ON THE CORE OF AN ECONOMY WITH ARBITRARY CONSUMPTION SETS AND ASYMMETRIC INFORMATION

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On the core of an economy with arbitrary consumption sets and asymmetric information

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Abstract

This paper analyses the properties of (strong) core allocations in a two-period asymmetric information economy that also involves both negligible and non-negligible agents as well as an infinite dimensional commodity space. Within this setup, we allow the consumption set of each agent to be an arbitrary subset of the commodity space that may not have any lower bound. Our first result deals with the robustness of the core and the strong core allocations with respect to the restrictions imposed on the size of the blocking coalitions in an economy with only non-negligible agents. The second result is a generalization of the first result to an economy that allows the simultaneous presence of negligible as well as non-negligible agents with the consideration of Aubin coalitions. Finally, we show that (strong) core allocations are coalitional fair in the sense that no coalition of negligible agents could redistribute among its members the net trade of any other coalition containing all non-negligible agents in a way that could assign a preferred bundle to each of its members, and vice versa.

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Keywords: Mixed Economy; Core; Vind's theorem; Coalitionally fair allocations.

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1 Introduction

The core of an economy is a solution concept that acknowledges the fact that coalitions of agents may corporate to improve their own welfare. In other words, for any allocation not belonging to the core, there is a coalition whose members achieve better commodity bundles that the non-core allocation by redistributing their initial endowments with themselves. In a classical exchange economy with a continuum of agents, the core coincides with the set of competitive allocations, refer to Aumann [2]. However, the equivalence theorem fails to hold, in general, if there are some non-negligible market participants in addition to the negligible ones (see Schitovitz [28]). The purpose of the paper is to study the core allocations in the exchange economy embodying a large number of agents some of which are non-negligible. Note that the market participants become non-negligible due to the following two reasons: (i) first reason is some agents may be endowed with an *exceptional* initial endowments, because their initial ownership of commodities are sufficiently large with respect to the total market endowment. This is typical in monopolistic or, more generally, in oligopolistic markets; and (ii) the second reason is, while the initial endowment is spread over of continuum of negligible agents, some of them may join forces and decide to act as a single agent in the form of cartels, syndicates, or similar institutions.

The economic activity is taken into account uncertainty, where agents subscribe to contracts at the time $\tau = 0$ (ex-ante) that are contingent upon the realized state of nature at time $\tau = 1$ (ex-post), in a way so that their expected payoff is maximized. In this paper, we consider an infinite dimensional commodity space, as it arises naturally due to several reasons: modeling allocations over an infinite time horizon, and economies with commodity differentiation, among others. We refer to Mas-colell and Zame [24] for more details. Our primary focus is an ordered Banach space having a non-empty positive interior. One major issue that arises while dealing with the main results is that Lyapunov's convexity theorem fails to hold in the exact form. The consumption set for each agent in each state to assumed to be an arbitrary subset of the commodity space, which may not have any lower bound. Thus, not only the private information restricts the trade of individuals in the ex-ante stage the structure of the consumption sets prevent us to apply strong monotonicity condition at certain bundles.

In the above setup, we study the veto power of arbitrary-sized coalitions for non-core allocations and the coalitional fairness of the core allocations. This significantly extends the scope of the theory, incorporating much larger class of models as it involves the four aspects together: negligible as well as non-negligible agents, infinite-dimensional commodity spaces, uncertainty with asymmetric information and arbitrary consumption set do not necessarily have lower bounds.

Extensions of the Schmeidler-Vind theorems: For an atomless economy with restricted consumption sets and asymmetric information, we investigate the size of the blocking coalition for a non-core allocation in our setup. This type of investigation goes back to the seminal contributions of Schmeidler [29] and Vind [33] in a framework with the positive cone of the Euclidean space as the consumption sets of agents and without uncertainty. More precisely, Schmeidler [29] showed that if a feasible allocation is not the core of the economy then it can be blocked by coalitions of small measures. Thus, the core (in particular, the set of competitive allocations) can be implemented only through the formation of small coalitions. Schmeidler's idea of a blocking mechanism was further extended by Vind [33] by showing that for any feasible allocation outside the core of an economy then for any measure ε less than the measure of the grand coalition there is a coalition S whose measure is exactly ε such that the non-core allocation is blocked by S. One of the implications of this theorem is normative in the following sense: as an arbitrarily large size of coalitions are entitled to block each non-core allocation, the core can be seen as a solution supported by an arbitrarily large majority of agents. Later, these results were extended to several frameworks by Bhowmik [4], Bhowmik and Cao [5, 6], Bhowmik and Graziano [8], Evren and Hüsseinov [13], Graziano and Romaniello [17], Hervés-Beloso et al. [19], Hervés-Beloso et al. [21], Pesce [26, 27] among others. Recently, Bhowmik and Graziano [9] have extended this result in a setting where agents' consumption sets are arbitrary subsets (without any lower bound restrictions) of a finite-dimensional space and ex-ante trades are defined in terms of some general restrictions. Such restrictions include two different scenarios: asymmetric information economies and asset market economies. In the present paper, we generalize the above result of Bhowmik and Graziano [9] to an economy with an infinite dimensional commodity space but only consider an asymmetric information scenario. Not only that, our paper also extends all of the above results in the following direction.

- First, we consider an ex-ante strong core allocation, which is a feasible allocation that cannot be weakly blocked by a non-null coalition. By definition, the ex-ante strong core allocation is an ex-ante core allocation, but the converse fails to hold, in general. Nevertheless, by adopting continuity and strong monotonicity of preferences, one can readily verify that two core notions are the same in a classical economy without uncertainty with asymmetric information in which the positive cone is the consumption set of each agent. However, in a model that involves either uncertainty with asymmetric information or arbitrary consumption sets, such a conclusion cannot be immediately drawn in the presence of continuity and strong monotonicity of preferences. The present paper deals with this issue and formulates a set of sufficient conditions that ensures the equivalence of two core notions in our framework. As a consequence, our Vind's theorem is also valid for the ex-ante strong core in a framework of an infinite dimensional commodity space.

- We show that the core of a mixed economy coincides with the core of an atomless economy derived by splitting each atom into a continuum of small agents, and vice versa. In view of this result and Vind's theorem for atomless economies, we can generalize Vind's theorem to a mixed economy by considering generalized coalitions. It is worthwhile to point out that this result is the first generalization of Vind's theorem in an economy with asymmetric information and a finite-dimensional commodity space, where the feasibility is defined as exact and the consumption set for each agent is an arbitrary subset of the commodity space.

Coalitional fairness of ex-core allocations: Next, we investigate the coalitional fairness of core allocations, which is property of equity, introduced by Gabszewicz in his seminal paper Gabszewicz [15], in which bundle comparisons are allowed between coalitions of agents according to the concept of coalitional envy.[1] According to Gabszewicz [15], an allocation is coalitionally unfair if a coalition is treated under the allocation in a discriminatory way by the market. More generally, an allocation is coalitionally fair if no coalition could benefit from achieving the net trade of some other coalition, which means under coalitional fairness, no coalition could redistribute among its members the net trade of any other coalition in a way that could assign a preferred bundle to each of its members. It is well-known that a core allocation is not necessarily coalitionally fair in a mixed economy, refer to Gabszewicz [15]. Thus, we restricts ourselves to coalitions containing either no large agents or all of them, and show that any core allocation is coalitionally fair in the sense that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents or vice versa. Therefore, despite their privileged initial position, large agents can not enforce a core allocation because this would render the allocation unfair towards some coalition of small agents, and vice versa. Related research in this direction either focuses on a finite-dimensional commodity space or an infinite dimensional commodity space with the positive cone as the consumption set of each agent, see Bhowmik [4], and Bhowmik and Graziano [9]. Thus, our result generalizes the above results to centain extent.

[1] See also Schmeidler and Vind [30] and Varian [32].

2 Description of the model

We consider a standard pure exchange economy with uncertainty and asymmetric information. We assume that the economic activity takes place over two periods $\tau = 0, 1$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where Ω is a finite set denoting all possible states of nature at time $\tau = 1$ and the σ-algebra $\mathscr{F}$ denotes all events. At time $\tau = 0$ (ex-ante stage) there is uncertainty about the state of nature that will be realized at time $\tau = 1$ (ex-post stage). At the ex-ante stage, agents arrange contract on redistribution of their initial endowments. At $\tau = 1$, agents carry out previously made agreements, and consumption takes place[2].

Economic agents: The space of **economic agents** is described by a complete probability space $(T, \mathscr{T}, \mu)$, where T represents the set of agents, the σ-algebra $\mathscr{T}$ represents the collections of allowable coalitions whose economic weights on the market are given by μ. Since $\mu(T) < \infty$, the set T of agents can be decomposed in the disjoint union of an atomless sector T_0 of non-influential (small or negligible) agents and the set T_1 of influential (large or non-negligible) agents, which is the union of at most countable family $\{A_1, A_2, \cdots\}$ of atoms of μ. Abusing notation, we also denote by T_1 the collection $\{A_1, A_2, \cdots\}$. Thus, the space of agents not only allow us to investigate in a unified manner the markets that are competitive and the markets that are not, but also deal with the simultaneous action of influential and non-influential agents. This general representation permits to cover simultaneously the case of an economy with a finite set of agents (when T_0 is empty and T_1 is finite), the case of an atomless economy (when T_1 is empty), the case of mixed markets in which an ocean of negligible agents coexists with few influential agents (when both T_0 and T_1 have positive measure). Moving from this representation, we can also identify two relevant subfamilies from $\mathscr{T}$ by defining

$$\mathscr{T}_0 := \{S \in \mathscr{T} : S \subseteq T_0\} \text{ and } \mathscr{T}_1 := \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

Thus, $\mathscr{T}_0$ is a subfamily of $\mathscr{T}$ containing no atoms whereas $\mathscr{T}_1$ is a subfamily of $\mathscr{T}$ containing all atoms. Finally, we denote by

$$\mathscr{T}_2 := \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ formed by coalitions containing either no atoms or all atoms.

Commodity Spaces: The **commodity space** in our model is an ordered separable Banach space with the interior of the positive cone is non-empty. We denote by $\mathbb{Y}$ the

[2]For simplicity, we assume that there are no endowments and thus no consumption at $\tau = 0$. Hence, agents are only concerned with allocating their second period ($\tau = 1$) endowments.

commodity space of our economy whereas the notation $\mathbb{Y}_+$ is employed to denote the positive cone of $\mathbb{Y}$. Let $\mathbb{Y}_{++}$ be the interior of $\mathbb{Y}_+$.

Defining an economy: We introduce a mixed economy with uncertainity and asymmetric information, and an ordered separable Banach space whose positive cone has non-empty interior as the commodity space.

Definition 2.1. An **economy** is defined as $\mathscr{E} := \{(X_t, \mathscr{F}_t, u_t, e(t,\cdot), \mathbb{P}_t) : t \in T\}$ with the following specifications:

(A) $X_t : \Omega \rightrightarrows \mathbb{Y}$ denotes the (state-contingent) consumption set of agent $t \in T$[3];

(B) $\mathscr{F}_t$ is the σ-algebra generated by a measurable partition $\mathscr{P}_t$ of Ω (i.e. $\mathscr{P}_t \subseteq \mathscr{F}$) denoting the private information of agent t;

(C) $u_t : \Omega \times \mathbb{Y} \to \mathbb{R}$ is the state-dependent utility function of agent t;

(D) $e(t,\cdot) : \Omega \to \mathbb{Y}$ is the random initial endowment of agent t;

(E) $\mathbb{P}_t : \Omega \to [0,1]$ is the prior of agent t.

Available Information and Expected Utilities: The family of all paritions of Ω is denoted by $\mathfrak{P}$. Since Ω is finite, $\mathfrak{P}$ has only finitely many different elements: $\mathscr{P}_1, \cdots, \mathscr{P}_n$. We assume that $T_i := \{t \in T : \mathscr{P}_t = \mathscr{P}_i\}$ is $\mathscr{T}$-measurable for all $1 \leq i \leq n$. For every $1 \leq i \leq n$, define $\mathscr{G}_i$ to be the set of all functions $\varphi : \Omega \to \mathbb{Y}$ such that φ is $\mathscr{P}_i$-measurable.[4] For any $x : \Omega \to \mathbb{Y}$, define the **ex-ante expected utility** of agent t by the usual formula

$$V_t(x) = \sum_{\omega \in \Omega} u_t(\omega, x(\omega))\mathbb{P}_t(\omega).$$

We now state our main assumptions to be used throughout the paper.

Assumptions: Consider an economy $\mathscr{E}$ as defined in Definition 2.1.

($\mathbf{A}_1$) For all $(t,\omega) \in T \times \Omega$, $X_t(\omega)$ is a closed convex cone.

($\mathbf{A}_2$) The correspondence $\mathbf{\Theta} : T \times \Omega \rightrightarrows \mathbb{Y}$, defined by $\mathbf{\Theta}(t,\omega) := X_t(\omega)$, is such that $\mathbf{\Theta}(\cdot,\omega)$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$.

[3] Notice that we do not impose non-negative constraints on consumption sets. Thus, short sales are allowed.

[4] By $\mathscr{P}_i$-measurability, we mean the measurability with respect to the σ-algebra generated by $\mathscr{P}_i$.

($\mathbf{A}_3$) The mapping $e(\cdot,\omega) : T \to \mathbb{Y}$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$ and $e(t,\omega)$ is an interior point of $X_t(\omega)$ for all $\omega \in \Omega$.

($\mathbf{A}_4$) The mapping $\varphi : T \to [0,1]^\Omega$, defined by $\varphi(t) = \mathbb{P}_t$, is $\mathscr{T}$-measurable.

($\mathbf{A}_5$) For all $(t,\omega) \in T_1 \times \Omega$, $u_t(\omega,\cdot)$ is quasi-concave.

($\mathbf{A}_6$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,\cdot)$ is continuous and for all $x \in \mathbb{Y}$, $t \mapsto u_t(\omega,x)$ is $\mathscr{T}$-measurable.

($\mathbf{A}_7$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,y) > u_t(\omega,x)$ for all $x,y \in X_t(\omega)$ with $y \geq x$ and $x \neq y$.

($\mathbf{A}_8$) For all $(t,\omega) \in T \times \Omega$, $x \in X_t$ and $\varepsilon > 0$, there is an $y \in \mathscr{G}_t \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x + y \in X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$[5].

($\mathbf{A}'_8$) For all $(t,\omega) \in T \times \Omega$, $x \in X_t$ and $\varepsilon > 0$, there is an $y \in \bigcap\{\varepsilon\mathscr{G}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x + y \in \mathrm{int}X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$.

Remark 2.2. The assumptions in ($\mathbf{A}_1$)-($\mathbf{A}_7$) are standard in the literature of general equilibrium in economies with asymmetric information and/ or restricted consumption sets. Assumptions ($\mathbf{A}_8$) and ($\mathbf{A}'_8$) are satisfied under the monotonity assumption whenever $X_t(\omega) = \mathbb{Y}_+$ for all $(t,\omega) \in T \times \Omega$.

3 The Schmeidler-Grodal-Vind theorems

Our aim in this section is to introduce the ex-ante (Aubin) core allocations in an economy with a mixed measure space of agents by considering ordinary (generalized) coalitions and providing characterizations of ex-ante (Aubin) core allocations by means of the size of coalitions in the sense of Schmeidler [29], Grodal [18], and Vind [33] either in an economy containing a continuum of negligible agents or in a economy which comprised of both negligible and non-negligible agents.

3.1 Defining (Aubin) core allocations

In this subsection, we introduce the notion of **ex-ante (Aubin) core** for a two-period economy with uncertainty either by considering strong blocking or weak blocking. We assume implicitly that the trade takes place at time $\tau = 0$ and that contracts are binding: they are carried out after the resolution of uncertainty and there is no possibility

[5] $\mathbb{B}(0,\varepsilon)$ denotes the closed ball centered at 0 and radius ε in $\mathbb{Y}$.

of their renegotiation. Moreover, the consumption of each agent is compatible with her private information. In what follows, we deal with the relationship between the two different notions of core allocations. We start introducing the concept of an allocation, which is a specification of the amount of commodities assigned to each agent.

Definition 3.1. An **allocation** in $\mathscr{E}$ is a Bochner integrable function $f : T \times \Omega \to \mathbb{Y}$ such that

(i) $f(t, \omega) \in X_t(\omega)$ for all $(t, \omega) \in T \times \Omega$; and

(ii) $f(t, \cdot) \in \mathscr{G}_i$ for all $(t, \omega) \in T_i \times \Omega$ and all $1 \leq i \leq n$.

It is said to be **feasible** if $\int_T f(\cdot, \omega) d\mu = \int_T e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$. We assume that e is an allocation.

An element of $\mathscr{T}$ with positive measure is interpreted as an **ordinary coalition** or simply, a **coalition** of agents. Each $S \in \mathscr{T}$ can be regarded as a function $\chi_S : T \to \{0, 1\}$, defined by

$$\chi_S(t) := \begin{cases} 1, & \text{if } t \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\chi_S(t)$ means the degree of membership of agent $t \in T$ to the coalition S. Following this interpretation for an ordinary coalition, it is natural to introduce a family of **generalized coalitions** as follows (see [25]). To this end, for any function $\gamma : T \to \mathbb{R}$, define the **support** of the function γ as

$$S_\gamma = \{t \in T : \gamma(t) \neq 0\}.$$

An **Aubin** or a **generalized coalition** of $\mathscr{E}$ is a simple, measurable function $\gamma : T \to \mathbb{R}$ whose support has a positive measure. It is worthwhile to point out that $\gamma(t)$ represents the share of resources employed by agent t. By identifying $S \in \mathscr{T}$ with χ_S, we can treat S as a generalized coalition. The **weight of a generalized coalition** γ, denoted by $\mu^A(\gamma)$, is given by $\mu^A(\gamma) = \int_T \gamma \, d\mu$. For any ordinary coalition, this weight simply coincides with the measure of the coalition itself.

Our first notion of (Aubin) core aims to study the blocking mechanism under the assumptions that a coalition deviates from a proposed allocation if its members guarantee a *strictly better* commodity bundle for themselves by the redistribution.

Definition 3.2. An allocation f is **ex-ante blocked by a generalized coalition** γ if there is an allocation g such that $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S_γ, and

$$\int_T \gamma g(\cdot, \omega) d\mu = \int_T \gamma e(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. The **ex-ante Aubin core** of $\mathscr{E}$, denoted by $\mathscr{C}^A(\mathscr{E})$, is the set of feasible allocations that are not ex-ante blocked by any generalized coalition. If the generalized coalitions are replaced with ordinary coalitions, the corresponding set of allocations is called the **core** of $\mathscr{E}$, denoted by $\mathscr{C}(\mathscr{E})$.

The next formalization of core differs from the earlier one in the sense that agents within a blocking generalized coalition are not *worse-off* by the re-distribution whereas some are *strictly better-off*. To formally define, we consider a **sub-coalition** of a generalized coalition γ is a generalized coalition ρ such that $S_\rho \subseteq S_\gamma$.

Definition 3.3. An allocation f is **ex-ante weakly blocked by a generalized coalition** γ if there is a sub-coalition ρ of γ and an allocation g such that

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_ρ;

(ii) $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ μ-a.e. on S_γ; and

(iii) $\int_T \gamma g(\cdot,\omega)d\mu = \int_T \gamma e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$.

The **ex-ante Aubin strong core** of $\mathscr{E}$, denoted by $\mathscr{C}^{AS}(\mathscr{E})$, is the set of feasible allocations that are not ex-ante weakly blocked by any generalized coalition. If the generalized coalitions are replaced with ordinary coalitions, the corresponding set of allocations is called the **ex-ante strong core** of $\mathscr{E}$, denoted by $\mathscr{C}^S(\mathscr{E})$.

Recognized that if an allocation f is ex-ante blocked by a generalized coalition γ then it is also ex-ante weakly blocked by the same coalition. For the converse, we additionally assume in our next result that if an allocation f is ex-ante weakly blocked by a generalized coalition γ via some allocation g and if ρ is a sub-coalition of γ in which members of S_ρ strictly prefer g to f then the information available to both coalitions is the same, i.e, $\mathbb{I}_{S_\gamma} = \mathbb{I}_{S_\rho}$, where $\mathbb{I}_G := \{i : \mu(G_i) > 0\}$ for any ordinary coalition G in which $G_i := G \cap T_i$ for all $1 \leq i \leq n$. The basic intuition is that members belonging to R_i, where $R := S_\gamma \setminus S_\rho$ and $i \in \mathbb{I}_R$, can be allocated $\mathscr{P}_i$-measurable consumption bundles that give higher utilities by reducing the utility level of the members of $S_\rho \cap T_i$ due to continuity and strong monotonicity. However, such an argument cannot be made easily in arbitrary consumption sets. In what follows, we establish this result in a continuum economy by showing that if an allocation is ex-ante Aubin blocked by a generalized coalition then it can also be blocked by an ordinary coalition. In this regard, Lemma 6.1 and Lemma 6.2 in Appendix play vital roles.

Proposition 3.4. *Let $\mathscr{E}$ be a continuum economy satisfying* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*. Suppose that γ is a generalized coalition and g is an allocation such that $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ μ-a.e. on S_γ. Assume further that the measurable set*

$$B := \{t \in S_\gamma : V_t(g(t,\cdot)) > V_t(f(t,\cdot))\}$$

has strictly positive measure and $\mathbb{I}_{S_\gamma} = \mathbb{I}_B$. Then there are coalitions E, R, an element $\lambda_0 \in (0,1)$, an element $\eta > 0$, and an allocation y such that

(i) $R \subseteq E \subseteq S_\gamma$ *and* $\mathbb{I}_R = \mathbb{I}_E = \mathbb{I}_{S_\gamma}$;

(ii) $\int_E (y-e)\,d\mu = \lambda_0 \int_T \gamma(g-e)\,d\mu$;

(iii) $y(t,\cdot) + z \in X_t$ *for all* $z \in \mathbb{B}(0,\eta)^\Omega$ *and μ-a.e. on R; and*

(iv) $V_t(y(t,\cdot)+z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta)^\Omega$ *and μ-a.e. on R.*

Proof. The proof of the proposition is relegated to Appendix. □

Corollary 3.5. *For a continuum economy $\mathscr{E}$ satisfying* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *and an allocation f, assume that f is ex-ante weakly blocked by a generalized coalition γ via some allocation g satisfying $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_ρ for some sub-coalition ρ of γ satisfying $\mathbb{I}_{S_\gamma} = \mathbb{I}_{S_\rho}$. Then there is a coalition E such that f is blocked by E.*

Remark 3.6. Notice that, in the proof of Proposition 3.4, the number λ_0 can be chosen sufficiently close to 1. Moreover, it also follows that, if γ is replaced with ordinary coalition S, then the coalition E can be choosen so that $\mu(E) \geq \lambda_0 \mu(S)$.

In view of the above proposition, we have the following theorem whose proof is immediate.

Theorem 3.7. *Suppose that $\mathscr{E}$ is a continuum economy satisfying* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*. Then* $\mathscr{C}^A(\mathscr{E}) = \mathscr{C}(\mathscr{E})$

3.2 The size of blocking coalitions in a continuum economy

In this subsection, we address the issues related to the size of a blocking coalition, extending the corresponding results of Schmeidler [29], Grodal [18] and Vind [33] to the case of a continuum economy with arbitrary consumption sets and private information.

Extending the Schmeidler theorem: The insight of Schmeidler theorem was that, in a continuum economy, if a feasible non-core allocation is blocked by some coalition S then it can also be blocked by a coalition of any given measure less than that of

S. The immediate implication of this theorem includes the fact that the core (and thus, the set of competitive allocations) can be implemented by the formation of small coalitions only. In what follows, we extend this result to our framework. This definitely extends the corresponding result of Bhowmik and Graziano [9] to a certain extent. It is worthwhile to point out that the techniques adopted in the proof of Bhowmik and Graziano [9] are not appropriate in our setup of infinitely many commodities. Thus, in order to obtain the Schmeidler theorem in our framework, we first establish the following proposition. This proposition can be considered an extension of the Lyapunov convexity theorem.

Proposition 3.8. *Let $\mathscr{E}$ be a mixed economy and let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *be satisfied. Suppose that ψ, f and g are allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition $S \in \mathscr{T}_0$ and $0 < \delta < 1$. Assume further that $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$ for some sub-coalition R of S satisfying $\mathbb{I}_R = \mathbb{I}_S$. Then there are an $\eta_0 > 0$, two coalitions B and C, and an allocation φ such that*

(i) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ *and* $\mu(B) = \delta\mu(S)$;

(ii) $\varphi(t,\omega) + z(\omega) \in X_t(\omega)$ *for all* $z(\omega) \in \mathbb{B}(0,\eta_0)$ *and* $(t,\omega) \in C \times \Omega$;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta_0)^\Omega$ *and μ-a.e. on C;*

(iv) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ *μ-a.e. on $B \setminus C$; and*

(v) $\int_B (\varphi(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_S (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$ *for all* $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

The following theorem is an immediate implication of the above proposition, which extends Schmeidler's [29] theorem to our framework.

Theorem 3.9. *Consider a continuum economy $\mathscr{E}$ and assume that the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *are satisfied. Let f be an allocation of $\mathscr{E}$ blocked by some coalition S. Then, for any $\varepsilon \in (0, \mu(S))$, there is a coalition R such that $\mu(R) = \varepsilon$ and f is blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

Extending the Grodal Theorem: Given an $\varepsilon > 0$, it was shown in [**?**] for an atomless economy that the blocking coalition S can be chosen as a union of finitely many disjoint sub-coalitions, each of which having measure and diameter less than ε. A coalition whose measure and diameter are less than ε intuitively means that the coalition consists of relatively *few* agents and that the agents in the coalition resemble one another in chosen characteristics. Next, we extend this result to a club economy with an atomless measure space of agents, where the consumption sets of agents' are arbitrary subsets of an ordered Banach space having non-empty interior of the positive cone.

Theorem 3.10. *Let $\mathscr{E}$ be a continuum economy such that $\mathscr{E}$ satisfies $(\mathbf{A}_1)$-$(\mathbf{A}_8)$. Suppose further that T is endowed with a pseudometric which makes T a separable topological space such that $\mathscr{B}(T) \subseteq \mathscr{T}$. If f is an allocation and R is a coalition blocking f then there exists some $\alpha > 0$ such that any coalition $S \subseteq R$ satisfying $\mu(R \setminus S) < \alpha$ blocking f. Furthermore, for any $f \notin \mathscr{C}(\mathscr{E})$ and any $\varepsilon, \delta > 0$, there exists a coalition S with $\mu(S) \leq \varepsilon$ blocking f and $S = \bigcup_{i=1}^{n} S_i$ for a finite collection of coalitions $\{S_1, \cdots, S_n\}$ with diameter of S_i smaller than δ for all $i = 1, \cdots, n$.*

Extending the Vind theorem: Vind's theorem (refer to [33]) states that, in a continuum economy, if a feasible allocation is not in the core of the economy then there is a blocking coalition of any given measure less than the measure of the grand coalition. Thus, the core allocations (and hence, the competitive allocations) can also be characterized by means of coalitions of arbitrarily large sizes. We now intend to show a similar result in our framework. To this end, we first establish the following result, which claims that if an allocation is blocked by a coalition S via some allocation g then there is another allocation h in which everyone is better off than what she gets under f. This Proposition extends the corresponding results in Bhowmik and Cao [7] and Hervés-Beloso and Moreno-García [22].

Proposition 3.11. *Let $\mathscr{E}$ be a continuum economy such that the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ are satisfied. Suppose that f and g are two allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S with $g(t,\omega)$ being an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$ for some sub-coalition R of S satisfying $\mathbb{I}_R = \mathbb{I}_S$. Then, for any $0 < \delta < 1$, there exists some allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, $h(t,\cdot)$ is an interior point of X_t for all $t \in G$ for some sub-coalition G of S with $\mathbb{I}_G = \mathbb{I}_S$, and*

$$\int_S h(\cdot,\omega) d\mu = \int_S (\delta g(\cdot,\omega) + (1-\delta) f(\cdot,\omega)) d\mu$$

for all $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

Corollary 3.12. Consider now a mixed economy where all large agents have continuous and quasi-concave utility functions. For any large agent A and $x, y \in X_A$, if

$V_A(y) > V_A(x)$ and $0 < \delta < 1$ then, by Lemma 5.26 of Aliprantis and Border [1], we have $V_A(\delta y + (1-\delta)x) > V_A(x)$. In view of this, the conclusion of Proposition 3.11 can be obtained in a mixed model.

Next, we formulate a version of Vind's (1972) theorem on blocking by an arbitrary coalition.

Theorem 3.13. *Consider a continuum economy $\mathscr{E}$ in which the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_8')$ are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}(\mathscr{E})$. Then for any $\varepsilon \in (0,1)$, there is some coalition R such that $\mu(R) = \varepsilon$ and f is blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

Remark 3.14. We now complete the proof by replacing the assumption $[\mathbf{A}_8']$ with $\mathbb{I}_S = \mathbb{I}_T$ and $[\mathbf{A}_8]$. Let $D := T \setminus S$. For each $i \in \mathbb{I}_D$, there is some coalition $F_i \in \mathscr{T}_{D_i}$ such that $\mu(F_i) = \delta\mu(D_i)$ and

$$b_i(\omega) := \delta \int_{D_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu - \int_{F_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \eta\delta\mu(C_i)\right).$$

Define $z : C \times \Omega \to \mathbb{Y}$ by letting $z(t,\omega) := \frac{b_i(\omega)}{\delta\mu(C_i)}$ if $(t,\omega) \in R_i \times \Omega$ and $i \in \mathbb{I}_D$; and $z(t,\omega) := 0$, otherwise. Let $\tilde{g} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\tilde{g}(t,\omega) := \begin{cases} g(t,\omega) - z(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

By Proposition 3.11, there exist some $\mathscr{G}$-assignment h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_S h(\cdot,\omega) d\mu = \int_S (\delta\tilde{g}(\cdot,\omega) + (1-\delta) f(\cdot,\omega)) d\mu$$

for all $\omega \in \Omega$. We define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by

$$y(t,\omega) := \begin{cases} \psi(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be raedily verified that f is blocked by the coalition $E := F \cup S$ via y. □

3.3 The size of blocking coalitions in a mixed economy

In this subsection, we generalize the main results of Subsection 3.2 to a mixed economy. To this end, we first associate $\mathscr{E}$ with an atomless economy $\widetilde{\mathscr{E}}$ and study the connection between the ex-ante (Aubin) core allocations of these two economies. This extends the result of Greenberg and Shitovitz [14] and some of its follow-up papers as mentioned in Section 1.

Interpretation via an atomless economy: Given the economy $\mathscr{E}$, the economy $\widetilde{\mathscr{E}}$ is obtained by *splitting* each large agent into a continuum of small agents whose characteristics are the same as that of large agent. Therefore, the space of agents of $\widetilde{\mathscr{E}}$, denoted by $(\widetilde{T}, \widetilde{\mathscr{T}}, \widetilde{\mu})$, satisfies the following: (i) $\widetilde{T}_0 = T_0$ and $\widetilde{\mu}(\widetilde{T}_1) = \mu(T_1)$, where $\widetilde{T}_1 := T \setminus T_0$; (ii) $\widetilde{\mathscr{T}}$ and $\widetilde{\mu}$ are obtained by the direct sum of $\mathscr{T}$ and μ restricted to T_0 and the Lebesgue atomless measure space over $\widetilde{T}_1$; and (iii) each atom A_i one-to-one corresponds to a Lebesgue measurable subset $\widetilde{A}_i$ of $\widetilde{T}_1$ such that $\mu(A_i) = \widetilde{\mu}(\widetilde{A}_i)$, where $\{\widetilde{A}_i : i \geq 1\}$ can be expressed as the disjoint union of the intervals $\{\widetilde{A}_i : i \geq 1\}$ given by $\widetilde{A}_1 := [\mu(T_0), \mu(T_0) + \mu(A_1))$, and

$$\widetilde{A}_i := \left[\mu(T_0) + \mu\left(\bigcup_{j=1}^{i-1} A_j\right),\ \mu(T_0) + \mu\left(\bigcup_{j=1}^{i} A_j\right)\right),$$

for all $i \geq 2$. Furthermore, the space of states of nature and the commodity space of $\widetilde{\mathscr{E}}$ are the same as those of $\mathscr{E}$. Finally, the characteristics $(\widetilde{X}_t \widetilde{\mathscr{F}}_t, \widetilde{u}_t, \widetilde{e}(t,\cdot), \widetilde{\mathbb{P}}_t)$ of each agent $t \in \widetilde{T}$ in $\widetilde{\mathscr{E}}$ is defined as follows:

$$\widetilde{X}_t := \begin{cases} X_t, & \text{if } t \in T_0; \\ X_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{\mathscr{F}}_t := \begin{cases} \mathscr{F}_t, & \text{if } t \in T_0; \\ \mathscr{F}_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{u}_t := \begin{cases} u_t, & \text{if } t \in T_0; \\ u_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{e}(t,\cdot) := \begin{cases} e(t,\cdot), & \text{if } t \in T_0; \\ e(A_i,\cdot), & \text{if } t \in \widetilde{A}_i, \end{cases}$$

and

$$\widetilde{\mathbb{P}}_t := \begin{cases} \mathbb{P}_t, & \text{if } t \in T_0; \\ \mathbb{P}_{A_i}, & \text{if } t \in \widetilde{A}_i. \end{cases}$$

We now introduce some notations for the rest of the section. To an allocation f in $\mathscr{E}$, we associate an allocation $\widetilde{f} := \Xi[f]$ in $\widetilde{\mathscr{E}}$, defined by

$$\widetilde{f}(t,\omega) := \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ f(A_i,\omega), & \text{if } (t,\omega) \in \widetilde{A_i} \times \Omega. \end{cases}$$

Reciprocally, for each allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, we define an allocation $f := \Phi[\widetilde{f}]$ in $\mathscr{E}$ such that

$$f(t,\omega) := \begin{cases} \widetilde{f}(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ \frac{1}{\widetilde{\mu}(\widetilde{A_i})} \int_{\widetilde{A_i}} \widetilde{f}(\cdot,\omega)\, d\widetilde{\mu}, & \text{if } t = A_i \text{ and } \omega \in \Omega. \end{cases}$$

Recognized that if f is a feasible allocation in $\mathscr{E}$ then $\Xi[f]$ is a feasible allocation in $\widetilde{\mathscr{E}}$. Similarly, for each feasible allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, the allocation $\Phi[\widetilde{f}]$ is feasible in $\mathscr{E}$.

We show that an allocation is in the ex ante core of a mixed economy assigns indifferent consumption plans to all large agents. This is due to the fact that all agents have the same characteristics.

Proposition 3.15. *Let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ *be satisfied for a mixed economy* $\mathscr{E}$. *Let R be a coalition in* $\mathscr{T}_1$[6] *having the same characteristics. If f is in the ex ante core of $\mathscr{E}$ then* $V_t(f(t,\cdot)) = V_t(\mathbf{x}_f)$ *μ-a.e. on R, where*

$$\mathbf{x}_f(\omega) := \frac{1}{\mu(R)} \int_R f(\cdot,\omega) d\mu$$

for all $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

Remark 3.16. If $\mathbb{Y}$ is finite dimensional then one can dispense with the assumption $(\mathbf{A}'_8)$. In fact, the assumption $(\mathbf{A}'_8)$ help us to apply Proposition 3.8 in the proof of Proposition 3.15. In the case of finite dimension, we can just use $(\mathbf{A}_8)$ and apply the Lyapunov convexity theorem instead of Proposition 3.8.

Lemma 3.17. *Let $\mathscr{E}$ be a continuum economy and let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_7)$ *be satisfied. Suppose that f and g are allocations such that* $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ *μ-a.e. on some coalition S and $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$ for some sub-coalition R of S satisfying* $\mathbb{I}_R = \mathbb{I}_S$. *Assume further that* $\mu(S \cap H) \geq \alpha$ *for some coalition H of $\mathscr{E}$ and some $\alpha > 0$. Then there are a coalition B and an allocation h such that f is blocked by B via h and* $\mu(B \cap H) = \alpha$.

[6] If T_1 is empty then R contains only negligible agents.

Proof. The proof of the lemma is relegated to Appendix. $\square$

Proposition 3.18. *Let $\mathscr{E}$ be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8')$. *If $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$ then $f := \Phi[\widetilde{f}] \in \mathscr{C}^A(\mathscr{E})$.*

Proof. The proof of the proposition is relegated to Appendix. $\square$

Proposition 3.19. *Let $\mathscr{E}$ be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8')$. *Suppose also that $R \in \mathscr{T}_1$ is a coalition having the same characteristics. Then $f \in \mathscr{C}(\mathscr{E}) \Rightarrow \widetilde{f} := \Xi[f] \in \mathscr{C}(\widetilde{\mathscr{E}})$ if either of the following two conditions are true:*

(i) *$R = T_1$ has at least two elements.*

(ii) *T_1 has exactly one element and $\mu(R \setminus T_1) > 0$.*

Proof. The proof of the proposition is relegated to Appendix. $\square$

Extending the Grodal Theorem: Next, an extension of Theorem 3.10 to an economy with a mixed measure space of agents is presented. Basically, we show that for any given $\varepsilon, \delta > 0$, there is a generalized coalition γ whose measure is less than ε and γ can be expressed as some of the pairwise disjoint generalized coalitions each of its diameters is less than δ. To this end, we say that two generalized coalitions γ_1 and γ_2 are **disjoint** if $(\gamma_1 \wedge \gamma_2)(t) := \min\{\gamma_1(t), \gamma_2(t)\} = 0$ for all $t \in T$. As a consequence of this, we have $S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$. Following Gerla and Volpe [?] (see also Bhowmik and Graziano [8]), the **diameter** of a generalized coalition γ is defined by

$$\operatorname{diam}(\gamma) := \sup\left\{\min\{\alpha, \beta\}\|a - b\| : \lambda_a^\alpha, \lambda_b^\beta \text{ are fuzzy points of } \gamma\right\},$$

where a **fuzzy point** λ_a^ξ is a function $\lambda_a^\xi : T \to (0, 1]$ for each $a \in T$ and $\xi \in (0, 1]$, such that $\lambda_a^\xi(t) = 0$ if $t \neq a$ and $\lambda_a^\xi(t) = \xi$ if $t = a$.

Theorem 3.20. *Let $\mathscr{E}$ be a mixed economy such that $\mathscr{E}$ satisfies* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$. *Suppose further that T is endowed with a pseudometric which makes T a separable topological space such that $\mathscr{B}(T) \subseteq \mathscr{T}$ and $f \notin \mathscr{C}^A(\mathscr{E})$. For any $\varepsilon, \delta > 0$, there exist a generalized coalition γ with $\mu^A(\gamma) \leq \varepsilon$ and a finite collection $\{\gamma_1, \cdots, \gamma_n\}$ of pairwise disjoint characteristics functions*[7] *of ordinary coalitions such that the diameter of γ_i smaller than δ and $S_{\gamma_i} \subseteq T_0$ for all $i \in \{1, \cdots, n\}$, f is blocked by γ and*

$$\gamma = \begin{cases} \sum_{i=1}^n \gamma_i + \sum_{k \in \mathbb{K}} \alpha_k \chi_{A_k}, & \text{if } \mathbb{K} \neq \emptyset; \\ \sum_{i=1}^n \gamma_i, & \text{if } \mathbb{K} = \emptyset, \end{cases}$$

where $\mathbb{K} := \{k : A_k \in S_\gamma\}$ and $\alpha_k \in (0, 1]$ if $k \in \mathbb{K}$.

[7] Thus, γ_i can be treated as an ordinary coalition.

Remark 3.21. Notice that each sub-coalition γ_i of $\gamma|_{T_0}$ is chosen as the set of agents sharing their full initial endowments and the diameter of γ_j is exactly the same as that of S_{γ_j}. Therefore, agents in γ_i have δ-similar characteristics in the ordinary sense, implying the second part of Theorem 3.10 as a simple corollary. For a mixed economy, the blocking coalition γ contains atmost finitely many atoms, which means that each sub-coalition of diameter δ is either δ-similar non-atomic agents or a single atom, not a neighborhood of points contained in an atom. Since an atom can be treated as δ-similar to itself for any $\delta > 0$, our approach for taking a neighborhood containing a single atom does not violate Grodal's requirements. Therefore, similar to Grodal [18]. Therefore, it can be considered as an extension Grodal's theorem to a mixed economy.

Extending the Vind Theorem: In view of above results and Theorem 3.13, one can readily derive the following result as in Bhowmik and Graziano [8].

Theorem 3.22. *Consider a mixed economy $\mathscr{E}$ in which the assumptions* $(\mathbf{A}_1)$*-*$(\mathbf{A}'_8)$ *are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}^A(\mathscr{E})$. Then for any $\varepsilon \in (0, 1)$, there is some Aubin coalition γ such that $\mu^{\mathrm{A}}(\gamma) = \varepsilon$ and f is Aubin blocked by γ.*

Proof. The proof of the theorem is relegated to Appendix. □

Remark 3.23. It is clear from the proof of Theorem **??** that for an $\varepsilon > 0$, there exists a generalized coalition γ such that f is blocked by γ, $\tilde{\mu}(\gamma) = \varepsilon$ and $\gamma(t) = 1$ if $t \in S_\gamma \cap T_0$. Thus, as in the case of atomless economies, non-atomic agents in S_γ use their full initial endowments. However, the atomic agents in γ only use parts of their initial endowments and the share α_i for an atomic agent A_i depends on the size of γ. So, Theorem **??** can be treated as an extension of that in an atomless economy.

4 Coalitional fairness of core allocations

In this section, we study the coalitional fairness of the ex ante core allocations. This means that the stability of an allocation under the coalitional improvement guarantees that it is also equitable in the sense that no coalition *envies* the net trade of any other disjoint coalition. The concept of a *coalitionally fair allocation* was first proposed by Gabszewicz in his seminal paper Gabszewicz [15] for an exchange economy, where an allocation is said to be **coalitionally fair** if no coalition can redistribute among its members the net trade of any other coalition, in such a way that each of them is better-off. It is worthwhile to pointing out that competitive equilibrium allocations are coalitionally fair, which also belongs to the core of the economy. For a classical economy with an atomless measure space of agents, Aumman's core-equivalence theorem guarantees that the set of coalitionally fair allocations coincides with the core of the economy. However, it is unclear to us whether such a result holds true whenever the consumption sets are not bounded from below and restrictions imposed on ex ante trade. On the other hand, in a classical mixed economy, it is well known (refer to Shitovitz [28]) that the core-equivalence theorem does not hold in general, and one should expect a kind of *exploitation* of small agents by large agents. Fortunately, Bhowmik and Graziano [9] obtained a partial result on the coalitional fairness of the core allocations in the sense that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents and vice versa in a framework similar to us but only for finitely many commodities. This result extends Theorem 2 in Gabszewicz [15], who established the result in a classical deterministic economy with finitely many commodities.

The first notion fairness requires that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents.

Definition 4.1. A feasible allocation f is called $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S \in \mathscr{T}_0$, $E \in \mathscr{T}_1$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $\displaystyle\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

In what follows, we show that any allocation in the ex-ante core is coalitionally fair in a way that no coalition of small agents can redistribute among its members the net trade of any other coalition containing all large agents, in such a way that each of them is better-off.

Theorem 4.2. *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_?)$ *be satisfied. Then any allocation in the ex-ante core of* $\mathscr{E}$ *is* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair.*

Proof. The proof of the theorem is relegated to Appendix. □

In the next notion fairness, the role of coalitions are opposite, i.e., no coalition containing all large agents envies the net trade of a disjoint coalition of small agents.

Definition 4.3. A feasible allocation f is called $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S \in \mathscr{T}_1$, $E \in \mathscr{T}_0$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $$\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$$

To prove that any ex-ante core allocation is $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair we establish the following lemma.

Lemma 4.4. *Assume that f and h are two allocations such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S. Then there exist $0 < \lambda, \eta < 1$, a sub-coalition R of S and an allocation y such that*

(i) *$y(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;*

(ii) *$V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on S;*

(iii) *$V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on $S \setminus R$; and*

(iv) *$\int_S (y - e)d\mu = (1-\lambda)\int_S (h-e)d\mu$.*

Proof. The proof of the theorem is relegated to Appendix. □

The following theorem demonstrates that any allocation in the ex-ante core is coalitionally fair in a way that no coalition comprised of all large agents can redistribute among its members the net trade of any other coalition containing only small agents, in such a way that each of them is better-off.

Theorem 4.5. *Let $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ be satisfied. Then any allocation in the ex-ante core is $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair.*

Proof. The proof of the theorem is relegated to Appendix. □

5 Concluding remarks

We have investigated the blocking of any allocation not belonging to the (strong) core of an atomless economy with asymmetric information and infinitely many commodities. It has been shown that the ex-ante (strong) core can be characterized by means of coalitions of a given size less than that of the grand coalition, extending the results in Schmeidler [29] and Vind [33]. Thus, it is enough to consider coalitions of either arbitrarily small sizes or arbitrarily large sizes to find the (strong) core of a continuum economy. It is further shown that Grodal's theorem (refer to Grodal [18]) holds true in a continuum economy. All of these results have also been carried out to a mixed economy

by considering generalized coalitions instead of standard coalitions. To do this, we associated an atomless economy with a mixed economy and showed that the Aubin (strong) core of the mixed economy is equivalent to the (strong) core of the atomless economy. Lastly, we proved that any (strong) core is coalitionally fair in the sense that no coalition of small agents envies the net trade of a coalition containing all large agents and vice versa. All of these results have been obtained without assuming the separability condition on the commodity space. The difficulties that arise in all of these results are due to the following facts: (i) the standard Lyapunov convexity theorem does not hold, it holds only in a weak form in an infinite dimensional commodity space; (ii) the consumption sets are arbitrary subsets of the commodity space which may not satisfy the free-disposal condition $X_t + \mathbb{Y}_+^{\Omega} \subseteq X_t$ for $t \in T$, thus the strong monotonicity condition may not be applied whenever required; and (iii) information asymmetry, which further restricts the consumptions of each agent. All of these difficulties are taken care of by establishing several key propositions, out of which Proposition 3.4 has its own interest. In fact, as a consequence of this proposition, we conclude that the ex-ante strong core is equivalent to the ex-ante core in a continuum economy.
We close with further remarks dealing with possible extensions and applications of our results.

Remark 5.1. Hervés-Beloso and Moreno-García [22] provides a characterization of Walrasian allocations in terms of robustly efficient allocations in a continuum economy. Later, it is extended by Bhowmik and Cao [7] to a mixed economy with asymmetric information and an ordered separable Banach space whose positive cone has an interior point by applying Vind's theorem and a result similar to Proposition 3.11. Thus, in view of Proposition 3.11 and Theorem 3.13, it would be interesting to know whether the main result of Bhowmik and Cao [7] can be extended to our framework.

6 Appendix

Lemma 6.1. *Suppose that f and g are two allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S with $g(t,\omega)$ being an interior point of $X_t(\omega)$ for all $(t,\omega) \in S \times \Omega$. Then for any $0 < \varepsilon < \mu(S)$, there are some $\eta > 0$ and a sub-coalition R of S such that*

(i) $\mu(R) > \mu(S) - \varepsilon$;

(ii) $g(t,\omega) + z(\omega) \in X_t(\omega)$ *for all* $z(\omega) \in \mathbb{B}(0,\eta)$ *and* $(t,\omega) \in R \times \Omega$; *and*

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta)^{\Omega}$ *and* μ-*a.e. on* R.

Proof. Define a correspondence $\mathbf{\Upsilon} : S \rightrightarrows \mathbb{R}_+$ by letting

$$\mathbf{\Upsilon}(t) := \left\{\eta \in (0,\infty) : g(t,\cdot) + z \in X_t \text{ and } V_t(g(t,\cdot)+z) > V_t(f(t,\cdot)) \text{ for all } z \in \mathbb{B}(0,\eta)^{\Omega}\right\}.$$

By the continuity of preferences and the fact that $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in S \times \Omega$, we have $\mathbf{\Upsilon}(t) \neq \emptyset$ μ-a.e. on S. As $\mathbf{\Upsilon}(t)$ is bounded from above, the function $\varphi : S \to \mathbb{R}_+$, defined by $\varphi(t) := \sup \mathbf{\Upsilon}(t)$, is well-defined. We show that φ is $\mathscr{T}_S$-measurable. To this end, first note that the function $\psi : S \times \mathbb{Y}^{\Omega} \to \mathbb{R}$, defined by $\psi(t,z) := V_t(g(t,\cdot)+z) - V_t(f(t,\cdot))$, is a Carathéodory function, and thus, it is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable. Define a correspondence $\mathbf{G} : S \rightrightarrows \mathbb{Y}^{\Omega}$ by letting

$$\mathbf{G}(t) := \left\{z \in \mathbb{Y}^{\Omega} : \psi(t,z) > 0\right\}.$$

It follows that $\mathbf{G}$ is non-empty valued and has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable graph, as $\mathrm{Gr}_{\mathbf{G}} = \psi^{-1}(0,\infty)$. Consider a correspondence $\mathbf{H} : S \rightrightarrows \mathbb{Y}^{\Omega}$ defined by

$$\mathbf{H}(t) := \left\{z \in \mathbb{Y}^{\Omega} : g(t,\omega) + z(\omega) \in X_t(\omega) \text{ for all } \omega \in \Omega\right\}.$$

Due to the closeness of $X_t(\omega)$, $\mathbf{H}(t)$ can be equivalently expressed as[8]

$$\mathbf{H}(t) = \left\{z \in \mathbb{Y}^{\Omega} : \mathrm{dist}(g(t,\omega)+z(\omega), X_t(\omega)) = 0 \text{ for all } \omega \in \Omega\right\}.$$

In view of the fact that $0 \in \mathbf{H}(t)$, we have $\mathbf{H}(t) \neq \emptyset$ μ-a.e. on S. Moreover, $\mathrm{Gr}_{\mathbf{H}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable as $\mathrm{Gr}_{\mathbf{H}} = y^{-1}(\{0\})$, where $y : S \times \mathbb{Y}^{\Omega} \to \mathbb{R}$ is defined by $y(t,\omega) := \mathrm{dist}(g(t,\omega)+z(\omega), X_t(\omega))$, is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable. Finally, define a correspondence $\mathbf{\Phi} : S \rightrightarrows \mathbb{Y}^{\Omega}$ such that $\mathbf{\Phi}(t) := \mathbf{G}(t) \cap \mathbf{H}(t)$ for all $t \in S$. As $0 \in \mathbf{\Phi}(t)$, we have $\mathbf{\Phi}(t) \neq \emptyset$ μ-a.e. on S. Moreover, $\mathrm{Gr}_{\mathbf{\Phi}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable. Analogously, the correspondnce $\mathbf{\Theta}_\eta : S \rightrightarrows \mathbb{Y}^{\Omega}$, defined by $\mathbf{\Theta}_\eta(t) := \mathbb{B}(0,\eta)^{\Omega}$, has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^{\Omega})$-measurable graph, for all $\eta > 0$. Thus,

$$\mathbf{\Upsilon}(t) = \{\eta \in \mathbb{R}_+ : \mathbf{\Theta}_\eta(t) \subseteq \mathbf{\Phi}(t)\} = \{\eta \in \mathbb{R}_+ : \mathbf{\Lambda}_\eta(t) = \emptyset\},$$

where $\mathbf{\Lambda}_\eta : S \rightrightarrows \mathbb{Y}^{\Omega}$, defined as $\mathbf{\Lambda}_\eta(t) := \mathbf{\Theta}_\eta(t) \cap (\mathbb{Y}^{\Omega} \setminus \mathbf{\Phi}(t))$, has $\mathscr{T}_S$-measurable graph. Finally, the $\mathscr{T}_S$-measurability of φ follows from the fact that for each $\alpha > 0$, we have

$$\{t \in S : \varphi(t) < \alpha\} = \bigcup_{\eta \in \mathbb{Q} \cap (0,\alpha)} \mathrm{Proj}_S \mathbf{\Lambda}_\eta.$$

For each $\eta \in \mathbb{Q} \cap (0,1)$, define $B_\eta := \{t \in S : \varphi(t) \geq \eta\}$. Thus, $\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$ is family of $\mathscr{T}_S$-measurable sets such that $B_\eta \subseteq B_{\eta'}$ if and only if $\eta \geq \eta'$ and $S \sim \bigcup\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$[9]. Let $\varepsilon \in (0,\mu(S))$. Then there is some $\eta_0 \in \mathbb{Q} \cap (0,1)$ such

[8] For any $x \in \mathbb{Y}$ and $A \subseteq \mathbb{Y}$, the distance between x and A, denoted by $\mathrm{dist}(x,A)$. defined as

$$\mathrm{dist}(x,A) := \inf\{\|x-y\| : y \in A\}.$$

[9] $C \sim D$ means $\mu(C \Delta D) = 0$, where $C \Delta D = (C \setminus D) \cup (D \setminus C)$.

that $\mu(B_{\eta_0}) > \mu(S) - \varepsilon$. Set $R := B_{\eta_0}$ and note that, for $t \in R$, as $\varphi(t) \geq \eta_0$, we have $\mathbb{B}(0, \eta_0)^{\Omega} \subseteq \mathbf{\Phi}(t)$. This completes the proof. $\square$

The following lemma on the convexity of vector measure is an application of the infinite dimensional version of the Lyapunov convexity theorem (refer to Uhl (1969)), whose proof can be found in Bhowmik and Cao (2013) and Evren and Hüsenov (2008).

Lemma 6.2. *Consider a continuum economy and assume that* $f \in L_1\left(\mu, \mathbb{Y}^{\Omega}\right)$. *Suppose also that* S, R *are two coalitions of* $\mathscr{E}$ *such that* $\mu(S \cap R) > 0$. *Then,*

$$H := \mathrm{cl}\left\{\left(\mu(B \cap R), \int_B f d\mu\right) : B \in \mathscr{T}_S\right\}$$

is a convex subset of $\mathbb{R} \times \mathbb{Y}^{\Omega}$. *Moreover, for any* $0 < \delta < 1$, *there is a sequence* $\{G_n\}_{n \geq 1} \subseteq \mathscr{T}_S$ *such that* $\mu(G_n \cap R) = \delta\mu(S \cap R)$ *for all* $n \geq 1$ *and*

$$\lim_{n \to \infty} \int_{G_n} f(\cdot, \omega) d\mu = \delta \int_S f(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$.

Proof of Proposition 3.4: It is given that

(i) $V_t(g(t, \cdot)) \geq V_t(f(t, \cdot))$ μ-a.e. on S_{γ}; and

(ii) $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ for all $t \in B$ and $\mathbb{I}_B = \mathbb{I}_{S_{\gamma}}$.

Define $\varphi : T \times \Omega \times (0, 1) \to \mathbb{Y}$ by letting $\varphi(t, \omega, \lambda) := \lambda g(t, \omega) + (1 - \lambda)e(t, \omega)$. By Lemma 5.28 in Aliprantis and Border [1], we conclude that $\varphi(t, \omega, \lambda)$ is an interior point of $X_t(\omega)$ for all $(t, \omega, \lambda) \in T \times \Omega \times (0, 1)$. For each $t \in B$, by the continuity of preference, there is an element $\lambda_t \in (0, 1)$ such that $V_t(\varphi(t, \cdot, \lambda)) > V_t(f(t, \cdot))$ for all $\lambda \geq \lambda_t$. For each $\lambda \in (0, 1) \cap \mathbb{Q}$, define $B_{\lambda} := \{t \in B : \lambda \geq \lambda_t\}$. Thus, $\{B_{\lambda} : \lambda \in \mathbb{Q} \cap (0, 1)\}$ is a family of $\mathscr{T}_B$-measurable sets such that $B_{\lambda} \subseteq B_{\lambda'}$ if and only if $\lambda \leq \lambda'$. Furthermore,

$$B \sim \bigcup\{B_{\lambda} : \lambda \in \mathbb{Q} \cap (0, 1)\}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mu(B_i) : i \in \mathbb{I}_B\}$. Therefore, there is an $\lambda_0 \in (0, 1) \cap \mathbb{Q}$ such that $\mu(B_{\lambda_0}) > \mu(B) - \varepsilon$, which implies $\mathbb{I}_{B_{\lambda_0}} = \mathbb{I}_B = \mathbb{I}_{S_{\gamma}}$. By Lemma 6.1, there are some $\eta > 0$ and a sub-coalition $\widehat{B}$ of B_{λ_0} such that

(a) $\mathbb{I}_{\widehat{B}} = \mathbb{I}_{B_{\lambda_0}}$;

(b) $\varphi(t, \omega, \lambda_0) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0, \eta)$ and $(t, \omega) \in \widehat{B} \times \Omega$; and

(c) $V_t(\varphi(t,\cdot,\lambda_0)+z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and $t \in \widehat{B}$.

Since γ is simple and measurable, there is a collection $\{Q_1, \cdots, Q_m\}$ of pairwise disjoint measurable sets such that $\gamma(t) := \gamma_j$ for some $\gamma_j \in [0,1]$ and all $t \in Q_j$. We define $\mathbb{J} := \{j : \gamma_j \neq 0\}$. So, the support of γ is given by

$$S_\gamma = \bigcup \{Q_j : j \in \mathbb{J}\}.$$

Let $\mathbb{K} := \{(i,j) \in \mathbb{I}_{\widehat{B}} \times \mathbb{J} : \mu(\widehat{B}_i \cap Q_j) > 0\}$. Pick an element $(i,j) \in \mathbb{K}$. By Lemma 6.2, there exists a sequence $\{G_n\}_{n\geq 1} \subseteq \mathscr{T}_{\widehat{B}_i \cap Q_j}$ of coalitions such that $\mu(G_n) = \gamma_j \mu(\widehat{B}_i \cap Q_j)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega))d\mu = \gamma_j \int_{\widehat{B}_i \cap Q_j} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \gamma_j \int_{\widehat{B}_i \cap Q_j} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega))\, d\mu - \int_{G_n} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega))d\mu,$$

satisfies $\xi_n \in \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Choose an integer $n_{ij} \geq 1$ such that

$$\xi_{n_{ij}}(\omega) \in \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3m}\right)$$

for all $\omega \in \Omega$. It follows that

$$\sum_{\{j:(i,j)\in\mathbb{K}\}} \xi_{n_{ij}}(\omega) \in \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3}\right).$$

We define

$$R := \bigcup \left\{G_{n_{ij}} : (i,j) \in \mathbb{K}\right\}.$$

Letting $F := S_\gamma \setminus \widehat{B}$, we note that $\mathbb{I}_F \subseteq \mathbb{I}_{S_\gamma}$. Define

$$\mathbb{M} := \{(i,j) \in \mathbb{I}_F \times \mathbb{J} : \mu(F_i \cap Q_j) > 0\}.$$

For any $(i,j) \in \mathbb{M}$, similar to above, there is a subcoalition H_{ij} of $F_i \cap Q_j$ such that $\mu(H_{ij}) = \lambda_0\gamma_j\mu(F_i \cap Q_j)$ and for all $\omega \in \Omega$,

$$b_{ij}(\omega) := \lambda_0\gamma_j \int_{F_i\cap Q_j} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu - \int_{H_{ij}} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3m}\right).$$

As a consequence, we have

$$\sum_{\{j:(i,j)\in\mathbb{M}\}} b_{ij}(\omega) \in \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3}\right).$$

Pick an $(i,j) \in \mathbb{M}$, and define

$$\mathbb{D}_i := \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3m}\right)^{\Omega}.$$

As in Lemma 6.1, the correspondence $\mathbf{F}_{ij} : H_{ij} \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_{ij}(t) := \{z \in \mathbb{D}_i : g(t,\cdot) + z \in X_t \text{ and } V_t(g(t,\cdot)+z) > V_t(f(t,\cdot))\},$$

is non-empty valued and has $\mathscr{T}_{H_{ij}} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph, which further implies the existence of a $\mathscr{T}_{H_{ij}}$-measurable selection ξ_{ij} of $\mathbf{F}_{ij}$. Define

$$\zeta_{ij} := \frac{1}{\mu(H_{ij})}\int_{H_{ij}} \xi_{ij} d\mu.$$

By properties of the Bochner integral (see Diestel and Uhl [11], Corollary 8, p. 48), one has $\zeta_{ij} \in \overline{\text{co}}\{\xi_{ij}(t) : t \in H_{ij}\}$[10], which, in view of the fact that $\mathbb{D}_i$ is closed and convex, immediately implies that $\zeta_{ij} \in \mathbb{D}_i$. Therefore, $\beta_{ij} := \zeta_{ij}\mu(H_{ij}) \in \mathbb{D}_i$. Consequently,

$$\sum_{\{j:(i,j)\in\mathbb{M}\}} \beta_{ij}(\omega) \in \mathbb{B}\left(0, \frac{\eta\mu(\widehat{B}_i)}{3}\right).$$

Let $C := \bigcup\{H_{ij} : (i,j) \in \mathbb{M}\}$. For each $i \in \mathbb{I}_{S_\gamma}$, let $x_i : \Omega \to \mathbb{Y}$ be a function defined by

$$x_i(\omega) := \begin{cases} \sum_{\{j:(i,j)\in\mathbb{K}\}} \xi_{n_{ij}}(\omega) + \sum_{\{j:(i,j)\in\mathbb{M}\}}[b_{ij}(\omega) - \beta_{ij}(\omega)], & \text{if } \omega \in \Omega \text{ and } i \in \mathbb{I}_F; \\ \sum_{\{j:(i,j)\in\mathbb{K}\}} \xi_{n_{ij}}(\omega), & \text{if } \omega \in \Omega \text{ and } i \notin \mathbb{I}_F. \end{cases}$$

It follows that $x_i(\omega) \in \mathbb{B}(0,\eta)$. Finally, we define a function $y : T \times \Omega \to \mathbb{Y}$ defined by[11]

$$y(t,\omega) := \begin{cases} \varphi(t,\omega,\lambda_0) + \frac{x_i(\omega)}{G_{n_{ij}}}, & \text{if } (t,\omega) \in G_{n_{ij}} \times \Omega \text{ and } (i,j) \in \mathbb{K}; \\ g(t,\omega) + \xi_{ij}(t,\omega), & \text{if } (t,\omega) \in H_{ij} \times \Omega \text{ and } (i,j) \in \mathbb{M}; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

[10] Here, $\overline{\text{co}}$ stands for the closed convex hull.
[11] $\xi(t,\omega)$ denotes the ω^{th}-coordinate of $\xi(t)$.

Recognized that y is an allocation with $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $E := C \cup R$. It can be readily verified that

$$\int_E (y(\cdot,\omega) - e(\cdot,\omega))d\mu = \lambda_0 \int_T \gamma(g(\cdot,\omega) - e(\cdot,\omega))d\mu.$$

For each $t \in R_i$, define

$$\eta_i := \min \left\{ \eta - \operatorname{dist}\left(0, \frac{x_i(\omega)}{G_{n_{ij}}}\right) : \omega \in \Omega \text{ and } (i,j) \in \mathbb{K} \right\}.^{12}$$

Let $\eta_0 := \min\{\eta_i : i \in \mathbb{I}_R\}$. As a consequence, we have $y(t,\cdot) + \mathbb{B}(0,\eta_0)^\Omega \subseteq X_t$ and $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ μ-a.e. on R. This completes the proof. $\square$

Proof of Proposition 3.8: Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mu(R_i) : i \in \mathbb{I}_R\}$. By Lemma 6.1, one can find an $\eta > 0$ and a sub-coalition C of R such that

(i) $\mu(C) > \mu(R) - \varepsilon$;

(ii) $g(t,\omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta)$ and $(t,\omega) \in C \times \Omega$; and

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C.

It follows from (i) that $\mathbb{I}_C = \mathbb{I}_R = \mathbb{I}_S$ and $\mathbb{I}_{S\setminus C} \subseteq \mathbb{I}_S$. Pick an $i \in \mathbb{I}_C$. By Lemma 6.2, there exists a sequence $\{G_n\}_{n\geq 1} \subseteq \mathscr{T}_{C_i}$ such that $\mu(G_n) = \delta\mu(C_i)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu,$$

satisfies $\xi_n \in \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Choose an $n_i \geq 1$ such that

$$\xi_{n_i}(\omega) \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i)}{2}\right)$$

for all $i \in \mathbb{I}_S$ and $\omega \in \Omega$. Similarly, for each $i \in \mathbb{I}_D$ (where $D := S \setminus C$), there is some $F_i \in \mathscr{T}_{D_i}$ such that $\mu(F_i) = \delta\mu(D_i)$ and

$$b_i(\omega) := \delta \int_{D_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{F_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i)}{2}\right).$$

[12] Note that $z(\cdot,\omega)$ is constant on R_i.

For each $\omega \in \Omega$, define $z_i(\omega) := b_i(\omega)$ if $i \in \mathbb{I}_D$; and $z_i(\omega) := 0$, if $i \in \mathbb{I}_S \setminus \mathbb{I}_D$. Analogously, define

$$K_i := \begin{cases} G_{n_i} \cup F_i, & \text{if } i \in \mathbb{I}_D; \\ G_{n_i}, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D. \end{cases}$$

Recognized that, for each $i \in \mathbb{I}_S$, we have

$$S_i := \begin{cases} C_i \cup D_i, & \text{if } i \in \mathbb{I}_D; \\ C_i, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D. \end{cases}$$

For each $i \in \mathbb{I}_S$, define a function $\varphi^i : K_i \times \Omega \to \mathbb{Y}$ such that

$$\varphi^i(t,\omega) := \begin{cases} g(t,\omega) + \frac{1}{\delta\mu(C_i)}(\xi_{n_i}(\omega) + z_i(\omega)), & \text{if } (t,\omega) \in G_{n_i} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $\varphi^i(t,\cdot)$ is $\mathscr{F}_i$-measurable. Furthermore, in light of (ii) and (iii), we have $\varphi^i(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in K_i \times \Omega$ and $V_t(\varphi^i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on K_i. Lastly, note that

$$\int_{K_i} (\varphi^i(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_{S_i} (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. Let $B := \bigcup\{K_i : i \in \mathbb{I}_S\}$ and

$$\eta_0 := \min\left\{\eta - \operatorname{dist}\left(0, \frac{1}{\delta\mu(C_i)}(\xi_{n_i}(\omega) + z_i(\omega))\right) : i \in \mathbb{I}_R \text{ and } \omega \in \Omega\right\}.$$

Thus, the function $\varphi : T \times \Omega \to \mathbb{Y}$, defined by $\varphi(t,\omega) := \varphi^i(t,\omega)$ for all $(t,\omega) \in T_i \times \Omega$, satisfies the requied properties for the above choices of B, C and η_0. □

Proof of Theorem 3.9: Pick an $\varepsilon \in (0, \mu(S))$. In view of Proposition 3.4 and Remark 3.6, we can choose two coalitions E, R and an allocation g such that (i) $\mu(E) > \varepsilon$; (ii) $R \subseteq E \subseteq S$ and $\mathbb{I}_R = \mathbb{I}_E = \mathbb{I}_S$; (ii) f is blocked by E via g; and (iii) $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$. Let $\delta \in (0,1)$ be such that $\varepsilon = \delta\mu(E)$. By Proposition 3.8 that there are an $\eta_0 > 0$, two coalitions B, C and an allocation φ such that

(i) $C \subseteq B \subseteq E$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_E$, and $\mu(B) = \delta\mu(E)$;

(ii) $\varphi(t,\omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta_0)$ and $(t,\omega) \in C \times \Omega$;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in (0,\eta_0)^\Omega$ and μ-a.e. on C;

(ii) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $B \setminus C$; and

(iii) $\int_B(\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = \delta \int_E(g(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Consequently, $\mu(B) = \varepsilon$ and $\int_B(\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = 0$ for all $\omega \in \Omega$. This means that f is blocked by B. □

Proof of Theorem 3.10: Let f be an allocation and R be a coalition blocking f. Let $\beta := \mu(R) - \frac{\alpha}{2}$. As in the proof of Theorem 3.9, there are an $\eta > 0$, two coalitions B, C and an allocation φ such that

(i) $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_R$ and $\mu(B) = \beta$;

(ii) $\varphi(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C; and

(iv) f is blocked by B via φ.

Let $i \in \mathbb{I}_B$. By absolute continuity of the Bochner integral, there exists some $\zeta_i > 0$ such that

$$\frac{2}{\mu(C_i)} \int_{G_i} (\varphi(\cdot,\omega) - e(\cdot,\omega))\, d\mu \in \mathbb{B}(0,\eta)$$

for all $\omega \in \Omega$ and $G_i \in \mathscr{T}_{B_i}$ satisfying $\mu(G_i) < \zeta_i$. Define

$$\alpha := \min\{2\zeta_i,\ \mu(C_i) : i \in \mathbb{I}_B\}.$$

For all $i \in \mathbb{I}_B$, pick any $D_i \in \mathscr{T}_{B_i}$ such that $\mu(B_i \setminus D_i) < \frac{\alpha}{2}$. Therefore, $\mu(C_i \cap D_i) > \frac{\mu(C_i)}{2}$. It follows that

$$x_i(\omega) := \frac{1}{\mu(C_i \cap D_i)} \int_{B_i \setminus D_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu \in \mathbb{B}(0,\eta)$$

for each $\omega \in \Omega$. Let $h_i : T \times \Omega \to \mathbb{Y}_+$ such that

$$h_i(t,\omega) := \begin{cases} \varphi(t,\omega) + x_i(\omega), & \text{if } (t,\omega) \in (C_i \cap D_i) \times \Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

By (ii) and (iii), it follows that $h_i(t,\cdot) \in X_t$ and $V_t(h_i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $C_i \cap D_i$. Furthermore,

$$\int_{D_i} (h_i(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{B_i} (\varphi(\cdot,\omega) - e(\cdot,\omega))\, d\mu$$

for all $\omega \in \Omega$. Let $S \in \mathscr{T}_R$ be a coalition such that $\mu(R \setminus S) < \alpha$. It follows that $\mu(B \setminus S) < \frac{\alpha}{2}$ and hence, $\mu(B_i \setminus S_i) < \frac{\alpha}{2}$ for all $i \in \mathbb{I}_B$. Consequently, for each $i \in \mathbb{I}_B$, there is an allocation h_i such that $V_t(h_i(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S_i, and

$$\int_{S_i} (h_i(\cdot, \omega) - e(\cdot, \omega))d\mu = \int_{B_i} (g(\cdot, \omega) - e(\cdot, \omega))\, d\mu$$

for all $\omega \in \Omega$. We consider an allocation $h : T \times \Omega \to \mathbb{Y}_+$, defined by $h(t, \omega) = h_i(t, \omega)$ if $(t, \omega) \in S_i \times \Omega$, $i \in \mathbb{I}_R$ and $h(t, \omega) = g(t, \omega)$, otherwise. Recognized that S blocks the allocation f via h.

For the second part, assume that $f \notin \mathscr{C}(\mathscr{E})$ and choose $\varepsilon, \delta > 0$. Applying Theorem 3.9, we find a coalition R with $\mu(R) = \varepsilon$ blocking f via some allocation g. Let $\{t_n : n \geq 1\}$ be a countable dense subset of R. For all $n \geq 1$, define

$$G_n := R \cap \mathbb{B}\left(t_n, \frac{\delta}{2}\right)$$

Letting $F_n := \bigcup\{G_k : 1 \leq k \leq n\}$ for all $n \geq 1$, we see that $\{F_n : n \geq 1\}$ is an ascending sequence and $R = \bigcup\{F_n : n \geq 1\}$. Thus, there is an $n_0 \geq 1$ such that $\mu(R \setminus F_{n_0}) < \delta$. This completes the proof. □

Proof of Proposition 3.11: Let $0 < \delta < 1$. In view of Proposition 3.8, there are an $\eta_0 > 0$, two non-null coalitions B and C, and an allocation φ such that

(i) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and $\mu(B) = \delta\mu(S)$;

(ii) $\varphi(t, \omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0, \eta_0)$ and $(t, \omega) \in C \times \Omega$;

(iii) $V_t(\varphi(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{B}(0, \eta_0)^\Omega$ and μ-a.e. on C;

(iii) $V_t(\varphi(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on $B \setminus C$; and

(iv) $\int_B (\varphi(\cdot, \omega) - f(\cdot, \omega))d\mu = \delta \int_S (g(\cdot, \omega) - f(\cdot, \omega))d\mu$ for all $\omega \in \Omega$.

Let $E := S \setminus B$ and define, for each $i \in \mathbb{I}_E$, the set

$$\mathbb{D}_i := \mathscr{G}_i \cap \mathbb{B}\left(0, \eta_0\mu(C_i)\right)^\Omega,$$

where $C_i := C \cap T_i$. As in Lemma 6.1, the correspondence $\mathbf{F}_i : E_i \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_i(t) := \{z \in \mathbb{D}_i : f(t, \cdot) + z \in X_t \text{ and } V_t(f(t, \cdot) + z) > V_t(f(t, \cdot))\},$$

has a $\mathscr{T}_{E_i} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph. By our stated assumptions, $\mathbf{F}_i(t) \neq \emptyset$ for all $t \in E_i$. By the Aumann-Saint-Beuve measurable selection theorem, there is a $\mathscr{T}_{E_i}$-measurable selection ξ_i of $\mathbf{F}_i$. Define

$$\zeta_i := \frac{1}{\mu(E_i)} \int_{E_i} \xi_i d\mu.$$

As in the proof of Proposition 3.4, one can show that $\zeta_i \in \mathbb{D}_i$. So, $\varepsilon_i := \zeta_i \mu(E_i) \in \mathbb{D}_i$ and

$$b_i := \frac{\varepsilon_i}{\mu(C_i)} \in \mathscr{G}_i \cap \mathbb{B}\left(0, \eta_0\right)^{\Omega}.$$

Let $h : S \times \Omega \to \mathbb{Y}$ be a function such that

$$h(t, \omega) := \begin{cases} \varphi(t, \omega) - b_i(\omega), & \text{if } (t, \omega) \in C_i \times \Omega \text{ and } i \in \mathbb{I}_E; \\ f(t, \omega) + \xi_i(t, \omega), & \text{if } (t, \omega) \in E_i \times \Omega \text{ and } i \in \mathbb{I}_E; \\ \varphi(t, \omega), & \text{otherwise.} \end{cases}$$

It is evident that h is an allocation and $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S. It can be readily verified that

$$\int_S h(\cdot, \omega) d\mu = \int_S (\delta g(\cdot, \omega) + (1 - \delta) f(\cdot, \omega)) d\mu$$

for all $\omega \in \Omega$. This completes the proof. $\square$

Proof of Theorem 3.13: Since f is a non-core allocation, there exists a coalition S and an allocation g such that f is blocked by S via g. Then for each $\varepsilon \in (0, \mu(S))$, by Theorem 3.9, there are a coalition R and an allocation φ such that $\mu(R) = \varepsilon$ and f is blocked by R via φ. If $\mu(S) = \mu(T)$, then there is nothing more to verify. Thus, we assume that $\mu(S) < \mu(T)$ and choose an $\varepsilon \in (\mu(S), \mu(T))$. Define

$$\delta := 1 - \frac{\varepsilon - \mu(S)}{\mu(T \setminus S)}.$$

By Lemma 6.1, one can find an $\eta_0 > 0$ and a sub-coalition C of B such that

(A) $\mathbb{I}_C = \mathbb{I}_B$;

(B) $g(t, \omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0, \eta_0)$ and $(t, \omega) \in C \times \Omega$; and

(C) $V_t(g(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{B}(0, \eta_0)^{\Omega}$ and μ-a.e. on C.

Define

$$\mathbb{D} := \bigcap \{\eta_0 \delta\mu(C)\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta_0\delta\mu(C)\right)^{\Omega}.$$

As in Lemma 6.1, the correspondence $\mathbf{F} : T \setminus S \rightrightarrows \mathbb{D}$, defined by

$$\mathbf{F}(t) := \{z \in \mathbb{D} : f(t,\cdot) + z \in \mathrm{int}X_t \text{ and } V_t(f(t,\cdot)+z) > V_t(f(t,\cdot))\},$$

is non-empty valued and has $\mathscr{T}_{T\setminus S} \otimes \mathscr{B}(\mathbb{D})$-measurable graph, which further implies the existence of a $\mathscr{T}_{T\setminus S}$-measurable selection ξ of $\mathbf{F}$. Define

$$\zeta := \frac{1}{\mu(T\setminus S)} \int_{T\setminus S} \xi d\mu.$$

As in the proof of Proposition 3.4, one can show that $\zeta \in \mathbb{D}$. So, $\varepsilon := \zeta\mu(T\setminus S) \in \mathbb{D}$ and

$$\gamma := \frac{\varepsilon}{\delta\mu(C)} \in \bigcap \{\eta_0 \mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta_0\right)^{\Omega}.$$

In view of Proposition 3.8 there exist a coalition F and an allocation ψ such that

(a) $\mu(F) = (1-\delta)\mu(T\setminus S)$;

(b) $V_t(\psi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on F; and

(c) $\int_F (\psi(\cdot,\omega) - e(\cdot,\omega))d\mu = (1-\delta)\int_{T\setminus S}(f(\cdot,\omega) + \xi(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Let $\widetilde{g} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\widetilde{g}(t,\omega) := \begin{cases} g(t,\omega) - \gamma(\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

By Proposition 3.11, there exist some allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_S h(\cdot,\omega)d\mu = \int_S (\delta\widetilde{g}(\cdot,\omega) + (1-\delta)f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. We define a function $y : T \times \Omega \to \mathbb{Y}$ by setting

$$y(t,\omega) := \begin{cases} \psi(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Recognized that y is an allocation with $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $E := F \cup S$. It can be readily verified that $\mu(E) = \varepsilon$ and

$$\int_E (y(\cdot,\omega) - e(\cdot,\omega))d\mu = (1-\delta)\int_T (f(\cdot,\omega) - e(\cdot,\omega))d\mu = 0.$$

This completes the proof. $\square$

Proof of Proposition 3.15: Denoting by X_R, $\mathscr{F}_R$, V_R, and $e_R(\cdot)$ the common values of X_t, $\mathscr{F}_t$, V_t, and $e(t,\cdot)$, respectively. Suppose, on contrary, that $V_R(\mathbf{x}_f) > V_R(f(t,\cdot))$ for all $t \in B$ for some sub-coalition B of R. Without loss of generality, we may assume that $\mu(R) < \mu(T)$. Otherwise, f will be be blocked by B via $\mathbf{x}_f$. This can be seen as follows:

$$\int_B \mathbf{x}_f d\mu = \frac{\mu(B)}{\mu(R)} \int_R \mathbf{x}_f d\mu = \frac{\mu(B)}{\mu(R)} \int_R e d\mu = \int_B e d\mu.$$

Therefore, we assume that $\mu(R) < \mu(T)$. Then there are an $\lambda \in (0,1)$ and a sub-coalition D of B such that

$$V_R(\lambda \mathbf{x}_f + (1-\lambda)e_R) > V_R(f(t,\cdot))$$

for all $t \in D$. By Lemma 5.28 of Aliprantis and Border [1], we have $\lambda \mathbf{x}_f + (1-\lambda)e_R$ is an interior point of X_R. It follows that there are an $\eta > 0$ and a sub-coalition E of D such that

$$V_R(\lambda \mathbf{x}_f + (1-\lambda)e_R - z) > V_R(f(t,\cdot))$$

for all $z \in \mathbb{B}(0,\eta)^\Omega$ and $t \in E$. Let $\delta \in (0,1]$ be such that $\mu(E) = \delta\mu(R)$. Define

$$\mathbb{D} := \bigcap \{\eta\mu(E)\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta\mu(E)\right)^\Omega.$$

As before, one can find an allocation $\xi : T_0 \times \Omega \to \mathbb{Y}$ such that

(i) $\xi(t,\cdot) \in \mathbb{D}$ μ-a.e. on T_0;

(ii) $f(t,\cdot) + \xi(t,\cdot) \in \mathrm{int} X_t$ μ-a.e. on T_0; and

(iii) $V_t(f(t,\cdot) + \xi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on T_0.

By Proposition 3.8, there exists a coalition $C \in \mathscr{T}_{T\setminus R}$ and an allocation φ such that

(A) $\mu(C) = \delta\mu(T \setminus R)$;

(B) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on C; and

(C) $\displaystyle\int_C (\varphi - e) d\mu = \lambda\delta \int_{T\setminus R} (f + \xi - e) d\mu.$

Define

$$\zeta := \frac{1}{\mu(T\setminus R)} \int_{T\setminus R} \xi d\mu.$$

As in the proof of Proposition 3.4, one can show that $\zeta \in \mathbb{D}$, which further implies $\alpha := \lambda\delta\zeta\mu(T \setminus R) \in \mathbb{D}$. Consequently,

$$\gamma := \frac{\alpha}{\mu(E)} \in \bigcap\{\eta\mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{B}\left(0, \eta\right)^{\Omega}.$$

Finally, we define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by

$$y(t, \omega) := \begin{cases} \lambda \mathbf{x}_f(\omega) + (1-\lambda)e_R(\omega) - \gamma(\omega), & \text{if } (t, \omega) \in E \times \Omega; \\ \varphi(t, \omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that $S := C \cup E$ blocks f via y. This is a contradiction. Hence, $V_R(f(t, \cdot)) \geq V_R(\mathbf{x}_f)$ μ-a.e. on R. Let G be a sub-coalition of R such that$V_R(f(t, \cdot)) > V_R(\mathbf{x}_f)$ for all $t \in G$. By applying Jensen's inequality, one obtains

$$V_R\left(\frac{1}{\mu(G)} \int_G f d\mu\right) > V_R(\mathbf{x}_f)$$

and

$$V_R\left(\frac{1}{\mu(R \setminus G)} \int_{R \setminus G} f d\mu\right) \geq V_R(\mathbf{x}_f).$$

Let $\alpha := \frac{\mu(G)}{\mu(R)}$. By Lemma 5.26 in Aliprantis and Border (2005), one has

$$\begin{aligned} V_R(\mathbf{x}_f) &= V_R\left(\frac{\alpha}{\mu(G)} \int_G f d\mu + \frac{1-\alpha}{\mu(R \setminus G)} \int_{R \setminus G} f d\mu\right) \\ &> V_R(\mathbf{x}_f), \end{aligned}$$

which is a contradiction. Therefore, $V_R(f(t, \cdot)) = V_R(\mathbf{x}_f)$ μ-a.e. on R. □

Proof of Lemma 3.17: By Lemma 6.1, there exists an $\eta > 0$ such that $g(t, \cdot) + z \in X_t$ and $V_t(g(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{B}(0, \eta)^{\Omega}$ and all $t \in R$. Let $\delta \in (0, 1]$ be a number such that $\alpha = \delta\mu(S \cap H)$. Pick an element $i \in \mathbb{I}_S$. By Lemma 6.2, there is a sequence $\{C_i^n : n \geq 1\} \subseteq \mathscr{T}_{S_i}$ such that $\mu(C_i^n \cap H) = \delta\mu(S_i \cap H)$ for all $n \geq 1$ and $\{x_i^n : n \geq 1\}$ converges to 0 in norm-topology, where

$$x_i^n := \delta \int_{S_i} (g - e) d\mu - \int_{C_i^n} (g - e) d\mu$$

for all $n \geq 1$. Let $n_0 \geq 1$ be such that

$$\frac{x_i^{n_0}}{\mu(C_i^{n_0})} \in \mathbb{B}(0, \eta)^{\Omega}$$

for all $i \in \mathbb{I}_S$. Define $G := \bigcup\{C_i^{n_0} : i \in \mathbb{I}_S\}$. Consider an allocation $h : T \times \Omega \to \mathbb{Y}$ defined that

$$h(t,\omega) := \begin{cases} g(t,\omega) + \frac{x_i^{n_0}}{\mu(C_i^{n_0})}, & \text{if } (t,\omega) \in C_i^{n_0} \times \Omega \text{ and } i \in \mathbb{I}_S; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that f is blocked by G via h and $\mu(G \cap H) = \alpha$. $\square$

Proof of Proposition 3.18: Let $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$. Suppose by the way of contradiction that $f := \Phi[\widetilde{f}] \notin \mathscr{C}^A(\mathscr{E})$. Consequently, there are an Aubin coalition γ and an allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_γ and

$$\int_{S_\gamma} \gamma g(\cdot,\omega)\, d\mu = \int_{S_\gamma} \gamma e(\cdot,\omega)\, d\mu$$

for all $\omega \in \Omega$. Define $\mathbb{J} := \{j : A_j \subseteq S_\gamma\}$. By Theorem 3.7, we may assume that $\mathbb{J} \neq \emptyset$. Consequently,

$$\int_{S_\gamma \cap T_0} \gamma(g-e)\, d\mu + \sum_{j \in \mathbb{J}} \gamma(A_j)\mu(A_j)(g(A_j) - e(A_j)) = 0.$$

In view of Proposition 3.4, there is an $r_0 \in (0,1)$ and an allocation y such that $u_t(y(t,\cdot)) > u_t(f(t,\cdot))$ μ-a.e. on $S_\gamma \cap T_0$ and

$$\int_{S_\gamma \cap T_0} (y-e)\, d\mu = r_0 \int_{S_\gamma \cap T_0} \gamma(g-e)\, d\mu.$$

By the Lyapunov convexity theorem, there is a sub-coalition $\widetilde{B_j}$ of $\widetilde{A_j}$ such that

$$\widetilde{\mu}(\widetilde{B_j}) = r_0 \gamma(A_j)\widetilde{\mu}(\widetilde{A_j}).$$

Define an allocation $\varphi : T \to \mathbb{Y}$ by letting

$$\varphi(t,\omega) := \begin{cases} \widetilde{g}(t,\omega), & \text{if } (t,\omega) \in \widetilde{B}_i \times \Omega \text{ and } i \in \mathbb{J}; \\ y(t,\omega), & \text{otherwise,} \end{cases}$$

where $\widetilde{g} := \Xi[g]$. Define

$$\widetilde{S} := (S_\gamma \cap T_0) \cup \bigcup\{\widetilde{B_j} : j \in \mathbb{J}\}.$$

It fdollows that

$$\int_{\widetilde{S}} (\varphi - e)\, d\mu = 0.$$

Pick an $j \in \mathbb{J}$. For a continuum economy $\mathscr{E}$ with $T_1 = \emptyset$ and $R \cap T_0 = \widetilde{A_j}$, we have in view of Proposition 3.15, the equaility $V_t(f(A_j, \cdot)) = V_t(\widetilde{f}(t, \cdot))$ μ-a.e. on $\widetilde{A_j}$. Therefore, μ-a.e. on $\widetilde{B_j}$, we have

$$V_t(\varphi(t, \cdot)) = V_{A_i}(g(A_i, \cdot)) > V_{A_i}(f(A_i, \cdot)) = V_{A_j}(\widetilde{f}(t, \cdot)).$$

Hence, $\widetilde{f}$ is blocked by $\widetilde{S}$ via φ, which leads to a contradiction. □

Proof of Proposition 3.19: First, we define $\mathbf{x}_{\widetilde{f}} : \Omega \to \mathbb{Y}$ by letting

$$\mathbf{x}_{\widetilde{f}}(\omega) := \frac{1}{\widetilde{\mu}(R)} \int_R \widetilde{f}(\cdot, \omega) d\widetilde{\mu}$$

for all $\omega \in \Omega$. Thus, consider a feasible allocation $\widetilde{f}^A : T \times \Omega \to \mathbb{Y}$ such that

$$\widetilde{f}^A(t, \omega) := \begin{cases} \widetilde{f}(t, \omega), & \text{if } (t, \omega) \in (T \setminus R) \times \Omega; \\ \mathbf{x}_{\widetilde{f}}(\omega), & \text{otherwise.} \end{cases}$$

In view of Proposition 3.15, we have $V_t(\widetilde{f}(t, \cdot)) = V_t(\widetilde{f}^A(t, \cdot))$ μ-a.e. on R. Suppose, by the way of contradiction, that $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. Thus, $\widetilde{f}^A$ is not in the core of $\widetilde{\mathscr{E}}$.

Case 1. $R = T_1$ and $|T_1| \geq 2$. Choose an element $A_0 \in T_1$ and let $\mu(A_0) = \varepsilon > 0$. By Theorem 3.13, $\widetilde{f}^A$ is blocked by a coalition $\widetilde{B}$ of $\widetilde{\mathscr{E}}$ with $\widetilde{\mu}(\widetilde{B}) = \widetilde{\mu}(T_0) + \varepsilon$, which gives $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) \geq \varepsilon$. Therefore, in the light of Lemma 3.17, there exist a coalition $\widetilde{E}$ and an assignment $\widetilde{y}$ such that $\widetilde{f}^A$ will be blocked by $\widetilde{E}$ via $\widetilde{y}$ and $\widetilde{\mu}(\widetilde{E} \cap \widetilde{T}_1) = \varepsilon$. Define a coalition S of $\mathscr{E}$ such that $S := (\widetilde{E} \cap T_0) \cup A_0$, and define a function $y : T \times \Omega \to \mathbb{Y}$ by

$$y(t, \omega) = \begin{cases} \widetilde{y}(t, \omega), & \text{if } (t, \omega) \in (T \setminus A_0) \times \Omega; \\ \frac{1}{\varepsilon} \int_{\widetilde{E} \cap \widetilde{T}_1} \widetilde{y}(\cdot, \omega) d\widetilde{\mu}, & \text{otherwise.} \end{cases}$$

Recognized that y is an allocation of $\mathscr{E}$ such that

$$\int_S y(\cdot, \omega) d\mu = \int_S e(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. Furthermore, by the quasi-concavity of V_{T_1}, we have $V_t(y(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S, which leads to a contradiction.

Case 2. $\mu(R \setminus T_1) > 0$. Define $C := R \cap T_0$. Since $\widetilde{f}^A$ is not in the core of $\widetilde{\mathscr{E}}$, by Proposition 3.4, we conclude that there are a coalition $\widetilde{B}$ and an allocation $\widetilde{y}$ such that $\widetilde{f}^A$ will be blocked by $\widetilde{B}$ via $\widetilde{y}$ and $\widetilde{y}(t, \cdot)$ is an interior point of X_t for all $t \in \widetilde{G}$ for some sub-coalition $\widetilde{G}$ of $\widetilde{B}$ satisfying $\mathbb{I}_{\widetilde{B}} = \mathbb{I}_{\widetilde{G}}$. If $\widetilde{B} \subseteq T_0$, there is noting more to verify.

Thereofore, we assume that $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) > 0$. Let $\varepsilon := \widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1)$. Define a function $\widetilde{y}^A : \widetilde{T} \times \Omega \to \mathbb{Y}$ by

$$\widetilde{y}^A(t,\omega) := \begin{cases} \frac{1}{\varepsilon}\int_{\widetilde{B}\cap\widetilde{T}_1} \widetilde{y}(\cdot,\omega)d\widetilde{\mu}, & \text{if } (t,\omega) \in (\widetilde{B} \cap \widetilde{T}_1) \times \Omega; \\ \widetilde{y}(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $V_{T_1}(\widetilde{y}^A(t,\cdot)) > V_{T_1}(\widetilde{f}^A(t,\cdot))$ μ-a.e. on $\widetilde{B}$ and

$$\int_{\widetilde{B}\cap T_0} (\widetilde{y}^A - e)d\widetilde{\mu} + \varepsilon(\widetilde{y}^A - e_{T_1}) = 0. \tag{6.1}$$

As $\mathbb{I}_{\widetilde{B}} = \mathbb{I}_{\widetilde{G}}$, one of the following must hold: $\mathbb{I}_R \subseteq \mathbb{I}_{\widetilde{G}}$. If $\widetilde{\mu}(C) \geq \varepsilon$ then we choose a coalition $\widehat{R} \subseteq C$ such that $\widetilde{\mu}(\widehat{R}) = \varepsilon$. Consequently, by Equation (6.1), we have

$$\int_{\widetilde{B}\cap T_0} (\widetilde{y}^A - e)d\widetilde{\mu} + \widetilde{\mu}(\widehat{R})(\widetilde{y}^A - e_{T_1}) = 0.$$

If $\widetilde{\mu}(C) < \varepsilon$ then first choose an $\alpha \in (0,1)$ such that $\widetilde{\mu}(C) = \alpha\varepsilon$. By Proposition 3.8, there are two coalitions $\widehat{K}$ and $\widehat{D}$ and an allocation φ such that $\widehat{K} \subseteq \widehat{D} \subseteq \widetilde{B} \cap T_0$ with $\mathbb{I}_{\widehat{K}} = \mathbb{I}_{\widehat{D}} = \mathbb{I}_{\widetilde{B}\cap T_0}$; $V_t(\varphi(t,\cdot)) > V_t(f(t\cdot))$ for all $t \in \widehat{D}$; $\varphi(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in \widehat{K} \times \Omega$; and

$$\int_{\widehat{D}} (\varphi - e)d\widetilde{\mu} = \alpha \int_{\widetilde{B}\cap T_0} (\widetilde{y}^A - e)d\widetilde{\mu}.$$

In view of Equation (6.1), we have $\mathbb{I}_{\widehat{K}\cup C} = \mathbb{I}_{\widehat{D}\cup C} = \mathbb{I}_{\widetilde{B}}$ and

$$\int_{\widehat{D}} (\varphi - e)d\widetilde{\mu} + \widetilde{\mu}(C)(\widetilde{y}^A - e_{T_1}) = 0.$$

Hence, in either of these cases, there are coalitions D, K, R and allocation ξ such that $K \subseteq D \subseteq \widetilde{B} \cap T_0$ and $N \subseteq C$ such that $\mathbb{I}_{K\cup N} = \mathbb{I}_{D\cup N} = \mathbb{I}_{\widetilde{B}}$ and

$$\int_D (\xi - e)d\widetilde{\mu} + \widetilde{\mu}(N)(\widetilde{y}^A - e_{T_1}) = 0.$$

If $\widetilde{\mu}(D \cap N) = 0$ then $D \cup N$ blocks the allocation $\widetilde{f}^A$ via ζ, where the allocation ζ is defined by

$$\zeta(t,\omega) = \begin{cases} \xi(t,\omega), & \text{if } (t,\omega) \in D \times \Omega; \\ \widetilde{y}^A(t,\omega), & \text{otherwise.} \end{cases}$$

If $\widetilde{\mu}(D \cap N) > 0$ then we define $E := (D \setminus N) \cup (N \setminus D)$ and $G := D \cap N$. Recognized that $\zeta(t,\omega)$ is an interior point of $X_t(\omega)$ for all $t \in H$ for some sub-coalition H of $K \cup N$

satisfying $\mathbb{I}_H = \mathbb{I}_{(K\cup N)\cap E}$. By Proposition 3.8, there is some coalition $F \subseteq E$ and an allocation h such that

$$\int_F (h-e)d\widetilde{\mu} = \frac{1}{2}\int_E (\zeta - e)d\widetilde{\mu}.$$

By Proposition 3.11, there exist an allocation ι and a sub-coalition V of G such that $\mathbb{I}_V = \mathbb{I}_G$, $V_t(\iota(t,\cdot)) > V_t(f(t,\cdot))$; and

$$\int_G (\iota - e)d\mu = \frac{1}{2}\int_G (\xi - e)d\mu + \frac{1}{2}\int_G (\widetilde{y}^A - e)d\mu.$$

Then $S := F \cup G$ blocks the allocation $\widetilde{f}^A$ via ψ, where the allocation ψ is defined by

$$\psi(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ \iota(t,\omega), & \text{otherwise.} \end{cases}$$

This contradicts with the fact that f is in the ex-ante core of $\mathscr{E}$. $\square$

Proof of Theorem 3.20: Let us choose $\varepsilon, \delta > 0$. Let $f \notin \mathscr{C}^A(\mathscr{E})$. Defining $\widetilde{f} := \Xi[f]$, we note that $f = \Phi[\widetilde{f}]$. Thus, by Proposition 3.18, we have $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. In view of Theorem 3.10,we have a coalition S with $\widetilde{\mu}(S) \le \varepsilon$ blocking f and $S = \bigcup_{i=1}^n S_i$ for a finite collection of coalitions $\{S^1, \cdots, S^n\}$ with diameter of S_i smaller than δ for all $i = 1, \cdots, n$. Let

$$B^1 := S^1 \text{ and } B^i = S^i \setminus \bigcup\{S^j : 1 \le j < i\}$$

for all $i \ge 2$. Define $G^i := B^i \cap T_0$ for each $i \in \{1, \cdots, n\}$ and note that

$$S = \bigcup\{G^i : 1 \le i \le n\} \cup (S \cap T_1).$$

Put,

$$\mathbb{I} := \left\{k : \widetilde{\mu}(\widetilde{A_k} \cap S) > 0\right\}.$$

Applying Theorem 3.10, we can find some $\eta > 0$ such that for any coalition F of S satisfying $\mu(S \backslash F) < \eta$ blocking f. Choose a finite subset $\mathbb{K}$ of $\mathbb{I}$ such that $\sum_{k\in\mathbb{I}\backslash\mathbb{K}} \mu(A_k) < \eta$. We define $R := S$ if $S \subseteq T_0$; and

$$R := \bigcup\{G^i : 1 \le i \le n\} \cup \bigcup\{\widetilde{A_k} \cap S : k \in \mathbb{K}\}.$$

Therefore, R is a coalition containing either no atom or finitely many atoms. For $\mathbb{K} \neq \emptyset$, let $\gamma : T \to [0,1]$ be an Aubin coalition such that

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in R \cap T_0; \\ \alpha_k, & \text{if } t = A_k,\ k \in \mathbb{K}; \\ 0, & \text{otherwise,} \end{cases}$$

and for $\mathbb{K} = \emptyset$, define an Aubin coalition $\gamma : T \to [0,1]$ such that

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in R \cap T_0; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\alpha_k := \frac{\widetilde{\mu}(\widetilde{A_k} \cap S)}{\widetilde{\mu}(\widetilde{A_k})}.$$

For all $1 \le i \le n$, let $\gamma_i : T \to [0,1]$ be an Aubin coalition such that $\gamma_i := \chi_{G^i}$. Thus, $\{\gamma_1, \cdots, \gamma_n\}$ is a finite collection of pairwise disjoint generalized coalitions and $S_{\gamma_i} \subseteq T_0$ for all $1 \le i \le n$. It follows from the definition of the diameter of a generalized coalition by taking $\alpha = \beta = 1$ that

$$\mathrm{diam}(\gamma_i) = \sup\{\|a-b\| : a, b \in S_{\gamma_i}\} = \mathrm{diam}(S_{\gamma_i}) < \delta.$$

Furthermore, it can be readily verified that

$$\gamma = \begin{cases} \sum_{i=1}^n \gamma_i + \sum_{k\in\mathbb{K}} \alpha_k \chi_{A_k}, & \text{if } \mathbb{K} \neq \emptyset; \\ \sum_{i=1}^n \gamma_i, & \text{if } \mathbb{K} = \emptyset, \end{cases}$$

This completes the proof. □

Proof of Theorem 3.22: Let f be a feasible allocation of $\mathscr{E}$ such that $f \notin \mathscr{C}^A(\mathscr{E})$ and let $\varepsilon \in (0,1)$. Letting $\widetilde{f} := \Xi[f]$, we note that $f = \Phi[\widetilde{f}]$. Thus, applying Proposition 3.18, one has $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. Therefore, in view of Theorem 3.13, one can find a coalition S and an allocation $\widetilde{g}$ in $\widetilde{\mathscr{E}}$ such that $\widetilde{\mu}(S) = \varepsilon$ and $\widetilde{f}$ is blocked by the coalition S via some allocation $\widetilde{g}$. Put $\mathbb{J} = \{j : \widetilde{\mu}(S \cap \widetilde{A_i}) > 0\}$. The rest of the proof is decomposed into two cases:

Case 1. $\mathbb{J} \neq \emptyset$. In this case, we have

$$\int_{S\cap T_0} \widetilde{g}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \int_{S\cap\widetilde{A_j}} \widetilde{g}\, d\widetilde{\mu} = \int_{S\cap T_0} \widetilde{e}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \int_{S\cap\widetilde{A_j}} \widetilde{e}\, d\widetilde{\mu}.$$

For each $j \in \mathbb{J}$, choose some $\gamma_j \in (0,1]$ such that $\widetilde{\mu}(S \cap \widetilde{A_j}) = \gamma_i \mu(A_i)$ and define

$$g_j := \frac{1}{\widetilde{\mu}(S \cap \widetilde{A_j})} \int_{S\cap\widetilde{A_j}} \widetilde{g}\, d\widetilde{\mu}.$$

By Jensen's inequality, we have $V_{A_j}(g_j) > V_{A_j}(f(A_j))$ for all $j \in \mathbb{J}$ and

$$\int_{S\cap T_0} \widetilde{g}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \gamma_j g_j \mu(A_j) = \int_{S\cap T_0} \widetilde{e}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \gamma_j e_j \mu(A_j).$$

Define an allocation $g : T \to \mathbb{Y}_+$ by

$$g(t) := \begin{cases} \widetilde{g}(t), & \text{if } t \in S \cap T_0; \\ g_j, & \text{if } t = A_j \text{ and } j \in \mathbb{J}; \\ f(t), & \text{otherwise,} \end{cases}$$

and an Aubin coalition $\gamma : T \to [0,1]$ by

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in S \cap T_0; \\ \gamma_j, & \text{if } t = A_j \text{ and } j \in \mathbb{J}; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_γ and

$$\int_T \gamma g(\cdot,\omega)\, d\mu = \int_T \gamma e(\cdot,\omega)\, d\mu$$

for all $\omega \in \Omega$. Furthermore, note that

$$\int_T \gamma\, d\mu = \mu(S \cap T_0) + \sum_{j \in \mathbb{J}} \int_{A_j} \gamma_j\, d\mu = \widetilde{\mu}(S) = \varepsilon.$$

Case 2. $\mathbb{J} = \emptyset$. Analogous to **Case 1**, one can show that f is blocked by an Aubin coalition γ via g, where the function $g : T \to \mathbb{Y}_+$ is defined by

$$g(t) := \begin{cases} \widetilde{g}(t), & \text{if } t \in S \cap T_0; \\ f(t), & \text{otherwise,} \end{cases}$$

and the Aubin coalition $\gamma : T \to [0,1]$ is defined by

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in S \cap T_0; \\ 0, & \text{otherwise.} \end{cases}$$

Recognized that $\int_T \gamma\, d\mu = \widetilde{\mu}(S \cap T_0) = \widetilde{\mu}(S) = \varepsilon$. □

Proof of Theorem 4.2: Let f be in the ex-ante core of $\mathscr{E}$. Assume by the way of contradiction that f is not $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$-fair. This means that there exist two disjoint elements $S \in \mathscr{T}_0$, $E \in \mathscr{T}_1$ and an allocation g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $\displaystyle\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

By Proposition 3.8, there are an $\eta_0 > 0$, two coalitions B and R, and an allocation $\widetilde{g}$ such that

(1) $R \subseteq B \subseteq S$ and $\mathbb{I}_R = \mathbb{I}_B = \mathbb{I}_S$;

(2) $\widetilde{g}(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on R;

(3) $V_t(\widetilde{g}(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on R;

(4) $V_t(\widetilde{g}(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on $B \setminus R$; and

(5) $\int_B (\widetilde{g}(\cdot,\omega) - e(\cdot,\omega))d\mu = \frac{1}{2}\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Define $G := S \setminus B$ and

$$\mathbb{D}_i := \bigcap \left\{ \frac{\eta_0 \mu(R_i)}{6} \mathscr{G}_j : 1 \le j \le n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0 \mu(R_i)}{6}\right)^\Omega.$$

for each $i \in \mathbb{I}_G$, where $R_i := R \cap T_i$. Pick an $i \in \mathbb{I}_G$. Then there is some coalition $C_i \in \mathscr{T}_{G_i}$ such that

$$b_i := \frac{1}{3}\int_{G_i} (f - e)\, d\mu - \int_{C_i} (f - e)\, d\mu \in \mathbb{B}\left(0, \frac{\eta_0 \mu(R_i)}{6}\right)^\Omega.$$

As in Lemma 6.1, the correspondence $\mathbf{F}_i : C_i \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_i(t) := \{ z \in \mathbb{D}_i : f(t,\cdot) + z \in X_t \text{ and } V_t(f(t,\cdot) + z) > V_t(f(t,\cdot)) \},$$

is non-empty valued and has $\mathscr{T}_{C_i} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph, which further implies the existence of a $\mathscr{T}_{C_i}$-measurable selection ξ_i of $\mathbf{F}_i$. Define

$$\zeta_i := \frac{1}{\mu(C_i)} \int_{C_i} \xi d\mu.$$

As in the proof of Proposition 3.4, one can show that $\zeta_i \in \mathbb{D}_i$. So, $\varepsilon_i := \zeta_i \mu(C_i) \in \mathbb{D}_i$. For each $i \in \mathbb{I}_G$, let $z_i : R_i \times \Omega \to \mathbb{Y}$ be a function define by

$$z_i(t,\omega) := \frac{3}{2\mu(R_i)}(b_i(\omega) - \varepsilon_i(\omega))$$

for all $(t,\omega) \in R_i \times \Omega$. Thus,

$$z_i(t,\cdot) \in \bigcap \left\{ \frac{\eta_0}{2} \mathscr{G}_j : 1 \le j \le n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{2}\right)^\Omega$$

for all $t \in R_i$ with $i \in \mathbb{I}_G$. Consider an allocation $\varphi : T \times \Omega \to \mathbb{Y}$ defined by

$$\varphi(t,\omega) := \begin{cases} \widetilde{g}(t,\omega) + z_i(t,\omega), & \text{if } (t,\omega) \in R_i \times \Omega \text{ and } i \in \mathbb{I}_G; \\ \widetilde{g}(t,\omega), & \text{otherwise.} \end{cases}$$

Firstly, note that $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B. Furthermore, by (ii) and (iii), we have

$$\varphi(t,\omega) + \mathbb{B}\left(0, \frac{\eta_0}{2}\right) \subseteq X_t(\omega)$$

for all $(t,\omega) \in R \times \Omega$.

Case 1. $\mu(S \cup E) = \mu(T)$. By Proposition 3.11, there is an allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B, and

$$\int_B (h-e)d\mu = \frac{2}{3}\int_B (\varphi - e)d\mu + \frac{1}{3}\int_B (f-e)d\mu.$$

Define

$$C := \bigcup\{C_i : i \in \mathbb{I}_G\} \text{ and } K := B \cup C.$$

Let $\psi : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\psi(t,\omega) := \begin{cases} f(t,\omega) + \xi(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that

$$\int_K (\psi - e)d\mu = \frac{1}{3}\int_{S\cup E} (f-e)d\mu = 0,$$

which contradicts with the fact that f is in the ex-ante core of $\mathscr{E}$.

Case 2. $\mu(S \cup E) < \mu(T)$. Define $Q := T \setminus (S \cup E)$. Applying an argument similar to that in the proof of Theorem 3.13, one can show that there exists an element

$$c \in \bigcap\left\{\frac{\eta_0\mu(R)}{3}\mathscr{G}_j : 1 \le j \le n\right\} \cap \mathbb{B}\left(0, \frac{\eta_0\mu(R)}{3}\right)^{\Omega}$$

such that $c = \displaystyle\int_Q \gamma d\mu$, where

(a) $\gamma(t) \in \bigcap\left\{\frac{\eta_0}{3}\mathscr{G}_i : 1 \le j \le n\right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{3}\right)^{\Omega}$;

(b) $f(t,\cdot) + \gamma(t) \in \text{int}X_t$; and

(c) $V_t(f(t,\cdot)+\gamma(t)) > V_t(f(t,\cdot))$

μ-a.e. Q. By applying Proposition 3.8, one obtains a coalition D and an allocation κ such that $V_t(\kappa(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on D and

$$\int_D (\kappa - e)d\mu = \frac{1}{3}\int_Q (f+\gamma-e)d\mu$$

for all $\omega \in \Omega$. Let $\widetilde{\varphi} : T\times\Omega \to \mathbb{Y}$ be an allocation such that

$$\widetilde{\varphi}(t,\omega) := \begin{cases} \varphi(t,\omega) - \frac{3}{2\mu(R)}c(\omega), & \text{if } (t,\omega)\in R\times\Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $\widetilde{\varphi}(t,\omega) \in X_t(\omega)$ for all $(t,\omega)\in R\times\Omega$ and $V_t(\widetilde{\varphi}(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on R. By Proposition 3.11, there is an allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_B (h-e)d\mu = \frac{2}{3}\int_B (\widetilde{\varphi}-e)d\mu + \frac{1}{3}\int_B (f-e)d\mu.$$

Let

$$C := D \cup \bigcup\{C_i : i \in \mathbb{I}_G\} \text{ and } K := B\cup C$$

Let $\psi : T\times\Omega\to\mathbb{Y}$ be an allocation such that

$$\psi(t,\omega) = \begin{cases} f(t,\omega)+\gamma(t,\omega), & \text{if } (t,\omega)\in C\times\Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Therefore, as in **Case 1**, ψ is blocked by K via ψ. This is a contradiction. $\square$

Proof of Lemma 4.4: For any $i \in \mathbb{I}_S$ and $r \geq 1$, define

$$S_i^r := \left\{ t \in S_i : e(t,\omega) + \mathbb{B}\left(0, \frac{1}{r}\right) \subseteq X_t(\omega) \text{ for all } \omega\in\Omega \right\}.$$

It follows that $\{S_i^r : r \geq 1\}$ is an increasing sequnce of $\mathscr{T}$-measurable sets and $\lim_{r\to\infty}\mu(S_i \setminus S_i^r) = 0$ for all $i \in \mathbb{I}_S$. Pick an interger r_0 such that $\mu(S_i^{r_0}) > \frac{2\mu(S_i)}{3}$ for all $i\in\mathbb{I}_S$. Let $\{\eta_m : m\geq 1\}\subseteq (0,1)$ be a sequence of real numbers converging to 0, and $b\in\mathbb{Y}_{++}$ be such that $b \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)$. Consider a function $h^m : S\times\Omega\to\mathbb{Y}$ defined by

$$h^m(t,\omega) := (1-\eta_m)h(t,\omega) + \eta_m(e(t,\omega)-2b).$$

Put,

$$B^m := \left\{ t\in S : V_t(h^k(t,\cdot)) > V_t(f(t,\cdot)) \text{ for all } k \geq m \right\}.$$

By ($\mathbf{A}_4$) and ($\mathbf{A}_6$), the mapping $\xi^k : S \to \mathbb{R}$, defined by

$$\xi^k(t) := V_t(h^k(t,\cdot)) - V_t(f(t,\cdot)),$$

is $\mathscr{T}$-measurable and so is B^m. It is obvious that $\{B^m : m \geq 1\}$ is an ascending sequence and $S \sim \bigcup\{B^m : m \geq 1\}$. Define $c := \min\{\mu(S_i) : i \in \mathbb{I}_S\}$, and choose some $\upsilon > 0$ such that

$$\frac{2}{c}\int_R (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for any $R \in \mathscr{T}$ with $R \subseteq S$ and $\mu(R) < \upsilon$. Let $m_0 \geq 1$ be an integer such that $\mu(S \setminus B^{m_0}) < \min\{\upsilon, \frac{d}{4}\}$, where $d := \min\{\mu(S_i^{r_0}) : i \in \mathbb{I}_S\}$. In view of this, we get

$$\mu(S_i^{r_0} \cap B^{m_0}) > \frac{3\mu(S_i^{r_0})}{4} > \frac{\mu(S_i)}{2} \geq \frac{c}{2}$$

and

$$\frac{2}{c}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for all $i \in \mathbb{I}_S$, which further implies

$$\frac{1}{\mu(S_i \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega.$$

for all $i \in \mathbb{I}_S$. We are ready to choose $\lambda := \eta_{m_0}$. As in the proof of Proposition 3.4, one can show that

$$\frac{1}{\mu(S_i \setminus B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathscr{G}_i$$

for all $i \in \mathbb{I}_S$. Recognized that $\mu(S_i \setminus B^{m_0}) < \mu(S_i^{r_0} \cap B^{m_0})$ for each $i \in \mathbb{I}_S$. Convexity of $\mathscr{G}_i$ and $0 \in \mathscr{G}_i$ further yield that

$$\frac{1}{\mu(S_i^{r_0} \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for all $i \in \mathbb{I}_S$. Thus,

$$x_i := \frac{1}{\mu(S_i^{r_0} \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu$$

satisfies $x_i \in \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{1}{r_0}\right)^\Omega$ for all $i \in \mathbb{I}_S$, and thus, by the definition of $S_i^{r_0}$, we have $e(t,\omega) - x_i(\omega) \in X_t(\omega)$ for all $(t,\omega) \in S_i^{r_0} \times \Omega$ and $i \in \mathbb{I}_S$. For all $i \in \mathbb{I}_S$, consider an assignment $g_i : S_i \times \Omega \to \mathbb{Y}$ defined by

$$g_i(t,\omega) := \begin{cases} (1-\lambda)h(t,\omega) + \lambda(e(t,\omega) - x_i(\omega)), & \text{if } (t,\omega) \in (S_i^{r_0} \cap B^{m_0}) \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It is obvious that $g_i(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in S_i \times \Omega$ and $g_i(t,\cdot) - e(t,\cdot) \in \mathscr{G}_i$ for all $t \in S_i$. As $g_i(t,\omega) \gg h^{m_0}(t,\omega)$ for all $t \in S_i^{r_0} \cap B^{m_0}$, we have $V_t(g_i(t,\cdot)) > V_t(h^{m_0}(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in S_i^{r_0} \cap B^{m_0}$. Therefore, $V_t(g_i(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in S_i$, and

$$\int_{S_i} (g_i - e) d\mu = (1-\lambda) \int_{S_i} (h - e) d\mu.$$

for all $i \in I$. Thus, the allocation $y : T \times \Omega \to \mathbb{Y}$, defined by

$$y(t,\omega) := \begin{cases} g_i(t,\omega), & \text{if } (t,\omega) \in S_i \times \Omega, i \in I; \\ h(t,\omega), & \text{otherwise,} \end{cases}$$

satisfies the required condition. $\square$

Proof of Theorem 4.5: Let f be not in the ex-ante core of $\mathscr{E}$. Suppose, on contrary, that it is not $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair, which means that there exist two disjoint elements $S \in \mathscr{T}_1$, $E \in \mathscr{T}_0$ and an allocation g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $\displaystyle\int_S (g(\cdot,\omega) - e(\cdot,\omega)) d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega)) d\mu.$

By Lemma 3.8, there exist $0 < \lambda, \eta < 1$, a sub-coalition R of S with $\mathbb{I}_R = \mathbb{I}_S$ and an $\mathscr{G}$-assignment y such that

(1) $y(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;

(2) $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;

(3) $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on $S \setminus R$; and

(4) $\displaystyle\int_S (y - e) d\mu = (1-\lambda) \int_S (g - e) d\mu.$

By combining (ii) and (4), we have

$$\int_S (y - e) d\mu = (1-\lambda) \int_E (f - e) d\mu.$$

This implies that

$$\int_S (y - e) d\mu + \lambda \int_E (f - e) d\mu + \int_{T \setminus E} (f - e) d\mu = 0.$$

As a consequence, we have

$$\frac{1}{2}\int_S (y-e)d\mu + \frac{1}{2}\int_S (f-e)d\mu + \frac{\lambda}{2}\int_E (f-e)d\mu + \frac{1}{2}\int_{T\setminus(S\cup E)} (f-e)d\mu = 0.$$

Applying an argument similar to that in the proof of Theorem 4.2, one can show that there exists a function $\xi : E\times\Omega \to \mathbb{Y}$ such that

(i) $\xi(t,\cdot) \in \bigcap\left\{\frac{\eta\mu(R)}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta\mu(R)}{2}\right)^{\Omega}$;

(ii) $f(t,\cdot)+\xi(t,\cdot) \in \mathrm{int}X_t$; and

(iii) $V_t(f(t,\cdot)+\xi(t,\cdot)) > V_t(f(t,\cdot))$

for all $t\in E$. Again, by Lemma 3.8, there exist a sub-coalition B of S with $\mathbb{I}_B = \mathbb{I}_E$ and an $\mathscr{G}$-assignment φ such that

$$\int_B (\varphi - e)d\mu = \frac{\lambda}{2}\int_E (f+\xi-e)d\mu.$$

Define $c := \int_E \xi d\mu$ and note that

$$c \in \bigcap\left\{\frac{\eta\mu(R)}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta\mu(R)}{2}\right)^{\Omega}.$$

It follows that

$$\gamma := \frac{c}{\mu(R)} \in \bigcap\left\{\frac{\eta}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta}{2}\right)^{\Omega}.$$

Consider an allocation $\widetilde{y} : T\times\Omega\to\mathbb{Y}$ defined by

$$\widetilde{y}(t,\omega) = \begin{cases} y(t,\omega)-\gamma(\omega), & \text{if } (t,\omega)\in R\times\Omega; \\ y(t,\omega), & \text{otherwise.} \end{cases}$$

It is obvious that $V_t(\widetilde{y}(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S. Furthermore, by (ii) and (iii), we have

$$\widetilde{y}(t,\omega) + \mathbb{B}\left(0, \frac{\eta}{2}\right) \subseteq X_t(\omega)$$

for all $(t,\omega)\in R\times\Omega$.

Case 1. $\mu(S\cup E) = \mu(T)$. By Proposition 3.11, there exists an allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B, and

$$\int_S (h-e)d\mu = \frac{1}{2}\int_S (\widetilde{y}-e)d\mu + \frac{1}{2}\int_S (f-e)d\mu.$$

Let $\psi : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\psi(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{if } (t,\omega) \in B \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that $K := B \cup S$ blocks f via ψ, which leads to a contradiction.

Case 2. $\mu(S \cup E) < \mu(T)$. As in the proof of **Case 2** of Theorem 4.2, one can derive a contradiction. □

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