{l \in F} \beta_{il} W_{ijl} +$$

$$+ \sum_{m \in C} \sum_{n \in F} \gamma_{imn} (X_{ijm} W_{ijn}$$

$\pi_{ij} = \frac{e^{v_{ij}}}{\sum_{k \in M} e^{v_{ik}}}$, $i, j \in M \to$

probability of switching.

***

ANY THING GOES[11] $\to$ under a number of general assumpts. (convexity) $\to$ an equilibrium must $\exists$ and the no. of equilibria is typically finite;
$\searrow$ we've already shown $\to$
can actually occur.

$\underbrace{\mathbf{D}}_{L \times L} z(p) \to$ a matrix $\to \omega + z(p) >> 0$

$$\sum_k \frac{\partial z_l(p)}{\partial p_k} \cdot p_k = 0, \text{ for } \forall l \wedge p;$$

$$\sum_k p_k \cdot \frac{\partial z_k(p)}{\partial p_l} = -z_l(p), \forall l, p.$$

$$\underbrace{\mathbf{D}}_{L \times L} z(p) = \sum_{i=1}^{I} \left[S_i(p, p \cdot \omega_i) - \mathbf{D}_{w_i} x_i(p, p \cdot \omega_i) \cdot z_i(p)' \right]$$

$$\underbrace{\mathbf{D}_{w_i}}_{L \times L} x_i(p, p \cdot \omega_i) \cdot z_i(p)' = \begin{pmatrix} \frac{\partial x_{1i}(p, p \cdot \omega_i)}{\partial w_i} \cdot z_{1i}(p) & \dots & \frac{\partial x_{1i}(p, p \cdot \omega_i)}{\partial w_i} \cdot z_{L1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_{Li}(p, p \cdot \omega_i)}{\partial w_i} \cdot z_{1i}(p) & \dots & \frac{\partial x_{Li}(p, p \cdot \omega_i)}{\partial w_i} \cdot z_{Li}(p) \end{pmatrix}$$

$\mathbf{D}_{w_i} \longrightarrow$ is of rank 1
$\searrow$ wealth effect of consumer i can hurt at most one direction of the price change.

PROP. 17.E.1: $I < L$;
$\to$ for any equilibrium price vector p
$\exists$ some direction of price change $dp \neq 0$
such that $p \cdot dp = 0 \to dp$ is not propor=
tional to p and $dp \cdot \mathbf{D} z(p) \leq 0$.

[11] back to mas colell, chapt. 17.E

PROP. 17.E.2: given a price vector p,
let $z \in \mathbb{R}^L$ be an arbitrary vector
and $\mathbf{A}$ an arbitrary $L \times L$ matrix
satisfying an aggregate excess demand
function $z(.)$ such that $z(p) = z$
and $\mathbf{D}z(p) = \mathbf{A}$.

***

PROP. 17.E.3: suppose that $z(.)$ is a con=
tinuous function defined on:
$\mathbb{P}_\epsilon = \{p \in \mathbb{R}^L_+ : p_l/p_{l'} \geq \epsilon \text{ for } \forall l, l'\}$
and with values in $\mathbb{R}^L$.
assume that, in addition, $z(.)$ is homo=
geneous of degree zero and satisfies
walras' law $\rightarrow \exists$ an economy
of L consumers whose aggregate
excess demand function
coincides w/$z(p)$ in the
domain $\mathbb{P}_\epsilon$

***

PROP. 17.E.4: for any $N \geq 1$,
suppose that we assign to each $n = 1, \ldots, N$
a price vector p^n, normalized to $||p^n|| = 1$
and $\underbrace{\mathbf{A}_n}_{L \times L}$ matrix of rank $L - 1$,
satisfying $\underbrace{\mathbf{A}_n}_{L \times L} \cdot \underbrace{\mathbf{p}^n}_{L \times 1} = \underbrace{\mathbf{0}}_{L \times 1}$ and $\underbrace{\mathbf{p}^{n'}}_{1 \times L} \cdot \underbrace{\mathbf{A}_n}_{L \times L} = \underbrace{\mathbf{0}}_{1 \times L}$.
suppose also the index formula
$\sum_n (-1)^{L-1} - \text{sign}|\hat{\mathbf{A}}_n| = +1 \rightarrow$ holds;
if $L = 2 \rightarrow > 0$ and < 0 index
equilibria alternate $\rightarrow$
$\rightarrow$ there $\exists$ an economy w/L consumers
s.t. the aggregate excess demand
$z(.)$ has the properties:
(i) $z(p) = 0$ for $||p|| = 1$ if and only if
$p = p^n$ for some n;
(ii) $\underbrace{\mathbf{D}}_{L \times L} z(p^n) = \underbrace{\mathbf{A}_n}_{L \times L}$ for every n.
uniqueness$\rightarrow$ $Y^* := \{y \in \mathbb{R}^L : p \cdot y \leq 0 \wedge p' \cdot y \leq 0\}$.
$\rightarrow$ set of equilibrium price vectors is convex.

***

PROP. 17.F.2:suppose the excess demand
function $z(.)$ is such that, for any constant
returns technology Y, the economy formed by
$z(.)$ and Y has a unique (normalized)
equilibrium price vector $\rightarrow z(.)$ sa=

tisfies the WARP.else, if $z(.)$
satisfies WARP $\rightarrow$ the set of equilibrium
prices $\rightarrow$ convex.

***

PROP. 17.BB.1: an allocation $(x_1^*, \ldots, x_I^*; y_1^*, \ldots, y_J^*)$ and a price system $p \neq 0 \rightarrow$ are a walrasian equilibrium if:

1. $\forall j \rightarrow p \cdot y_j \leq p \cdot y_j^*$, $\forall y_j \in Y_j$;
2. $\forall i \rightarrow p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} \cdot p \cdot y_j^*$
   and if $x_i \succ x_i^* \rightarrow p \cdot x_i \geq p \cdot \omega_i + \sum_j \theta_{ij} \cdot p \cdot y_j^*$;
3. $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

***

CHAPTER 19 - GENERAL EQUILIBRIUM UNDER UNCERTAINTY - 19.B A MARKET ECONOMY W/CONTINGENT COMMODITIES

states of the world;
Arrow - Debreu equilibria;
self - fulfilling or rational expectations;
spot trade $\rightarrow$ Radner equilibrium
arbitrage $\rightarrow$ assets
incomplete markets
market value maximization $\rightarrow$ share holders' objective.

L physical commodities;
I consumers;
J firms.

$\rightarrow$ **uncertain preferences** $\rightarrow$
over S possible states of the world[12]
$s = 1, \ldots, S$; state - contingent commodity;
$\searrow$ state - contingent vector.

DEF. 19.B1: for every physical commo= dity $l = 1, \ldots, L$ and states $s = 1, \ldots, S$, a unit of state - contingent commodity l is a title to receive a unit of the physical good l if and only if s occurs $\rightarrow$ a state - contingent commodity vector is specified by:

$$\underbrace{\mathbf{x}}_{LS \times 1} = (x_{11}, \ldots, x_{L1}, \ldots, x_{1s}, \ldots, x_{LS}) \in \mathbb{R}^{LS}$$

$\rightarrow$ understood as an entitlement to receive

[12]states is, of course, meant to be like economic scenarios or business conditions, not geographical states.

the commodity vector $(x_{1s}, \dots, x_{Ls})$
if state s occurs.

$\to$ **contingent commodity vector** $\to$
$\to$ a collection of random variables,
the l^{th} r.v. being $(x_{l1}, \dots, x_{lS})$.

consumer $i = 1, \dots, I$,
$\omega_i = (\omega_{11i}, \dots, \omega_{L1i}, \dots, \omega_{1Si}, \dots, \omega_{LSi}) \in \mathbb{R}^{LS}$
$\to$ is state s occurs $\to$ cons. i has endowment
vector $(\omega_{1si}, \dots, \omega_{Lsi}) \in \mathbb{R}^L$.

the preferences of consumer i may also
depend on the state of the world
$\to$ e.g. $\to$ consumer's employment of wine.
$\searrow$ prefs. $\to$ defined over contingent com=
modity vectors. $\succsim_i \to X_i \subseteq R^{LS}$.

***

19.C ARROW DEBREU EQUILIBRIUM

economy with uncertainty:

$\to$ set of states of the world S;
$\to$ consmpt. set $X_i \subset \mathbb{R}^{LS}$;
$\to$ endowment vector $\omega_i \in \mathbb{R}^{LS}$;
$\to$ preference relation $\succsim_i$ on X_i, $\forall i = 1, \dots, I$;
$\to$ production set $Y_j \subset \mathbb{R}^{LS}$;
$\to$ profit shares $(\theta_{j1}, \dots, \theta_{jI})$, $\forall j = 1, \dots, J$.

$\exists$ a market for every contingent commo=
dity ls; markets open before the resolution
of uncertainty, at $t = 0$, say.
price of the good $\to p_{ls}$;
all agents have to be able
to recognize the state of the world.

$\to$ **symmetric information** $\exists$.

DEF. 19.C.1: an allocation
$(x_1^*, \dots, x_I^*; y_1^*, \dots, y_J^*) \in X_1 \times \dots \times X_I \times Y_1 \times \dots \times Y_J \subset \mathbb{R}^{LS(I+J)}$
and a system of prices for the contingent commodities $p = (p_{11}, \dots, p_{LS})$
represents an Arrow - Debreu equilibrium if:

1. for $\forall j \to y_j^* : p \cdot y_j^* \geq p \cdot y_j$, $\forall y_j \in Y_j$;
2. for $\forall i \to x_i^*$: is maximal in the budget set
$\left\{x_j \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} \cdot y_j^*\right\}$;
3. $\sum_{i=1}^I x_i^* = \sum_{j=1}^J y_j^* + \sum_{i=1}^I \omega_i$.

***

19.D SEQUENTIAL TRADE

the arrow - debreu framework $\to$ reliable
illustration of the power of general
equilibrium theory.

we take $X_i = \mathbb{R}_+^{LS}$ for every j.

$t = 0 \wedge t = 1 \rightarrow t = 0, 1$, two period model; at $t = 0 \rightarrow \nexists$ information.

$(x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI} \rightarrow$ arrow-debreu equilibrium allocation w/prices $(p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS} \rightarrow$ market for delivery of goods at $t = 1 \rightarrow$ $\rightarrow$ **forwards markets**[13].

at $t = 1$, the state of the world is revealed; contracts are executed and $\forall i$ gets $x_{si}^* = \{x_{1si}^*, \ldots, x_{Lsi}^*\} \in \mathbb{R}^L$.

after this but before the actual consump= tion $\rightarrow$ of x_i^*, markets for the phy= sical goods were open at $t = 1 \rightarrow$ **spot markets**[14] $\rightarrow$ no incentive to trade in these markets...

$x_{si} = (x_{1si}, \ldots, x_{Lsi})$ for $i = 1, \ldots$:
$\sum_{i=1}^{I} x_{si} \leq \sum_{i=1}^{I} \omega_{si} \wedge (x_{1i}^*, \ldots, x_{si}, \ldots, x_{Si}^*) \succsim_i$
$\succsim_i (x_{1i}^*, \ldots, x_{si}^*, \ldots, x_{Si}^*)\ \forall i$.

w/at least one strict preference.

$\rightarrow$ pareto optimality of the arrow - debreu equilibrium $\rightarrow$ allocation $\rightarrow (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$

isn't pareto optimal $\rightarrow$ contradiction!

***

prices for the l contingent commodities
$\rightarrow$ states $s : p_s = (p_{1s}, \ldots, p_{Ls})$;
$p_s \rightarrow$ system of spot prices at $s \rightarrow$
for $(x_{s1}^*, \ldots, x_{sI}^*)$, a null excess de=
mand for all traders $\rightarrow$ clears markets.
$\searrow \mathcal{U}_i(x_{1i}, \ldots, x_{Si})$ is a utility fun=
ction for $\succsim_i$ and $(x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$
maximizes $\rightarrow U_i(x_{1i}^*, \ldots, x_{si}, \ldots, X_{Si}^*)$
s.t. $\sum_s p_s \cdot (x_{si} - \omega_{si}) \leq 0 \rightarrow \forall s$,
x_{is}^* maximizes $\mathcal{U}_i(x_{1i}^*, \ldots, x_{si}, \ldots, x_{Si}^*)$
s.t. $p_s \cdot (x_{si} - \omega_{si}) \leq p_s \cdot (x_{si}^* - \omega_{si})$
$\rightarrow p_s \cdot (x_{si} - x_{si}^*) \leq 0$.
at $t = 0 \rightarrow$ consumers have expectations

[13]**mercato delle operazioni per consegna differita** (fernando picchia, IL NUOVO ECONOMICS AND BUSINESS, zanichelli 1990, seconda edizione.) - un qualsiasi mercato nel quale vengono trattati titoli, valute o beni per consegna futura. si veda anche la voce **forward contracting** $\rightsquigarrow$ qualsiasi operazione commerciale in relazione alla quale il venditore e il compratore si accordano per la consegna, ad una specifica data futura, di una determinata qualità e quantità di beni, il cui prezzo può essere stabilito in anticipo o al momento della consegna. si tratta, in genere, di intermediari che trattano opzioni di acquisto o di vendita di titoli o di merci, al fine di far incontrare venditori e compratori per consegne differite (*ibidem*).

[14]**mercato a pronti**, **mercato per contanti** $\rightsquigarrow$ nelle borse merci, il termine indica le vendite di derrate che vengono concluse per pagamento immediato o, al più tardi, al momento in cui il compratore entra in possesso dei documenti rappresentativi delle merci. anche la consegna si intende che deve essere immediata. nel mercato delle valute, il termine indica quella parte delle contrattazioni relative a consegna e pagamento immediati di valute estere. in senso più lato, il termine indica un mercato nel quale vengono trattati i prodotti primari disponibili, costituito da venditori in possesso dei beni che intendono vendere e compratori disposti ad acquistarli per pagamento e consegna immediati.

$\rightarrow$ spot prices $\rightarrow$ at $t = 1$, for $\forall s \in S$;
$\underbrace{p_s \in \mathbb{R}^S}_{\text{exp. price vector}}$ at $t = 0$; $\underbrace{(p_1, \ldots, p_S)}_{\text{exp. prices}} \in \mathbb{R}^{LS}$
$t = 0 \rightarrow \mathbf{q} = (q_1, \ldots, q_S) \in \mathbb{R}^S$.
faced w/prices $\mathbf{q} \in \mathbb{R}^S_{t=0}$ and exp. spot prices
$(p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ at $t = 1 \rightarrow$ every consumer i
formulates a consumption plan $\rightarrow$
$(z_{1i}, \ldots, z_{Si}) \in \mathbb{R}^S$ for contingent
commodities at $t = 0$; and
a set of spot market consumpt.
plans $\rightarrow (x_{1i}, \ldots, x_{Si}) \rightarrow \mathbb{R}^{LS}$
for the $\neq s = 1, \ldots, S$
that may occur at $t = 1$.

PROBL. 19.D.1
$\mathcal{U}(.) \rightarrow$ utility function for $\succsim_i$,

$$\max_{\substack{(x_{1i},\ldots,x_{Si}) \in \mathbb{R}^{LS} \\ (z_{1i}\ldots,z_{Si}) \in \mathbb{R}^S}} \left\{\mathcal{U}_i(x_{1i}, \ldots, x_{Si})\right\} \text{ such that}$$
(i) $\sum_{s=1}^{S} q_s z_{si} \leq 0$
(ii) $p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} \cdot z_{si}$,
$\forall i = 1, \ldots, I$

***

DEF: 19.D.1: a collection formed by a price vector $\mathbf{q} = (q_1, \ldots, q_S) \in \mathbb{R}^S$ for contingent first good commodities at $t = 0 \rightarrow$;
$\rightarrow$ a spot price vector $\rightarrow p_s = (p_{1s}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$ at $t = 1$ constitutes a radner equilibrium 1982 if

1. for $\forall i \rightarrow$ the consumption plans z_i^*, x_i^* solve PROBL. 19.D.1;
2. $\sum_i z_{si}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_i$ for every s.

$\rightarrow$ sequence of budget sets, at various dates - states.

PROP. 19.D.1: we have
(i) if the allocation $x^* \in \mathbb{R}^{LSI}_{++}$ and the contingent commodities' price vector $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$ constitute an arrow - debreu equilibrium $\rightarrow$ $\exists$ prices $\mathbf{q} \in \mathbb{R}^S_{++}$ for contingent first good commodities and consumption plans for these commodities $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{SI}$ such that the consumption plan x^*, z^*, the prices $\mathbf{q}$ and the spot prices $(p_1, \ldots, p_S)$ are a radner equil.;
(ii) if the consumption plans $x^* \in \mathbb{R}^{LSI}$, $z \in \mathbb{R}^{SI}$ and prices $\mathbf{q} \in \mathbb{R}^S_{++}$, $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}_{++}$ are a radner

equilibrium $\to \exists$ multipliers
$(\mu_1, \ldots, \mu_S) \in \mathbb{R}^S_{++} : x^*$ and
the contingent commodities
price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}_{++}$
$\to$ is an arrow - debreu equilibrium
$\to \mu_s :=$ value at $t = 0$, of a dollar
at $t = 1$ and states.

***

19.E ASSET MARKETS

the S contingent commodities $\to$
$\to$ transferring wealth across the states
of the world $\to$ functioning of the assets
market. theoretical structure.

an asset := a title to receive either
physical goods or dollars; at $t = 1$,
$\to$ PAY OFFS of an asset $\to$ RETURNS.

real $\to$ physical goods;
financial $\to$ paper money;

DEF. 19.E.1: a unit of asset, or security
is a title to receive an amount r_s of
good 1 at date $t = 1$ if state s occurs.
an asset is therefore characterized
by its return vector $\mathbf{r} = (r_1, \ldots, r_S) \in \mathbb{R}^S$.

DEF. 19.E.2: a collection formed
by a price vector $\mathbf{q} = (q_1, \ldots, q_K) \in \mathbb{R}^K$
for assets traded at $t = 0$, a spot price
vector $\mathbf{p} = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s,
and, for every consumer i, portfolio
plans $\mathbf{z}_i^* = (z_{1i}^*, \ldots, z_{Ki}^*) \in \mathbb{R}^K$ at
$t = 0$ and consumption plans
$\mathbf{x}_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at
$t = 1 \to$ radner equilibrium if:

(i) for every i, the consmpt. plans x_i^*,
z_i^* solve the problem;

$$\max_{\substack{(x_{1i},\ldots,x_{Si}) \in \mathbb{R}^{LS}_+ \\ (z_{1i},\ldots,z_{Ki}) \in \mathbb{R}^K}} \left\{\mathcal{U}_i(x_{1i}, \ldots, x_{Si})\right\} \text{ such that}$$

(a) $\sum_k q_k \cdot z_{ki} \leq 0$;

(b) $p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \sum_k p_{1s} z_{ki} r_{Sk}$, $\forall s$;

(ii) $\sum_i z_{ki}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_i \ \forall k, s$.

in the budget set $\to$ wealth of consumer i
at state s is the sum of the spot
value of his initial endowment.

return matrix $\rightarrow S \times K$ matrix
whose k^{th} column is the return
vector of the k^{th} asset.

its generic sk entry is
$r_{sk} \rightarrow$ the return of asset k
in state s.

$\mathcal{B}_i(p, q, \mathcal{B}) = \{x \in \mathbb{R}_+^{LS}$: for some portfolio $z_i \in \mathbb{R}^K$
we've $q \cdot z_i \leq 0$ and

$$\left.\begin{pmatrix} p_1 \cdot (x_{1i} - \omega_{1i}) \\ \vdots \\ p_S \cdot (x_{Si} - \omega_{Si}) \end{pmatrix} \leq \begin{bmatrix} r_{11} & \dots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{S1} & \dots & r_{SK} \end{bmatrix} z_i = \mathbf{R} z_i \right\}$$

assmpt.: unlimited short sales are possible.
$\rightarrow$ knowledge of the return matrix $\mathbf{R}$ suffices
to place significant restrictions on the
asset price vector $\mathbf{q} = (q_1, \dots, q_K)$.

PROP. 19.E.1: assume that every return
vector is non-negative and non-zero.
$\rightarrow r_k \geq 0$ and $r_k \neq 0$ for all k. then,
for every column vector $\mathbf{q} \in \mathbb{R}^K$
of asset prices arising in a
radner equilibrium $\rightarrow$ we can
find the multipliers $\mu = (\mu_1, \dots, \mu_S) \geq 0$
such that $q_k = \sum_{s=1}^{S} \mu_s r_{sk}$ for all k
$q' = \mu \cdot R$.

PROOF:

1. call the system $(q) \in \mathbb{R}^K$ of asset prices
arbitrage free if $\exists$ no portfolio $\mathbf{z} = (z_1, \dots, z_K)$ such that
$\mathbf{q} \cdot \mathbf{z} \leq 0$, $\mathbf{R} \cdot \mathbf{z} \geq \mathbf{0}$ and $\mathbf{R} \cdot \mathbf{z} \neq \mathbf{0} \rightarrow \nexists$ any portfolio that is
budgetary feasible and that yields a non-negative return
in every state and a strictly positive return
in some state.

whether an asset price vector
is arbitrage free or not depends only on the returns of the assets
and not on preferences.
preferences $\rightarrow$ strongly monotone
$\rightarrow$ equilibrium asset price vector $\rightarrow$
$\mathbf{q} \in \mathbb{R}^K \leftrightarrow \mathbf{q} \rightarrow$ must be arbitrage free.

LEMMA 19.E.1: if the asset price
vector $\mathbf{q} \in \mathbb{R}^K$ is arbitrage free $\rightarrow \exists$ a vector
of multipliers $\mu = (mu_1, \dots, \mu_S) \geq 0$:
$\underbrace{\mathbf{q}'}_{1 \times K} = \underbrace{\mu}_{1 \times S} \cdot \underbrace{R}_{S \times K}$.

PROOF: non-zero returns $\rightarrow$ an arbitrage
free price vector $\mathbf{q} : q_k > 0$, $\forall k = 1, \dots, K$.
$\rightarrow$ w/out loss of generality $\rightarrow$ no row of $\underbrace{\mathbf{R}}_{S \times K}$
has all of its entries $= 0$.
$V = \{v \in \mathbb{R}^S : V = \mathbf{R} \cdot \mathbf{z}$ for some $\mathbf{z} \in \mathbb{R}^K$, w/$\mathbf{q} \cdot \mathbf{z} = 0\}$.

$\rightarrow$ convex set.

arbitrage freeness $\rightarrow V \bigcap \{\mathbb{R}^S_+ \backslash \{0\}\} = \emptyset$. both V and $\mathbb{R}^S_+ \backslash \{0\}$ are convex sets and the origin $\in V \rightarrow$ separating hyperplane theorem $\rightarrow$ to get a non zero vector $\mu' = (\mu'_1, \ldots, \mu'_S)$ such that $\mu' \cdot \mathbf{v} \leq \mathbf{0}$ for $\forall v \in V$ and $\mu' \cdot \mathbf{v} \geq 0$ for $\forall v \in \mathbb{R}^S$.

it must be that $\mu' \geq 0$.

row vector $\mathbf{q}' \propto \mu' \cdot \mathbf{R} = 0$.

entries of μ' and $\mathbf{R}$ are all non-negative and no row of $\mathbf{R}$ is null.

$\rightarrow \mu : \mathbf{R} \geq 0$ and $\mu' \cdot \mathbf{R} = \mathbf{0}'$. if $\mathbf{q}$ is not $\propto \mu' \cdot \mathbf{R} \rightarrow \bar{\mathbf{z}} \in \mathbb{R}^K$: $\mathbf{q} \cdot \bar{\mathbf{z}} = \mathbf{0} \wedge \mu' \cdot \mathbf{R}\bar{\mathbf{z}} \geq \mathbf{0} \rightarrow v \in V$

and $\mu' \cdot \mathbf{v} \neq 0 \rightarrow$ contradiction!

$\mathbf{q}' \propto \mu' \cdot \mathbf{R} \rightarrow \mathbf{q}' = \alpha\mu' \cdot \mathbf{R}$ $\forall \alpha > 0$, $\alpha \in \mathbb{R}$; $\mu = \alpha\mu' \rightarrow \blacksquare$.

2. $\mathcal{U}_i(X_{1i}, \ldots, X_{Si}) = = \sum_{s=1}^S \pi_{si} u_{si}(x_{si})$ $\rightarrow$ bernoulli utility $u_{si}(.) \rightarrow$ concave, strictly $\nearrow$ and differentiable. $v_{si}(p_s, w_{si}) \rightarrow$ $\rightarrow$ indirect utility.

radner equilibrium $\rightarrow$ asset prices $\mathbf{q} = (q_1, \ldots, q_K) \rightarrow$ asset spot prices $\rightarrow$ $\mathbf{p} = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$. unlimited short sales are possible $\rightarrow$ optimal portfolio choice $z_i^* \in \mathbb{R}^K$ of any consumer i is necessa= rily interior, and $w_s^* = p_s\omega_{si} +$ $\sum_k r_{sk} z_{ki}^* \rightarrow$ F.O.C.s, for $\alpha_i > 0$;

$\forall k = 1, \ldots, K$.

$\mu_{si} = \frac{\pi_{si}}{\alpha_i} \cdot \frac{\partial v_{si}(p_s, w_{si}^*)}{\partial w_{si}}$;

$q_j = \sum_s \mu_{si} r_{sk} \rightarrow \mu = (\mu_1, \ldots, \mu_S)$

$\rightarrow$ choosing any consumer i and

letting $\mu_s = \mu_{si}$, the mar=

ginal utility of wealth

at state s of consumer i weighted by

π_{si}/α_i. $\alpha_i :=$ lagrange

multiplier of the budget

constraint at $t = 0 \rightarrow$ mar=

ginal utlilty of wealth

at $t = 0$. for any i

$\mu_i :=$ ratio of the expected

utility at $t = 0$ of one

extra unit of wealth

at $t = 1 \rightarrow$ intertemporal
MRS and state s.
$\neq i \rightarrow \neq \mu_i = (\mu_{i1}, \ldots, \mu_{iS}) \rightarrow \neq \mu'_s$
$\max \sum_{s=1}^{S} \pi_{si} v_{si}(p_1, p_s\omega_s + \sum_k r_{sk} z_{ki})$ s.t. $\sum_k q_k z_{ki} \leq 0$.

DEF: 19.E.3: an asset structure with an $S \times K$ return matrix $\mathbf{R}$ is complete if rank $R = S \leftrightarrow$ if $\exists$ some subset of S assets with linearly independent returns.

ex.

$$\underbrace{\mathbf{R}}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \text{rank}(\mathbf{R}) = 3 \blacksquare$$

***

PROP. 19.E.2: suppose that an **asset structure** is **complete**. then:

(i) if the **consmpt. plans** $\mathbf{x}^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ and **price vector** $(p_1, \ldots, p_s) \in \mathbb{R}^{LS}_{++}$ constitutes an **arrow-debreu equilibrium** $\rightarrow \exists$ **asset prices** $\mathbf{q} \in \mathbb{R}^K_{++}$ and **portfolio plans** $\mathbf{z}^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ s.t. the **consmpt. plans** $\mathbf{x}^*$, **portfolio plans** $\mathbf{z}^*$, **assets prices** $\mathbf{q}$, and **spot prices** $(p_1, \ldots, p_S)$ constitute a **radner equilibrium** ;

(ii) **conversely** , if the consmpt. plans $\mathbf{x}^* \in \mathbb{R}^{LSI}$, portfolio plans $\mathbf{z}^* \in \mathbb{R}^{KI}$, and prices $\mathbf{q} \in \mathbf{R}^K_{++}$, $(p_1, \ldots, p_S) \in \mathbb{R}^{KI}$ constitute a radner equilibrium,
$\rightarrow \exists$ multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^{LS}_{++}$ such that the consumption plans $\mathbf{x}^*$ and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}$ are an arrow-debreu equilibrium.

the **multiplier** μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s; $p_{1s} = 1 \rightarrow$ the numeraire.
the linear space: Range $\mathbf{R} = \{v \in \mathbb{R}^S :$
$: \underbrace{\mathbf{v}}_{S \times 1} = \underbrace{\mathbf{R}}_{S \times K} \cdot \underbrace{\mathbf{z}}_{K \times 1}$ for some $\mathbf{z} \in \mathbb{R}^K\} \subset \mathbb{R}^S$.

$\rightarrow$ the set of wealth vectors that can be spanned
by means of two different asset structures to give rise to the same linear space.

***

PROP. 19.E.3: suppose that asset price vector $\mathbf{q} \in \mathbb{R}^K$, the spot prices $\mathbf{p} = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$, the consumption plans $\mathbf{x}^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}_{++}$ and the portfolio plans $(z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ are a radner equilibrium for an asset structure with $S \times K$ return matrix $\mathbf{R}$. let $\mathbb{R}'$ be the $S \times K$ return matrix of a second asset structure. if Range $\mathbf{R'}$ = Range $\mathbf{R}$, $\rightarrow \mathbf{x}^*$ is still the consmpt. allocation of a radner equilibrium in the economy with the second asset structure.

PROOF: the asset prices satisfy the arbitrage condition $\mathbf{q'} = \mu \times \mathbf{R}$, for some $\mu \in \mathbb{R}^S_*$. denote $\mathbf{q'} = [\mathbf{q} \cdot \mathbf{R'}]'$; we claim that if Range($\mathbf{R}$) = Range($\mathbf{R'}$) $\rightarrow$ we can find $z'_i \in$ Range($\mathbf{R'}$) such that $\mathbf{R}z_i = \mathbf{R'}z'_i$. but $\rightarrow \mathbf{q'}z'_i = \mu \cdot \mathbf{R'z'_i} = \mu \cdot \mathbf{Rz_i} =$ $= \mathbf{q} \cdot \mathbf{z_i} \leq 0 \rightarrow x_i \in \mathcal{B}_i(\mathbf{p}, \mathbf{q'}, \mathbf{R'})$.
the converse statement

[if $x_i \in \mathcal{B}_i(\mathbf{q}, \mathbf{p'}, \mathbf{R'}) \rightarrow x_i \in \mathcal{B}_i(\mathbf{q}, \mathbf{p}, \mathbf{R})$] $\rightarrow$ same.

asset prices $\mathbf{q'}$;

spot prices $\mathbf{p} = (p_1, \dots, p_S)$

consmpt. allocations $\rightarrow \mathbf{x}^*$

radner equilibrium $\rightarrow$ in the economy with asset structure having return ma= trix $\mathbf{R'} \rightarrow$ find portfolios $(z'_1, \dots, z'_I) \in \mathbb{R}^{KI}$ such that $\sum_i z'_i = 0$ and, for $\forall$ consumer i, the vector of across-states wealth transfers $\rightarrow m_i = (p_i \cdot (x^*_{1i} - \omega_{1i}), \dots)$ $p_S \cdot (x^*_{Si} - \omega_{Si})' \rightarrow$ is s.t. $m_i = R'_i z'_i$ by strong monotonicity of preference.

we've $m_i = \mathbf{Rz'_i} \forall i \rightarrow m_i \in$ Range($\mathbf{R}$) $\rightarrow m_i \in$ Range($\mathbf{R'}$) for $\forall i$.

choose $z'_1, \dots, z'_{I-1} : m_i = \mathbf{R'z'_i}$, $\forall i = 1, \dots, I$. let $z'_I = -z'_1 - \dots - z'_{I-1} \rightarrow$ $\sum_i z_i = 0$ and also

$m_I = -(m_1 + \dots m_{I-1}) =$

$= -\mathbf{R'}(z'_1 + \dots + z'_{I-1})$

$= \mathbf{R'}z'_I$. ■

an asset is redundant if its deletion does not affect the linear space Range $\mathbf{R}$ of spannable wealth transfers $\leftarrow$ if its return vector is a linear combination of return vector the remaining assets.

such an asset can be priced just by looking at the matrix of returns and prices of other assets.

assets $\rightarrow$ **short term** $\rightarrow$ **payoff at** $t = 0$;

$\rightarrow$ **long term** $\rightarrow$ **payoff at** $t = T$;

$S(tE) \rightarrow$ # of successes at tE;

$k(tE) \rightarrow$ # of assets available at tE;

$\underbrace{R}_{S \times K}(tE) \times k(tE) \rightarrow R(tE) = S(tE) \rightarrow$ by completeness.

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19.E INCOMPLETE MARKETS:

→ informational asymmetries;

sunspots → preferences admit an expec= ted utility representation and the set of states is such that → a. the probability estimates for the ≠ states are the same for i, i' and s; and the states do not affect the fundamentals of the economy; ↔ the bernoulli utility functions and the endow= ments of every consumer i are uniform across states, i.e. $u_{si}(.) = u_i(.)$ and $\omega_{si} = \omega_i$ for all s.

sunspot set → sunspot equilibrium, does the radner eq. allocation can assign varying consmpt. across states.

magill and shafer, 1991.
sunspot problem → influence on fun= damentals that could make an unobser= vable signal have an effect on prices. the utility functions $u(.)$ are strictly concave ↔ consumers are strictly risk averse → any pareto optimal allocation $(x_1, \ldots, x_I) \in \mathbb{R}^{LSI}$ must be uniform across states → for $\forall i$ → $x_{1i} = x_{Li} = \cdots = x_{si} = \cdots = x_{Si}$. for every i and s, we re= place the consmpt. bundle of cons. i in state s → $x_{si} \in \mathbb{R}^L_{++}$ by the expected consmpt. bundle of this cons. $\bar{x}_i = \sum_s \pi_s x_{si} \in \mathbb{R}^L_+$. the new allocation is state independent and also feasible, b/cause:
$\sum_i \bar{x}_i = \sum_i \sum_s x_{si} =$
$= \sum_s \pi_s (\sum_i x_{si}) \leq \sum_s \pi_s (\sum_i \omega_i) = \sum_i \omega_i$
by concavity of $u(.)$ → no consumer is worse-off:

$\sum_s \pi_s u_i(\bar{x}_i) = u_i(\bar{x}_i) = u_i(\sum_s \pi_s x_{si}) \geq$
$\geq \sum_s \pi_s u_i(x_{si})$, for every $i = 1, \ldots, I$.

due to the pareto optimality of $(x_1, \ldots, x_I)$ → the above weak ≥ → = →

$u_i(\bar{x}_i) = \sum_s \pi_s u_i(x_{si}).\forall i.$

strict concavity of $u_i(.) \to x_{si} = x_i, \forall s.$

parto allocation $(x_1, \ldots, x_I)$ is state independent + first welfare theorem → → if a system of complete markets over state S can be organized → the equilibria are sunspot free → con= ampt. ≡ uniform across states. → traders.

if ∄ a complete set of insurance opp. → the above conclusions doesn't hold true. it's not possible for some radner equi= librium allocations to be dependent on the state → fail pareto optimality. consumers expect ≠ prices at ≠ states → self fulfilling expectations arise. ex. ∄ assets → $K = 0$ → a sys. of spot prices $(p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ is a radner equilibrium iff every p_s is a walrasian equil. price vector for the spot economy, defined by $\{u_i(.), \omega_i\}_{i=1}^{I}$; if this economy admits several walrasian equilibria → by selecting ≠ equilibrium price vectors → for ≠ states → we get a sunspot equilibrium. → a pareto inefficient rad= ner equilibrium.

the radner equilibrium allocations need not being pareto optimal → there may ∃ reallocations of consumption that make all consumers at least as well of and at least one consumer strictly better off → this need not imply that a welfare authority who's as constrained in interstate transfers as market is and can achieve a pareto optimum.

$t = 1$ → ∃ a single commodity per state; → the amount of consumption good that any consumer gets in the ≠ states is entirely determined by the portfolio z_i.

$x_{si} = \sum_k z_{ki} r_{sk} + \omega_{st}.$

$\mathcal{U}_i^*(z_i) = \mathcal{U}_i(z_{1i}, \ldots, z_{Ki})$

$= \mathcal{U}_i(\sum_k z_{ki} r_{1k} + \omega_{1i}, \ldots, \sum_k z_{ki} r_{Sk} + \omega_{Si})$

→ utility induced by portfolio z_i.

DEF. 19.F.1: the asset allocation $(z_1, \ldots, z_I) \in \mathbb{R}^{KI}$ is constrained pareto

optimal if it is feasible (i.e. $\sum_{i=1}^{I} z_i \leq 0$)
and there is no other feasible asset
allocation $(z'_1, \ldots, z'_I) \in \mathbb{R}^{KI}$ such that
$\mathcal{U}_i^*(z'_1, \ldots, z'_I) \leq \mathcal{U}_i^*(z_1, \ldots, z_I)$, $\forall i = 1, \ldots, I$.
with at least one inequality strict.
in this $L = 1$ context, the utility
maximization problem of cons. i becomes:

$\max_{z_i \in \mathbb{R}^K} \mathcal{U}_i^*(z_{1i}, \ldots, z_{Ki})$ s.t. $q \cdot z_i \leq 0$.

suppose that $z_i^* \in \mathbb{R}^K$ for $i = 1, \ldots, I$,
is a family of solutions to these individual
problems, for the asset price vector $\mathbf{q} \in \mathbb{R}^K$.
$\rightarrow$ $\mathbf{q} \in \mathbb{R}^K$ is a radner equilibrium price
if and only if $\sum_{i=1}^{I} z_i^* \leq 0$. this has be=
come now a perfectly conventional
equilibrium probles w/K commo=
dities $\rightarrow$ apply the first welfare
theorem.

PROP. 19.F.1.: suppose there are
two periods and only one consumption
good in $t = 2$; then, any radner equilibrium
is constrained pareto optimal,
in the sense that there are no possible
redistributions of assets in $t = 1$ that
leaves every consumer as well off
and at least one consumer strictly
better off.

with one consmpt. good $\rightarrow$ $\nexists$ any
possibility of trade.

$t = 2$ prices don't matter.

$\rightarrow$ all consmpt. takes place here.

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19.G FIRM BEHAVIOUR IN GENERAL EQUILIBRIUM MODELS UNDER UNCERTAINTY

production and firms $\rightarrow$ more difficult in a
context of possibly incomplete markets.
objectives of the firm $\rightarrow$ diamond, 1967.

$t = 0, 1$; S $\rightarrow$ possible states at $t = 1$.
L physical commodities, traded in the spot
markets of $t = 1$; K assets traded at time $t = 0$;
no consmpt. at $t = 0$; returns on the assets
are in physical amounts of the good 1 (nume=
raire). $\underbrace{\mathbf{R}}_{S \times K}$ $\rightarrow$ return matrix.

a firm produced a random amount of nume=
raire at $t = 1$; $(a_1 \ldots, a_S)$ $\rightarrow$ state contin=
gent levels of production of the firm.
$\exists$ shares $\theta_i \geq 0$, $\sum_{i=1}^{I} \theta_i = 1$, giving the
proportion of the firm that belongs

to cons. i. the firm is an asset
w/return vector $\mathbf{a} = (a_1, \ldots, a_S)$ whose
shares are tradeable in the financial
markets at $t = 0$.

the firm can choose its random
production plan. $\mathbf{A} \subset \mathbb{R}^S$ of possible
choices of return vectors $(a_1, \ldots, a_S) \in \mathbf{A}$
of the firm. $\mathbf{a} \in \mathbf{A}$ is chosen before
the financial markets of $t = 0$ open.
$\rightarrow$ decision is made by initial share=
holders;

DEF. 19.G.1: a set $\mathbf{A} \subset \mathbb{R}^S$ of random variables
is spanned by a given asset structure
if every $\mathbf{a} \in \mathbf{A}$ is in the range of the
return matrix $\mathbf{R}$ of the asset structure
$\leftrightarrow$ if every $\mathbf{a} \in \mathbf{A}$ can be expressed
as a linear combination of the
variable asset returns.

if we assume, first, that $\mathbf{A}$ is spanned
by $\mathbf{R}$, second, that we're dealing
with a small project $\rightarrow$ $\mathbf{a} \in \mathbf{A}$ are small
relative to the size of the economy,
$a_s / || \sum_i \omega_i ||$ is small for all s

$\rightarrow$ we're almost justified in taking
the equilibrium spot prices $\mathbf{p} = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$
and asset prices $\mathbf{q} = (q_1, \ldots, q_K) \in \mathbb{R}^K$
as constants independent of the
particular production plan chosen
by the firm.

for the assets prices $\mathbf{q} \in \mathbb{R}^K$ the market
value $V(\mathbf{a}, \mathbf{q})$ of any production plan
$\mathbf{a} \in \mathbf{A}$ can be computed by arbitrage
$\rightarrow$ if $\mathbf{a} = \sum_k \alpha_k r_k \rightarrow V(\mathbf{a}, \mathbf{q}) = \sum_k \alpha_k q_k$.

if the firm is added as a new asset
to the given list of assets, and each
production plan $\mathbf{a} \in \mathbf{A}$ is priced at its
arbitrage value $V(\mathbf{a}) \rightarrow$ any budget
feasible production plan of any cons.
can be reached w/out purchasing any
shares of the firm.

$\rightarrow$ for the fixed asset price $\mathbf{q} \in \mathbb{R}^K$
and spot prices $\mathbf{p} = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$,
the budget constraint of cons. i is
$\mathcal{B}_{ai} = \{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{LS}_+$: $\exists$ a portfolio
$z_i \in \mathbb{R}^K$ such that $p_S(x_{is} - \omega_{is}) \leq \sum_k p_{is} \cdot r_{sk} \cdot z_{ki}$ for $\forall s$,
and $\mathbf{q} \cdot z_i \leq \theta_i V(\mathbf{a}, \mathbf{q})\}$.

at constant prices, every consumer-owner
i.e. any i with $\theta_i \geq 0$ faced with the choice
between two production plans $\mathbf{a}, \mathbf{a}' \in \mathbf{A}$,

will prefer the one with higher market value $\rightarrow V(\mathbf{a}, \mathbf{q}) \geq V(\mathbf{a}', \mathbf{q}) \rightarrow \mathcal{B}\mathbf{a}_i' \subset \mathcal{B}\mathbf{a}_i$

$\rightarrow$ market value maximization $\rightarrow$ unani= mous desire of the firm's intial owners.

if $\mathbf{A}$ is non - spannable by the given asset.

structure $\rightarrow$ two serious difficulties $\rightarrow$price quoting, common to any commodity innovation problem. w/out spanning $\rightarrow$ the value of a production plan $\mathbf{a} \in \mathbf{A}$ cannot be computed from current prices of assets simply by arbitrage.

$\rightarrow$ value isn't implicitly quoted in the economy.

price taking $\rightarrow$ unlimited short - sales $\rightarrow$ discontinuity in the plausibility of the price taking assmpt. if a new asset $\mathbf{a} \in \mathbf{A}$ isn't generated by the current asset structure $\rightarrow$ its availability $\nearrow$ the span of available wealth transfers by one whole extra dimension $\rightarrow$ substantial impact w/possibly dramatic effects on prices. $(\omega_{si} + (\omega_i \mathbf{a}_s, 0, \ldots, 0)) \in \mathbb{R}^L$ for every s.

19.H IMPERFECT INFORMATION

up to now, we've concentrated the analysis upon a model where spot trading for goods occurs under perfect information $\rightarrow$ about the state of the world. $\rightarrow$ (a)symmetric.

$t = 1$, only.

$s = 1, \ldots, S$ states of the world.

single spot market $\rightarrow$ simplest case. a first commodity is traded against a second good $\rightarrow$ money $\rightarrow L = 2$; price of the second good ≈ 1; $\mathbf{p} \in \mathbb{R} \rightarrow$ price of the non-monetary commodity; $i = 1, \ldots, I$ consumers.

$\pi = (\pi_{1i}, \ldots, \pi_{Si}) \rightarrow$ probabilities over the states, $\mathbf{x} = (z_{1i}, \ldots, x_{Si}) \in \mathbb{R}^{2S}$ $\rightarrow$ random consmpt. vector $\rightarrow$ evaluated by consumer i according to an ext. V.N. - M. utility function:

$\mathcal{U}(x_i) = \sum_{s=1}^{S} \pi_{si} u_{si}(x_{si})$,

$u_{si}(.) :=$ bernoulli utility function in state s.

consumer i has an intitial, state de= pendent endowment vector $\omega_i = (\omega_{1i}, \ldots, \omega_{Si}) \in \mathbb{R}^{2S}$, and a signal function $\sigma_i(.)$ to every state $s \in S$.

the state s occurs at the beginning of the period $\rightarrow$ we assume that consumer then receives the initial endowment ω_{si} and the signal $\sigma_i(s) \in \mathbb{R}$. the consumer can distinguish two states $\rightarrow$ s and $s' \in S$ if and only if $\sigma_i(s) \neq \sigma_i(s')$ $\rightarrow$ information partitions instead of explicit signal functions. info associated with $\sigma_i(.) \rightarrow \mathcal{S}_i$.

$\rightarrow \{s \in S : \sigma_i(s) = c\}$, letting $c \in \mathbb{R}$ run over all possible values.

endowments have to be measurable w.r.t. the signal function; $\omega_{si} = \omega_{s'i}$ whenever $\sigma_i(s) = \sigma_i(s') \rightarrow \omega_{si} := \omega_{\sigma_i(s)i}$ the endowments of of goods of cons. i do not reveal him information about the state of the world that's not already given by the signal. spot market operates after signal are emitted $\rightarrow$ consumption takes place.

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symmetric information $\rightarrow$ if any two states, s and $s' \in S$, are distinguishable by every others, consumer $k \rightarrow \sigma_i(s) \neq \sigma_i(s')$ if and only if $\sigma_k(s) \neq \sigma_k(s')$.

all consumers share the same signal function. $\sigma_i(.) = \sigma(.)$ for all $i = 1, \ldots, I$.

$\sigma(.) :=$ public signal.

state s occurs $\rightarrow \sigma(s)$ is received by the consumers $\rightarrow$ initial endowment $\omega_{\sigma(s)i}$. from the signal and the prior probabilities $\pi_i = (\pi_{1i}, \ldots, \pi_{Si}) > 0$.

$\pi_{s'i}|\sigma(s) = \frac{\pi_{s'i}}{\Sigma\{s'':\sigma(s'')=\sigma(s)\}\pi_{s''i}}$ $\rightarrow$ the posterior probability for any s' with $\sigma(s') = \sigma(s)$; and $\pi_{s'i}|\sigma(s') = 0$ otherwise.

the utility of a consmpt. bundle $x_i \in \mathbb{R}^2$ conditional on the signal$\sigma(s)$ is then $u_i(x_i|\sigma(s)) = \sum_{s'}\{(\pi_{s'}|\sigma(s))\}\mathcal{U}_{s'i}(x_i)$ $\rightarrow$ we've, conditional on s, perfectly well specified spot economy.

price-taking assmpt. $\rightarrow$ an equilibrium price will be generated.

$p[\sigma(s)] \in \mathbb{R}$.

DEF. 19.H.1: the signal function $\sigma' : S \rightarrow \mathbb{R}$ is at least as informative as $\sigma : S \rightarrow \mathbb{R}$ is $\sigma(s) \neq \sigma(s') \rightarrow \sigma'(s) \neq \sigma'(s')$ for any pair s, s'. it is more infor=

mative if, also, $\sigma'(s) \neq \sigma'(s') \rightarrow$ for some pairs s, s' with $\sigma(s) = \sigma(s')$.

$\rightarrow$ the more informative information partition $\rightarrow$ welfare improvement. $\rightarrow$ exp. utilities of the $\neq$ consumers under $\sigma(.)$ and under $\sigma'(.)$ when the expectation is evaluated before it occurs.

decision problem of a single consumer $i \rightarrow$ suppose the spot price $p \in \mathbb{R}$ and the consumer wealth $\omega_i \in \mathbb{R}$ are given and independent on the state of the world.

for any signal function $\sigma(.)$, the consumer forms a consmpt. plan $x_i^{\sigma(.)} \in \mathbb{R}^{2S} \rightarrow$ subject to the restrictions that $x_{si}^{\sigma(.)} = \sigma_{s'i}$ whenever $\sigma(s') = \sigma(s)$, the consumer forms a consumption $x_{si}^{\sigma(.)}$ in his budget set that max exp. utility conditional on the signal $\sigma(s)$. the ex ante utility of the information signal function $\sigma(.)$ is thus $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(.)})$.

PROP. 19.H.1: in the single consumer problem if the signal function $\sigma'(.)$ is at least as informative as $\sigma(.) \rightarrow$ the ex ante utility derived from $\sigma'(.)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma'(.)})$ is at least as large as the ex ante utility derived from $\sigma(.)$, $\sum_s \pi_{si} u_{si}(x_{si}^{\sigma(.)})$.

PROOF. $\forall \sigma(.)$, $x_i^{\sigma(.)}$ solves
$\max \sigma_s \pi_{si} u_{si}(x_{si})$ s.t.
$x_i \in \mathcal{B}_i^{\sigma(.)} = \{x_i \in \mathbb{R}^{2S} : p \cdot x_{1si} + x_{2si} \leq \omega_i$
$\forall s$, and $x_{si} = x_{s'i}$ whenever $\sigma(s) = \sigma(s')\}$.
if $\sigma'(.)$ is at least as informative as $\sigma(.) \rightarrow$
$\mathcal{B}_S^{\sigma(.)} \subset \mathcal{B}_i^{\sigma'(.)}$.
this result doesn't apply to a set of interacting decision makers.

asymmetric information $\rightarrow$ information is symmetric $\rightarrow$ the signal functions $\sum_{i=1}^{I} \sigma_i(.)$ are private and not necessarily the same across consumers. when s occurs, every consumer observes $\sigma_i(s)$ and uses his signal function $\sigma_i(.)$ to update probabilities and utility functions $\rightarrow$ an economy to which we can associate in the usual way a spot market clearing price written as $\mathbf{p}(\sigma_1(s), \ldots, \sigma_I(s))$. the price $\mathbf{p}(\sigma_1, \ldots, \sigma_S)$ depends on all individual signals $\rightarrow$ the price aggregates the information of all market participants $\rightarrow$ the price function $\mathbf{p}(s) = \mathbf{p}(\sigma_1(s), \ldots, \sigma_I(s))$ need not be measurable w.r.t. the single signal functions $\sigma_i(.)$ $\rightarrow$ two states $s, s' \in S$ are not distingushable by consumer $i \leftrightarrow \sigma_i(s) = \sigma_I(s')$, but are distinguished by the market $\rightarrow \mathbf{p}(\sigma_1(s), \ldots, \sigma_i(s)) \neq$ $\neq \mathbf{p}(\sigma_1(s'), \ldots, \sigma_I(s'))$.

DEF. 19.H.2: the price function $\mathbf{p}(.)$ is a rational expectation equilibrium price function if, for every s, $\mathbf{p}(s)$ clears the spot market when every i knwos that $s \in \mathbb{E}_{\mathbf{p}(s),\sigma_i(s)}$ and, therefore, evaluated commodity bundles $x_i \in \mathbb{R}^2$ according to the bayesian updated utility function $\rightarrow \sum_{s=1}^{S}(\pi_{s',i}|\mathbf{p}(s), \sigma_i(s))\mathbf{u}_{s'i}(x_i)$.

all individual signal functions are known to all consumers abd for every state, the vec= tor of signal values $\sigma_1(s), \ldots, \sigma_I(s)$ is made public $\rightarrow$ is usable by all consumers $\rightarrow$ to update probabilities and utilities; the market - clearing price function $\hat{\mathbf{p}}(s) = \hat{\mathbf{p}}(\sigma_1(s), \ldots, \sigma_I(s)$ thus generated is called the pooled information equilibrium price function. if the values of $\hat{\mathbf{p}}(.)$ distinguish all possible values of $(\sigma_1, \ldots, \sigma_I)$, $\leftrightarrow$ if $\hat{\mathbf{p}}(s) \neq \hat{\mathbf{p}}(s')$ whenever $\sigma_i(s) \neq$ $\neq \sigma_i(s')$ for some s, s', and $i \rightarrow$ the price function $\hat{\mathbf{p}}(.)$ is fully revealing.

if the pooled information eq. price function is fully revealing $\rightarrow$ it must be a rational expectations equilibrium price function $\rightarrow$ for any s_i, $\hat{\mathbf{p}}(s)$ is determined under the assmpt. that every i knows that $s \in \{s' : \sigma_k(s') = \sigma_k(s)$ for all $k\}$. $\{s' : \sigma_k(s') = \sigma_k(s)$ for all $k\} = \{s' : \hat{\mathbf{p}}(s') = \hat{\mathbf{p}}(s)\}$.

for any $s \rightarrow \hat{\mathbf{p}}(s)$ is a market clearing price when every i knows that $s \in \mathbb{E}_{\hat{\mathbf{p}}(s),\sigma_i(s)} \rightarrow \hat{\mathbf{p}}(.)$ is a rational expectations equilibrium price funct.

if the pooled information equilibrium price function is fully revealing

→ the pooled info. used by consumers
need not be obtained by violating
any privacy constraint → simply
derived from the public price
signals.

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CHAPTER 20: EQUILIBRIUM AND TIME

extension of competitive equilibrium
theory to an intertemporal setting.
1. particular case of the general
equilibrium theory developed previously;
2. stressing the structure of time;
open-ended infinity of time;
production takes time; stationa=
rity and time separability.

DETERMINISTIC VERSION OF THE THEORY

stokey and lucas, 1989 → general theory.
$\sum_t = \sum_{t=0}^{+\infty} = \lim_{T\to+\infty} \sum_{t=0}^{T}$.

20. INTERTEMPORAL UTILITY.

$t = 0, 1, 2, \dots$
$C = (c_0 \dots, c_t, \dots)$ → consmpt. streams;
$c_t \in \mathbb{R}^L_+$, $c_t \geq 0$, trajectory, program;
$\sup ||c_t|| < +\infty$ → bounded streams.
preferences over consmpt. streams;
$V(c) = \sum_{t=0}^{+\infty} \delta^t u(c_t)$ $(\star)$
$\delta < 1$ → discount factor;
$u(.) : \mathbb{R}^L \to \mathbb{R}^L_+$; strictly ↗ and concave;
$c' = (c_0, c_1', \dots)$ → T-period backwards shift;
$c_t' = c_{t+T}$ for all $t \geq 0$
(i) **time impatience** → utility's discounted; $\delta \in [0, 1)$ → if $\mathbf{c} = (c_0, c_1, \dots, c_t, \dots)$ is a
non-zero consmpt. stream → the forward shifted
consumption stream $\mathbf{c}' = (0, c_0, c_1, \dots, c_{t-1}, \dots)$
is strictly worse than $\mathbf{c}$;
→ very helpful assmpt. for guaran=
teeing that a bounded consmpt. stream
has a finite utility value → allowing
us to compare any such two streams;
→ distant future does not matter for
current decisions;
(ii) **stationarity** → $V(c) = \sum_{t=0}^{+\infty} u_t(c_t)$.
→ a special case of $(\star)$; consider two
consmpt. streams → $\mathbf{c} \neq \mathbf{c}'$ such that
$c_t = c_t'$ for $t \leq T-1$;
$V(\mathbf{c}) \geq V(\mathbf{c}')$ iff $V(\mathbf{c}^T) \geq V(\mathbf{c}'^T)$.
utility functions → $V(\mathbf{c}) = \sum_{t=0}^{+\infty} \delta^t u(c_t)$
violate the stationarity assmpt.;
preferences over the future are ⊥ age
of the consumer;

(iii) **additive separability** $\to$ at any date T
the induced ordering of consmpt. streams
is independent on the consmpt. streams
beginning at $T+1$. past cons. create HABITS
and ADDICTIONS $\to V(\mathbf{c}) = \sum_{t=0}^{+\infty} u_c(c_{t-1}, c_t)$

$z_t \to$ habit variables and household
production technology $\to$ using an input
vector c_{t-1} at $t-1 \to$

(iv) **length of the period** $\to$

enjoyment of current consumption
is independent on the consumption
in other periods; length of the
period $\to$ high enough for prices
to hold constant;

(v) **recursive utility** $\to V(\mathbf{c}) = U(c_0) + \delta V(c_1)$
for any consmpt. stream $\mathbf{c} = (c_0, c_1, \ldots, c_t, \ldots)$
$u = u(c_0) \to$ current utility; $V = V(c_1) \to$
future utility $\to MRS$ btwn current
and future utility $\to \delta \to$ indep.
of the levels of current and future
utility. recursive utility model $\to$
$\to$ koopmans, 1960 $\to$

$u \geq 0 \to$ current utility;

$V \geq 0 \to$ future utility;

$u(c_t) \to$ current utility function $\to$

$G(u, V) \to$ aggregator function

indifference curves in the (u, V) plane
aren't straight lines $\to$ utility of a
consmpt. stream $\to \mathbf{c} = (c_0, c_1, \ldots, c_t, \ldots)$

$$\to V(\mathbf{c}) = G(u(c_0), V(c^1)) =$$
$$= G(u(c_0), G(u(c_1), V(c^2))) = \ldots$$

$\to$ assmpt. of impatience

(vi) **altruism** $\to V(\mathbf{c}) = u(c_0) + \delta \cdot V(c^1) \to$
multigenerational inerpretation of the single
consumer problem $\to$ if generations
live one period $\to$ generation 0 enjoying
consmpt. according to $u(c_0)$, but caring
also about the future utility $V(c^!) \to$ of the
next generation $\to \delta V(c^1) \to V(\mathbf{c}) = u(c_0) + \delta V(c^1) \to$ overall utility. entire dynasty
behaves as a signle individual. $\delta < 1 \to$
members of the current generation care
about their children.

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20.C INTERTEMPORAL PRODUCTION AND EFFICIENCY

$\to t = 0, 1, 2, \ldots, +\infty$

in $\forall t$, there are L commodities;
$L = 2$, for simplicity; commodities
as labour services $\to$ a generalized

consmpt.-investment good.
goods are non-durables.
$\rightarrow$ storage technology $\rightarrow$
to transport utility
over time.
production technology.
$\rightarrow$ set.
the production possibilities of time t
are entirely determined by the prod.
decisions of time $t-1$.
tech. possibilities at time t will be entirely
determined by a production set
$Y \subset \mathbb{R}^{2L}$ whose generic entries, or
producton plans are written $\mathbf{y} = (y_a, y_b)$,
the indices a and b are mnemonic for "after" and "before".
production plans in Y cover two periods.
$y_b \in \mathbb{R}^L$ and $y_a \in \mathbb{L}$
negative entries $\rightarrow$ inputs;
positive entries $\rightarrow$ outputs;
(i) Y is closed and convex;
(ii) $Y \cap \mathbb{R}^{2L} = \{0\} \leftrightarrow$ no free lunch;
(iii) $Y - \mathbb{R}^{2L}_{+} \subset Y \leftrightarrow$ free disposal;
production takes time;
(iv) if $\mathbf{y} = (y_b, y_a) \in Y \rightarrow (y_b; 0) \in Y \rightarrow$
possibility of truncation.
not producing in the last period
is a possibility.

RAMSEY[15] SOLOW[16] MODEL.
two commodities $\rightarrow$ investment + consumption goods + labour.
$F(K, L)$; any amount $k \geq 0$ and of $l \geq 0 \rightarrow$ production function
assigns $\rightarrow$ the total amount $F(K, L)$ of consmpt. - invest. good available.
$Y = \{(-k, -l, x, 0) : k \geq 0, l \geq 0, x \leq F(k, l)\} \in \mathbb{R}^4_{+}$.
$\rightarrow$ labour is a primary factor $\rightarrow$ it
cannot be produced. ■

COST OF ADJUSTMENT MODEL
$L = 3 \rightarrow$ capacity;
$\rightarrow$ a consumption good;
$\rightarrow$ labour
k and $l \rightarrow$ amounts if investmed capacity
and labour $\rightarrow$ at $t = 0 \rightarrow F(k, l)$ units
of consmpt. good $\rightarrow$ output in the
last period $\rightarrow$ output can be transformed
into invested capacity $\rightarrow$ at $t = T$,
at the cost of $k' + \gamma(k' - k)$ units
of consmpt. good for k' units
of capacity $\rightarrow \gamma(.) :=$ convex

[15] 1928
[16] 1956

function : $\gamma(k'-k) = 0$ for $k' < k$.
$\gamma(k'-k) > 0$ for $k' > k$.
$\gamma(k'-k) :=$ cost of adjustment capacity upward in a given period relative to the previous period.
$Y = \{(-k, 0, -l, k', x, 0), k \geq 0, l \geq 0, k' \geq 0, x \leq F(k,l)$
$-k' - \gamma(k'-k)\} - \mathbb{R}^6_+$. ■

TWO - SECTOR MODEL

$Y = \{(-k, 0, -l, k', x, 0) : k \geq 0, l \geq 0, k' \geq 0$
$x \leq G(k, l, k')\} - \mathbb{R}^6_+$

the investment and consumption good needn't be perfect substitutes → uzawa, 1964. if they are → back to the ramsey - solow model.

if $G(k, lk') = F(k,l) - k' \rightarrow CAC$[17].

DEF. 20.C.1: the list $(y_0, y_1, \ldots, y_t, \ldots)$ is a production path or trajectory, or program, if $y_t \in Y \subset \mathbb{R}^{2L}, \forall t$.
aalong a production path $(y_0, \ldots, y_t, \ldots)$ there is overlap in time indices over which the production pans y_{t-1} and y_t are defined.

both $y_{a,t-1} \in \mathbb{R}^L$ and $y_b t \in \mathbb{R}^L$ are plans, for input use or output production at date t.

at every t, a net input-output vector equal to $y_{a,t-1} + y_{bt} \in \mathbb{R}^L$
at $t = 0$, we've $t_{a,-1} = 0$.

think of the technology at every t as being run by a distinct firm, and on $\hat{y}_t$ as an infinite sequence w/non-zero entries only in the t and $t-1$ places, then $\sum_{t=0}^{+\infty} \hat{y}_t$ is the aggregate production plan.

w/an infinite horizon, we've a countable infinity of commodities and of firms, instead of only a finite no.

DEF. 20.C.2: the production path $(y_0, \ldots, y_t, \ldots)$ is efficient if there is no other production plan $(y'_0, \ldots, y'_t, \ldots)$ such that $y_{a,t-1} + y_{bt} \leq y'_{a,t-1} + y'_{bt}$, for all t, and equality does not hold for at least one t.

a price vector → $(p_0, p_1, \ldots, p_t, \ldots), p_t \in \mathbb{R}^L$; → present - value prices; given a path $(y_0, y_1, \ldots, y_t, \ldots)$ and price sequence $(p_0, \ldots$ $\ldots, p_t, \ldots) \rightarrow$ the profit level associated

[17] convex adjustment costs.

with the production plan at t is
$\pi = p_t \cdot y_{bt} + p_{t+1} \cdot y_{at}$

DEF. 20.C.3: the production path $\{y_t\}_{t=0}^{+\infty}$ is myopically or short run profit maximizing for the price sequence $\{p_t\}_{t=0}^{L}$ if, for every t, we've:

$p_t \cdot y_{bt} + p_{t+1} \cdot y_{a,t+1} \geq p_t \cdot y'_{bt} + p_{t+1} \cdot y'_{at}$
for all $y'_t \in Y$.

prices $(p_0, \dots, p_t, \dots)$ capable of sustaining a path $(y_1, \dots, y_t, \dots)$ as myopically profit maximizing are often called malinvaud prices
→ malinvaud, 1953.

we don't require $\sum_{t=0}^{\infty} p_t(y_{a,t-1} + y_{bt}) < \infty$.
capital over accumulation → impedes the application of the first welfare theorem. → efficiency → yes,
if the present value of production plan becomes insignificant as $t \to +\infty$.
if the present value of the period t production plan for $t+1$ goes to zero
↔ $p_{t+1} \cdot y_{at} \to 0$ as $t \to +\infty$.
→ transversality condition.

PROP. 20.C.1: suppose the production path $(y_0, \dots, y_t, \dots)$ is myopically profit maximizing w.r.t. the price sequence $(p_0, \dots, p_t, \dots) >> 0$; suppose also that the production path and the price sequence satisfy the transversality condition $p_{t+1} \cdot y_{at} \to 0$;
then the path $(y_0, \dots, y_t, \dots)$
is efficient.
→ a modified version of the first welfare theorem
→ (i) is there a system of malinvaud prices $(p_0, \dots, p_t, \dots)$ for $(y_0, \dots, y_t, \dots,)$ that is a sequence of $\{p_t\}_{t=0}^{+\infty}$ w.r.t. which $\{y_t\}_{t=0}^{+\infty}$ is myopically π[18] maximizing?
(ii) if the answer (i) is yes,
can we conclude that the pair $\{y_t\}_{t=0}^{+\infty}$; $\{p_t\}_{t=0}^{+\infty}$ satisfies the tvc?

RAMSEY - SOLOW MODEL

a path → sequence $\{k_t, l_t, c_t\}_{t=0}^{+\infty}$
→ capital, labour and consumption;
$k_{t+1} + c_{t+1} = F(k_t, l_t)$
capital path $(k_0, \dots, k_t, \dots)$
(q_t, w_t) → prices of the two commodities at time t; profits at t are $q_{t+1}F(k_t, l_t) - q_t k_t - w_t l_t$;
FONC: $\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t)$ and $\frac{w_t}{w_{t+1}} = \nabla_2 F(k_t, l_t)$
supporting prices for any feasible capital path
→ TVC → $q_{t+1}F(k_t, l_t) \to 0$
if the sequence of productions $F(k_t, l_t)$ is bounded → $q_t \to 0$. a set of sufficient conditions → for efficiency of a feasible and bounded capital sequence $(k_0, \dots, k_t, \dots)$

[18] profit

is that there $\exists$ a sequence of output prices $\{q_t\}_{t=0}^{+\infty}$ such that:
$\frac{q_t}{q_{t+1}} = \nabla_1 F(k_t, l_t)$ for all $t = 0, \dots, +\infty$
and $q_t \to 0$;
due to the chance of capital over accumulation the original definition is not sufficient for efficiency.
$\to \sum_{t=0}^{+\infty} \frac{1}{q_t} = +\infty$. ■

COST OF ADJUSTMENT MODEL

a production at $t-1 \to k_{t-1}, l_{t-1}, k_t, c_t$.
$\to$ prices $\to q_{t-1}, w_{t-1}, q_t, s_t$. profits are:
$s_t(F(k_{t-1}), l_{t-1}) - k_t - \gamma(k_t - k_{t-1}) +$
$+ q_t k_t - q_{t-1}k_{t-1} - w_{t-1}l_{t-1}$.

first order conditions:

(i) $q_t = s_t(1 + \gamma'(k_t - k_{t-1}))$;

(ii) $q_{t-1} = s_t(\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1}))$;

(i) + (ii) $\frac{q_{t-1}}{q_t} = \frac{\nabla_1 F(k_{t-1}, l_{t-1}) + \gamma'(k_t - k_{t-1})}{1 + \gamma'(k_t - k_{t-1})}$;

if $CAC = 0 \leftrightarrow \gamma(.) = 0 \to$ back to the ramsey - solow model.

b/cause of efficiency $\to \forall y_t \in$ boundary $\{Y\}$ smoothness $\to \forall t$, the product set Y has a single outward normal $\mathbf{q}_t = (q_{bt}, q_{at})$ at $y_t \to$ at $y_t \in Y$, all the MRT of inputs and outputs are uniquely identified.
$\forall t \to q_{a,t-1} = \beta q_{bt}$, $\beta > 0$
move from $y_{a,t-1}$ to $y'_{a,t-1}$
and from y_{bt} to y'_{bt} ↯.

$\{p_t\}_{t=0}^{+\infty} \to$ by induction.
$p_0 = q_{b0}$; suppose now that prices $\{p_t\}_{t=0}^{T}$ are already determined and that every y_t up to $t = T-1$ is myopically profit maximizing for these prices. because of the FOCs for π_t max. at $T-1$, we've $p_t = \alpha q_{a,T-1}$ for some $\alpha > 0$. we know $q_{a,T-1} = \beta q_{bt}$,
$\to$ if we put $p_{T+1} = \alpha\beta q_{aT} \to$
$(\mathbf{p}_T, \mathbf{p}_{T+1}) = (\alpha\beta q_{bT}; \alpha\beta q_{aT}) \propto$
$\propto \mathbf{q}_T = (q_{bT}, q_{aT}) \to y_T$ is π_t max.
for $(\mathbf{p_T}, \mathbf{p_{T+1}})$.
production path is short-run efficient.
separating hyperplane theorem.
limit operation at the horizon $\to +\infty$.

20.D EQUILIBRIUM: THE ONE CONSUMER CASE .

we bring consmpt. and production sides together $\to$ studying equilibrium in an intertemporal setting.
an economy is specified by a short-term production

technology $Y \subset \mathbb{R}^{2L}$, a utility
function $u(.)$ defined on $\mathbb{R}^L_+$,
a discount factor $\delta < 1$, and a bounded
sequence of infinite endowments $\{\omega_t\}_{t=0}^{+\infty}$, $\omega_t \in \mathbb{R}^L_+$.

we assume that Y satisfies hypotheses (i),...,(iv) $\rightarrow$ time impatience, stationarity, additive separability, length of period $\rightarrow$ and that $u(.)$ is strictly concave, differentiable and has strictly positive marginal utilities throughout its domain.

prices are sequences $\{p_t\}_{t=0}^{+\infty}$ with $p_t \in \mathbb{R}^L_+$, prices of a complete system of forward markets occurring simultaneously at $t = 0$ or as the correctly anticipated prices of a spot market.

let's consider only bounded price sequences $\rightarrow ||p_t|| \rightarrow 0$ given a production path $\{y_t\}_{t=0}^{+\infty}$, $y_t \in Y$, the induced stream of consumption $\{c_t\}_{t=0}^{+\infty}$ is given by $c_t = y_{a,t-1} + y_{bt} + \omega_t$

if $c_t \geq 0$, $\forall t$, $\rightarrow \{y_t\}_{t=0}^{+\infty}$ is feasible. $\rightarrow$ sustaining non-negative consmpt. $\forall t$, from now on we restrict all our production paths and consmpt. streams to be bounded given a production path $\{y_t\}_{t=0}^{+\infty}$, and a price sequence $\{p_t\}_{t=0}^{\infty}$ $\rightarrow$ the induced stream of profits $\{\pi_t\}_{t=0}^{+\infty}$is given by:
$\pi_t = p_t \cdot y_{bt} + p_{t+1} \cdot y_{at+1}$ for $\forall t$;
fixing T and rearranging terms of:
$\sum_{t \leq T} p_t \cdot c_t = \sum_{t \leq T} p_t \cdot (y_{a,t-1} + y_{bt} + \omega_t)$
$\rightarrow$ the transversality condition is equivalent to the overall value of consumption not being strictly $>$ wealth.

DEF. 20.D.1: the (bounded) production path $\{y_t\}_{t=0}^{+\infty}$, $y_t^* \in Y$ and the (bounded) price sequence $\{p_t\}_{t=0}^{+\infty}$ constitutes a walrasian (or competitive) equilibrium if:

(i) $c_t^* = y_{b,t-1}^* + y_{a,t}^* + \omega_t \geq 0$

(ii) $\forall t$, $\pi_t = p_t \cdot y_{bt}^* + p_{t+1} \cdot y_{at}^* \geq p_t \cdot y_b + p_{t+1} \cdot y_a$ for all $\mathbf{y} = (y_b, y_a) \in Y$;

(iii) the $\{\mathbf{c}_t\}_{t=0}^{+\infty} \geq 0$ solve the problem $\max_{\mathbf{c}_t} \sum_{t=0}^{+\infty} \delta^t u(\mathbf{c}_t)$ s.t. $\sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \mathbf{c}_t \leq \sum_{t=0}^{+\infty} \pi_t + \sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \omega_t$;

(i) $\rightarrow$ feasibility requirement;

(ii) $\rightarrow$ myopic profit maximization;

(iii) $\rightarrow$ budget constraint $\rightarrow$ comple= teness assmpt. $\rightarrow \exists$ a forward market for every commodity at every date or assets (money, e.g.) are available that are capable of transferring purchasing power across time.

strict monotonicity $\rightarrow$

$\mathbf{w} = \sum_{t=0}^{+\infty} \pi_t + \sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \omega_t < +\infty$.

at the equilibrium consumption, the budget constraint must hold with equality.

PROP. 20.D.1: suppose that the (bounded) production plan $(y_0^*, \dots, y_t^*, \dots)$ and the (bounded) price sequence $(p_0, \dots, p_t, \dots)$ constitute a walrasian equilibrium $\rightarrow$ the tvc $p_{t+1} \cdot y_{at}^*$ holds.

PROOF: $c_t^* = y_{a,t-1}^* + y_{bt}^* + \omega_t$;
$\sum_{t=0}^{T} \{\pi_t + p_t \cdot \omega_t\} - \sum_{t=0}^{T} p_t \cdot c_t = p_{T+1} \cdot y_{aT}$
each sum on the l.h.s. converges to
$\mathbf{w} < +\infty$ as $t \rightarrow +\infty \rightarrow p_{T+1} \cdot y_{aT} \rightarrow 0$. ■

DEF. 20.D.2: the consmpt. stream $\{c_t\}_{t=0}^{+\infty}$ is myopically or short - run utility maximizing in the budget set determined by $\{p_t\}_{t=0}^{+\infty}$ and $\mathbf{w} < +\infty$ if utility cannot be $\nearrow$ by a new consmpt. stream that merely transfers purchasing power between some two consecutive periods.

$\{\mathbf{c}_t\}_{t=0}^{+\infty} >> 0$ is short run utility maximizing for $\{\mathbf{p}_t\}_{t=0}^{+\infty}$ and $\mathbf{w} < +\infty$ if and only if it satisfies $\sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \mathbf{c}_t = \mathbf{w}$ and the FOCs
$\forall t$, $\exists \lambda_t > 0 : \lambda_t \mathbf{p}_t = \nabla u(\mathbf{c}_t)$ and
$\lambda_t \mathbf{p}_{t+1} = \delta \nabla u(\mathbf{c}_{t+1}) \rightarrow \lambda_t \mathbf{p}_t = \nabla u(\mathbf{c}_t)$
and $\lambda_{t-} \mathbf{p}_t = \delta \nabla u(\mathbf{c}_{t+1})$

$\lambda_{t-1} = \delta \lambda_t \rightarrow \lambda_0 = \delta^t \lambda_t$;

$\lambda = \lambda_0 \rightarrow$ for some λ, $\lambda p_t = \delta \nabla u(c_t)$

for all t. (■)

PROP. 20.D.2: if the consmpt. stream $\{\mathbf{c}_t\}_{t=0}^{+\infty}$ satisfies $\sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \mathbf{c}_t = \mathbf{w} < +\infty$ and (■) $\rightarrow$ it's utility maximizing in the budget set determined by $\{\mathbf{p}_t\}_{t=0}^{+\infty}$ and $\mathbf{w}$.

since a walrasian equilibrium is myopically profit maximizing and satisfies the tvc $\rightarrow$ its produc= tion is efficient.

$\max \sum_{t=0}^{+\infty} \delta^t u(c_t)$ s.t. $c_t = y_{a,t-1} + y_{at} + \omega_t \geq 0$ and $y_t \in Y$, $\forall t$. ▲

DEF. 20.D.3: any walrasian equilibrium path $(y_0*, \dots, y_t^*, \dots)$ solves the planning problem ▲.

proof: $\mathcal{B} \rightarrow$ budget set determined by the walrasian equilibrium price sequence $(p_0, \dots, p_t, \dots)$ and wealth
$\mathbf{w} = \sum_{t=0}^{+\infty} \pi_t + \sum_{t=0}^{+\infty} \mathbf{p}_t \cdot \omega_t$, where
$\pi_t = \mathbf{p}_t \cdot \mathbf{y}_{bt}^* + \mathbf{p}_{t+1} \cdot \mathbf{y}_{a,t+1}^*$, for all t.
$\mathcal{B} = \{(c_0', \dots, c_t', \dots) : \mathbf{c}_t' \geq 0$ for all t and
$\sum_{t=0}^{\infty} \mathbf{p}_t \cdot \mathbf{c}_t' \leq \mathbf{w}\} \rightarrow \mathbf{c}_t^* = \mathbf{y}_{a,t-1}^* + \mathbf{y}_{bt}^* +$
$+\omega_t \rightarrow$ is a maximal in the budget set.

$\rightarrow$ any feasibe path $(y_0'', \dots, y_t'', \dots)$,
any path for which $y_t'' \in Y$ and
$\mathbf{c}_t'' = \mathbf{y}_{a,t-1}'' + \mathbf{y}_{bt}'' + \omega_t \geq 0$, $\forall t$
must yield a consmpt. stream
in $\mathcal{B}$. $\forall T$,

$\sum_{t\leq T} \mathbf{p}_t \cdot \mathbf{c}''_t = \sum_{t\leq T-1}(\mathbf{p}_t \cdot \mathbf{y}''_{bt} + \mathbf{p}_{t+1} \cdot \mathbf{y}''_{at})+$
$+\mathbf{p}_t \cdot \mathbf{y}''_{bT} + \sum_{t\leq T} \mathbf{p}_t\omega_t$
by the chance of truncation of consmpt. plans → we've $(\mathbf{y}''_{bt}, 0) \in Y$.
by short - run profit maximization,
$\mathbf{p}_t \cdot \mathbf{y}''_{bt} \leq \pi_t$ and $\mathbf{p}_t \cdot \mathbf{y}''_{bt} + \mathbf{p}_{t+1} \cdot \mathbf{y}''_{at} \leq$
$\leq \pi_t$ for all $t \leq T-1$. ■

PROP. 20.D.4: suppose the bounded path $(y^*_0, \dots, y^*_t, \dots)$ solves the planning pro= blem ▲ and that it yields strictly positive consmpt. (for some $\epsilon > 0$), $c_{lt} = y^*_{l,a,t-1} + y^*_{lbt} + \omega_{lt} > \epsilon$ for $\forall l, t$.
→ the path is a walrasian equilibrium w.r.t. some price sequence $(p_0, \dots, p_t, \dots)$.

PROP. 20.D.5: suppose there is a uniform bound on the consmpt. streams generated by all feasible paths.
→ ▲ attains a maximum → ∃ a feasible path that yields utility at least as large as the utility corresponding to any other feasible path.

PROP. 20.D.6: the planning problem has at most one consmpt. stream solution.

PROOF: maximum of a strictly concave function in a convex set is unique → topology.

$\sum_{t=0}^{+\infty} \delta^t u(\mathbf{c}_t) = \sum_{t=0}^{+\infty} \delta^t u(\mathbf{c}'_t) = \gamma$

$\{\mathbf{c}_t\}_{t=0}^{+\infty}$ and $\{\mathbf{c}'_t\}_{t=0}^{+\infty}$

$\mathbf{y}''_t = \frac{\mathbf{y}_2}{2} + \frac{\mathbf{y}'_t}{2}$

$\mathbf{c}''_t = \frac{\mathbf{c}_2}{2} + \frac{\mathbf{c}'_t}{2}$

$\sum_{t=0}^{+\infty} \delta^t u(c''_t) \geq \gamma$

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COMPUTATION OF EQUILIBRIUM and EULER EQUATIONS

$(N+1)$-sector model.
N → capital goods; labour and a consmpt. good; labour endowment → fixed through time → a function $G(k, k')$ gives the total amount of consmpt. gottable at any t if the invest. in capital at $t-1$ is $k \in \mathbb{R}^N$ → the investment at t is required to be $k' \in \mathbb{R}^N_+$ fixed at the level endogenously given by initial endowment →
$A \subset \mathbb{R}^N \times \mathbb{R}^N$ → region of pairs

$(k, k') \in \mathbb{R}^{2N}$ compatible w/non-
negative consmpt.
$u(G(k, k')) = u(k, k')$
$A \to$ convex; $u(.) \to$
strictly concave.
$u(k_{-1}, k_0) = 0 \to$ initial value.
$\max \sum_{t=0}^{+\infty} \delta^t u(k_{t-1}, k_t)$ s.t. **
$(k_{t-1}; k_t) \in A$, $\forall t$, $k_0 = \bar{k}_0$
concavity of $u(.) \to$ uniqueness
of the solution.
$\forall t \geq 1 \to k_t \in \mathbb{R}^N \to \delta^t u(k_{t-1}, k_t) +$
$+\delta^{t+1} u(k_t, k_{t+1}) \to \{k_t\}_{t=0}^{+\infty} \to$
$\frac{\partial u(k_{t-1}, k_t)}{\partial k'_n} + \delta \frac{\partial u(k_t, k_{t+1})}{\partial k_n} = 0$, $\forall n \leq N$, $\forall t \geq 1$
$\nabla_2 u(k_{t-1}, k_t) + \delta \nabla_1 u(k_t, k_{t+1}) = 0$, $\forall t \geq 1$.
$\to$ vector notation.
$\to$ euler equation of ▲.

RAMSEY SOLOW MODEL

$L_t = 1$, $\forall t$; $u(k, k') = u[F(k) - k']$
$A = \{(k, k') : k' \leq F(k)\} \to$ E.E.:
$-u'(F(k_{t-1}) - k_t) + \delta u'(F(k_t) - k_{t+1}) \cdot$
$\cdot F'(k_t) = 0$, $\forall t \geq 1 \leftrightarrow \frac{u'(c_t)}{\delta u'(c_{t+1})} = F(k_t)$, $\forall t \geq 1$.

COST OF ADJUSTMENT TECHNOLOGY

$L_t = 1 \to$ normalized
$\max \sum_{t=0}^{+\infty} \big[F(k_{t-1}) - k_t - \gamma(k_t - k_{t-1})\big]$
E.E. $\to -1 - \gamma'(k_t - k_{t-1}) + \delta\big[F'(k_t) + \gamma'(k_{t+1} - k_t)\big] = 0$
for all $t \geq 1 \to$ the marginal cost of a
unit of investment in capacity at t
equals the discounted value of the mar=
ginal product of capacity at t + the
marginal savings in the cost of capa=
city expansion at $t + 1 \to$ iterating from $t + 1$
$1 + \gamma'(k_1 - k_0) = \sum_{t \geq 1} \delta^t \big(F'(k_t) - 1\big)$
$\to$ the q - theory of investment. ■

a path $(k_0, \dots, k_t, \dots) \leftrightarrow \{k_t\}_{t=0}^{+\infty}$
$\to$ satisfying the euler necessary
equations $\to$ from them and the
concavity of $u(.,.) \to$ the euler
equations are also sufficient to
guarantee that the trajectory
cannot be improved upon.
euler equations $\to$ necessary
and sufficient for short - run
optimization.

for long - run optimization
they require a regularity
property on the path.
the path $\{k_0, \dots, k_t, \dots\}$
is strictly interior if

it stays strictly away
from the boundary of the
admissible region $A \rightarrow$ if there $\exists \epsilon > 0$
such that for every t, $\exists$ an ϵ - neigh=
bourhood of (k_t, k_{t+1}) entirely contained
in A.

PROP. 20.D.7: suppose that the path
$(\bar{k}_0, \ldots, \bar{k}_t, \ldots)$ is bounded, is strictly
interior and satisfies the euler eqs. $\rightarrow$
$\rightarrow$ it solves the intertemporal
optimization problem of the
representative consumer.

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POLICY FUNCTION AND VALUE FUNCTION

$\psi(k)$ and $V(k)$; given an initialì condition $k_0 = k$,
the max. value attained by ** is denoted $V(k)$,
and, if $(k_0, k_1, \ldots, k_t, \ldots)$ is the unique trajectory
solving **, with $k_0 = k$, then we put $\psi(k) = k_1$;
$\rightarrow \psi(k) \in \mathbb{R}^N$ is the vector of
optimal levels of investment $\rightarrow$
of capital; at $t = 1$, when the levels
of capital at $t = 0$ are k.

if the path $\{\hat{k}_t\}_{t=0}^{+\infty}$ solves **
for $k_0 = \hat{k}_0 \rightarrow$, $\forall T$,the path
$(\hat{k}_T, \ldots, \hat{k}_{T+t}, \ldots)$ solves ** for $k_0 = \hat{k}_T$.
if $(k_0, \ldots, k_t, \ldots)$ solves ** $\rightarrow$

$k_{t+1} = \psi(k_t)$, $\forall t = 0, \ldots, +\infty$

$\rightarrow$ the optimal path can be computed
from knowledge of k_0 and the policy
function $\psi(.)$. but how to find $\psi(.)$?

1. EULER EQUATIONS $\rightarrow$ iterative procedure
for finding $\psi(k)$; fix $k_0 = k$ and consider
the equation corresponding to k_1;
with k_0 given, we've N equations
in the $2N$ unknowns $k_1 \in \mathbb{R}^N$ and
$k_2 \in \mathbb{R}^N \rightarrow$ there are N degrees of
freedom. we try to fix k_1 arbitrarily
and we use then the N euler equations
at $t = 1$ to solve for the remaining
k_2 unknowns.
suppose that such a solution
k_2 is found by the strict concavity
of $u(.) \rightarrow$ a solution $\exists$ and is unique;
we can repeat the process $\rightarrow$
the N euler equations for period 2
are exactly determined, now $\rightarrow$
both k_1 and k_2 are given, but we've
still the N variables k_3 correspon=

ding to $t = 3 \to$ with which we can try to satisfy the N equations of $t = 2 \to$ three possibilities $\to$
a. the process breaks down some= where $\to$ given k_{t-1} and k_t, there is no solution $k_{t+1} \leftrightarrow (k_t, k_{t+1}) \in A$
; b. we generate an unbounded sequence;
c. we generate a bounded and strictly interior solution $\to$ sequence $(k_0, k_1, \dots, k_t, \dots)$.

in case c., we've found an optimum $\to$ $\to$ optimum is unique $\to$ given k_0, the third possibility (trajectory starting at k_0, with k_1 interior and bounded) $\to$can occur for at most one value of k_1. if it occurs $\to$ this value is precisely $\psi(k_0)$.

$\to$ solve the difference equation $\to$ w/initial condition (k_0, k_1) and for fixed k_0, search for an initial condition k_1 $\to$ generating a bound= ded infinite path.

RAMSEY - SOLOW MODEL

$F(k) = 2k \to$ linear technology;
$u(c_t) = \sum_{t=0}^{+\infty} \left(\frac{1}{2}\right)^t \ln c_t \to$ utility;
$u(k_{t-1}, k_t) = \ln(2k_{t-1} - k_t)$
$k_{t+1} = 3k_t - 2k_{t-1}$.
$\to k_t = k_0 + (k_1 - k_0)(2^t - 1) \to$ solution

if $k_1 > k_0 \to k_t$ is unbounded $\to$ the only value is $k_1 = k_0$.
$\to c_t = 2k_{t-1} - k_t = 2k_0 - k_1$;
for $k_1 > k_0 \to$ path compatible with the euler equation but not optimal. ■

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2. DYNAMIC PROGRAMMING

recursivity of the optimum problem.
■ $V(K) = \max_{K' \text{with} (K,K') \in A} u(K, K') + \delta V(K')$
and obtains $\psi(K)$, the vector K' that solves ■;
now the problem is to find the value funct. $V(.)$.
under some general conditions $\to$ (if $V(.)$ is bounded) $\to$ the value funct. is the only funct. that solves ■
$\to$ when viewed as a functional equation
$\to V(.)$ is the onkly functional for which
■ is true $\forall K$ and that $\exists$ some well - known and quite effective algorithms for solving such equations for $V(.)$.

(i) the value funct. $V(K)$ is concave;
(ii) $\forall$ perturbation parameter $z \in \mathbb{R}^N$
with $(k+z, \psi(k)) \in A$, we've
$V(k+z) \geq u(k+z, \psi(k)) + \delta V(\psi(k))$
$N = 1$ and $(k, \psi(k))$ is interior to A.
$V(k) = u(k, \psi(k)) + \delta V(\psi(k))$.
$V'(k) = \nabla_1 u(k, \psi(k))$ and,
$V''(k) \geq \nabla^2_{11} u(k, \psi(k))$.

***

$k_1 > k_0 \rightarrow$ capital over accumulation.
along the optimal path the value function is maximized by the utility of a single period adjustments.

***

M.M LINEAR PROGRAMMING $\rightarrow$ special cases of constrained optimization problems. constraints and objective functions are linear $\rightarrow$

$$\underbrace{\mathbf{x}}_{N \times 1} = (x_1, \ldots, x_N)'$$

$$\max_{\{\underbrace{\mathbf{x}}_{N \times 1}\}} f_1 x_1 + \cdots + f_N x_N \quad \text{s.t.}$$

$$\begin{array}{l} a_{11}x_1 + \cdots + a_{1N}x_N \leq c_1 \\ \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{k1}x_1 + \cdots + a_{kN}x_N \leq c_k \end{array}$$

$$\leftrightarrow \max_{\{\mathbf{x} \in \mathbb{R}^N_+\}} \quad \underbrace{\mathbf{f}}_{k \times N} \cdot \underbrace{\mathbf{x}}_{N \times 1} \quad \text{s.t.} \quad \underbrace{\mathbf{A}}_{k \times N} \cdot \underbrace{\mathbf{x}}_{N \times 1} \leq \underbrace{\mathbf{c}}_{k \times 1}$$

dual problem $\rightarrow$ minimization problem with k variables and N constraints:

$$\min_{\{\underbrace{\lambda}_{k \times 1} \geq 0\}} c_1 \lambda_1 + \cdots + c_k \lambda_k \quad \text{s.t.}$$

$$\begin{array}{l} a_{11}\lambda_1 + \cdots + a_{k1}\lambda_k \geq f_1 \\ \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{1N}\lambda_1 + \cdots + a_{kN}\lambda_k \geq f_N \end{array}$$

$$\leftrightarrow \min \underbrace{\mathbf{c}}_{N \times k} \cdot \underbrace{\lambda}_{k \times 1} \quad \text{s.t.} \quad \underbrace{\mathbf{A}'}_{N \times k} \cdot \underbrace{\lambda}_{k \times 1} \geq \underbrace{\mathbf{f}}_{N \times 1}$$

$\underbrace{\mathbf{x}^*}_{N \times 1} \rightarrow$ satisfies $\bullet$;

$\underbrace{\lambda^*}_{k \times 1} \rightarrow$ satisfies $\blacktriangledown$

$$\mathbf{f} \cdot \mathbf{x}^* \leq (\mathbf{A}' \cdot \lambda^*) \cdot \mathbf{x}^* =$$

$$= \underbrace{\lambda}_{k \times 1} \cdot (\underbrace{\mathbf{A}}_{k \times N} \cdot \underbrace{x}_{N \times 1}) \leq \lambda \cdot \mathbf{c} = \mathbf{c} \cdot \lambda$$

the solution to the primal problem can be no larger than the solution to the dual problem.

$\rightarrow$ duality theorem of linear programming
$\rightarrow (\lambda_1, \ldots, \lambda_k) \rightarrow$ langrange multipliers.

THEO. M.M.1: suppose the primal problem
• attains a maximum at $v \in \mathbb{R}$; v is also
the minimum value attained by
the dual problem ▼.

M.N. DYNAMIC PROGRAMMING.

$\rightarrow \{x_t\}_{t=0}^{+\infty} \rightarrow$ optimality problems.
stokey and lucas, 1989. $u : A \times A \rightarrow \mathbb{R}$
a continuous function, over a non-empty,
compact set $A \subset \mathbb{R}^N$.

$\delta \in (0, 1)$; $\mathbf{z} \in A \rightarrow$ a vector of initial
conditions for $\{x_t\}_{t=0}^{+\infty}$.

$\max_{\{x_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \delta^t u(x_t, x_{t+1})$ s.t.
$\mathbf{x}_t \in \mathbf{A}$, $\forall t = 0, 1, \ldots, +\infty$
$\mathbf{x}_0 = \mathbf{z}$

a maximizer sequence $\exists \rightarrow \exists$ a max
for $v(\mathbf{z})$; $v : A \rightarrow \mathbb{R}$ is the value
function of * $\rightarrow$ continuous.
A is convex and $u(.)$ is concave.
$\rightarrow v(.)$ is also concave.

$\forall \mathbf{z} \in A \rightarrow$ Bellman equation
$v(\mathbf{z}) = \max_{\mathbf{z}' \in A} u(\mathbf{z}, \mathbf{z}') + \delta v(\mathbf{z}')$
$\rightarrow$ the value funct. is the only
maximizer.

THEO. M.N.1: $f : A \rightarrow \mathbb{R}$ is a continuous
function $\rightarrow$ s.t. $\forall \mathbf{z} \in A$, the Bellman equation
is fulfilled $\rightarrow f(\mathbf{z}) = \max_{\mathbf{z}' \in A} u(\mathbf{z}, \mathbf{z}') + \delta f(\mathbf{z}')$, $\forall \mathbf{z} \in A$
$f(.) \equiv v(.) \leftrightarrow f(\mathbf{z}) = v(\mathbf{z})$, $\forall \mathbf{z} \in A$.
$f_0 : A \rightarrow \mathbb{R}$; $f_0(\mathbf{z}') :=$ trial evaluation
function, then try $f_1(.)$;

$f_1(\mathbf{z}) = \max_{\mathbf{z}' \in A} u(\mathbf{z}, \mathbf{z}') + \delta f_0(\mathbf{z}')$
if $f_0(.) = f_1(.) \rightarrow f_0(.)$ satisfies the
bellman equation $\rightarrow f_0(.) = v_0(.)$.

$\{f_r(s)\}_{r=0}^{+\infty} \rightarrow$ sequence of functions.
$\{\bar{x}_t\}_{t=0}^{+\infty}$ is a trajectory solving the * $\rightarrow$
$\forall t \geq 1$, the decisions taken at t must be optimal;
$\max_{x_t \in A} u(\bar{x}_{t-1}, x_t) + \delta u(x_t, \bar{x}_{t+1})$

$\rightarrow \frac{\partial u(\bar{x}_{t-1}, \bar{x}_t)}{\partial x_{N+n}} + \delta \frac{\partial u(\bar{x}_t, \bar{x}_{t+1})}{\partial x_n} = 0$;
$\forall n = 1, \ldots, N \rightarrow u(., .)$ has $2N$ argu=
ments; the N variables of the initial period
$+N$ of the subsequent. $\rightarrow$ euler equations of *. ■

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20. STATIONARY PATHS, INT. RATES AND GOLDEN RULES

study of steady states → bliss 1975, gale 1973.

DEF. 20.E.1: a production path $(y_0, \ldots, y_t, \ldots)$ is stationary or a steady state of there is a production plan $\bar{y} = (\bar{y}_b, \bar{y}_a)$ such that $y_t = \bar{y}$ for all $t \geq 0$.
stationary path → $(\bar{y}_0, \ldots, \bar{y}_t, \ldots)$ → ≪the stationary path $\bar{y}$≫→ ∃ a price vector $\mathbf{p}_0 \in \mathbb{R}^L$ and an $\alpha \geq 0$ such that the path is myopically profit maximizing for the price sequence $(\mathbf{p}_0, \alpha\mathbf{p}_0, \ldots, \alpha^t\mathbf{p}_o)$.

DEF. 20.E.2: suppose the stationary path $(\bar{y}, \ldots, \bar{y}_t, \ldots)$, $\bar{y} \in Y$ is myopically supported by proportional prices with rate of interest r, then the path is efficient if $r > 0$ and ineff=ficient is $r < 0$.

$r = 0$ → stationary equilibrium paths → differentiability and interiority → and stationarity → path $\{\bar{y}\}_{t=0}^{+\infty}$ that's also and equilibrium → can be supported only by the price sequence $p_t = \delta^t \nabla u(\bar{c})$

where $\bar{c} = \bar{y}_b + \bar{y}_a$;

a stationary equilibrium is supported by a price sequence embodying a proportionality factor equal to the discount factor δ or with rate of interest $r = \frac{1-\delta}{\delta}$.

***

PROP. 20.E.2: a stationary production path that is myopically supported by proportional prices $p_t = \alpha^t p_0$ with $\alpha = \delta$, is called a modified golden rule path → a stationary production path myopically supported by constant prices $p_t = p_0$ is called a golden rule path.

depending on the technology and discount factor δ →

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REGRESSION DISCONTINUITY DESIGN FOR FRB's TRADEMARK valuation

basic discontinuous regression framework[19] $\rightarrow$

$$\underbrace{\mathbf{y}_t}_{(T\times 1)} = \underbrace{\alpha_t}_{(T\times 1)} + \underbrace{\beta_t}_{(1\times T)} \times \underbrace{\mathbf{D}_t}_{(T\times 1)} + \underbrace{\gamma_t}_{(3\times T)} \times \underbrace{\mathbf{X}_t}_{(T\times 3)} + \underbrace{\epsilon_t}_{(T\times 1)}$$

$$\underbrace{\mathbf{y}_t}_{T\times 1} = \begin{pmatrix} ricavi_{1998} \\ ricavi_{1999} \\ \vdots \\ ricavi_{2020} \end{pmatrix}; \quad \underbrace{\mathbf{D}}_{T\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \quad \rightarrow \mathbf{D} = \left\{ \begin{array}{ll} 0 & \text{if no trademark registered;} \\ 1 & \text{if regist. trademark;} \end{array} \right]$$

$$\underbrace{\mathbf{X}}_{T\times 3} = \begin{pmatrix} immobilizz_{1998} & costi\_sviluppo_{1998} & spese\_legali_{1998} \\ \vdots & \vdots & \vdots \\ immobilizz_{2020} & costi\_sviluppo_{2020} & spese\_legali_{2020} \end{pmatrix}$$

$\mathbf{D}_t \rightarrow$ dummy for treatment status; or $D \in [0;1] \rightarrow$ with each valiue of $D = 1 \rightarrow$ registration of trademark in a given state.

[19] the cutoff point is the time the trademark was registered; given that the trademark has been registered in multiple countries $\rightarrow$ we should adopt a framework for RDD with multiple cutoff rules, one per region of registration $\rightarrow$ e.g. Italy, the European Union, United States, Canada, Brazil, Argentina, China, India, Israel, etc.