Utility maximization analysis of an emerging firm: a bordered Hessian approach

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Abstract

In this paper, method of Lagrange multipliers is used to investigate the utility function; subject to two constraints: budget constraint, and coupon constraint, and to verify that the utility is maximized. An economic model of an emerging firm has been developed here by considering four commodity variables. In the study, determinant of the 6×6 bordered Hessian matrix is operated to verify the utility maximization. Two Lagrangian multipliers are used here, as devices of optimization procedures, during the mathematical calculation. In this article, an attempt has been taken to achieve optimal result by the application of scientific method of optimization.

Keywords: Bordered Hessian, commodity, Lagrange multipliers, utility maximization

JEL Codes: C51, C61, C67, L23, L97, I31

1. Introduction

At present mathematical modeling in economics is an essential part; especially during the investigation of optimization policy [Samuelson, 1947; Carter, 2001]. The concept of utility was developed in the late 18th century by the English moral philosopher Jeremy Bentham (1748-1832) and English philosopher John Stuart Mill (1806-1873) [Bentham, 1780; Gauthier, 1975; Read, 2004]. The property of a commodity that enables to satisfy human wants is called utility [Bentham, 1780; Mohajan & Mohajan, 2022a,b]. Individuals seek to obtain the highest level of satisfaction from their purchasing goods [Stigler, 1950; Kirsh, 2017]. In the society, utility directly influences the demand and supply of the firms [Fishburn, 1970].

In multivariate calculus, Lagrange multipliers method is a very useful and powerful technique, which transforms a constrained problem to a higher dimensional unconstrained problem [Islam et al., 2009a,b,; Mohajan, 2021 b,c]. To increase the utility of the consumers we have included coupon system among the consumers. The individuals in the society can collect coupons and buy the essential commodities on priority basis [Islam et al., 2010, 2011; Moscati, 2013; Mohajan & Mohajan, 2022b]. In the society, utility maximization is a blessing both for humankind and the firm; as it provides maximum profit to the firms and in parallel increases welfare of the society [Eaton & Lipsey, 1975].

In this study, we have included four commodity variables to analyze the optimization techniques. We have operated the study with the determinant of 6×6 Hessian matrix, where four commodity variables, their corresponding four price vectors and four types of coupon number system are applied. In the analysis we have considered two commodities are of unit amount, and later we have considered four commodities are of unit amount. Finally, we have considered that prices and number of coupons of a pair of commodities are same, and ultimately we have proved that utility is maximized [Mohajan, 2021a].

2. Literature Review

Two American economists John V. Baxley and John C. Moorhouse have discussed the utility maximization method through the mathematical formulation, where they have introduced an explicit example of optimization [Baxley & Moorhouse, 1984]. Qi Zhao and his coauthors have proposed multi-product utility maximization as a general approach to the recommendation driven by economic principles [Zhao et al., 2017].

Pahlaj Moolio and his coauthors have used a Lagrange multiplier to form and improve economic models. They have also taken attempts to develop and solve optimization problems [Moolio et al., 2009]. Well-known mathematician Jamal Nazrul Islam and his coauthors have discussed utility maximization and other optimization problems by considering reasonable interpretation of the Lagrange multipliers. In their studies they have examined the behavior of the firm/organizations by analyzing comparative static results [Islam et al., 2009a,b, 2010, 2011]. Lia Roy and her coworkers have discussed cost minimization policy of an industry with detail mathematical formulation [Roy et al., 2021].

Two authors Jannatul Ferdous and Haradhan Kumar Mohajan have tried to calculate a profit maximization problem from sale items of an industry [Ferdous & Mohajan, 2022]. Devajit Mohajan and Haradhan Kumar Mohajan have consulted on profit maximization problem. They have used four variable inputs, such as capital, labor, principal raw materials, and other inputs in an industry [Mohajan & Mohajan, 2022a]. They have also discussed the utility maximization of an organization by considering two constraints: budget constraint and coupon constraint [Mohajan & Mohajan, 2022b].

Earlier, Haradhan Kumar Mohajan has used three inputs, such as capital, labor and other inputs for the sustainable production of a factory of Bangladesh [Mohajan, 2021b]. He also discussed the utility maximization of Bangladeshi consumers, where he has followed the optimization techniques [Mohajan, 2021a]. He and his coauthors have scrutinized on optimization problems for the social welfare [Mohajan et al., 2013].

3. Methodology of the Study

Methodology in any creative research is the organized and meaningful procedural works that follow scientific methods efficiently [Kothari, 2008]. In this study, we have considered two Lagrange multipliers λ_1 and λ_2; and have applied the determinant of 6×6 Hessian matrix [Mohajan, 2022a,b]. In this study, we have considered an economic world where there are only four commodities. We have started our study with 4-dimensional constrained problem, and later we have added two Lagrange multipliers. Then the Lagrangian function becomes a 6-dimensional unconstrained problem that maximizes utility function [Mohajan, 2017a, 2022b].

To prepare this paper, we have followed both qualitative and quantitative research approaches [Mohajan, 2018, 2020]. We have tried our best to maintain the reliability and validity, and also have tried to cite references properly both in the text and reference list [Mohajan, 2013, 2014a,b, 2017b, 2020]. In this paper, we have depended on the secondary data sources of optimization. We have taken help from the published journal articles, printed books of famous authors, conference papers, internet, websites, etc. [Islam et al., 2012a,b; Mohajan, 2018, 2022c; Mohajan & Mohajan, 2022d-f].

4. Objective of the Study

The core objective of this study is to verify utility maximization policy of an emerging firm. The other trivial objectives are as follows:

- to expand the determinant of bordered Hessian matrix properly, and
- to provide mathematical calculations accurately.

5. An Economic Model

We consider four commodities of a firm as; X_1, X_2, X_3, and X_4 [Mohajan & Mohajan, 2022b]. Let a consumer wants to buy only α_1, α_2, α_3, and α_4 amounts from these four commodities. In this model, we consider that the consumer spends all of his/her income to purchase these four commodities, and also submits all of his/her coupons. Let us consider a utility function as follows [Islam et al., 2010; Mohajan & Mohajan, 2022b]:

$$u(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha_1\alpha_2\alpha_3\alpha_4\,. \tag{1}$$

The budget constraint of the consumer is [Moolio et al., 2009],

$$B(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=p_1\alpha_1+p_2\alpha_2+p_3\alpha_3+p_4\alpha_4 \tag{2}$$

where p_1, p_2, p_3, and p_4 are the prices (in dollars) of per unit of commodity of X_1, X_2, X_3, and X_4, respectively. Now the coupon constraint is [Roy et al., 2021; Ferdous & Mohajan, 2022],

$$K(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=c_1\alpha_1+c_2\alpha_2+c_3\alpha_3+c_4\alpha_4 \tag{3}$$

where c_1, c_2, c_3, and c_4 are the coupons necessary to buy a unit of commodity of α_1, α_2, α_3, and α_4, respectively.

Using utility function from (2) and (3) in (1) we get Lagrangian function [Mohajan & Mohajan, 2022b,d],

$$\begin{aligned} v(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\mu_1,\mu_2)=&\ \alpha_1\alpha_2\alpha_3\alpha_4+\lambda_1(B-p_1\alpha_1-p_2\alpha_2-p_3\alpha_3-p_4\alpha_4)\\ &+\lambda_2(K-c_1\alpha_1-c_2\alpha_2-c_3\alpha_3-c_4\alpha_4) \end{aligned} \tag{4}$$

where λ_1 and λ_2 Lagrangian multipliers. Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility function. Using the necessary conditions of multivariate calculus for maximization in equation (4) we yield [Islam et al., 2011; Mohajan, 2021a];

$$v_{\mu_1} = B - p_1\alpha_1 - p_2\alpha_2 - p_3\alpha_3 - p_4\alpha_4 = 0, \tag{5a}$$

$$v_{\mu_2} = K - c_1\alpha_1 - c_2\alpha_2 - c_3\alpha_3 - c_4\alpha_4 = 0, \tag{5b}$$

$$v_1 = \alpha_2\alpha_3\alpha_4 - \mu_1 p_1 - \mu_2 c_1 = 0, \tag{5c}$$

$$v_2 = \alpha_1\alpha_3\alpha_4 - \mu_1 p_2 - \mu_2 c_2 = 0, \tag{5d}$$

$$v_3 = \alpha_1\alpha_2\alpha_4 - \mu_1 p_3 - \mu_2 c_3 = 0, \text{ and} \tag{5e}$$

$$v_4 = \alpha_1\alpha_2\alpha_3 - \mu_1 p_4 - \mu_2 c_4 = 0. \tag{5f}$$

Using equations (1) to (5a-f) we can express α_1, α_2, α_3, and α_4 as follows [Mohajan & Mohajan, 2022b]:

$$\alpha_1 = \frac{Bc_2 - Cp_2}{c_2 p_1 + c_1 p_2} + \frac{c_3 p_2 - c_2 p_3}{c_2 p_1 + c_1 p_2}\alpha_3 + \frac{c_4 p_2 - c_2 p_4}{c_2 p_1 + c_1 p_2}\alpha_4 \tag{6}$$

$$\alpha_2 = \frac{Cp_1 - Bc_1}{c_2 p_1 + c_1 p_2} - \frac{c_1 p_3 + c_3 p_1}{c_2 p_1 + c_1 p_2}\alpha_3 - \frac{c_1 p_4 + c_4 p_1}{c_2 p_1 + c_1 p_2}\alpha_4 \tag{7}$$

where $c_2 p_1 + c_1 p_2 \neq 0$.

For the simplicity let, $\alpha_3 = \alpha_4 = 1$, i.e., these commodities have one unit each; the amount of other two commodities can be accounted as;

$$\alpha_1 = \frac{(c_3 + c_4 - K)p_2 + (B - p_3 - p_4)c_2}{c_2 p_1 + c_1 p_2}, \tag{8}$$

$$\alpha_2 = \frac{-(c_3 + c_4 - K)p_1 - (B + p_3 + p_4)c_1}{c_2 p_1 + c_1 p_2}. \tag{9}$$

Since α_1 and α_2 are amount of commodities, so that they are of course positive. Equation (6) indicates that $\alpha_1 > 0$, if $c_3 + c_4 > K$ and $B > p_3 + p_4$. On the other hand, equation (7) indicates that $\alpha_2 > 0$, if $K > c_3 + c_4$ and $(K - c_3 - c_4)p_1 > (B + p_3 + p_4)c_1$.

Multiplying (8) and (9) we get;

$$\alpha_1\alpha_2 = \frac{-(K - c_3 - c_4)^2 p_1 p_2 - \left\{B^2 - (p_3 + p_4)^2\right\}c_1 c_2}{(c_2 p_1 + c_1 p_2)^2} \times \frac{\left\{B(c_2 p_1 + c_1 p_2) + (p_3 + p_4)(c_1 p_2 - c_2 p_1)\right\}}{(K - c_3 - c_4)^{-1}}. \tag{10}$$

Since α_1 and α_2 are positive, and their product $\alpha_1\alpha_1$ is also positive. Equation (10) indicates that $c_3+c_4>K$ and $B>p_3+p_4$. Equations (8) to (10) show that $c_3+c_4>K$ and $B>p_3+p_4$ in our economic model for $\alpha_1>0$, $\alpha_2>0$, and $\alpha_1\alpha_1>0$.

Now taking second order and cross-partial derivatives in (5a-f) we obtain;

$$B_1=p_1,\quad B_2=p_2,\quad B_3=p_3,\quad B_4=p_4.$$
$$K_1=c_1,\quad K_2=c_2,\quad K_3=c_3,\quad K_4=c_4. \tag{11}$$

$$v_{11}=0,\quad v_{12}=v_{21}=\alpha_3\alpha_4,\quad v_{13}=v_{31}=\alpha_2\alpha_4,$$
$$v_{14}=v_{41}=\alpha_2\alpha_3,\quad v_{22}=0,\quad v_{23}=v_{32}=\alpha_1\alpha_4,$$
$$v_{24}=v_{42}=\alpha_1\alpha_3,\ v_{33}=0,\ v_{34}=v_{43}=\alpha_1\alpha_2,\quad v_{44}=0. \tag{12}$$

Now we consider the bordered Hessian [Roy et al., 2021; Mohajan & Mohajan, 2022a],

$$|H|=\begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{12} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{43} & v_{44} \end{vmatrix}. \tag{13}$$

In our model, the number of constraints is two and commodity variables are four. For utility to be maximized, the determinant of the 6×6 Hssian matrix must be negative, i.e., $|H|<0$. Now taking the expansion of (13) we get [Islam et al., 2010; Mohajan & Mohajan, 2022a,c],

$$|H|=-B_1\begin{vmatrix} 0 & 0 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{12} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{22} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{32} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{42} & v_{43} & v_{44} \end{vmatrix}+B_2\begin{vmatrix} 0 & 0 & -K_1 & -K_3 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{13} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{23} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{33} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{43} & v_{44} \end{vmatrix}$$
$$-B_3\begin{vmatrix} 0 & 0 & -K_1 & -K_2 & -K_4 \\ -B_1 & -K_1 & v_{11} & v_{12} & v_{14} \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{24} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{34} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{44} \end{vmatrix}+B_4\begin{vmatrix} 0 & 0 & -K_1 & -K_2 & -K_3 \\ -B_1 & -K_1 & v_{11} & v_{12} & v_{13} \\ -B_2 & -K_2 & v_{21} & v_{22} & v_{23} \\ -B_3 & -K_3 & v_{31} & v_{32} & v_{33} \\ -B_4 & -K_4 & v_{41} & v_{42} & v_{43} \end{vmatrix}$$

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=-B_1\left\{-K_2\begin{vmatrix}-B_1 & -K_1 & v_{13} & v_{14}\\ -B_2 & -K_2 & v_{23} & v_{24}\\ -B_3 & -K_3 & v_{33} & v_{34}\\ -B_4 & -K_4 & v_{43} & v_{44}\end{vmatrix}+K_3\begin{vmatrix}-B_1 & -K_1 & v_{12} & v_{14}\\ -B_2 & -K_2 & v_{22} & v_{24}\\ -B_3 & -K_3 & v_{32} & v_{34}\\ -B_4 & -K_4 & v_{42} & v_{44}\end{vmatrix}-K_4\begin{vmatrix}-B_1 & -K_1 & v_{12} & v_{13}\\ -B_2 & -K_2 & v_{22} & v_{23}\\ -B_3 & -K_3 & v_{32} & v_{33}\\ -B_4 & -K_4 & v_{42} & v_{43}\end{vmatrix}\right\}
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+B_2\left\{-K_1\begin{vmatrix}-B_1 & -K_1 & v_{13} & v_{14}\\ -B_2 & -K_2 & v_{23} & v_{24}\\ -B_3 & -K_3 & v_{33} & v_{34}\\ -B_4 & -K_4 & v_{43} & v_{44}\end{vmatrix}+K_3\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{14}\\ -B_2 & -K_2 & v_{21} & v_{24}\\ -B_3 & -K_3 & v_{31} & v_{34}\\ -B_4 & -K_4 & v_{41} & v_{44}\end{vmatrix}-K_4\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{13}\\ -B_2 & -K_2 & v_{21} & v_{23}\\ -B_3 & -K_3 & v_{31} & v_{33}\\ -B_4 & -K_4 & v_{41} & v_{43}\end{vmatrix}\right\}
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-B_3\left\{-K_1\begin{vmatrix}-B_1 & -K_1 & v_{12} & v_{14}\\ -B_2 & -K_2 & v_{22} & v_{24}\\ -B_3 & -K_3 & v_{32} & v_{34}\\ -B_4 & -K_4 & v_{42} & v_{44}\end{vmatrix}+K_2\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{14}\\ -B_2 & -K_2 & v_{21} & v_{24}\\ -B_3 & -K_3 & v_{31} & v_{34}\\ -B_4 & -K_4 & v_{41} & v_{44}\end{vmatrix}-K_4\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{12}\\ -B_2 & -K_2 & v_{21} & v_{22}\\ -B_3 & -K_3 & v_{31} & v_{32}\\ -B_4 & -K_4 & v_{41} & v_{42}\end{vmatrix}\right\}
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+B_4\left\{-K_1\begin{vmatrix}-B_1 & -K_1 & v_{12} & v_{13}\\ -B_2 & -K_2 & v_{22} & v_{23}\\ -B_3 & -K_3 & v_{32} & v_{33}\\ -B_4 & -K_4 & v_{42} & v_{43}\end{vmatrix}+K_2\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{13}\\ -B_2 & -K_2 & v_{21} & v_{23}\\ -B_3 & -K_3 & v_{31} & v_{33}\\ -B_4 & -K_4 & v_{41} & v_{43}\end{vmatrix}-K_3\begin{vmatrix}-B_1 & -K_1 & v_{11} & v_{12}\\ -B_2 & -K_2 & v_{21} & v_{22}\\ -B_3 & -K_3 & v_{31} & v_{32}\\ -B_4 & -K_4 & v_{41} & v_{42}\end{vmatrix}\right\}
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=B_1K_2\left\{-B_1\begin{vmatrix}-K_2 & v_{23} & v_{24}\\ -K_3 & v_{33} & v_{34}\\ -K_4 & v_{43} & v_{44}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{23} & v_{24}\\ -B_3 & v_{33} & v_{34}\\ -B_4 & v_{43} & v_{44}\end{vmatrix}+v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{23}\\ -B_3 & -K_3 & v_{33}\\ -B_4 & -K_4 & v_{43}\end{vmatrix}\right\}
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-B_1K_3\left\{-B_1\begin{vmatrix}-K_2 & v_{22} & v_{24}\\ -K_3 & v_{32} & v_{34}\\ -K_4 & v_{42} & v_{44}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{22} & v_{24}\\ -B_3 & v_{32} & v_{34}\\ -B_4 & v_{42} & v_{44}\end{vmatrix}+v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\right\}
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\left.+v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{24}\\ -B_3 & -K_3 & v_{34}\\ -B_4 & -K_4 & v_{44}\end{vmatrix}-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\right\}-B_3K_2\left\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{24}\\ -K_3 & v_{31} & v_{34}\\ -K_4 & v_{41} & v_{44}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{21} & v_{24}\\ -B_3 & v_{31} & v_{34}\\ -B_4 & v_{41} & v_{44}\end{vmatrix}\right.
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-v_{14}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\Bigg\}\quad +B_3K_4\Bigg\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{22}\\ -K_3 & v_{31} & v_{32}\\ -K_4 & v_{41} & v_{42}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{21} & v_{22}\\ -B_3 & v_{31} & v_{32}\\ -B_4 & v_{41} & v_{42}\end{vmatrix}-v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\Bigg\}
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-B_4K_1\Bigg\{-B_1\begin{vmatrix}-K_2 & v_{22} & v_{23}\\ -K_3 & v_{32} & v_{33}\\ -K_4 & v_{42} & v_{43}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{22} & v_{23}\\ -B_3 & v_{32} & v_{33}\\ -B_4 & v_{42} & v_{43}\end{vmatrix}+v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{23}\\ -B_3 & -K_3 & v_{33}\\ -B_4 & -K_4 & v_{43}\end{vmatrix}-v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{22}\\ -B_3 & -K_3 & v_{32}\\ -B_4 & -K_4 & v_{42}\end{vmatrix}\Bigg\}
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+B_4K_2\Bigg\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{23}\\ -K_3 & v_{31} & v_{33}\\ -K_4 & v_{41} & v_{43}\end{vmatrix}+K_1\begin{vmatrix}-B_2 & v_{21} & v_{23}\\ -B_3 & v_{31} & v_{33}\\ -B_4 & v_{41} & v_{43}\end{vmatrix}-v_{13}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\Bigg\}-B_4K_3\Bigg\{-B_1\begin{vmatrix}-K_2 & v_{21} & v_{22}\\ -K_3 & v_{31} & v_{32}\\ -K_4 & v_{41} & v_{42}\end{vmatrix}
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+K_1\begin{vmatrix}-B_2 & v_{21} & v_{22}\\ -B_3 & v_{31} & v_{32}\\ -B_4 & v_{41} & v_{42}\end{vmatrix}-v_{12}\begin{vmatrix}-B_2 & -K_2 & v_{21}\\ -B_3 & -K_3 & v_{31}\\ -B_4 & -K_4 & v_{41}\end{vmatrix}\Bigg\}
$$

$$
=-B_1^2K_2^2v_{34}^2+B_1^2K_2K_4v_{23}v_{34}+B_1^2K_2K_3v_{24}v_{34}+B_1B_2K_1K_2v_{34}^2-B_1B_4K_1K_2v_{23}v_{34}+B_1B_3K_1K_2v_{24}v_{34}
$$

$$
-B_1B_2K_2K_4v_{13}v_{34}+B_1B_4K_2^2v_{13}v_{34}+B_1B_3K_2K_4v_{13}v_{24}-B_1B_4K_2K_3v_{13}v_{24}-B_1B_2K_2K_3v_{14}v_{34}
$$

$$
+B_1B_3K_2K_3v_{14}v_{34}+B_1B_3K_2K_4v_{14}v_{23}-B_1B_4K_2K_3v_{14}v_{23}+B_1^2K_2K_3v_{24}v_{34}-B_1^2K_3^2v_{24}^2+B_1^2K_3K_4v_{24}v_{23}
$$

$$
-B_1B_2K_1K_3v_{24}v_{34}+B_1B_3K_1K_3v_{24}^2-B_1B_4K_1K_3v_{24}v_{23}-B_1B_2K_3K_4v_{12}v_{34}-B_1B_4K_2K_3v_{12}v_{34}
$$

$$
-B_1B_3K_3K_4v_{12}v_{24}+B_1B_4K_3^2v_{12}v_{24}+B_1B_2K_3^2v_{14}v_{24}-B_1B_2K_3K_4v_{14}v_{23}-B_1B_3K_2K_3v_{14}v_{24}
$$

$$
+B_1B_4K_2K_3v_{14}v_{23}+B_1^2K_2K_4v_{23}v_{34}+B_1^2K_3K_4v_{23}v_{24}-B_1^2K_4^2v_{23}^2-B_1B_2K_1K_4v_{23}v_{34}-B_1B_3K_1K_4v_{23}v_{24}
$$

$$
+B_1B_4K_1K_4v_{23}^2+B_1B_2K_3K_4v_{12}v_{34}-B_1B_2K_2K_4v_{12}v_{34}+B_1B_3K_4^2v_{12}v_{23}-B_1B_4K_3K_4v_{12}v_{23}
$$

$$
-B_1B_2K_3K_4v_{13}v_{24}+B_1B_2K_4^2v_{13}v_{23}+B_1B_3K_2K_4v_{13}v_{24}-B_1B_4K_2K_4v_{13}v_{23}+B_1B_2K_1K_2v_{34}^2
$$

$$
-B_1B_2K_1K_4v_{23}v_{34}-B_1B_2K_1K_3v_{24}v_{34}-B_1B_2K_2K_3v_{14}v_{34}-B_1B_2K_3^2v_{12}v_{14}+B_1B_2K_3K_4v_{12}v_{13}
$$

$$
+B_1B_2K_3^2v_{24}v_{14}+B_1B_2K_3K_4v_{13}v_{24}-B_2^2K_1^2v_{34}^2+B_2B_4K_1^2v_{23}v_{34}+B_2B_3K_1^2v_{24}v_{34}+B_2^2K_1K_4v_{13}v_{34}
$$

$$
-B_2B_4K_1K_2v_{13}v_{34}-B_2B_3K_1K_4v_{13}v_{24}+B_2B_3K_1K_3v_{13}v_{24}+B_2^2K_1K_3v_{14}v_{34}-B_2B_3K_1K_2v_{14}v_{34}
$$

$$
+B_2B_3K_1K_4v_{14}v_{23}-B_2B_4K_1K_3v_{14}v_{23}-B_1B_2K_2K_3v_{14}v_{34}-B_1B_2K_3K_4v_{12}v_{34}+B_1B_2K_3^2v_{24}v_{14}
$$

$$
-B_1B_2K_3K_4v_{24}v_{13}+B_2^2K_1K_4v_{14}v_{34}-B_2B_3K_1K_3v_{12}v_{34}-B_2B_3K_1K_3v_{24}v_{12}+B_2B_4K_1K_3v_{24}v_{14}-B_2^2K_3^2v_{14}^2
$$

$$
+B_2B_3K_2K_3v_{14}v_{34}-B_2B_3K_3K_4v_{12}v_{14}+B_2B_4K_3^2v_{12}v_{14}-B_1B_2K_2K_4v_{13}v_{34}-B_1B_2K_3K_4v_{12}v_{34}
$$

$$
-B_1B_2K_3K_4v_{14}v_{23}+B_1B_2K_4^2v_{23}v_{13}+B_2^2K_1K_4v_{13}v_{34}-B_2B_3K_1K_4v_{12}v_{34}+B_2B_3K_1K_4v_{14}v_{23}
$$

$$
-B_2B_4K_1K_4v_{23}v_{13}+B_2^2K_3K_4v_{13}v_{14}-B_2^2K_4^2v_{13}^2-B_2B_3K_2K_4v_{13}v_{14}+B_2B_4K_2K_4v_{13}^2+B_2B_3K_4^2v_{13}v_{12}
$$

$$
-B_2B_4K_3K_4v_{13}v_{12}-B_1B_3K_1K_2v_{24}v_{34}+B_1B_3K_1K_3v_{24}^2-B_1B_3K_1K_4v_{24}v_{23}+B_2B_3K_1^2v_{24}v_{34}-B_3^2K_1^2v_{24}^2
$$

$$
+B_3B_4K_1^2v_{23}v_{24}-B_2B_3K_1K_4v_{12}v_{34}+B_3B_4K_1K_2v_{12}v_{34}+B_3^2K_1K_4v_{12}v_{24}-B_3B_4K_1K_3v_{12}v_{24}
$$

$$
+B_2B_3K_1K_3v_{14}v_{24}-B_2B_3K_1K_4v_{14}v_{23}+B_3^2K_1K_2v_{14}v_{24}-B_3B_4K_1K_2v_{14}v_{23}+B_1B_3K_2^2v_{14}v_{34}
$$

$$
-B_1B_3K_2K_4v_{12}v_{34}-B_1B_3K_2K_3v_{14}v_{24}+B_1B_3K_2K_4v_{13}v_{24}-B_2B_3K_1K_2v_{14}v_{34}+B_3B_4K_1K_2v_{12}v_{34}
$$

$$
+B_3^2K_1K_2v_{14}v_{24}-B_3B_4K_1K_2v_{13}v_{24}+B_2B_3K_2K_3v_{14}^2-B_2B_3K_2K_4v_{13}v_{14}-B_3^2K_2^2v_{14}^2+B_3B_4K_2^2v_{13}v_{14}
$$

$$
+B_3^2K_2K_4v_{12}v_{14}-B_3B_4K_2K_3v_{12}v_{14}+B_1B_3K_2K_4v_{13}v_{24}-B_1B_3K_2K_4v_{14}v_{23}-B_1B_3K_3K_4v_{12}v_{24}
$$

$$
+B_1B_3K_4^2v_{12}v_{23}-B_2B_3K_1K_4v_{13}v_{24}+B_2B_3K_1K_4v_{14}v_{23}+B_3^2K_1K_4v_{12}v_{24}-B_3B_4K_1K_4v_{12}v_{23}
$$

$$
-B_2B_3K_3K_4v_{12}v_{14}+B_2B_3K_4^2v_{12}v_{13}+B_3^2K_2K_4v_{12}v_{14}-B_3B_4K_2K_4v_{12}v_{13}-B_3^2K_4^2v_{12}^2+B_3B_4K_3K_4v_{12}^2
$$

$$- B_1B_4K_1K_2v_{23}v_{34} \quad - B_1B_4K_1K_3v_{24}v_{23} \quad + B_1B_4K_1K_4v_{23}^2 \quad + B_2B_4K_1^2v_{23}v_{34} \quad + B_3B_4K_1^2v_{23}v_{24} \quad - B_4^2K_1^2v_{23}^2$$

$$+ B_2B_4K_1K_3v_{12}v_{34} \quad + B_3B_4K_1K_2v_{12}v_{34} \quad - B_3B_4K_1K_4v_{12}v_{23} \quad + B_3^2K_1K_3v_{12}v_{23} \quad + B_2B_4K_1K_3v_{13}v_{24}$$

$$- B_2B_4K_1K_4v_{13}v_{23} \quad - B_3B_4K_1K_2v_{13}v_{23} \quad + B_4^2K_1K_2v_{13}v_{24} \quad - B_2B_4K_1K_2v_{13}v_{34} \quad + B_3B_4K_1K_2v_{12}v_{34}$$

$$- B_3B_4K_1K_2v_{13}v_{23} \quad + B_4^2K_1K_2v_{13}v_{23} \quad - B_2B_4K_2K_3v_{13}v_{14} \quad + B_2B_4K_2K_4v_{13}^2 \quad + B_3B_4K_2^2v_{13}v_{14} \quad - B_4^2K_2^2v_{13}^2$$

$$- B_1B_4K_2K_3v_{13}v_{24} \quad + B_1B_4K_2K_4v_{14}v_{23} \quad + B_1B_4K_3^2v_{12}v_{14} \quad - B_1B_4K_3K_4v_{12}v_{13} \quad + B_2B_4K_1K_3v_{13}v_{24}$$

$$- B_2B_4K_1K_3v_{14}v_{23} \quad - B_3B_4K_1K_3v_{12}v_{24} \quad + B_4^2K_1K_3v_{12}v_{23} \quad + B_2B_4K_3^2v_{12}v_{14} \quad - B_2B_4K_3K_4v_{12}v_{13}$$

$$- B_3B_4K_2K_3v_{12}v_{14} + B_4^2K_2K_3v_{12}v_{13}$$

$$= - p_1^2c_1^2\alpha_1^2\alpha_2^2 \quad + p_1^2c_2c_4\alpha_1^2\alpha_2\alpha_4 \quad + p_1^2c_2c_3\alpha_1^2\alpha_2\alpha_3 \quad + p_1p_2c_1c_2\alpha_1^2\alpha_2^2 \quad - p_1p_4c_1c_2\alpha_1^2\alpha_2\alpha_4$$

$$+ p_1p_3c_1c_2\alpha_1^2\alpha_2\alpha_3 \quad - p_1p_2c_2c_4\alpha_1\alpha_2^2\alpha_4 \quad + p_1p_4c_2^2\alpha_1\alpha_2^2\alpha_4 \quad + p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4$$

$$- p_1p_2c_2c_3\alpha_1\alpha_2^2\alpha_4 \quad + p_1p_3c_2c_3\alpha_1\alpha_2^2\alpha_3 \quad + p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_1^2c_2c_3\alpha_1^2\alpha_2\alpha_3$$

$$- p_1^2c_3^2\alpha_1^2\alpha_2^2 \quad + p_1^2c_3c_4\alpha_1^2\alpha_3\alpha_4 \quad - p_1p_2c_1c_3\alpha_1^2\alpha_2\alpha_3 \quad + p_1p_3c_1c_3\alpha_1^2\alpha_3^2 \quad - p_1p_4c_1c_3\alpha_1^2\alpha_3\alpha_4$$

$$- p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_3c_3c_4\alpha_1\alpha_3^2\alpha_4 \quad + p_1p_4c_3^2\alpha_1\alpha_3^2\alpha_4 \quad + p_1p_2c_3^2\alpha_1\alpha_2\alpha_3^2$$

$$- p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_3c_2c_3\alpha_1\alpha_2\alpha_3^2 \quad + p_1p_4c_2c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_1^2c_2c_4\alpha_1^2\alpha_2\alpha_4 \quad + p_1^2c_3c_4\alpha_1^2\alpha_3\alpha_4$$

$$- p_1^2c_4^2\alpha_1^2\alpha_4^2 \quad - p_1p_2c_1c_4\alpha_1^2\alpha_2\alpha_4 \quad - p_1p_3c_1c_4\alpha_1^2\alpha_3\alpha_4 \quad + p_1p_4c_1c_4\alpha_1^2\alpha_4^2 \quad + p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4$$

$$- p_1p_2c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_1p_3c_4^2\alpha_1\alpha_3\alpha_4^2 \quad - p_1p_4c_3c_4\alpha_1\alpha_3\alpha_4^2 \quad - p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_1p_2c_4^2\alpha_1\alpha_2\alpha_4^2$$

$$+ p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_4c_2c_4\alpha_1\alpha_2\alpha_4^2 \quad - p_1p_2c_2c_3\alpha_1\alpha_2^2\alpha_3 \quad - p_1p_2c_3^2\alpha_2\alpha_3^2\alpha_4 \quad + p_1p_2c_3c_4\alpha_2\alpha_3\alpha_4^2$$

$$+ p_1p_2c_3^2\alpha_1\alpha_2\alpha_3^2 \quad + p_1p_2c_3c_4\alpha_1\alpha_2^2\alpha_4 \quad - p_2^2c_1^2\alpha_1^2\alpha_2^2 \quad + p_2p_4c_1^2\alpha_1^2\alpha_2\alpha_3 \quad + p_2p_3c_1^2\alpha_1^2\alpha_2\alpha_3 \quad + p_1p_2c_1c_2\alpha_1^2\alpha_2^2$$

$$- p_1p_2c_1c_4\alpha_1^2\alpha_2\alpha_4 \quad - p_1p_2c_1c_3\alpha_1^2\alpha_2\alpha_3 \quad + p_2^2c_1c_4\alpha_1\alpha_2^2\alpha_4 \quad - p_2p_4c_1c_2\alpha_1\alpha_2^2\alpha_4 \quad - p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_2p_3c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2^2c_1c_3\alpha_1\alpha_2^2\alpha_3 \quad - p_2p_3c_1c_2\alpha_1\alpha_2^2\alpha_3 \quad + p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4$$

$$- p_1p_2c_2c_3\alpha_1\alpha_2^2\alpha_3 \quad - p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_1p_2c_3^2\alpha_1\alpha_2\alpha_3^2 \quad - p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2^2c_1c_4\alpha_1\alpha_2^2\alpha_3$$

$$- p_2p_3c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_3c_1c_3\alpha_1\alpha_3^2\alpha_4 \quad + p_2p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2^2c_3^2\alpha_2^2\alpha_3^2 \quad + p_2p_3c_2c_3\alpha_1\alpha_2^2\alpha_3$$

$$- p_2p_3c_3c_4\alpha_2\alpha_3^2\alpha_4 \quad + p_2p_4c_3^2\alpha_2\alpha_3^2\alpha_4 \quad - p_1p_2c_2c_4\alpha_1\alpha_2^2\alpha_4 \quad - p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_2c_3c_4\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_1p_2c_4^2\alpha_1\alpha_2\alpha_4^2 \quad + p_2^2c_1c_4\alpha_1\alpha_2^2\alpha_4 \quad - p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_4c_1c_4\alpha_1\alpha_2\alpha_4^2$$

$$+ p_2^2c_3c_4\alpha_2^2\alpha_3\alpha_4 \quad - p_2^2c_4^2\alpha_2^2\alpha_4^2 \quad - p_2p_3c_2c_4\alpha_2^2\alpha_3\alpha_4 \quad + p_2p_4c_2c_4\alpha_2^2\alpha_4^2 \quad + p_2p_3c_4^2\alpha_2\alpha_3\alpha_4^2$$

$$- p_2p_4c_3c_4\alpha_2\alpha_3\alpha_4^2 \quad - p_1p_3c_1c_2\alpha_1^2\alpha_2\alpha_3 \quad + p_1p_3c_1c_3\alpha_1^2\alpha_3^2 \quad - p_1p_3c_1c_4\alpha_1^2\alpha_3\alpha_4 \quad + p_2p_3c_1^2\alpha_1^2\alpha_2\alpha_3$$

$$- p_3^2c_1^2\alpha_1^2\alpha_3^2 \quad + p_3p_4c_1^2\alpha_1^2\alpha_3\alpha_4 \quad - p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_3^2c_1c_4\alpha_1\alpha_3^2\alpha_4$$

$$- p_3p_4c_1c_3\alpha_1\alpha_3^2\alpha_4 \quad + p_2p_3c_1c_3\alpha_1\alpha_2\alpha_3^2 \quad - p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_3^2c_1c_2\alpha_1\alpha_2\alpha_3^2 \quad - p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_1p_3c_2^2\alpha_1\alpha_2^2\alpha_3 \quad - p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_3c_2c_3\alpha_1\alpha_2\alpha_3^2 \quad + p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_3c_1c_1\alpha_1\alpha_2^2\alpha_3$$

$$+ p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_3^2c_1c_2\alpha_1\alpha_2\alpha_3^2 \quad - p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2p_3c_2c_3\alpha_2^2\alpha_3^2 \quad - p_2p_3c_2c_4\alpha_2^2\alpha_3\alpha_4$$

$$- p_3^2c_2^2\alpha_2^2\alpha_3^2 \quad + p_3p_4c_2^2\alpha_2^2\alpha_3\alpha_4 \quad + p_3^2c_2c_4\alpha_2\alpha_3^2\alpha_4 \quad - p_3p_4c_2c_3\alpha_2\alpha_3^2\alpha_4 \quad + p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4$$

$$- p_1p_3c_2c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_1p_3c_3c_4\alpha_1\alpha_3^2\alpha_4 \quad + p_1p_3c_4^2\alpha_1\alpha_3\alpha_4^2 \quad - p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4 \quad + p_2p_3c_1c_4\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_3^2c_1c_4\alpha_1\alpha_3^2\alpha_4 \quad - p_3p_4c_1c_4\alpha_1\alpha_3\alpha_4^2 \quad - p_2p_3c_3c_4\alpha_2\alpha_3^2\alpha_4 \quad + p_2p_3c_4^2\alpha_2\alpha_3\alpha_4^2 \quad + p_3^2c_2c_4\alpha_2\alpha_3^2\alpha_4$$

$$- p_3p_4c_2c_4\alpha_2\alpha_3\alpha_4^2 \quad - p_3^2c_4^2\alpha_3^2\alpha_4^2 \quad + p_3p_4c_3c_4\alpha_3^2\alpha_4^2 \quad - p_1p_4c_1c_2\alpha_1^2\alpha_2\alpha_4 \quad - p_1p_4c_1c_3\alpha_1^2\alpha_3\alpha_4$$

$$+ p_1p_4c_1c_4\alpha_1^2\alpha_4^2 \quad + p_2p_4c_1^2\alpha_1^2\alpha_2\alpha_4 \quad + p_3p_4c_1^2\alpha_1^2\alpha_3\alpha_4 \quad - p_4^2c_1^2\alpha_1^2\alpha_4^2 \quad + p_2p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4$$

$$+ p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_3p_4c_1c_4\alpha_1\alpha_3\alpha_4^2 \quad + p_3^2c_1c_3\alpha_1\alpha_3\alpha_4^2 \quad + p_2p_4c_1c_3\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_4c_1c_4\alpha_1\alpha_2\alpha_4^2$$

$$- p_3p_4c_1c_2\alpha_1\alpha_2\alpha_4^2 \quad + p_4^2c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_2p_4c_1c_2\alpha_1\alpha_2^2\alpha_4 \quad + p_3p_4c_1c_2\alpha_1\alpha_2\alpha_3\alpha_4 \quad - p_3p_4c_1c_2\alpha_1\alpha_2\alpha_4^2$$

$$
\begin{aligned}
&+ p_4^2 c_1 c_2 \alpha_1 \alpha_2 \alpha_4^2 \quad - p_1 p_4 c_2 c_3 \alpha_1 \alpha_2^2 \alpha_4 \quad + p_1 p_4 c_2 c_4 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \quad + p_1 p_4 c_3^2 \alpha_2 \alpha_3^2 \alpha_4 \quad - p_1 p_4 c_3 c_4 \alpha_1 \alpha_3 \alpha_4^2 \\
&- p_2 p_4 c_2 c_3 \alpha_2^2 \alpha_3 \alpha_4 \quad + p_2 p_4 c_2 c_4 \alpha_2^2 \alpha_4^2 \quad + p_3 p_4 c_2^2 \alpha_2^2 \alpha_3 \alpha_4 \quad - p_4^2 c_2^2 \alpha_2^2 \alpha_4^2 \quad + p_2 p_4 c_1 c_3 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
&- p_2 p_4 c_1 c_3 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \quad - p_3 p_4 c_1 c_3 \alpha_1 \alpha_3^2 \alpha_4 \quad + p_4^2 c_1 c_3 \alpha_1 \alpha_3 \alpha_4^2 \quad + p_2 p_4 c_3^2 \alpha_2 \alpha_3^2 \alpha_4 \quad - p_2 p_4 c_3 c_4 \alpha_2 \alpha_3 \alpha_4^2 \\
&- p_3 p_4 c_2 c_3 \alpha_2 \alpha_3^2 \alpha_4 + p_4^2 c_2 c_3 \alpha_2 \alpha_3 \alpha_4^2 \\
&= \left(- p_1^2 c_1^2 - p_1^2 c_3^2 - p_2^2 c_1^2 + 2 p_1 p_2 c_1 c_2\right) \alpha_1^2 \alpha_2^2 \left(+ 2 p_1^2 c_2 c_3 - 2 p_1 p_2 c_1 c_3 + p_2 p_4 c_1^2 + 2 p_2 p_3 c_1^2\right) \alpha_1^2 \alpha_2 \alpha_3 \\
&\left(- p_3^2 c_1^2 + 2 p_1 p_3 c_1 c_3\right) \alpha_1^2 \alpha_3^2 \left(+ p_1^2 c_2 c_4 + p_2^2 c_1 c_4 - 2 p_1 p_2 c_1 c_4 - p_1 p_4 c_1 c_2 + p_2 p_4 c_1^2\right) \alpha_1^2 \alpha_2 \alpha_4 \\
&\left(+ 2 p_1 p_4 c_1 c_4 - p_1^2 c_4^2 - p_4^2 c_1^2\right) \alpha_1^2 \alpha_4^2 \quad \left(- p_3^2 c_2^2 - p_2^2 c_3^2 + p_2 p_3 c_2 c_3\right) \alpha_2^2 \alpha_3^2 \quad \left(- p_4^2 c_2^2 - p_2^2 c_4^2 + 2 p_2 p_4 c_2 c_4\right) \alpha_2^2 \alpha_4^2 \\
&\left(+ p_3 p_4 c_3 c_4 - p_3^2 c_4^2\right) \alpha_3^2 \alpha_4^2 + \left(3 p_1 p_3 c_2 c_4 - 7 p_1 p_2 c_3 c_4 - 2 p_2 p_3 c_1 c_4 + 2 p_2 p_4 c_1 c_3 + 3 p_3 p_4 c_1 c_2\right) \alpha_1 \alpha_2 \alpha_3 \alpha_4 \\
&\left(+ 2 p_1^2 c_3 c_4 - 2 p_1 p_3 c_1 c_4 - 2 p_1 p_4 c_1 c_3 + 2 p_3 p_4 c_1^2\right) \alpha_1^2 \alpha_3 \alpha_4 \\
&\left(+ 2 p_1 p_2 c_3^2 - 2 p_1 p_3 c_2 c_3 + p_1 p_2 c_3^2 + p_2 p_3 c_1 c_3 + 2 p_3^2 c_1 c_2\right) \alpha_1 \alpha_2 \alpha_3^2 \\
&\left(+ 2 p_1 p_2 c_4^2 - p_1 p_4 c_2 c_4 - 2 p_2 p_4 c_1 c_4 - 2 p_3 p_4 c_1 c_2 + p_4^2 c_1 c_2\right) \alpha_1 \alpha_2 \alpha_4^2 \\
&\left(- 2 p_1 p_2 c_2 c_4 - p_1 p_2 c_2 c_3 + p_1 p_4 c_2^2 + p_1 p_2 c_3 c_4 - 2 p_2 p_4 c_1 c_2 + 2 p_2^2 c_1 c_4 - p_1 p_4 c_2 c_3\right) \alpha_1 \alpha_2^2 \alpha_4 \\
&\left(+ p_1 p_3 c_2 c_3 - 2 p_1 p_2 c_2 c_3 + p_2^2 c_1 c_3 - 2 p_2 p_3 c_1 c_2 + p_2^2 c_1 c_4 + p_2 p_3 c_2 c_3 + p_1 p_3 c_2^2\right) \alpha_1 \alpha_2^2 \alpha_3 \\
&\left(- 2 p_3 p_4 c_1 c_3 - 2 p_1 p_3 c_3 c_4 + p_1 p_4 c_3^2 - p_2 p_3 c_1 c_3 + 2 p_3^2 c_1 c_4\right) \alpha_1 \alpha_3^2 \alpha_4 \\
&\left(+ 2 p_1 p_3 c_4^2 - 2 p_1 p_4 c_3 c_4 - 2 p_3 p_4 c_1 c_4 + p_4^2 c_1 c_3 + p_3^2 c_1 c_3\right) \alpha_1 \alpha_3 \alpha_4^2 \\
&\left(+ 2 p_2 p_4 c_3^2 + 2 p_3^2 c_2 c_4 - 2 p_3 p_4 c_2 c_3 - 2 p_2 p_3 c_3 c_4 + p_1 p_4 c_3^2 - p_1 p_2 c_3^2\right) \alpha_2 \alpha_3^2 \alpha_4 \\
&\left(+ p_1 p_2 c_3 c_4 + 2 p_2 p_3 c_4^2 - 2 p_2 p_4 c_3 c_4 - p_3 p_4 c_2 c_4 + p_4^2 c_2 c_3\right) \alpha_2 \alpha_3 \alpha_4^2 \\
&\left(+ p_2^2 c_3 c_4 - 2 p_2 p_3 c_2 c_4 + 2 p_3 p_4 c_2^2 - p_2 p_4 c_2 c_3\right) \alpha_2^2 \alpha_3 \alpha_4 . \qquad (14)
\end{aligned}
$$

We use $p_1 = p_3$ and $p_2 = p_4$ where pair of prices are same, and $c_1 = c_3$ and $c_2 = c_4$, where pair of coupon numbers are same, and every term contains $p_1 p_2 c_1 c_2$, i.e. we use $p_1^2 = p_2^2 = p_3^2 = p_3^2 = p_1 p_2$ and $c_1^2 = c_2^2 = c_3^2 = c_3^2 = c_1 c_2$, where the terms contain square, then (14) becomes;

$$
\begin{aligned}
|H| = &-3 p_1 p_2 c_1 c_2 \; - 5 p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \\
&- 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - 2 p_3 p_4 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \\
&- 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \; - 2 p_1 p_2 c_1 c_2 \\
&- 2 p_1 p_2 c_1 c_2 - 2 p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 - p_1 p_2 c_1 c_2 \\
&- p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; - p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \\
&+ 3 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_3 p_4 c_3 c_4 \; + p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \\
&+ p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \\
&+ 2 p_1 p_2 c_1 c_2 + 2 p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 + 2 p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 + p_1 p_2 c_1 c_2 \\
&+ 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \; + p_1 p_2 c_1 c_2 \; + 2 p_1 p_2 c_1 c_2 \\
&+ p_1 p_2 c_1 c_2 + 2 p_1 p_2 c_1 c_2 .
\end{aligned}
$$

$$|H| = -62\,p_1 p_2 c_1 c_2 + 60\,p_1 p_2 c_1 c_2 = -2\,p_1 p_2 c_1 c_2 < 0\,. \qquad (15)$$

Since the Lagrangian function contains two constraints, and we have operated the 6×6 bordered Hessian with four variables; therefore, for utility maximization Hessian need to be negative. From (15) we observe that Hessian is negative, i.e., $|H| < 0$, and therefore utility $v(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \mu_1, \mu_2)$ obtained in (4) is definitely maximized.

6. Conclusions

In this study we have tried to verify the utility maximization policy of an emerging firm. We have used two constraints: budget constraint and coupon constraint; and consequently, we have applied two Lagrange multipliers during the mathematical calculations of optimization. We have operated the research analysis with four commodity variables and applied the determinant of bordered Hessian matrix. In one stage, we face difficulties working with four commodity variables. Then for simplicity we have considered two commodities equal to unity. Later, we have measured all commodities are of unit amount, and prices of two commodities are same, and also two types of coupon numbers are same. Throughout the study we have tried to introduce mathematical calculations in some details. We hope future researchers will try to solve the optimization problem more efficiently, and they will develop their models more fruitfully.

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