Diversification benefits of commodities in portfolio allocation: A dynamic factor copula approach

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Abstract

This study provides a thorough analysis of the dynamics of volatility and dependence between seven international equity and 20 commodity markets across different sectors, highlighting the hedging role played by the latter. We explain volatility using a specification that distinguishes between the short and long term, while the dynamics of the dependence structure, or copula, are modeled by means of a latent factor structure, which can be split into commodity sectors such that there is homogeneous dependence within each sector. The dynamic of both models is captured through a score-driven specification. Moreover, we solve the risk aversion portfolio optimization problem to determine the existence of diversification benefits when constructing portfolios made up of a mix of commodities and stock markets. The main results of the study show that the dependence between the commodity and equity markets is variable over time and that the diversification potential of commodity markets is limited. Further, the factor copula approach is the best specification in terms of Sharpe ratio independent of portfolio settings for the different rebalancing periods.

Keywords: volatility; commodity markets; dynamic factor copula; dependence; portfolio optimization

1. Introduction

The increasing globalization of the financial system, together with the constant fluctuations in the international economy, has generated an increase in the levels of volatility and interdependence in the different stock markets. Modern portfolio theory suggests that individuals can benefit from portfolio diversification by choosing financial instruments with low dependence or correlation (Boubaker and Sghaier, 2013; Berger and Uddin, 2016; Wen at al. 2021). From this perspective, commodity markets represent a promising asset class through which to achieve such diversification effects, depending on the degree of integration of these markets (Aloui et al., 2013; Hammoudeh et al., 2014; Creti, et al., 2013; Daskalaki and Skiadopoulos, 2011; Shahzad et al. 2019; Cotter et al. 2017; Gagnon et al. 2020). In particular, recent research shows that the co-movement between stocks and commodities from different sectors began to increase after the subprime crisis, a result that suggests growing integration between these markets (Wen et al., 2012; Delatte and Lopez, 2013; Han et al., 2022). However, many of these co-movements are attributed to technological changes, production, and transportation costs, in conjunction with the growing demand for renewable energy (Choi and Hammoudeh, 2010; Sadorsky, 2012; Kang et al., 2017; Ji et al., 2018; Han and Li, 2022).

The most recent literature on this topic uses copula theory to show the increasing co-movement between the commodity and equity markets (Ojeda, 2020; Tiwari et al. 2020, Aepli et al. 2017). For instance, Delatte et al. (2013) state that, already by 2003, this joint co-movement had extended to all classes of commodities and became unequivocally stronger after the subprime crisis. A similar result was presented by Creti et al. (2013), who found strong time-varying correlations between commodity and stock markets, mainly after the subprime crisis. However, they also found evidence for the existence of commodities that offer an alternative means of achieving the benefits of diversification, even in times of financial turbulence. Hammoudeh et al. (2014) also describe opportunities involving the use of commodities to achieve the benefits of portfolio diversification. They show that the incorporation of petrochemical and grain commodities into a portfolio provides risk management strategies since their correlations with the stock market increase during periods of rise and decline in stock prices in bearish financial markets. However, none of these studies simultaneously consider the joint analysis of all these markets.

Recent literature shows the significant changes which have taken place in financial risk modeling in commodity markets as a result of the search for econometric specifications capable of jointly modeling this type of asset to understand the interdependence between the different sectors and markets, for the delivery of information useful for risk management in commodity-based portfolios (Ohashi and Okimoto, 2016; Albulescu et al. 2020; Vedenov and Power, 2022). However, it was found that elucidating the dependence structure between a large number of financial assets is a complex task, mainly due to the problem of dimensionality (Oh and Patton, 2017). Recently, Oh and Patton (2018) introduced a factor copula model with score-driven dynamics whose main advantage is its flexibility in modeling the joint dynamics of financial time series by means of a common dynamic factor.

This research aims to study the volatility and dependence structure existing among a set of commodity and stock market indices to determine the existence of benefits in the diversification and management of financial risk derived from a mixed investment. In the case of the commodities, we utilize the individual components of the S&P Goldman Sachs Commodity Index (GSCI), which is broadly diversified across the spectrum of commodity markets. The studied commodities are grouped into the energy, precious and industrial metals, and soft (agriculture and livestock) sectors. Moreover, we utilize seven equity indices belonging to the Morgan Stanley Capital International

(MSCI) World Index. These indices represent the performance of the broad equity universes of individual countries with a strong emphasis on index liquidity and investability (Hung and Shiu, 2016).

The contribution of this research is threefold: i) We analyze each of the commodities and stock markets' volatilities, investigating whether there are similarities between each of the groups; ii) We jointly model the dependence structure between these markets through a dynamic factor copula model, analyzing whether the latent factors that capture the dependence between the different markets can be associated with commodity or stock market sectors; iii) Using the results of previous estimations, we solve a mean-variance portfolio optimization problem with risk aversion to determine the existence of diversification benefits when building portfolios composed of a combination of commodities and equities.

The methodologies utilized to capture the volatility and dependence among the return series are the Beta-Skew-t-EGARCH introduced by Harvey and Sucarrat (2014) and a dynamic factor copula model proposed by Oh and Patton (2018). The main characteristic of both models is that their dynamics are obtained through the score of the conditional log-likelihood. Thus, the predictive and updated mechanisms are computed iteratively through a simple forward recursion function.

The results show short- and long-term patterns for the different markets analyzed in terms of estimated volatility. The most volatile sectors are the energy commodity and stock markets, although these are also the most homogeneous. Related to the degree of dependence between each of the sectors, we observe that a factor copula model with a heterogeneous structure seems to be the most appropriate. We observethat the diversification potential of commodity markets is limited compared to a portfolio composed solely of equity markets. However, we observe that soft commodities offer an attractive alternative means of achieving portfolio diversification effects, for long rebalancing periods of time and low risk aversion, given their weaker relationship with the latent common factor. Similarly, in the minimum variance portfolio the best strategy is obtained by incorporating a mix of precious metal commodities into the stock market portfolio. By contrast, the energy commodity and stock markets exhibit a more significant dependence.

The remainder of this paper is organized as follows. Section 2 introduces the methodology considered to estimate volatility and the factor copula model. Section 3 describes the data and provides the empirical results and analyses. Finally, Section 4 concludes the paper.

2. Methodology

We divide the methodology into three stages. First, we capture the volatility of equities and commodities through the Beta-Skew-t-EGARCH model (Harvey and Sucarrat, 2014), whose main advantage is that it allows us to decompose volatility into short- and long-term components. In the second stage, we capture the dependence between the marginal distributions using the dynamic factor copula model proposed by Oh and Patton (2018). This approach makes it possible to reduce the complexity of modeling the dependence among marginals in high dimensions through a single common latent factor. Finally, we solve the mean-variance portfolio optimization problem based on the marginals' specifications and the factor copula model introduced in the two previous stages.

2.1 Marginal distribution model

Let y_t denote the daily return on an asset at time t. The Beta-Skew-t-EGARCH specification is defined as follows:

$$y_t = \exp(\lambda_t)\,\varepsilon_t = \sigma_t\varepsilon_t, \qquad \varepsilon_t \sim st(0,\sigma_\varepsilon^2,\nu,\gamma), \quad \nu > 2, \quad \gamma \in (0,\infty), \tag{1}$$

$$\lambda_t = \omega + \lambda_{1,t}^{\dagger} + \lambda_{2,t}^{\dagger},$$

$$\lambda_{1,t}^{\dagger} = \phi_1\lambda_{1,t-1}^{\dagger} + k_1 u_{t-1}, \quad |\phi_1| < 1,$$

$$\lambda_{2,t}^{\dagger} = \phi_2\lambda_{2,t-1}^{\dagger} + k_2 u_{t-1} + k^{*}\mathrm{sgn}(-y_{t-1})(u_{t-1}+1), \quad |\phi_2| < 1, \quad \phi_1 \neq \phi_2$$

where σ_t is the conditional volatility and $\varepsilon_t = \varepsilon_t^{*} - \mu_\varepsilon^{*}$ is the conditional error. Both ε_t and ε_t^{*} have a marginal skew-t distribution, with scale parameters σ_ε^2, degrees of freedom ν, and asymmetry parameter γ. However, ε_t and ε_t^{*} have a mean value equal to zero and μ_ε^{*}, respectively. In particular, the probability density function of a non-centered skew-t random variable is as follows:

$$f(\varepsilon_t^*|\gamma) = \frac{2}{\gamma + \gamma^{-1}} f\left(\frac{\varepsilon_t^*}{\gamma^{\text{sgn}(\varepsilon_t^*)}}\right) \tag{2}$$

The parameters ϕ_1 and k_1 capture the dynamics of the short-term log-volatility, while ϕ_2 and k_2 capture the dynamics of the long-term log-volatility. Finally, k^* is the leverage parameter and the conditional score u_t is given by:

$$u_t = \frac{\left(\varepsilon_t^{*2} - \mu_\varepsilon^* \varepsilon_t^*\right)(\nu + 1)}{\varepsilon_t^{*2} + \nu\gamma^{2\text{sgn}(\varepsilon_t^*)}} - 1. \tag{3}$$

Once the volatility has been modeled and the standardized residuals of each series of returns have been determined, we apply the factor copula model with score-driven dynamics.

2.2 Factor copula model with score-driven dynamics

Let $\boldsymbol{Y_t} = (Y_{1t}, \dots, Y_{Nt})$ be a set of random variables of dimension N with joint distribution function $\boldsymbol{F_t}(y_1, \dots, y_N|\boldsymbol{\mathcal{H}_t})$ and conditional copula $\boldsymbol{C_t}(F_{1t}(y_1|\boldsymbol{\mathcal{H}_t}), \dots, F_{Nt}(y_N|\boldsymbol{\mathcal{H}_t})|\boldsymbol{\mathcal{H}_t})$**,** where $\boldsymbol{\mathcal{H}_t} = \{Y_{1j}, \dots, Y_{Nj} \,:\, \forall\, j < t\}$ represents the information set relating to the history of the stochastic process. On the other hand, we assume that there exists a set of latent random variables $\boldsymbol{X_t} = (X_{1t}, \dots, X_{Nt})$, where each marginal follows the following common factor model:

$$X_{it} = \delta_{it}(\theta_\delta)Z_t + \varepsilon_{it}, \qquad i = 1, 2, \dots, N \tag{4}$$

where $\delta_{it}(\theta_{\boldsymbol{\delta}})$ is a factor loading associated with the common factor in time t, $Z_t \sim F_{Z_t}(\theta_Z)$ and $\varepsilon_{it} \sim F_{\varepsilon_i}(\theta_\varepsilon)$ are univariate cumulative probability distributions of the common factor and idiosyncratic variables, respectively, and $Z_t \perp \varepsilon_i$. In the empirical analysis, we consider that Z_t follows a skew-t-Student distribution with degrees of freedom $\nu_Z \in (2, \infty]$ and a skew parameter

$\psi_Z \in (-1,1)$, whereas ε_{it} uses a t-Student distribution with degrees of freedom $\nu_\varepsilon \in (2,\infty]$. Finally, $\theta_{\boldsymbol{\delta}}, \theta_\varepsilon$, and θ_Z are parameter vectors involved in the dynamics of the common factor and the probability distribution functions, respectively.

The most important assumption in the factor copula specification is that both the observed ($\boldsymbol{Y_t}$) and the latent ($\boldsymbol{X_t}$) sets of random variables share the same copula function, while the marginal distributions need not be the same. This assumption allows high flexibility in high dimensions, while it does not involve the estimation of a high number of parameters.

We define $U_{it} = G_{it}(x_i)$ and $X_{it} = G_{it}^{\leftarrow}(u_i)$ as the conditional probability integral transformation and its inverse. Then, the density function of the copula of the latent variable $\boldsymbol{X_t}$ is given by:

$$\boldsymbol{c}(u_1, \dots, u_N) = \frac{\boldsymbol{g_t}\left(F_{1t}^{\leftarrow}(u_1), \dots, F_{Nt}^{\leftarrow}(u_N)\right)}{g_{1t}\left(F_{1t}^{\leftarrow}(u_1)\right) \times \cdots \times \; g_{Nt}\left(F_{Nt}^{\leftarrow}(u_N)\right)} \tag{5}$$

where $\boldsymbol{g_t}(x_1, \dots, x_N)$ and $g_{it}(x_i)$ are the joint and marginal density functions of $\boldsymbol{X_t}$, respectively. Using the property of independence between the random variables Z_t and ε_{it}, Oh and Patton (2018) show that the complexity of estimating a factor copula in N dimensions is reduced to numerically solving a pitatory and integral in one dimension as follows:

$$g_{it}(x_i) = \int_0^1 f_{\varepsilon_i}\left(x_i - \delta_{it} F_{Z_t}^{-1}(m)\right) dm \tag{6}$$

$$G_{it}(x_i) = \int_0^1 F_{\varepsilon_i}\left(x_i - \delta_{it} F_{Z_t}^{-1}(m)\right) dm \tag{7}$$

$$\boldsymbol{g_t}(x_1, \dots, x_N) = \int_0^1 \prod_{i=1}^{N} f_{\varepsilon_i}\left(x_i - \delta_{it} F_{Z_t}^{-1}(m)\right) dm \tag{8}$$

where $m \equiv F_{Z_t}(z)$ is a change in variable used to obtain bounded integrals (for further details, see the Appendix in Oh and Patton, 2018). Finally, the only parameter that has not yet been defined is the dynamic component associated with the common factor. Oh and Patton (2018) incorporate a dynamic structure into the factor loading δ_{it} using a score-driven specification:

$$\ln \delta_{i,t} = \omega_i + \beta \ln \delta_{i,t-1} + \alpha s_{i,t-1} \qquad i = 1,2,\dots,N \tag{9}$$

where $s_{it} = \partial \ln \boldsymbol{c}(u_1,\dots,u_N;\ \delta_{it}, \nu_z, \psi_z, \nu_\varepsilon)/\partial\, \delta_{it}$ is the score of the log-observation copula density, which is obtained through numerical differentiation; ω_i is a vector of coefficients of length N, and α and β are unrestricted parameters. Note that the total number of parameters for this specification is $N+2$. Alternatively, if we assume that the stochastic process δ_{it} is strictly stationary, Eq.(9) can be reduced to:

$$\ln \delta_{i,t} = \mathbb{E}[\ln \delta_{i,t}](1-\beta) + \beta \ln \delta_{i,t-1} + \alpha s_{i,t-1}, \qquad i = 1,2,\dots,N \tag{10}$$

where $\mathbb{E}[\ln \delta_{i,t}]$ can be previously estimated in nonparametric form as follows:

$$\widehat{\mathbb{E}}[\ln \boldsymbol{\delta_t}] = \mathop{\mathbf{arg\,min}}_{\boldsymbol{a}} \boldsymbol{m_T}(\boldsymbol{a})' \boldsymbol{m_T}(\boldsymbol{a}) \tag{11}$$

with $m_T(\boldsymbol{a}) = \text{vech}\{\phi(G(\exp a))\} - \widehat{\boldsymbol{\rho}}_{\boldsymbol{u}}^{\boldsymbol{s}}$; $\widehat{\boldsymbol{\rho}}_{\boldsymbol{u}}^{\boldsymbol{s}}$ is the Spearman correlation matrix estimated from $\boldsymbol{u} = (u_1,\dots,u_N)$; and $\phi(G(\delta)) = \boldsymbol{\rho_x}$ is the half-vectorization of the Spearman correlation matrix for the set of variables $\boldsymbol{X_t} = (X_{1t},\dots,X_{Nt})$ given by Eq. (4) with factor loadings δ_{it}. Replacing these estimates in Eq. (9), the dynamics of the factor copula model with score-driven dynamics are fully described through a set of only two parameters $\theta_{\boldsymbol{\delta}} = [\beta, \alpha]'$, independent of the number of dimensions.

2.3 Portfolio optimization

In the empirical analysis, we consider the portfolio selection problem by applying a dynamic version of the portfolio theory of Markowitz (1952), with the results of the estimates of the Beta-Skew-t-EGARCH model for the marginals and the factor copula model with score-driven dynamics. To this end, we consider a mean-variance portfolio optimization problem with risk aversion:

$$\arg\max_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{r} - \xi \boldsymbol{w}'\boldsymbol{Q}\boldsymbol{w} \qquad s.t. \ \sum_i w_i = 1, \ \ w_i \geq 0 \tag{12}$$

where $\boldsymbol{r}$ is the return vector, $\boldsymbol{w}$ is the vector of portfolio holdings or weights, $\boldsymbol{Q}$ is the covariance matrix, and ξ is a risk aversion parameter that specifies a trade-off between risk and expected return, according to the degree of aversion of the investor. As ξ increases from 0 to ∞, the optimal mean-variance portfolio moves along the return-variance efficient frontier from the maximum return portfolio, through the maximum Sharpe ratio portfolio, to the minimum risk portfolio, which is specified as follows:

$$\arg\min_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{Q}\boldsymbol{w} \qquad\qquad s.t. \quad \boldsymbol{r}\boldsymbol{w} = \boldsymbol{r_p}, \qquad \sum_i w_i = 1, \ \ w_i \geq 0 \tag{13}$$

where r_p is the profitability of the target portfolio. Therefore, the smaller (larger) the value of ξ, the less (more) important the risk term will be. Thus, the return and risk of the optimal mean-variance portfolio both decrease as the risk aversion increases.

In order to solve the mean-variance portfolio optimization problem, we construct the covariance matrix according to the following stages. First, the volatilities σ_{it} are extracted for each marginal of the Beta-Skew-t-EGARCH model defined in Eq. (1). Second, the pair-wise linear correlation coefficients ρ_{ijt} are obtained from the estimation of the factor copula model. These include the dynamic factor loadings δ_{it} and δ_{jt} and the variances σ_z^2 and σ_ε^2 of the probability distribution functions of the common factor and the idiosyncratic variables, respectively.

$$\rho_{ijt} = \frac{\delta_{it}\delta_{jt}\sigma_z^2}{\sqrt{(\delta_{it}^2\sigma_z^2 + \sigma_\varepsilon^2)(\delta_{jt}^2\sigma_z^2 + \sigma_\varepsilon^2)}}. \tag{14}$$

Finally, by combining the estimators of σ_{it} and ρ_{ijt} we obtain a time-varying covariance matrix as follows:

$$\boldsymbol{Q_t} = \begin{bmatrix} \sigma_{1t}^2 & \cdots & \rho_{1Nt}\sigma_{1t}\sigma_{Nt} \\ \vdots & \ddots & \vdots \\ \rho_{N1t}\sigma_{Nt}\sigma_{1t} & \cdots & \sigma_{Nt}^2 \end{bmatrix}. \tag{15}$$

In order to obtain the optimal investment weights, through the resolution of the optimal portfolio problem, an in-sample period of one month of observations (approximately 21 days) is used. In addition, we consider rebalancing periods of the constituents of the portfolio of 5, 10, and 21 days.

3. Empirical application

This section analyzes the dynamic structure of volatility and dependence on the proposed markets to determine later opportunities for diversification in the composition of a portfolio due to the incorporation of different types of commodities.

3.1 Data and summary statistics

The data consist of the daily returns of 20 individual components of the S&P GSCI (Goldman Sachs Commodity Index) and seven equity indices that belong to the MSCI World Index (Morgan Stanley Capital International). In particular, the analyzed commodities correspond to the most liquid commodity futures and provide a high level of diversification with low correlations to other asset classes, minimizing the effects of highly idiosyncratic events (Hung and Shiu, 2016). These are grouped into the following sectors: energy (crude oil, diesel, heating oil, and natural gas), precious metals (gold and silver), industrial metals (aluminum, copper, zinc, and nickel), and soft (cocoa, coffee, corn, cotton, Kansas wheat, wheat, soybeans, sugar, live cattle, and dead cattle). Meanwhile, the MSCI indices cover approximately 85% of the free float-adjusted market capitalization in each country and they are classified into developed markets (Germany, France, the United Kingdom, and the USA) and emerging markets (Taiwan, China, and South Korea). These indices are designed to measure the performance of the large and mid-cap segments of most of the equity universe in each country. The intuition behind our analyses is that, for example, soft and energy commodities generally respond the most when emerging markets dominate world growth. On the other hand, when industrialized economies dominate world growth, the industrial metals sector responds typically more than the soft commodities.

In the empirical analysis in the portfolio selection problem, we use the daily US 3-month treasury bill rate as a proxy for the risk-free rate. The data were retrieved from Bloomberg. The sample period was December 31, 2004 to December 31, 2019. Data generated prior to December 31, 2014 were used for the estimation, leaving the last five years for backtesting. The sample period included various phases of the boom and bust cycle for both stock and commodity markets. Table 1 presents some descriptive statistics regarding the return series, calculated as $R_t = 100\ln(P_t/P_{t-1})$, where P_t is the price at time t. We see that the average returns for most assets are close to zero. In addition, both commodity and stock markets exhibit the typical non-normality of financial time series confirmed by the Jarque–Bera test statistics. Furthermore, most of the returns are positively skewed and display excess kurtosis. In relation to unconditional volatility, natural gas price returns are identified as extremely volatile.

Concerning the Ljung–Box test for autocorrelation, the results are mixed, so the conditional mean modeling does not seem to play a fundamental role in these data. On the other hand, we found strong evidence of conditional heteroscedasticity for all of the returns analyzed through the ARCH-Lagrange multiplier (LM) test. Finally, the augmented Dickey–Fuller test, related to the presence of unit roots, showed that all the time series are stationary at a significance level of 1%.

3.2 Volatility Modeling

Table 2 shows the results of the estimation of the Beta-Skew-t-EGARCH model with two components for each of the time series. One of the advantages of this model is that it can mimic the long memory patterns commonly shown by the autocorrelation function of the absolute values of the returns (Harvey and Sucarrat, 2014). In particular, we observe high persistence of long-term volatility through the parameter ϕ_1, with values close to 1. Meanwhile, although high, the coefficients ϕ_2 , which capture short-term persistence, are, in most cases, smaller in magnitude than those observed in the long term (i.e., $\phi_2 < \phi_1 < 1$). As a consequence of this last result, the model is well-identified and is stationary. The long- and short-term ARCH effects, captured by the coefficients k_1 and k_2, are low, implying a weak response to external volatility shocks.

Concerning the leverage coefficient k^*, this is positive and significant in stock markets, so negative shocks seem to accentuate short-term volatility more than positive shocks of the same magnitude. However, these results are mixed in the case of commodity markets. In particular, we found a positive return-volatility relationship for natural gas, cocoa and coffee. These results are in line with previous studies (see for instance Baur and Dimpfl, 2018; Chen and Mu, 2021). There is also evidence of significant negative skew for most markets, although this is much more pronounced for stock markets, leading to the common interpretation that large negative returns are followed by higher volatility.

Another impressive result is that both the unconditional and conditional skewness were the same for all markets. Finally, the estimated degrees of freedom of the skew-t conditional density function suggest that most of the returns exhibit heavy tails. In particular, the commodity markets denominated as precious metals (gold and silver) show the lowest degrees of freedom on average, followed by the commodity markets for industrial metals and stock markets.

Figure 1 shows the behavior of the volatility grouped by sector during the analysis period. The metals sector (industrial and precious) follows a similar trajectory, with gold being the commodity that exhibits the lowest volatility. Similarly, except for natural gas, all other markets show a very similar trend in the energy sector. In soft commodities, the results are much more heterogeneous, with feeder and live cattle displaying the most similar and least volatile behavior. On the other hand, stock markets show the most homogeneous volatility at the sector level, demonstrating the significant integration that exists between international stock markets and the fewer opportunities for diversification between these same markets. Finally, all sectors show an increase in volatility during the subprime crisis that is reduced from 2010 onwards.

3.3 Factor Copula Modeling

The joint dependence dynamic is captured through three dynamic factor copula models adjusted to the pseudo-uniform marginals [0,1], obtained by employing the integral probability transformed of the standardized residuals.

The first specification corresponds to a heterogeneous structure, which measures each series' dependence with the common factor (Z_t). The second specification is the block structure, grouping the return series according to ex-ante information so that four sectors are considered: metals (industrial and precious), energy, soft, and stock markets. The last specification is the equidependence structure, which imposes absolute homogeneity on the series analyzed so that the dependence between each variable with the factor is identical.

Table 3 summarizes the results of the estimates for the specifications of the dynamic factor copula model. According to AIC, the heterogeneous structure specification fits better than the other copula structures, even when a higher number of parameters are involved. The above indicates that a heterogeneous structure of financial time series provides greater flexibility than conditioning the time series on a single parameter or a group of predefined commodities. Therefore, subsequent applications should take into consideration the results relating to the heterogeneous structure for the factor copula model.

The inverse values of the estimated degrees of freedom are small, ranging from $\nu_z^{-1} = 0.124$ for the common factor and $\nu_\varepsilon^{-1} = 0.121$ for the random innovations, which implies that the degrees of freedom involved in the specification are less than 10. The common factor asymmetry is negative, indicating a high degree of tail dependence among the negative returns on equity analyzed. This finding is similar to the results obtained by Bertels and Ziegelmann (2016). Furthermore, we observe a high temporal persistence of the factor loading processes through the parameter β in Eq. (9), which is very close to 1.

The factor loadings under the heterogeneous structure are shown in Figure 2 for the in- and out-of-sample data. The model has not been re-estimated. For the estimation of the out-of-sample dynamic factor loadings, we used the estimates of the in-sample parameters. We observe that in terms of magnitude, the energy and metals commodity markets are the groups with the highest levels of dependence on the latent factor, followed by the stock markets. By contrast, soft commodities present dynamic factor loadings in a lower magnitude range. In most markets, it is possible to distinguish between the subprime crisis and the European debt crisis during the period of 2007–2013.

On the one hand, the equity markets that showed a higher degree of homogeneity of dependence on the latent factor were the energy commodities as well as the stock markets of Europe and the USA. The commodity markets that showed a weak relationship with the latent factor were gold and silver in the metals markets, live and feeder cattle in soft commodities, and heating oil in the energy commodities sector. In the case of stock markets, a weak relationship with the latent factor was observed in the Asian markets of Taiwan, China, and South Korea. This last result is in line with the results of Berger and Uddin (2016). They find that the dependence between the S&P 500 and natural gas was characterized by higher degrees of freedom during the period after the subprime financial crisis.

3.4 Portfolio Optimization based on Factor Copula Model

In this section, we present the results of the portfolio optimization problem described in Section 2.3. We define different types of portfolios to reach high diversification. The portfolio AA considers all assets—both commodities and stock markets. The portfolio IM&SM includes industrial metals and stock markets, while the portfolio PM&SM considers precious metals and stock markets. The E&SM portfolio comprises the energy commodities and stock markets. The portfolio S&SM considers the so-called soft commodities and stock markets. Finally, the portfolio SM is only composed of stock markets.

For the specification of the covariance matrix in Eq. (15), we use three alternatives. The smoothed approach simply relies on the average of the estimated covariance matrix during the in-sample $\boldsymbol{Q} = \overline{\boldsymbol{Q}}$. The instantaneous approach uses the last observation at time *t* from the time-varying covariance matrix estimated during the in-sample period $\boldsymbol{Q} = \boldsymbol{Q}_t$. Finally, the standard approach involves a simple rolling window to estimate the covariance matrix of raw returns with a calibration period of one month $\boldsymbol{Q} = \boldsymbol{Q}_{\boldsymbol{r}}$. The key idea of these three covariance matrix specifications is that the most recent performance would repeat in the subsequent period.

The sample period to optimize each portfolio was from January 1, 2015, to December 31, 2019. The determination of the optimal investment weights was achieved by resolving the optimal portfolio problem with minimal variance and risk aversion, using Eqs. (12) and (13), respectively. A window of one month of observations (approximately 21 days) was used to solve the optimal portfolio problem, considering rebalancing periods of the portfolio constituents of 5, 10, 15, and 21 days. With these estimations, we calculate the average of the expected returns for each portfolio, the standard deviation, and the Sharpe ratio, for the following month. We use the Sharpe ratio to compare the different specifications. Further, we use different degrees of risk aversion captured by the parameter ξ, which varies between 0.1 and ∞ in our empirical exercise. The above implies that the optimal portfolio moves along the efficient frontier that yields the highest return for each risk level, i.e., from the maximum return portfolio to the minimum variance portfolio. Note that we do not consider short sales and transaction costs.

Table 4 summarizes these results. Four important results emerge at first glance. First, there are diversification benefits when commodities are included in portfolio strategies when the risk aversion parameter increases ($\xi \geq 5$) and the rebalancing period is of 10 days. In particular, for the portfolio with minimal variance ($\xi \to \infty$), the best strategy is obtained by the PM&SM portfolio. Second, the instantaneous approach offers, on average across all the portfolio settings, the best specification in terms of Sharpe ratio for the rebalancing period of 5 days, while the smoothed approach exhibits slightly higher Sharpe ratios for rebalancing periods of 10 and 15 days. Third, as the rebalancing periods increase, lower Sharpe ratios are obtained when the risk aversion parameter increases, tending towards the portfolio with minimal variance. Fourth, the SM portfolios give the highest Sharpe ratio for most cases. More specically, for 5 days rebalancing, SM portfolios appear to have the highest Sharpe ratio regardless of the value of the risk aversion parameter. Moreover, when the rebalancing period is of 15 days, SM portfolios seem to have the highest Sharpe ratio for the risk aversion parameter $\xi \geq 1$. Further, for the SM portfolios, the standard approach gives the best results with the highest Sharpe ratio.

Concerning the rebalancing periods, we observe that the best diversification strategy in terms of the Sharpe ratio at 5 days is for the SM portfolio, followed by the PM&SM and E&SM portfolios. Similar results are displayed for the rebalancing period of 10 days, with the exception that for the portfolio with minimal variance ($\xi \to \infty$), the best strategy is obtained by the PM&SM portfolio. For the rebalancing period of 21 days with low-risk aversion parameters ($\xi \leq 0.5$), the best diversification strategy is obtained with the S&SM portfolio for both copula approaches utilized. Finally, for risk aversion parameter $\xi \geq 1$, the SM and PM&SM portfolios exhibit the highest Sharpe ratios.

Regarding the different covariance matrix specifications, both copula approaches (smoothed and instantaneous) show, on average across all the portfolio settings, better performance in the diverse portfolio strategies and rebalancing periods, either in terms of expected values, standard deviations, and therefore, their Sharpe ratios.

An interesting result is the poor performance of the AA portfolio. One would expect portfolios consisting of all assets to perform better on average than other portfolios restricted to a subset of stocks. However, this is not the case in the empirical exercise carried out. We believe that the explanation lies mainly in the classical mean-variance portfolio selection for large-dimension optimization problems. The standard formulation leads to the recurring estimation of a covariance matrix (and its inverse) using only the most recent data, which can be very unstable when the number of asset returns increases, or when they are highly correlated (see for instance Dai and Wang, 2019; Kremer et al., 2020). Consequently, a slight change in the covariance matrix of the portfolio returns can have a big impact in the optimal portfolio weights. Despite the above difficulty, the dynamic factor copula specification shows a better performance in terms of Sharpe ratio than the standard approach.

Figures 3 and 4 show the optimal weights for every asset or sector using the heterogeneous specification with an instantaneous covariance matrix for a rebalancing or holding period of 21 days and risk aversion parameter $\xi = 1$. For most portfolio strategies, we observe that stocks exhibit more weight in the backtesting period. For the AA portfolio, which considers all assets, we observe that large weights are given for stock markets, followed by soft, precious, and industrial metals commodities. In the case of the IM&SM portfolio, the USA and aluminum are the two assets with the highest weight. Similar results were obtained for the PM&SM portfolio, where gold is a commodity that plays an important role. In the case of the E&SM portfolio, the highest weights correspond to stocks, although low natural gas participation in the optimal portfolio composition is also observed. The highest proportion of commodities included in the portfolio composition is displayed by the S&SM portfolio, with around 50% of the optimal weights, being the participation of each of the soft commodities heterogeneous. Finally, the SM portfolio is mainly driven by higher participation of USA, UK, Taiwan, and South Korea stock markets.

4. Conclusions

This article analyzes the dynamic behavior of volatility and the joint dependence between the financial returns of seven stock markets of the MSCI Inc. index and twenty sub-indices of commodities of the S&P GSCI. The model proposed to capture the volatility dynamics is the Beta-

Skew-t-EGARCH model, while to determine the dependence between markets, we use a factor copula specification with score-driven dynamics. Both specifications seem to capture the stylized factors present in the analyzed returns in terms of goodness of fit. In particular, we found short- and long-term patterns for the different sectors examined. The most volatile sectors, though they are also the most homogeneous, are energy commodities and stock markets. In terms of the degree of dependence between each of the sectors, we once again observed that energy commodities and stock markets are the ones that exhibit a stronger relationship with the latent factor copula. By contrast, the equity markets that are less related to the latent factor seem to be soft commodities.

Finally, we studied the potential to obtain portfolio diversification benefits presented by commodities by solving a portfolio optimization problem. Overall, the results show that the diversification potential is limited compared to a portfolio composed solely of equity markets, at least for the sample analysed. In the case of necessarily including commodities during the portfolio construction, there is some potential to reduce the volatility, mainly regarding the soft market sector and the natural gas market in terms of energy commodities with low-risk aversion. For the portfolio with minimal variance, the best strategy is obtained by adding precious metal commodities to a traditional portfolio. These results are in line with those obtained by Arouri et al. (2010); Cheng and Tu (2013), and Hammoudeh et al. (2014).

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Appendix

A. Tables

	Aluminum	Copper	Nickel	Zinc	Gold	Silver	Crude Oil	Gasoil	Natural Gas
Min.	-8.26	-10.38	-13.70	-11.13	-9.81	-19.49	-13.07	-11.13	-13.80
Mean	-0.002	0.027	0.001	0.023	0.038	0.031	0.007	0.009	-0.032
Max.	5.927	11.902	13.158	9.328	8.590	12.471	13.341	12.887	17.129
Std dev.	1.505	1.914	2.405	2.145	1.254	2.248	2.107	1.997	2.902
Skewness	-0.253	-0.070	0.006	-0.171	-0.406	-0.899	-0.217	-0.199	0.261
Kurtosis	1.858	3.725	2.765	2.113	5.236	6.537	4.338	4.244	2.021
Ljung–Box	10.32	24.04*	9.99	15.84	22.70*	10.67	25.39*	25.45*	27.11*
Jarque–Bera	405.13*	1517.40*	834.90*	500.88*	3063.60*	5015.00*	2075.00*	1983.10*	476.11*
ARCH (12)	424*	510*	512*	569*	1198*	1358*	499*	505*	559*
ADF test	-18.80*	-18.06*	-19.49*	-19.25*	-19.08*	-18.96*	-19.64*	-19.57*	-19.72*

	Heating Oil	Cocoa	Coffe	Corn	Cotton	Kansas Wheat	Wheat	Soybeans	Sugar
Min.	-9.68	-9.78	-11.25	-8.12	-7.13	-8.99	-9.79	-7.34	-12.37
Mean	0.012	0.024	0.018	0.025	0.013	0.023	0.024	0.024	0.019
Max.	9.907	8.344	10.853	8.663	6.940	8.097	15.599	6.432	8.184
Std dev.	1.905	1.750	1.956	1.931	1.753	1.891	2.058	1.599	2.055
Skewness	-0.064	-0.329	0.071	-0.015	-0.179	0.007	0.141	-0.251	-0.325
Kurtosis	2.780	3.124	2.083	1.658	1.257	1.711	2.918	2.033	2.698
Ljung–Box	11.26	7.03	15.66	16.01	27.65*	8.87	12.19	9.46	13.98
Jarque–Bera	846.09*	1112.80*	476.37*	300.54*	187.00*	319.98*	938.48*	479.27*	840.91*
ARCH (12)	535*	950*	676*	574*	376*	577*	752*	593*	719*
ADF test	-19.07*	-19.24*	-19.59*	-19.64*	-18.89*	-19.18*	-19.23*	-18.36*	-19.35*

	Feeder Cattle	Live Cattle	Taiwan	China	South Korea	UK	France	Germany	USA
Min.	-3.20	-3.24	-7.17	-12.84	-20.67	-10.43	-11.57	-9.64	-9.51
Mean	0.029	0.022	0.011	0.037	0.026	0.003	0.002	0.015	0.021
Max.	3.351	3.700	8.232	14.044	24.987	12.161	11.844	11.589	11.043
Std dev.	0.847	0.857	1.407	1.812	1.918	1.476	1.689	1.652	1.265
Skewness	-0.216	-0.092	-0.242	-0.017	-0.180	-0.108	-0.016	-0.041	-0.356
Kurtosis	1.060	1.127	3.262	7.442	2.045	9.761	6.796	6.093	11.502
Ljung–Box	35.40*	19.60**	27.59*	20.33**	18.08	60.72*	45.00*	16.99	56.24*
Jarque–Bera	143.38*	146.69*	1184.90*	6028.40*	45533.00*	10367.00*	5027.70*	4042.50*	14454.0*
ARCH (12)	451*	401*	627*	469*	623*	620*	707*	594*	455*
ADF test	-19.45*	-20.30*	-19.86*	-18.78*	-19.47*	-20.59*	-20.55*	-20.21*	-20.61*

Table 1: Descriptive statistics for daily returns of commodities and equity markets. The Ljung–Box test is significant with a lag of 10 days. ARCH is Engle's Lagrange multiplier test for conditional heteroskedasticity of order 12, and the augmented Dickey–Fuller test (ADF test) examines the null hypothesis that a unit root is present in the time series sample. * and ** indicate the level of statistical significance at 1% and 5%.

	Aluminum	Copper	Nickel	Zinc	Gold	Silver	Crude Oil	Gasoil	Natural Gas
ω	0.1354	0.2651	0.5861	0.2787	-0.1825	0.4154	0.6790	0.6746	0.9133
	(0.141)	(0.124)	(0.102)	(0.165)	(0.101)	(0.095)	(0.189)	(0.219)	(0.063)
ϕ_1	0.9977	0.9915	0.9936	0.9991	0.9914	0.9863	0.9984	0.9994	0.9783
	(0.003)	(0.004)	(0.041)	(0.001)	(0.003)	(0.014)	(0.002)	(0.001)	(0.007)
ϕ_2	0.9656	0.8028	0.9936	0.7562	0.3991	0.9857	0.9833	0.9774	0.6627
	(0.032)	(0.065)	(0.022)	(0.089)	(0.120)	(0.024)	(0.008)	(0.011)	(0.125)
k_1	0.0112	0.0449	-1.8377	0.0158	0.0357	0.4465	0.0177	0.0164	0.0442
	(0.008)	(0.007)	(0.012)	(0.003)	(0.005)	(0.285)	(0.007)	(0.005)	(0.007)
k_2	0.0179	-0.0365	1.8652	0.0047	-0.0590	-0.4148	0.0058	0.0082	-0.0416
	(0.009)	(0.014)	(0.007)	(0.009)	(0.016)	(0.287)	(0.010)	(0.008)	(0.012)
k^*	-0.0006	0.0409	0.0004	0.0303	0.0517	0.0006	0.0213	0.0237	-0.0253
	(0.004)	(0.008)	(0.003)	(0.008)	(0.011)	(0.003)	(0.004)	(0.005)	(0.007)
df	8.3507	7.4138	7.6894	8.9504	4.4967	4.2441	10.1786	8.7727	10.3258
	(1.189)	(0.999)	(1.087)	(1.493)	(0.419)	(0.398)	(1.791)	(1.339)	(1.881)
skew	0.9828	0.9751	1.0348	0.9714	0.9133	0.8807	0.9132	0.9130	1.0419
	(0.025)	(0.023)	(0.025)	(0.024)	(0.020)	(0.020)	(0.024)	(0.024)	(0.026)
LL	-4594.96	-4926.05	-5704.93	-5335.31	-3937.12	-5428.86	-5223.70	-5054.55	-6277.97
BIC	3.541	3.795	4.391	4.108	3.038	4.179	4.022	3.893	4.840
	Heating Oil	Cocoa	Coffee	Corn	Cotton	Kansas Wheat	Wheat	Soybeans	Sugar
ω	0.8157	0.2327	0.4247	0.4130	0.3513	0.3925	0.4115	0.2307	0.3889
	(0.193)	(0.100)	(0.049)	(0.074)	(0.092)	(0.098)	(0.134)	(0.069)	(0.129)
ϕ_1	0.9998	0.9947	0.9002	0.9853	0.9907	0.9930	0.9954	0.9847	0.9952
	(0.001)	(0.004)	(0.091)	(0.005)	(0.004)	(0.005)	(0.910)	(0.006)	(0.002)
ϕ_2	0.9698	0.9724	0.9769	0.6400	-0.1821	0.8883	0.9140	0.5036	-0.9909
	(0.018)	(0.047)	(0.013)	(0.263)	(0.412)	(0.067)	(0.058)	(0.174)	(0.007)
k_1	0.0143	0.0244	-0.0296	0.0360	0.0310	0.0246	0.0224	0.0356	0.0258
	(0.004)	(0.014)	(0.025)	(0.006)	(0.004)	(0.008)	(0.010)	(0.006)	(0.004)
k_2	0.0048	-0.0084	0.0369	0.0087	0.0194	0.0009	0.0047	-0.0216	0.0036
	(0.007)	(0.014)	(0.026)	(0.013)	(0.012)	(0.013)	(0.014)	(0.014)	(0.002)
k^*	0.0176	-0.0052	-0.0137	0.0240	0.0153	-0.0230	-0.0208	0.0332	-0.0009
	(0.005)	(0.004)	(0.004)	(0.010)	(0.010)	(0.008)	(0.007)	(0.010)	(0.001)
df	10.3372	5.0431	5.5988	7.1331	9.1416	8.6640	7.6147	6.8260	6.740
	(1.774)	(0.544)	(0.670)	(0.913)	(1.657)	(1.276)	(1.038)	(0.914)	(0.925)
skew	0.9691	0.9699	1.0146	1.0136	0.9526	1.0449	1.0442	0.9465	1.022
	(0.026)	(0.022)	(0.023)	(0.025)	(0.023)	(0.028)	(0.027)	(0.023)	(0.023)
LL	-5033.87	-4943.36	-5320.69	-5222.16	-4906.24	-5175.66	-5337.67	-4711.00	-5293.76
BIC	3.877	3.808	4.097	4.021	3.788	3.986	4.119	3.630	4.085
	Feeder Cattle	Live Cattle	Taiwan	China	South Korea	UK	France	Germany	USA
ω	-0.2689	-0.2764	0.0057	0.1875	0.2682	-0.2053	0.0499	0.0556	-0.3012
	(0.053)	(0.068)	(0.103)	(0.116)	(0.106)	(0.186)	(0.124)	(0.113)	(0.137)
ϕ_1	0.9801	0.9883	0.9928	0.9916	0.9908	0.9970	0.9911	0.9887	0.9950
	(0.012)	(0.005)	(0.003)	(0.003)	(0.004)	(0.003)	(0.004)	(0.005)	(0.003)
ϕ_2	0.9678	0.9866	0.8189	0.7345	0.9018	0.9188	0.8603	0.8428	0.9355
	(0.017)	(0.005)	(0.043)	(0.099)	(0.018)	(0.021)	(0.023)	(0.030)	(0.009)
k_1	0.0429	0.1078	0.0305	0.0396	0.0419	0.0241	0.0411	0.0481	0.0286
	(0.073)	(0.109)	(0.006)	(0.006)	(0.009)	(0.007)	(0.008)	(0.008)	(0.006)
k_2	-0.0265	-0.0936	-0.0240	-0.0214	-0.0200	0.0087	-0.0271	-0.0391	0.0039
	(0.075)	(0.109)	(0.011)	(0.013)	(0.012)	(0.011)	(0.011)	(0.012)	(0.009)
k^*	0.0159	0.0085	0.0465	0.0566	0.0511	0.0564	0.0701	0.0668	0.0818
	(0.004)	(0.002)	(0.008)	(0.010)	(0.005)	(0.006)	(0.008)	(0.008)	(0.006)
df	9.0770	7.9151	6.8576	7.7563	8.3318	8.9950	8.4933	7.8821	6.1257
	(1.663)	(1.295)	(0.945)	(1.183)	(1.321)	(1.363)	(1.183)	(1.088)	(0.694)
skew	0.9249	0.9578	0.9232	0.9519	0.8847	0.8985	0.9183	0.9164	0.8386
	(0.023)	(0.023)	(0.022)	(0.023)	(0.022)	(0.024)	(0.024)	(0.024)	(0.021)
LL	-3162.18	-3189.67	-4200.18	-4621.49	-4687.23	-3963.00	-4424.38	-4398.09	-3368.43
BIC	2.444	2.471	3.246	3.570	3.620	3.064	3.418	3.398	2.608

Table 2: Beta-Skew-t-EGARCH model estimation results with two components (short and long term). The in-sample estimation period is from December 31, 2004 to December 31, 2014. Standard errors in parenthesis are presented below the estimated

parameters. LL corresponds to the log-likelihood at the estimated parameters, while BIC corresponds to the Bayesian Information Criteria.

Factor Copula Specifications						
	Equidependence		Block Equidependence		Heterogeneous Dependence	
	par $\times 100$	(s.e.$\times 100$)	par $\times 100$	(s.e.$\times 100$)	par $\times 100$	(s.e.$\times 100$)
ω_1	-4.104	(0.080)	-0.167	(0.058)	-0.016	(0.003)
ω_2			-1.022	(0.054)	-0.003	(0.003)
ω_3			-3.163	(0.004)	-0.041	(0.004)
ω_4			-1.174	(0.007)	-0.018	(0.003)
ω_5					-0.060	(0.005)
ω_6					-0.038	(0.004)
ω_7					-0.016	(0.004)
ω_8					-0.014	(0.004)
ω_9					-0.163	(0.012)
ω_{10}					-0.027	(0.004)
ω_{11}					-0.109	(0.007)
ω_{12}					-0.095	(0.006)
ω_{13}					-0.076	(0.006)
ω_{14}					-0.093	(0.006)
ω_{15}					-0.086	(0.007)
ω_{16}					-0.084	(0.007)
ω_{17}					-0.060	(0.005)
ω_{18}					-0.108	(0.007)
ω_{19}					-0.229	(0.026)
ω_{20}					-0.149	(0.011)
ω_{21}					-0.116	(0.008)
ω_{22}					-0.093	(0.007)
ω_{23}					-0.099	(0.007)
ω_{24}					-0.012	(0.004)
ω_{25}					-0.014	(0.004)
ω_{26}					-0.016	(0.004)
ω_{27}					-0.704	(0.051)
α	2.422	(0.181)	7.071	(0.058)	12.365	(0.125)
β	93.404	(0.085)	98.623	(0.036)	99.903	(0.005)
ν_z^{-1}	28.111	(0.317)	19.331	(0.144)	12.410	(0.272)
ν_ε^{-1}	11.711	(0.318)	16.778	(0.058)	12.146	(0.160)
ψ_z	-11.509	(0.340)	-2.501	(0.098)	-5.502	(0.373)
LL	6315.72		7117.11		8044.03	
AIC	-12619.44		-14224.21		-16078.06	

Table 3: This table presents the log-likelihood (LL) at the estimated parameters, as well as the Akaike Information Criteria (AIC), for a variety of factor copula models. Parameters and standard errors are multiplied by 100 for ease of exposition. Standard errors are presented in parenthesis.

Strategies	Risk	5 days									10 days									21 days								
		Standard (Q_r)			Smoothed ($\bar{Q}$)			Instantaneous(Q_t)			Standard (Q_r)			Smoothed ($\bar{Q}$)			Instantaneous(Q_t)			Standard (Q_r)			Smoothed ($\bar{Q}$)			Instantaneous(Q_t)		
	ξ	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR	PR	SD	SR
AA	0.1	-6.97	1.55	-4.50	-7.52	1.63	-4.63	**-5.88**	**1.57**	**-3.73**	-4.16	1.58	-2.64	**-4.12**	**1.69**	**-2.45**	-4.12	1.61	-2.56	0.00	1.59	0.00	-0.01	1.60	-0.01	**0.41**	**1.61**	**0.25**
IM&SM		**1.68**	**1.19**	**1.41**	0.74	1.18	0.62	0.82	1.17	0.70	1.43	1.16	1.23	2.02	1.16	1.74	**2.20**	**1.16**	**1.90**	**2.22**	**1.12**	**1.99**	1.61	1.11	1.45	1.84	1.10	1.67
PM&SM		1.12	0.99	1.13	**1.11**	**0.94**	**1.18**	0.85	0.94	0.91	0.20	0.93	0.21	1.16	0.92	1.25	**1.36**	**0.92**	**1.48**	0.57	1.02	0.55	0.36	1.00	0.36	**0.56**	**0.99**	**0.57**
E&SM		**-1.37**	**1.45**	**-0.94**	-3.14	1.51	-2.08	-2.08	1.45	-1.43	**1.53**	**1.59**	**0.97**	0.40	1.65	0.24	1.01	1.60	0.63	0.60	1.73	0.35	1.50	1.76	0.85	**1.57**	**1.77**	**0.89**
S&SM		-2.84	1.36	-2.09	-2.10	1.38	-1.52	**-1.99**	**1.34**	**-1.49**	-2.52	1.40	-1.81	**-1.72**	**1.41**	**-1.22**	-2.18	1.37	-1.59	**2.59**	**1.22**	**2.13**	2.49	1.22	2.03	2.43	1.20	2.02
SM		**3.11**	**0.89**	**3.49**	2.55	0.88	2.91	2.72	0.88	3.10	1.89	0.88	2.14	2.28	0.87	2.62	**2.54**	**0.87**	**2.91**	**1.55**	**0.95**	**1.64**	0.89	0.93	0.95	0.79	0.93	0.85
AA	0.5	-3.98	0.95	-4.19	-4.81	0.97	-4.94	**-3.81**	**0.94**	**-4.07**	**-1.41**	**0.93**	**-1.52**	-2.23	0.97	-2.30	-2.67	0.94	-2.85	1.95	0.99	1.97	1.54	1.00	1.54	**2.08**	**0.95**	**2.18**
IM&SM		**1.86**	**0.86**	**2.15**	0.43	0.88	0.50	1.41	0.85	1.66	1.19	0.84	1.42	1.64	0.85	1.92	**1.88**	**0.84**	**2.24**	**1.43**	**0.82**	**1.73**	0.71	0.85	0.84	0.30	0.84	0.35
PM&SM		**1.75**	**0.73**	**2.39**	1.25	0.75	1.68	1.04	0.76	1.38	0.89	0.75	1.19	1.02	0.75	1.36	**1.26**	**0.75**	**1.68**	0.52	0.78	0.66	**1.05**	**0.80**	**1.31**	0.78	0.78	1.00
E&SM		**1.04**	**0.87**	**1.19**	0.64	0.86	0.75	0.85	0.87	0.99	**1.31**	**0.90**	**1.45**	0.52	0.90	0.58	0.65	0.92	0.70	**2.35**	**1.00**	**2.35**	1.09	0.99	1.10	1.69	0.98	1.72
S&SM		-1.23	0.85	-1.45	-1.86	0.89	-2.07	**-1.17**	**0.87**	**-1.35**	-1.74	0.87	-1.99	**-1.03**	**0.90**	**-1.15**	-1.56	0.86	-1.81	**2.40**	**0.81**	**2.94**	2.25	0.85	2.63	2.16	0.84	2.57
SM		**3.35**	**0.79**	**4.25**	2.55	0.77	3.30	2.66	0.78	3.43	2.20	0.79	2.80	2.18	0.77	2.84	**2.42**	**0.76**	**3.17**	**1.92**	**0.79**	**2.42**	1.18	0.79	1.50	1.01	0.79	1.28
AA	1	-2.52	0.75	-3.34	-3.20	0.77	-4.14	**-2.31**	**0.75**	**-3.09**	**-1.29**	**0.76**	**-1.69**	-1.57	0.76	-2.06	-1.88	0.74	-2.54	1.25	0.79	1.59	1.26	0.79	1.59	**1.65**	**0.77**	**2.15**
IM&SM		1.40	0.76	1.85	1.09	0.76	1.43	**1.80**	**0.74**	**2.42**	0.89	0.76	1.17	1.16	0.74	1.56	**1.36**	**0.74**	**1.85**	**1.57**	**0.74**	**2.11**	1.33	0.75	1.77	0.93	0.76	1.23
PM&SM		**1.66**	**0.64**	**2.61**	1.34	0.67	2.02	1.40	0.67	2.09	**1.35**	**0.67**	**2.02**	1.26	0.67	1.87	1.36	0.68	2.00	0.42	0.67	0.63	**1.53**	**0.71**	**2.15**	1.22	0.70	1.72
E&SM		1.54	0.77	2.00	1.76	0.74	2.38	**1.88**	**0.75**	**2.52**	**1.28**	**0.80**	**1.61**	0.97	0.75	1.30	0.97	0.77	1.26	**1.71**	**0.80**	**2.13**	1.20	0.83	1.44	1.50	0.82	1.82
S&SM		-0.66	0.71	-0.94	-1.13	0.74	-1.52	**-0.55**	**0.72**	**-0.76**	-1.39	0.73	-1.91	**-0.59**	**0.74**	**-0.80**	-0.83	0.72	-1.16	1.24	0.70	1.77	1.32	0.73	1.81	**1.48**	**0.72**	**2.05**
SM		2.75	0.73	3.74	2.70	0.72	3.74	**2.86**	**0.73**	**3.93**	1.98	0.74	2.67	2.01	0.72	2.80	**2.09**	**0.72**	**2.89**	**2.02**	**0.74**	**2.74**	1.55	0.73	2.10	1.46	0.74	1.97
AA	5	**-0.47**	**0.57**	**-0.82**	-0.97	0.52	-1.86	-0.49	0.51	-0.96	**-0.27**	**0.61**	**-0.44**	-0.28	0.53	-0.54	-0.39	0.53	-0.74	0.17	0.59	0.29	**0.75**	**0.55**	**1.37**	0.71	0.55	1.30
IM&SM		1.54	0.67	2.30	1.47	0.65	2.24	**1.93**	**0.65**	**2.95**	1.40	0.69	2.02	**1.49**	**0.65**	**2.27**	1.48	0.67	2.23	1.23	0.68	1.82	**1.48**	**0.67**	**2.21**	1.26	0.68	1.85
PM&SM		1.72	0.55	3.13	1.64	0.56	2.93	**1.80**	**0.57**	**3.18**	**1.93**	**0.57**	**3.38**	1.63	0.57	2.85	1.74	0.58	2.97	1.05	0.56	1.88	**1.67**	**0.58**	**2.86**	1.45	0.60	2.42
E&SM		1.83	0.69	2.65	1.98	0.66	3.00	**2.17**	**0.67**	**3.26**	1.22	0.72	1.70	**1.76**	**0.66**	**2.65**	1.78	0.67	2.64	1.63	0.70	2.33	**1.61**	**0.68**	**2.37**	1.61	0.69	2.32
S&SM		-0.50	0.60	-0.82	-0.15	0.56	-0.27	**0.12**	**0.56**	**0.22**	-0.81	0.62	-1.30	**0.28**	**0.57**	**0.49**	0.14	0.57	0.24	0.23	0.61	0.38	0.62	0.58	1.06	**0.63**	**0.58**	**1.09**
SM		**2.56**	**0.68**	**3.74**	2.21	0.67	3.28	2.47	0.68	3.64	1.96	0.70	2.80	**2.07**	**0.68**	**3.07**	2.03	0.68	2.96	**2.06**	**0.69**	**3.00**	1.76	0.69	2.57	1.73	0.70	2.49
AA	10	-0.07	0.55	-0.13	-0.35	0.49	-0.71	**0.05**	**0.49**	**0.11**	-0.12	0.58	-0.21	0.10	0.50	0.19	**0.12**	**0.50**	**0.24**	-0.24	0.57	-0.42	**0.58**	**0.52**	**1.12**	0.50	0.52	0.96
IM&SM		1.68	0.66	2.54	1.51	0.65	2.33	**1.93**	**0.65**	**2.98**	1.44	0.69	2.09	**1.52**	**0.65**	**2.35**	1.51	0.66	2.28	1.22	0.67	1.80	**1.38**	**0.66**	**2.08**	1.18	0.67	1.75
PM&SM		1.80	0.54	3.33	1.68	0.56	3.03	**1.86**	**0.56**	**3.33**	**1.96**	**0.56**	**3.49**	1.72	0.56	3.06	1.81	0.57	3.16	1.07	0.55	1.95	**1.65**	**0.57**	**2.88**	1.45	0.59	2.47
E&SM		1.90	0.69	2.76	1.98	0.66	3.01	**2.19**	**0.66**	**3.31**	1.29	0.71	1.80	**1.86**	**0.66**	**2.82**	1.88	0.67	2.82	1.39	0.70	2.00	**1.62**	**0.67**	**2.42**	1.58	0.68	2.32
S&SM		-0.35	0.60	-0.59	0.10	0.55	0.19	**0.35**	**0.54**	**0.65**	-0.34	0.61	-0.55	0.38	0.55	0.69	**0.40**	**0.56**	**0.71**	0.08	0.61	0.13	0.45	0.56	0.79	**0.47**	**0.57**	**0.83**
SM		**2.56**	**0.68**	**3.77**	2.14	0.67	3.18	2.39	0.68	3.53	2.08	0.70	2.99	**2.07**	**0.67**	**3.07**	2.02	0.68	2.96	**2.05**	**0.68**	**3.01**	1.75	0.68	2.56	1.72	0.69	2.49
AA	∞	**0.91**	**0.54**	**1.69**	0.34	0.48	0.71	0.74	0.48	1.54	0.41	0.57	0.72	0.47	0.49	0.96	**0.57**	**0.49**	**1.15**	-1.16	0.56	-2.06	**0.34**	**0.50**	**0.69**	0.21	0.50	0.41
IM&SM		1.97	0.66	2.99	1.59	0.65	2.45	**1.96**	**0.64**	**3.05**	1.49	0.69	2.17	**1.56**	**0.65**	**2.42**	1.52	0.66	2.32	1.12	0.68	1.66	**1.31**	**0.66**	**1.98**	1.02	0.67	1.53
PM&SM		1.86	0.54	3.46	1.79	0.55	3.24	**1.99**	**0.55**	**3.59**	**1.96**	**0.56**	**3.52**	1.83	0.56	3.30	1.91	0.57	3.38	1.25	0.54	2.31	**1.60**	**0.56**	**2.84**	1.39	0.57	2.43
E&SM		2.06	0.69	2.97	2.01	0.66	3.06	**2.28**	**0.66**	**3.46**	1.43	0.71	2.02	**1.95**	**0.66**	**2.97**	1.95	0.66	2.93	1.04	0.70	1.49	**1.62**	**0.66**	**2.44**	1.53	0.68	2.26
S&SM		0.02	0.60	0.03	0.39	0.54	0.73	**0.67**	**0.54**	**1.26**	0.32	0.61	0.53	0.49	0.54	0.89	**0.60**	**0.55**	**1.10**	-0.13	0.62	-0.20	**0.25**	**0.56**	**0.44**	0.24	0.56	0.43
SM		**2.58**	**0.68**	**3.80**	2.11	0.67	3.14	2.35	0.67	3.48	**2.19**	**0.69**	**3.16**	2.08	0.67	3.10	2.00	0.68	2.95	**2.01**	**0.68**	**2.97**	1.72	0.68	2.54	1.68	0.69	2.44
Mean		1.03	0.78	1.27	0.40	0.77	0.92	1.33	0.77	1.38	0.92	0.80	1.03	1.12	0.78	1.30	1.11	0.78	1.29	1.50	0.80	1.50	1.62	0.79	1.68	1.41	0.79	1.61

Table 4: Portfolio optimization results with risk aversion for the three specifications of the covariance matrix: standard, instantaneous, and smoothed. Only the last two specifications are based on the copula factor model. The analysis period runs from January 1, 2015 to December 31, 2019. The mean of the returns, the standard deviation, and the Sharpe ratio of the portfolio are represented by PR, SD and SR, respectively. The risk aversion coefficient is denoted by ξ. When $\xi = \infty$, we are dealing with the minimum risk portfolio problem. The best portfolio in terms of Sharpe ratio for the different specifications is highlighted in bold.

B. Figures

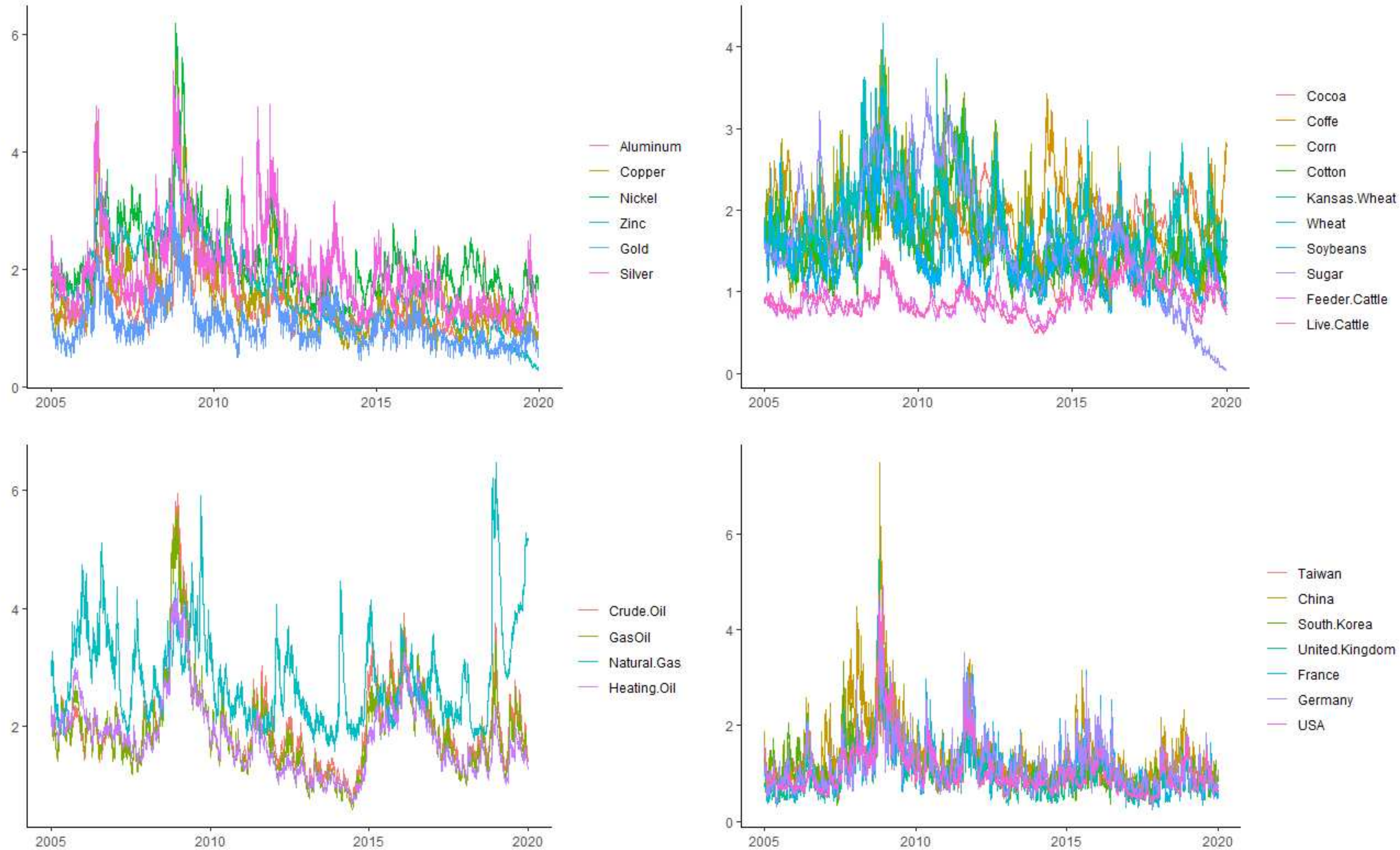

Figure 1: Volatility captured by the Beta-Skew-t-EGARCH model, with one and two components.

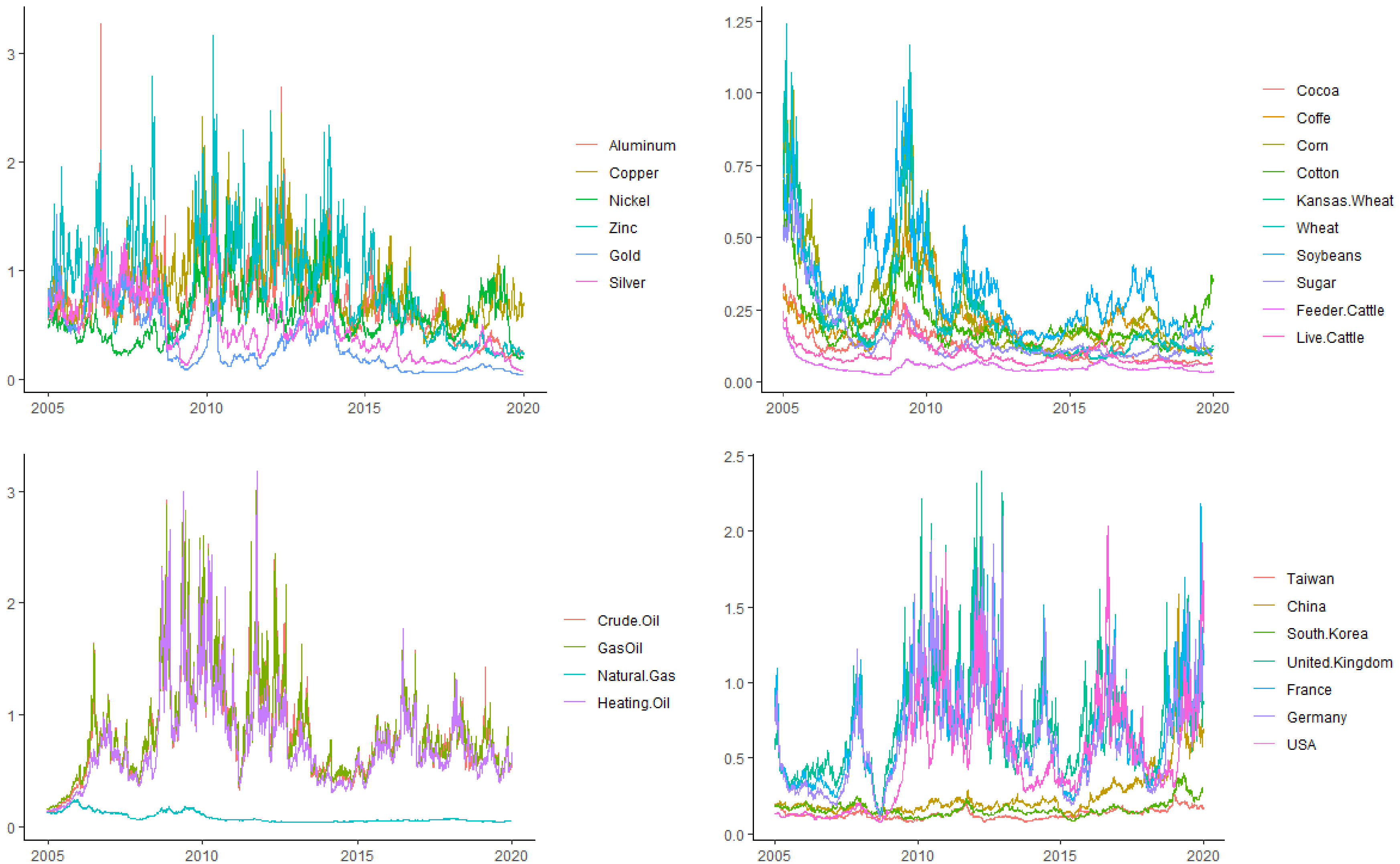

Figure 2: Factor loading across the time resulting from the skew t-t copula, with heterogeneous specification. Factor loadings are grouped according to type of commodity and stock markets.

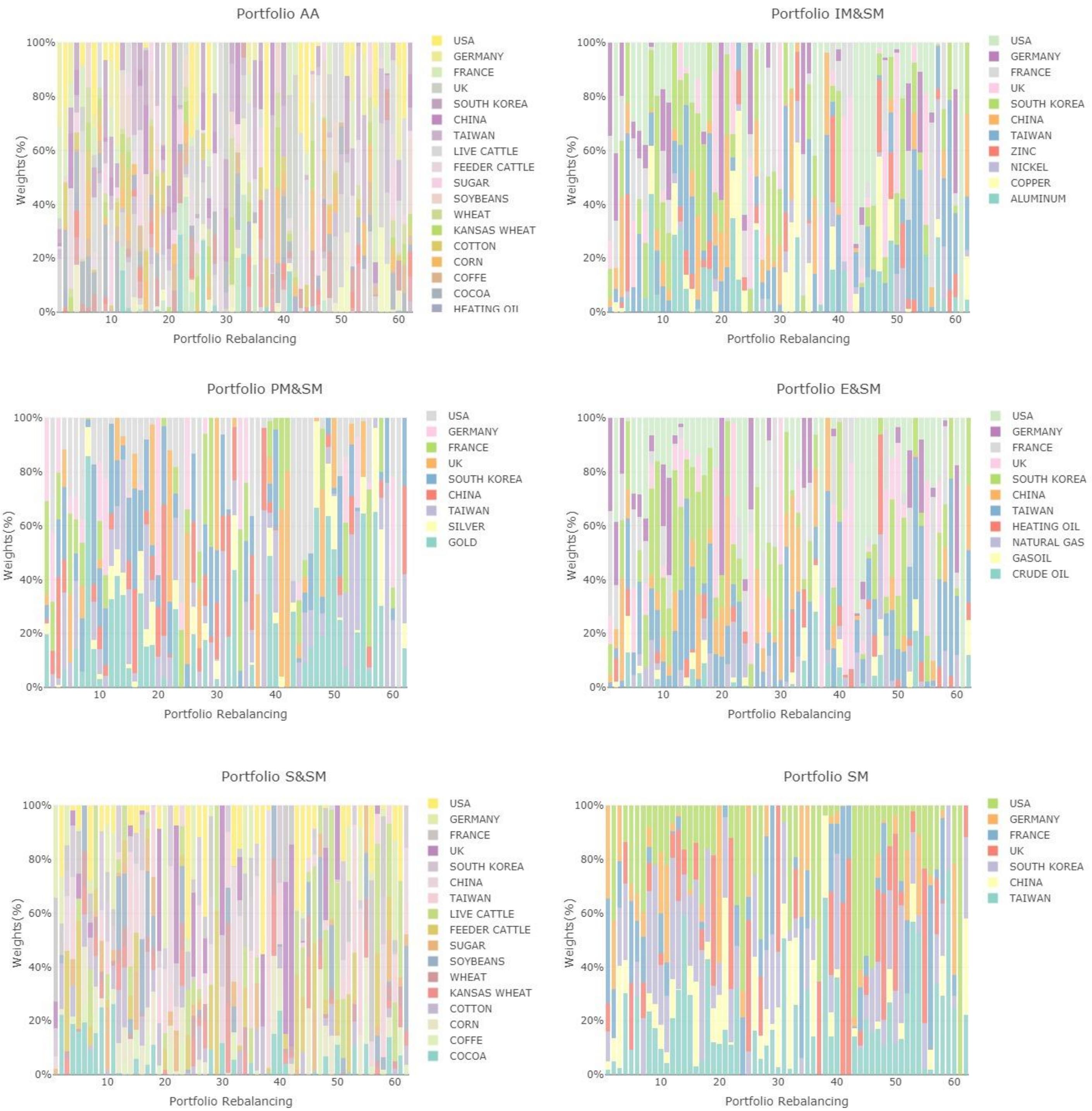

Figure 3: Optimal weights (%) for each proposed portfolio using the factor copula model, with heterogeneous specification. The rebalancing period is 21 days with risk of aversion ξ = 1.

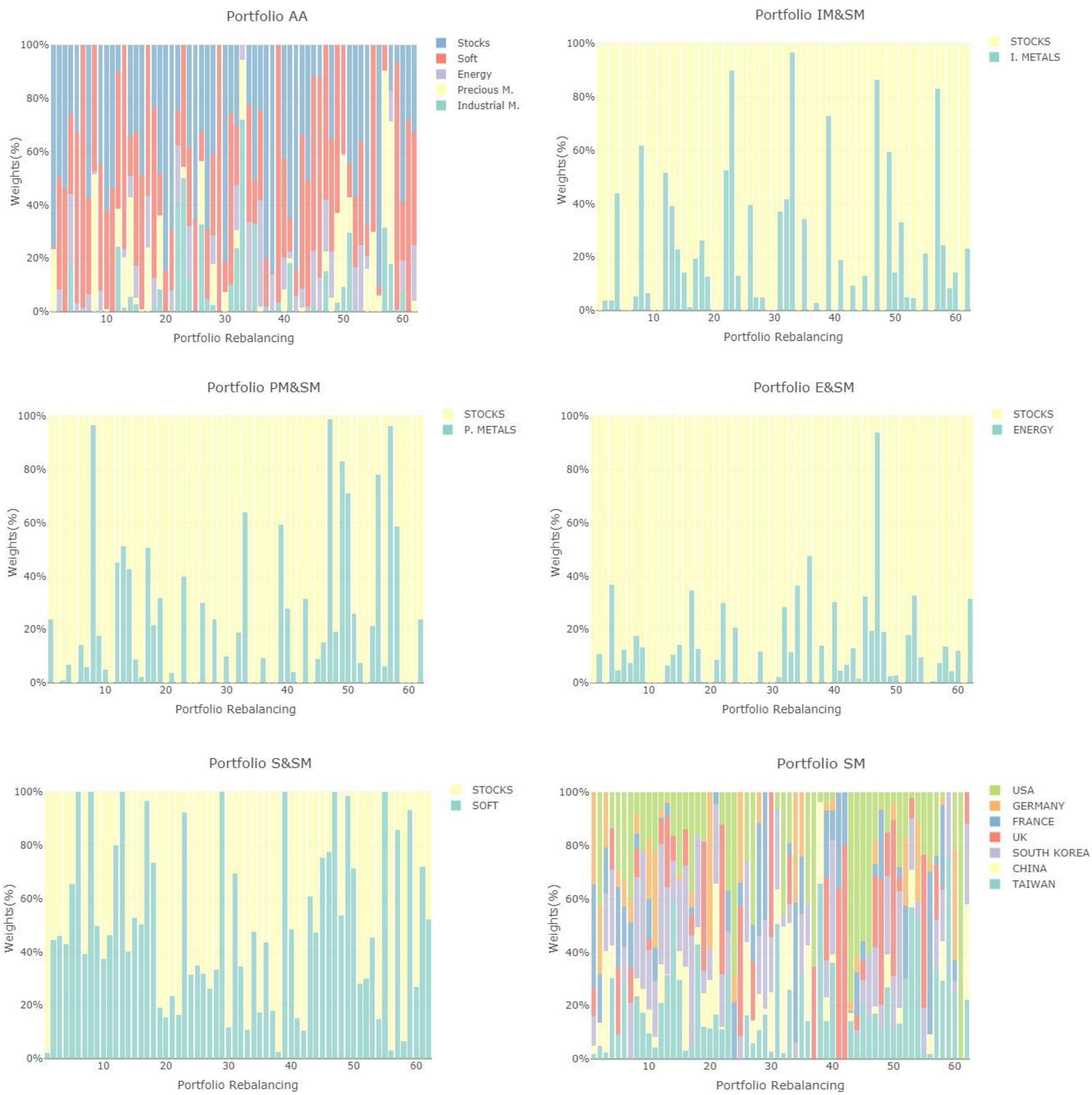

Figure 4: Optimal weights (%) for each portfolio strategy using the factor copula model, with heterogeneous specification, grouped across commodities and stock markets. The rebalancing period is 21 days with risk of aversion ξ = 1.