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Abstract

Around the turn of the Twentieth century, US and euro area long-term bond yields experienced a remarkable decline and remained at historically low levels even in the face of rising short-term rates (the so called "conundrum"). This unusual phenomenon has been analyzed by many researchers through the lens of macro-finance VARs and no-arbitrage term structure models. A commonly found result is that the decline in long-term rates was primarily driven by an unprecedented reduction in risk premia. I show that such result might be an artefact of the class of models employed to study the phenomenon. I propose an alternative model which suggests that, although risk premia played an important role in reducing bond yields, other two equally important forces were at play, i.e. a decline in the real natural rate of interest and a structural reduction in inflation expectations. I conclude that, after accounting for permanent shifts in the expectations about the future path of short-term rates, the dynamics of risk premia observed after the turn of the century have not been unusual if considered from an historical perspective.

The sharp decline in long-term interest rates experienced around the turn of the Twentieth century in the major industrialized countries has attracted considerable attention from both academic researchers and policy makers. An especially striking phenomenon has been the fact that long-term rates remained low even in the face of rising short-term rates, which former Federal Reserve chairman Alan Greenspan dubbed a "conundrum"[1].

A commonly held opinion is that the low level of long-term rates could be largely explained by a decline in the compensation for risk. Such opinion has been supported by much empirical work devoted to the measurement of bond risk premia and the analysis of their dynamics: see, for example, Kim and Wright (2005), Kremer and Rostagno (2006) and Backus and Wright (2007). More mixed evidence is provided by Rudebusch, Swanson and Wu (2006).

In this paper, I argue that models commonly employed to study recent developments in long-term rates might fail to capture permanent changes in the economy that have contributed to lower bond yields; as a consequence, such models might underestimate the level of risk premia in more recent years and overemphasize the severity of the bond yield conundrum.

Standard term-structure theory identifies two main drivers behind movements in long-term rates: changes in the expected future path of the short-term rate and fluctuations in risk premia. I provide some evidence that popular term-structure models, based on stationary vector autoregressions (VARs) [2], might be unable to detect long-lasting shifts in the expectations about the short-term rate. Such models tend to attribute most of the variability in long-term forward interest rates to forward premia and little or almost none to changing expectations. The technical reason for this problem is that estimated VARs often display a fast speed of convergence to their unique equilibrium and produce forecasts of the short-term rate at long horizons which remain roughly constant through time and are hardly different from their historical averages.

I propose to use a simple macroeconometric model in which permanent shifts in the natural real rate of interest[3], the growth rate of potential output and long-run inflation determine long-lasting changes in expectations about the short-term rate. Any permanent change in one of these three variables determines a new set of equilibrium values towards which all the other macroeconomic variables of interest tend to converge.

Estimates of the model suggest that in recent years a reduction in both

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[1] "Long-term interest rates have trended lower in recent months even as the Federal Reserve has raised the level of the target federal funds rate by 150 basis points. This development contrasts with most experience, which suggests that, other things being equal, increasing short-term interest rates are normally accompanied by a rise in longer-term yields... for the moment, the broadly unanticipated behavior of world bond markets remains a conundrum" (Testimony of federal Reserve Board Chairman Alan Greenspan to the U.S. Senate, February 16th, 2005).

[2] The same comments apply to VAR's continuos-time counterparts, such as multifactor Vasicek (1977) and Cox-Ingersoll-Ross (1985) models.

[3] In the model I propose, which is an extension of the model by Laubach and Williams (2003), the natural rate of interest is defined as in Wicksell (1936) as the real short-term interest rate consistent with a macroeconomic equilibrium in which output equals its potential and inflation is constant.

the real natural rate of interest and inflation expectations lowered long-term forecasts of the level of policy rates, both in the US and in the euro area. According to my model, this was an important cause of the fall in long-term bond yields, which, if overlooked, might lead to erroneously attribute the fall to a collapse of risk premia. The main inference to be drawn from the model is that, although risk premia did diminish, the fall was not dramatic and their current level is not unusual if considered from an historical perspective.

The paper is organized as follows: Section 1 discusses the challenges related to the estimation of risk premia and the possible drawbacks of standard econometric models; Section 2 presents the model I use to measure risk premia; Section 3 discusses some details of the estimation methodology; Section 4 contains some comments on the empirical evidence obtained with the model; the Appendix includes tables, figures and some technical details of the model.

1   Motivation

In this section I discuss the problems that might arise when one relies on standard stationary macro-finance VARs (and their continuous-time counterparts; see footnote 3) to estimate bond risk premia. A related discussion can be found in Kim and Orphanides (2005).

Among the many possible measures of the risk premium, I focus on the simplest and most straightforward: the forward premium, i.e. the difference between $f_{t,t+n}$ (the forward rate at which agents agree at time t to exchange funds at time $t + n$ for one period) and $\mathrm{E}_t\left[i_{t+n}\right]$ (the expected value of i_{t+n} given information available at time t, where i_{t+n} is the short-term rate at which agents will agree to exchange funds at time $t + n$ for the next period). Thus, the n-periods ahead forward premium $\varphi_{t,t+n}$ is defined as:

$$\varphi_{t,t+n} = f_{t,t+n} - \mathrm{E}_t\left[i_{t+n}\right] \tag{1}$$

Focusing on the forward premium is without loss of generality, because any bond yield can be decomposed into a sequence of forward rates at which agents agree today to exchange funds in the future.

Since forward rates $f_{t,t+n}$ are observable from market prices, the above equation makes clear that the whole challenge of measuring forward premia lies in accurately estimating unobservable expectations $\mathrm{E}_t\left[i_{t+n}\right]$ of short-term rates likely to prevail in the future.

In order to estimate $\mathrm{E}_t\left[i_{t+n}\right]$, a possibility would be to directly survey market participants' expectations about future levels of the interest rate. However, most existing studies rely on econometric models to estimate expectations about interest rates. A reason for doing so is that typically available survey data have limited historical depth, so that their time series properties and their reliability are difficult, if not impossible, to study; on the contrary, simple macro-econometric models use long time-series of readily available macroeconomic data, allowing the researcher to achieve sufficient historical perspective when studying the behavior of expectations and risk premia. Furthermore, some

papers (e.g.: Friedman - 1980; Froot - 1989) have provided evidence that the informative content of interest rate survey forecasts might be questionable.

Most of the econometric models which have been employed to analyze the term structure of interest rates share a similar structure: they specify the joint dynamics of the short-term interest rate and of a small number of other variables as a vector autoregression (or as a continuous-time model which can be discretized so as to yield a vector autoregression - see footnote 3). The other variables in the VAR are usually macroeconomic variables, such as inflation and the output gap, and variables explicitly or implicitly related to the shape of the yield curve. For example, Estrella and Mishkin (1997) and Evans and Marshall (1998) estimate VARs with yields of various maturities and macroeconomic variables. Numerous studies estimate no-arbitrage latent factor models where macroeconomic variables do not play any role (e.g.: Duffie and Kan - 1996; Dai and Singleton - 2000). Recently, a number of papers, starting with the seminal work by Ang and Piazzesi (2003), have jointly modeled both macroeconomic variables and (latent) variables related to the shape of the yield curve as VARs with full no-arbitrage restrictions (e.g.: Rudebusch and Wu - 2004; Hordal, Tristani and Vestin - 2006; Ang, Piazzesi and Wei - 2006).

All the aforementioned models can in principle be used to estimate expectations of the future path of short-term rates. However, as carefully illustrated by Kim and Orphanides (2005), estimates of long-horizon expectations provided by such models can be seriously misleading. While there is ample empirical and anectodal evidence that considerable structural changes took place in the major industrialized economies and reduced long-run expectations of the short rate (e.g.: Campbell and Viceira - 2001; Clarida, Gali and Gertler - 2000; Cogley and Sargent - 2001; Derby - 2004; Goto and Torus - 2003), most of these models provide estimates of long-horizon expectations which remain virtually unchanged across time. To capture structural breaks and produce enough variation in long-term expectations, estimated VARs should have at least one highly persistent factor with long half-life; however, it is well known that the very presence of such persistent factors makes estimation of VARs problematic, because of finite sample biases and inefficiencies of standard estimators, such as maximum likelihood ones. Most of the times, even if the true data-generating process features persistent factors, this is unlikely to show up in the estimates. Kim and Orphanides (2005), for example, estimate a standard no-arbitrage term structure model which, in principle, could allow for one or more persistent factors and show that, despite the presence of obvious structural breaks in their sample, the model is unable to detect them and produces estimates of long-run expectations wich are roughly constant across time. Furthermore, the choice of the sample lenght is bound to heavily influence the results and the level of estimated expectations. Below, I provide further evidence in this sense.

In Figure 1, I plot the 9-year ahead expectations of the short rate obtained from a standard macro-VAR with 12 lags of inflation, the output gap and the short rate, estimated with quarterly US data covering the period 1965-2006. The same figure also displays the plot of the 9-year ahead forward-rate. The difference between the two series is an estimate of the forward premium. It is

apparent that the measure of expectations thus obtained is very stable throughout the whole period: approximately 80 per cent of the values are between 6 and 7 per cent and the average expectation is 6.67 per cent, which almost coincides with the sample average of the short-term rate, which is 6.66. After the year 2000, model-implied expectations have on average been roughly equal to 6.5 per cent. This is in sharp contrast with statements from policy makers, who suggested a range of 3.5 to 4.5 per cent (see e.g.: Derby - 2004; Hoenig - 2005), and with survey forecasts (Kim and Orphanides - 2005). Reducing the sample period used to estimate the model, in order to mitigate the problems arising from structural breaks, does not seem to produce improvements: estimated expectations still remain roughly constant across time; however, their average shifts considerably, depending on the sample period I choose, pointing to a severe lack of robustness of the results with respect to sample choice.

If instead of a standard macro-VAR, I estimate a no-arbitrage VAR with both macroeconomic and latent variables (in the canonical form suggested by Pericoli and Taboga - 2008), which in principle should be able to identify persistent factors and capture structural change, I get results which are slightly better, in the sense that the model produces a somewhat greater variability of expectations (see Figure 2). Nonetheless, estimated expectations still look very stable (if compared with forward rates), they are at odds with market participants' and policymaker's perceptions during the most recent period and they also display limited robustness to sample choice.

Estimates for the euro area of both standard and no-arbitrage VARs (Figures 7 and 8) present drawbacks that are similar to those of the estimates carried out for the US and commented above.

As a solution to the aforementioned problems, I propose a simple model where some key variables are affected by permanent shocks which dynamically induce shifts in macroeconomic equilibria: long-run inflation expectations, the real natural rate of interest and the growth rate of potential output are identified as the sources of persistence which have a potential to capture structural change and generate the desired variation in long-term expectations. To impose persistence on the variables driving structural change, I model the corresponding stochastic processes as random walks, so that the resulting economic dynamics be not explosive, but rather allow for time-varying equilibria.

2   The model

My model is an extension of the model used by Laubach and Williams (2003) to estimate the natural rate of interest. According to the Wicksellian definition (Wicksell - 1936), the natural rate of interest is the real short-term interest rate consistent with a macroeconomic equilibrium in which output equals its potential and inflation is constant. The natural rate varies through time in response to structural changes in the economy: most theoretical frameworks underline its dependce on productivity growth, population growth, the subjective discount factor of individuals, their elasticity of intertemporal substitution and other

characteristics of agents' preferences and production technologies which tend to be subject to permanent shocks.

While Laubach and Williams do not specify an equation for the nominal short-term interest rate and estimate the real natural rate of interest only via estimation of an IS equation and an inflation equation, I add an equation for the nominal short-term interest rate to their model, so as to be able to recover interest rate dynamics and their dependence on the real natural rate of interest and other macroeconomic variables: thus, I have a device through which changes in the real natural rate of interest are transmitted to the nominal short-term rate and to expectations about its path in the long-run. Furthermore, while Laubach and Williams use proxies for inflation expectations, I use proper expectations, consistently derived within the model. As the equations to follow will clarify, this is a source of transmission of long-run shifts in inflation expectations.

The first building block of the model is a reduced form IS equation (see e.g. Rudebusch and Svensson - 1999):

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \frac{\alpha_r}{2} \sum_{j=1}^{2} \left(r_{t-j} - r_{t-j}^* \right) + \varepsilon_{x,t} \tag{2}$$

where x_t is the output gap, r_t is the ex-ante real policy rate, r_t^* is the real natural rate of interest and $\varepsilon_{x,t}$ is a serially uncorrelated shock. Provided α_r is negative, when the real policy rate is above (below) the natural rate, monetary policy is contractionary (expansionary) and output tends to decrease (increase).

The output gap is defined as:

$$x_t = y_t - y_t^* \tag{3}$$

where y_t and y_t^* are the logarithms of output and potential output respectively. The ex-ante real policy rate is the difference between the policy rate i_t and inflation expectations:

$$r_t = i_t - E_t \left[\pi_{t+1} \right] \tag{4}$$

Inflation π_t is determined by its own lags, lagged output gap and a serially uncorrelated shock:

$$\pi_t = \sum_{j=1}^{8} \beta_j \pi_{t-j} + \beta_x x_{t-1} + \varepsilon_{\pi,t} \tag{5}$$

The constraints $\sum_{j=1}^{8} \beta_j = 1$ and $\beta_j \geqslant 0, \forall j$ are imposed to ensure vertical-ity of the Phillips curve in the long-run (see e.g. Rudebusch and Svensson - 1999). Besides being able to capture rather rich inflation dynamics, the above specification of the Phillips curve is consistent with the Wicksellian definition of natural rate: together with equation (2) and (6), it implies that, absent any transitory shock to the economy and when the real short-term rate equals the natural rate, output converges to its potential and inflation converges to a constant level. Long-run verticality of the Phillips curve means that any level of

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inflation can be compatible with an economic equilibrium where output equals its potential. I stick to the original backward-looking specification proposed by Laubach and Williams (2003), because it allows for a closed-form solution of the model (see the Appendix).

Adjustments in the nominal short-term rate happen in accordance with the following equation:

$$i_t = \gamma i_{t-1} + (1 - \gamma) \left(i_{t-1}^n + \gamma_\pi \left(\overline{\pi}_{t-1} - \pi_{t-1}^* \right) + \gamma_x x_{t-1} \right) + \varepsilon_{i,t} \qquad (6)$$

where i_t^n is the nominal natural rate of interest, defined as:

$$i_t^n = r_t^* + E_t \left[\pi_{t+1} \right] \qquad (7)$$

π_t^* is long-run inflation:

$$\pi_t^* = \pi_{t-1}^* + \varepsilon_{\pi^*, t} \qquad (8)$$

and $\overline{\pi}_t$ is a measure of realized inflation, defined as:

$$\overline{\pi}_t = \frac{1}{4} \sum_{j=0}^{3} \pi_{t-j} \qquad (9)$$

Equation (6) could be interpreted as a smoothing policy rule (e.g.: Bjornland, Leitemo and Maih - 2006), whereby the central bank gradually raises interest rates when either output is above its potential or realized inflation is above the desired long-run level π_t^*. Provided γ_π and γ_x are strictly positive and γ is strictly less than 1, (6) guarantees that, in the absence of transitory shocks, the economy converges towards an equilibrium where output equals its potential, inflation is equal to its long-run level and the real short-term rate equals the real natural rate of interest. This allows for multiple equilibria that differ as to the level of inflation and the interest rate. This is the key feature of the model, which makes it suitable to reproduce the structural changes and the permanent shifts in long-run expectations not captured by standard stationary VARs. Specifying long-run inflation π_t^* as a random walk, I allow for permanent unpredictable changes in the equilibrium level of inflation: the variance of the innovation $\varepsilon_{\pi^*, t}$ determines the expected magnitude of such changes.

Potential output grows at a rate g_t

$$y_t^* = y_{t-1}^* + g_{t-1} \qquad (10)$$

which varies through time as a random walk:

$$g_t = g_{t-1} + \varepsilon_{g,t} \qquad (11)$$

Finally, the natural real rate of interest is the sum of two terms:

$$r_t^* = cg_t + z_t \qquad (12)$$

where c is a constant related to the elasticity of intertemporal substitution and z_t is a shock that captures other determinants of r_t^* such as the subjective discount factor. Also z_t is specified as a random walk:

$$z_t = z_{t-1} + \varepsilon_{z,t} \tag{13}$$

Equation (12) can be derived, for example, within a standard Solow or Ramsey growth model (e.g.: Barro and Sala-i-Martin - 1999), where it is related either to a balanced growth condition or to an optimality condition for savings.

In order to be able to jointly estimate the forward premium and the macroeconomic dynamics specified by the above equations, I add to the system an exogenous process for the forward premium:

$$\varphi_t = \varphi_{t-1} + \varepsilon_{\varphi,t} \tag{14}$$

The model can be written explicitly in companion form as a first order vector autoregression, by substituting inflation expectations with their value derived from (5). Details are reported in the Appendix.

3   Estimation strategy

I estimate the model by maximum likelihood, with both US and euro area quarterly data (see the Appendix for details on the data). I adopt the methodology proposed by Boivin and Giannoni (2005) for estimation in data-rich environments: all the variables in the model are treated as unobservable, in order to allow for measurement errors and for the possibility that many data series be available for a single economic concept[4]. An observation equation is defined for any data series, so that there can be multiple observation equations for a single variable. The observation equations are used to infer the values of the variables of the model via the Kalman filter.

As previously observed, my model can be written in companion form as a first order vector autoregression:

$$\xi_t = F\xi_{t-1} + v_t \tag{15}$$

where ξ_t is a column vector that contains all the variables of the model and some of their lags:

$$\xi_t = \begin{bmatrix} x_t & \pi_t & i_t & y_t^* & z_t & g_t & \pi_t^* & x_{t-1} & g_{t-1} & z_{t-1} & \pi_{t-1} & ... & \pi_{t-8} & i_{t-1} & \varphi_t \end{bmatrix}^\top \tag{16}$$

F is a matrix whose entries are determined by the solution of the model in the previous section (the functional form of the coefficients is reported in the Appendix) and v_t is a vector of error terms.

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[4] In what follows, I use the same terminology of Boivin and Giannoni (2005): an 'economic variable' is one of the variables in the model (whose concept is uniquely defined); a 'data series' is one of the many possible measures of the variable. For example, if inflation is a variable in the model, many data series could be used to measure it: CPI (consumer prices) inflation, core-CPI, GDP deflator, PCE (personal consumption expenditures) deflator, etc.

All the variables in ξ_t are treated as unobservable. Their values are estimated by the Kalman filter, using a set of observation equations:

$$y_t = H\xi_t + w_t \qquad (17)$$

where y_t is the vector of observed data series used to measure the unobservable variables, H is a matrix of loadings (see the Appendix for details) and w_t is a vector of error terms.

The procedure proposed by Boivin and Giannoni (2005) is motivated by the observation that: 1) many measures of economic variables are likely to be affected by measurement error; 2) there may be conceptual differences between model variables and the data series used to measure them; 3) there may be many different data series that correspond to a unique economic concept; 4) one might want to use data series which are proxies for unobservable or only partially observable variables. I choose to adopt Boivin and Giannoni's (2005) procedure, because I have many measures of inflation and all of them are potentially relevant to my model; furthermore, I have estimates and proxies of some unobservable variables (potential output, long-run inflation), which I want to exploit in order to achieve a better identification of unobservables.

The vector y_t includes: real GDP, an estimate of potential GDP, two different measures of inflation, the 3-month interest rate, the 9-year-ahead forward interest rate[5] and a consensus forecast of long-run inflation[6] (a detailed list of the data series and their sources, both for the US and for the euro area, is included in the Appendix).

4   Empirical evidence

From the estimates of the model[7] (estimated parameters are reported in Tables 2 and 3), similar pictures emerge for both the US and the euro area. Long-run inflation expectations and the real natural rate (Figures 4 and 10) underwent significant changes throughout the sample period. As a result, also long-run expectations of the short-term interest rate displayed a fair degree of variability (Figures 3 and 9). This is in sharp contrast with the evidence provided by traditional macro-VAR and no-arbitrage term-structure models (Figures 1, 2, 7 and 8), whereby expectations would have remained remarkably stable throughout the whole sample. My model suggests that, since the late Eighties, long-run expectations of the short-rate have considerably declined, both in the US and in the euro area, as a consequence of a progressive reduction in the real natural

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[5] It is standard practice in the literature to take the forward premium on the 9-year ahead forward rate as a measure of the risk premium embedded in long-term interest rates (e.g.: Kim and Orphanides - 2005).

[6] Both for the US and for the euro area, consensus forecasts of long-run inflation are available only for a subsample of the whole sample period. I nonetheless use them by augmenting the set of observation equations after they become available.

[7] In the tables and figures in the Appendix, the model described in sections 2 and 3 is referred to as "model with multiple equilibria", to distinguish it from traditional VAR models with a unique equilibrium.

rate and inflation expectations. Between 1990 and 2006, long-run inflation decreased from 4.50 to 1.95 in the euro area and from 4.40 to 2.30 in the US; the real natural rate decreased from 5.70 to 1.65 in the euro area and from 2.90 to 1.90 in the US. Taking these developments into account, I obtain estimates of the forward premium (Figures 5, 6, 11 and 12) which are materially different from those provided by standard models, especially as far as the last decade is concerned. Although the forward premium dropped sharply after 2002 in both economies, it fell to levels which are not at all unusual from an historical perspective and it is still broadly positive. At the end of 2006, the 9-year ahead forward premium was estimated to be 0.45 percentage points in the euro area and 1.90 points in the US. Hence, although previous studies correctly identify a fall in risk premia as an important cause of the recent pronounced reduction in bond yields, they probably tend to exaggerate the phenomenon, because they overlook shifts in expectations which hardly allow for a direct comparison of the current level of bond yields with that of ten or twenty years ago.

5   Conclusions

Decomposing forward interest rates into forward premia and expectations of spot interest rates that will prevail in the future is key to understanding the dynamics of long-term bond yields. I have shown that estimates of distant-horizon expectations provided by commonly employed VAR models display little or no variability and tend to attribute most of the variabilty in forward rates to forward premia. This is due to a lack of persistence of estimated shocks to short-term interest rates and macroeconomic variables. I propose a model that allows for more persistence and allows to capture substantial changes in distant-horizon expectations. Employing my model to study the historical evolution of long-term rates, I conclude that the sharp decline in long-term interest rates experienced around the turn of the Twentieth century in the major industrialized countries has been due to a progressive fall in expected future rates rather than to a collapse in risk premia, as other models seem to suggest.

6 Appendix

In order to write the model explicitly, I substitute inflation expectations in the IS equation and in the policy rule with their explicit expression derived from (5):

$$
\begin{aligned}
x_t &= \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \frac{\alpha_r}{2} \sum_{j=1}^{2} \left(r_{t-j} - r_{t-j}^* \right) \tag{18}\\[2mm]
&= \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \frac{\alpha_r}{2} \sum_{j=1}^{2} \left(i_{t-j} - E_{t-j}\left[\pi_{t-j+1} \right] - r_{t-j}^* \right)\\[2mm]
&= \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \frac{\alpha_r}{2} i_{t-1} + \frac{\alpha_r}{2} i_{t-2} - \frac{\alpha_r}{2} r_{t-1}^* - \frac{\alpha_r}{2} r_{t-2}^*\\[2mm]
&\quad - \frac{\alpha_r}{2}\left(\sum_{j=1}^{8} \beta_j \pi_{t-j} + \beta_x x_{t-1} \right) - \frac{\alpha_r}{2}\left(\sum_{j=2}^{9} \beta_{j-1} \pi_{t-j} + \beta_x x_{t-2} \right)\\[2mm]
&= \left(\alpha_1 - \frac{\alpha_r}{2}\beta_x \right) x_{t-1} + \left(\alpha_2 - \frac{\alpha_r}{2}\beta_x \right) x_{t-2} + \frac{\alpha_r}{2} i_{t-1} + \frac{\alpha_r}{2} i_{t-2} - \frac{\alpha_r}{2} z_{t-1} - \frac{\alpha_r}{2} z_{t-2}\\[2mm]
&\quad - \frac{\alpha_r}{2} cg_{t-1} - \frac{\alpha_r}{2} cg_{t-2} - \frac{\alpha_r}{2}\left(\beta_1 \pi_{t-1} + (\beta_1 + \beta_2) \pi_{t-2} + (\beta_2 + \beta_3) \pi_{t-3} \right)\\[2mm]
&\quad - \frac{\alpha_r}{2}\left((\beta_3 + \beta_4) \pi_{t-4} + (\beta_4 + \beta_5) \pi_{t-5} + (\beta_5 + \beta_6) \pi_{t-6} \right)\\[2mm]
&\quad - \frac{\alpha_r}{2}\left((\beta_6 + \beta_7) \pi_{t-7} + (\beta_7 + \beta_8) \pi_{t-8} + \beta_8 \pi_{t-9} \right)
\end{aligned}
$$

$$
\begin{aligned}
i_t &= \gamma i_{t-1} + (1 - \gamma)\left(i_{t-1}^n + \gamma_\pi \left(\overline{\pi}_{t-1} - \pi_{t-1}^* \right) + \gamma_x x_{t-1} \right) \tag{19}\\[2mm]
&= \gamma i_{t-1} + (1 - \gamma)\left(r_{t-1}^* + E_{t-1}\left[\pi_t \right] \right) + (1 - \gamma) \gamma_\pi \frac{1}{4} \sum_{j=1}^{4} \pi_{t-j}\\[2mm]
&\quad - (1 - \gamma) \gamma_\pi \pi_{t-1}^* + (1 - \gamma) \gamma_x x_{t-1}\\[2mm]
&= \gamma i_{t-1} + (1 - \gamma) cg_{t-1} + (1 - \gamma) z_{t-1} + (1 - \gamma)\left(\sum_{j=1}^{8} \beta_j \pi_{t-j} + \beta_x x_{t-1} \right)\\[2mm]
&\quad + (1 - \gamma) \gamma_\pi \frac{1}{4} \sum_{j=1}^{4} \pi_{t-j} - (1 - \gamma) \gamma_\pi \pi_{t-1}^* + (1 - \gamma) \gamma_x x_{t-1}\\[2mm]
&= \gamma i_{t-1} + (1 - \gamma) cg_{t-1} + (1 - \gamma) z_{t-1} + (1 - \gamma)\left(\beta_x + \gamma_x \right) x_{t-1} - (1 - \gamma) \gamma_\pi \pi_{t-1}^*\\[2mm]
&\quad + (1 - \gamma)\left(\left(\beta_1 + \gamma_\pi \frac{1}{4} \right) \pi_{t-1} + \left(\beta_2 + \gamma_\pi \frac{1}{4} \right) \pi_{t-2} + \left(\beta_3 + \gamma_\pi \frac{1}{4} \right) \pi_{t-3} \right)\\[2mm]
&\quad + (1 - \gamma)\left(\left(\beta_4 + \gamma_\pi \frac{1}{4} \right) \pi_{t-4} + \beta_5 \pi_{t-5} + \beta_6 \pi_{t-6} + \beta_7 \pi_{t-7} + \beta_8 \pi_{t-8} \right)
\end{aligned}
$$

Now, define the vector ξ_t as follows:

$$\xi_t = \begin{bmatrix} x_t & \pi_t & i_t & y_t^* & z_t & g_t & \pi_t^* & x_{t-1} & g_{t-1} & z_{t-1} & \pi_{t-1} & \ldots & \pi_{t-8} & i_{t-1} & \varphi_t \end{bmatrix}^\top$$

The whole model can be written in companion form as a first order vector autoregression:

$$\xi_t = F\xi_{t-1} + v_t$$

The non-zero entries of the companion matrix F are as follows ($F_{i,j}$ denotes the entry at the intersection of the i-th row and j-th column):

$F_{1,1} = \alpha_1 - \frac{\alpha_r}{2}\beta_x; F_{1,2} = -\frac{\alpha_r}{2}\beta_1; F_{1,3} = \frac{\alpha_r}{2}; F_{1,5} = -\frac{\alpha_r}{2};$
$F_{1,6} = -\frac{\alpha_r}{2}c; F_{1,8} = \alpha_2 - \frac{\alpha_r}{2}\beta_x; F_{1,9} = -\frac{\alpha_r}{2}c; F_{1,10} = -\frac{\alpha_r}{2};$
$F_{1,11} = -\frac{\alpha_r}{2}(\beta_1 + \beta_2); F_{1,12} = -\frac{\alpha_r}{2}(\beta_2 + \beta_3); F_{1,13} = -\frac{\alpha_r}{2}(\beta_3 + \beta_4);$
$F_{1,14} = -\frac{\alpha_r}{2}(\beta_4 + \beta_5); F_{1,15} = -\frac{\alpha_r}{2}(\beta_5 + \beta_6); F_{1,16} = -\frac{\alpha_r}{2}(\beta_6 + \beta_7);$
$F_{1,17} = -\frac{\alpha_r}{2}(\beta_7 + \beta_8); F_{1,18} = -\frac{\alpha_r}{2}\beta_8; F_{1,19} = \frac{\alpha_r}{2};$
$F_{2,1} = \beta_x; F_{2,2} = \beta_1; F_{2,11} = \beta_2; F_{2,12} = \beta_3; F_{2,13} = \beta_4; F_{2,14} = \beta_5;$
$F_{2,15} = \beta_6; F_{2,16} = \beta_7; F_{2,17} = \beta_8;$
$F_{3,1} = (1-\gamma)(\beta_x + \gamma_x); F_{3,2} = (1-\gamma)(\beta_1 + \gamma_\pi \frac{1}{4}); F_{3,3} = \gamma; F_{3,5} = (1-\gamma);$
$F_{3,6} = (1-\gamma)c; F_{3,7} = -(1-\gamma)\gamma_\pi; F_{3,11} = (1-\gamma)(\beta_2 + \gamma_\pi \frac{1}{4});$
$F_{3,12} = (1-\gamma)(\beta_3 + \gamma_\pi \frac{1}{4}); F_{3,13} = (1-\gamma)(\beta_4 + \gamma_\pi \frac{1}{4}); F_{3,14} = (1-\gamma)\beta_5;$
$F_{3,15} = (1-\gamma)\beta_6; F_{3,16} = (1-\gamma)\beta_7; F_{3,17} = (1-\gamma)\beta_8;$
$F_{4,4} = 1; F_{4,6} = 1; F_{5,5} = 1; F_{6,6} = 1; F_{7,7} = 1;$
$F_{8,1} = 1; F_{9,6} = 1; F_{10,5} = 1; F_{11,2} = 1;$
$F_{12,11} = 1; F_{13,12} = 1; F_{14,13} = 1; F_{15,14} = 1;$
$F_{16,15} = 1; F_{17,16} = 1; F_{18,17} = 1; F_{19,3} = 1; F_{20,20} = 1.$

All the remaining entries are zero.
The vector of error terms v_t is:

$$v_t = \begin{bmatrix} \varepsilon_{x,t} & \varepsilon_{\pi,t} & \varepsilon_{i,t} & 0 & \varepsilon_{z,t} & \varepsilon_{g,t} & \varepsilon_{\pi^*,t} & 0 & \ldots & 0 & \varepsilon_{\varphi,t} \end{bmatrix}^\top$$

I denote the variances of the non-zero terms by $\sigma_x^2, \sigma_\pi^2, \ldots, \sigma_\varphi^2$.
The non-zero entries of the matrix of loadings H, defining the observation equations are as follows:

$H_{1,2} = 1; H_{2,2} = 1; H_{3,1} = 1; H_{3,4} = 1; H_{4,4} = 1; H_{5,3} = 1;$
$H_{6,j} = \left(F^{36}\right)_{3,j}, j = 1, \ldots, 19$
$H_{6,20} = \left(F^{36}\right)_{3,20} + 1$
$H_{7,j} = \left(F^{40}\right)_{2,j}, j = 1, \ldots, 20$

All the remaining entries are zero. The first two equations correspond to two different inflation data series. The third and fourth equation correspond to output and potential output respectively. The fifth and sixth are the measurement equations for the short-term and the 9-year ahead forward interest rates, respectively. The seventh equation corresponds to long-run (10-year ahead) consensus inflation expectations.

The variances of the measurement errors in the observation equations are denoted by:

$$\tau_j = \mathrm{Var}\left(w_{j,t}\right), j = 1, ..., 7$$

Table 1 - Data and Data Sources

United States

Variable	Source	Sample Period	Acronym
US GDP (ar) cona	Datastream	1961-2006	USGDP...D
US CBO Forecast - Potential GDP (real) cona	Datastream	1961-2006	USFCGDPPD
US CPI - All urban: all items, sadj	Datastream	1961-2006	USCONPRCE
US CPI - All items less food & energy (core), sadj	Datastream	1961-2006	USCPCOREE
US Treasury Bill 2nd market 3 month - middle rate	Datastream	1961-2006	FRTBS3M
US Zero Coupon Curve (FED estimates)	BIS Databank (BIS_Macro)	1961-2006	HSLA,HSMA
Consumer Prices, Long Term Forecasts	Consensus Economics	1991-2006	N/A

Euro Area

Variable	Source	Sample Period	Acronym
Real GDP	Euro Area-Wide Model Database[8]	1970-2006	YER
Potential Output	Euro Area-Wide Model Database	1970-2006	YET
GDP at Factor Costs Deflator	Euro Area-Wide Model Database	1970-2006	YFD
HICP	Euro Area-Wide Model Database	1970-2006	HICP
Short-term interest rate	Euro Area-Wide Model Database	1970-2006	STN
Long-term interest rate	Euro Area-Wide Model Database	1970-2006	LTN
Consumer Prices, Long Term Forecasts	Consensus Economics	1991-2006	N/A

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[8] For a complete description of the dataset see Fagan, Henry and Mestre (2001).

Table 2
Parameter estimates - US

	Est.	St. Dev		Est.	St. Dev
α_1	1.1096	0.0539	c	1.5351	0.2363
α_2	-0.1893	0.0455	σ_x	0.0072	0.0001
α_r	-0.3705	0.1207	σ_π	0.0110	0.0004
β_1	0.3624	0.0732	σ_i	0.0055	0.0008
β_2	0	0.0565	σ_z	0.0054	0.0004
β_3	0.4911	0.0231	σ_g	0.0002	0.0001
β_4	0	0.0736	σ_{π^*}	0.0009	0.0001
β_5	0	0.0365	σ_φ	0.0138	0.0007
β_6	0.0402	0.1259	τ_1	0.0119	0.0007
β_7	0	0.0010	τ_2	0.0103	0.0011
β_8	0.1063	0.0956	τ_3	0.0018	0.0001
β_x	0.2434	0.0006	τ_4	0	0.0012
γ	0.2035	0.2802	τ_5	0	0.0028
γ_x	0.2134	0.1667	τ_6	0.0001	0.0002
γ_π	0.2413	0.2450	τ_7	0	0.0005

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Table 3
Parameter estimates - Euro Area

	Est.	St. Dev		Est.	St. Dev
α_1	1.5336	0.0173	c	0	0.0488
α_2	-0.4851	0.0229	σ_x	0.0034	0.0004
α_r	-0.1842	0.0168	σ_π	0.0036	0.0006
β_1	0.8054	0.0593	σ_i	0.0054	0.0003
β_2	0.1907	0.0344	σ_z	0.0034	0.0004
β_3	0.0038	0.0112	σ_g	0.0002	0.0003
β_4	0.0001	0.0202	σ_{π^*}	0.0009	0.0002
β_5	0	0.0047	σ_φ	0.0036	0.0007
β_6	0	0.0098	τ_1	0.0132	0.0009
β_7	0	0.0035	τ_2	0.0160	0.0005
β_8	0	0.0057	τ_3	0.0024	0.0001
β_x	0.0468	0.0004	τ_4	0.0001	0.0006
γ	0.6529	0.0075	τ_5	0	0.0002
γ_x	0.7798	0.0279	τ_6	0	0.0003
γ_π	0.3462	0.1056	τ_7	0	0.0003

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Figure 1 - Expected short rate 9-years ahead and forward rate – USA (65-06)
Standard Macro-VAR

Forward rate Expected short rate
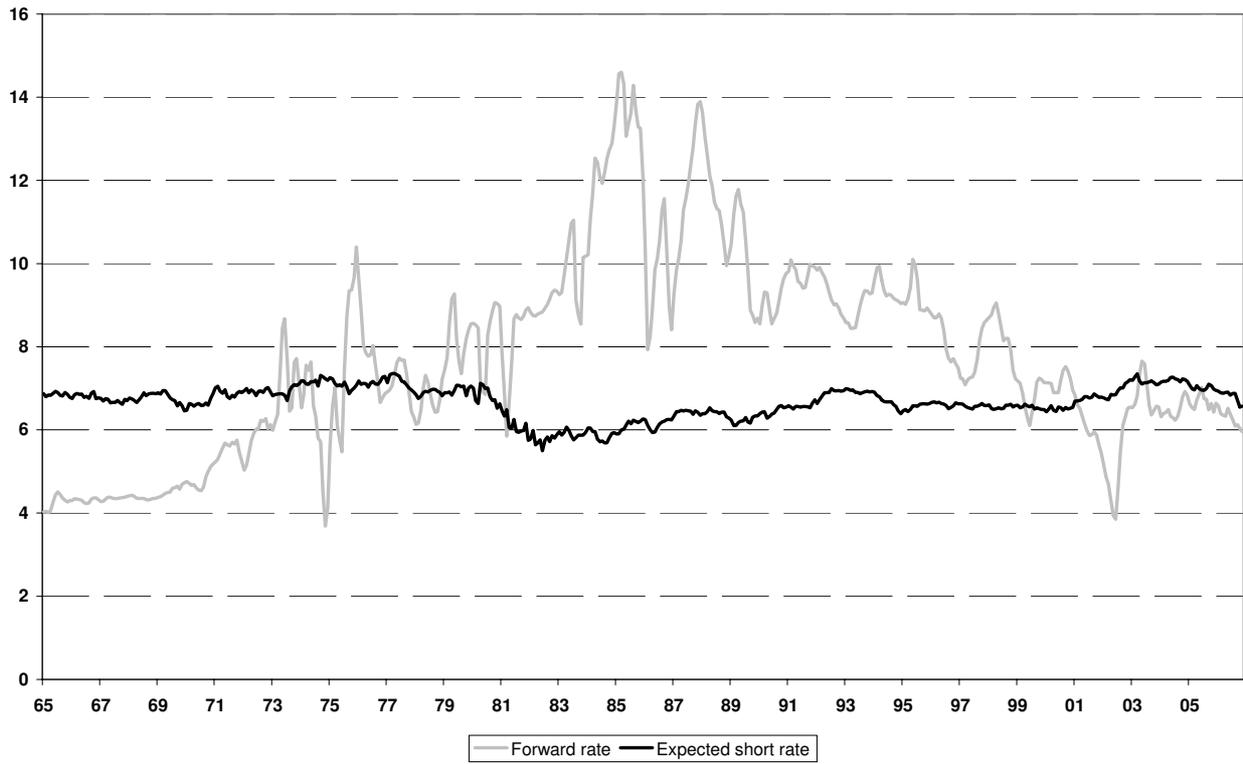
Figure 2 - Expected short rate 9-years ahead and forward rate – USA (65-06)
No-Arbitrage Macro-VAR with Latent Factors

Forward rate Expected short rate
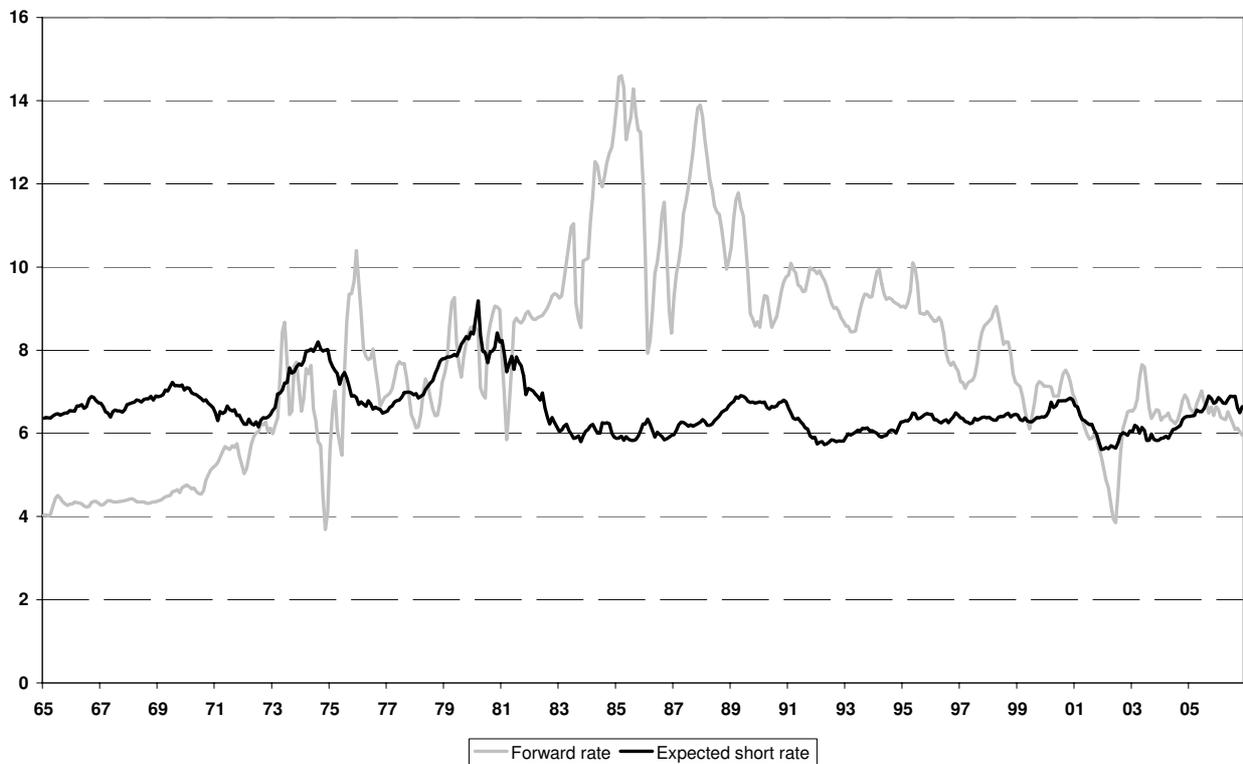
Figure 3 - Expected short rate 9-years ahead and forward rate – USA (65-06)
**Model with multiple equilibria**
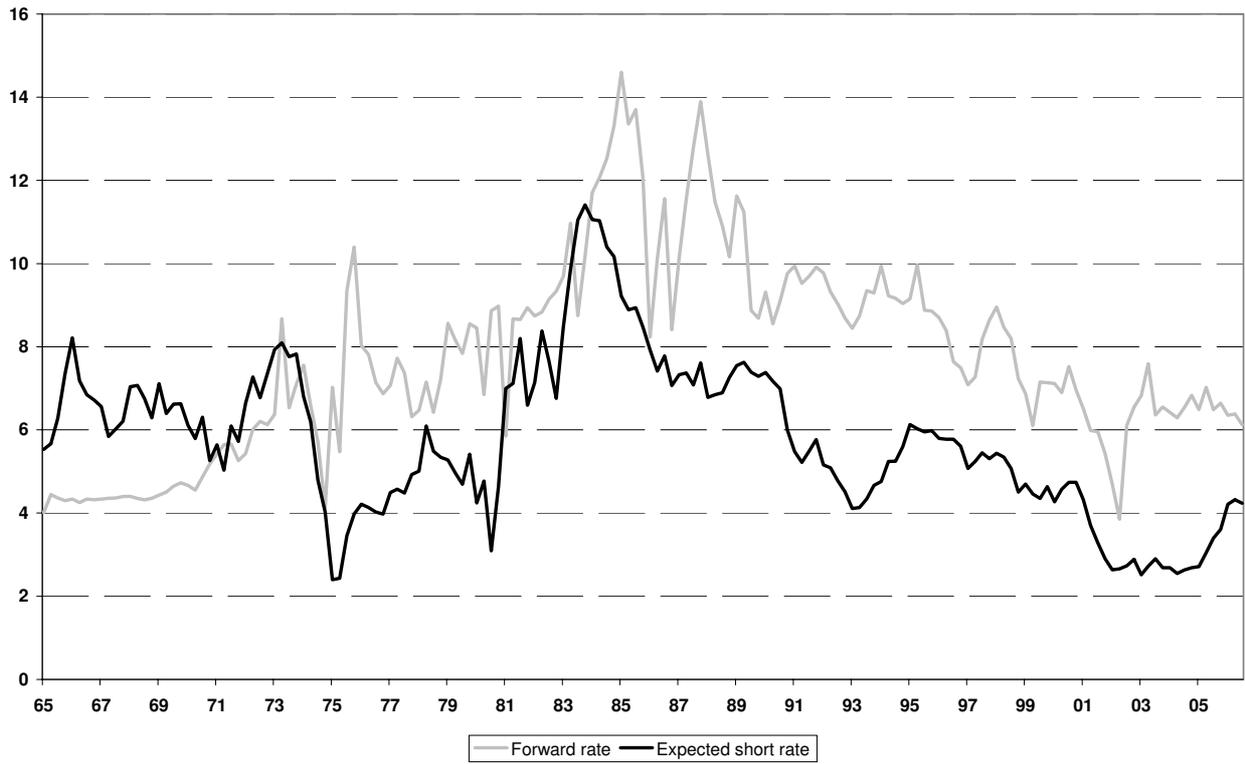
Figure 4 - Expected long-run inflation and the real natural rate – USA (65-06)
Model with multiple equilibria
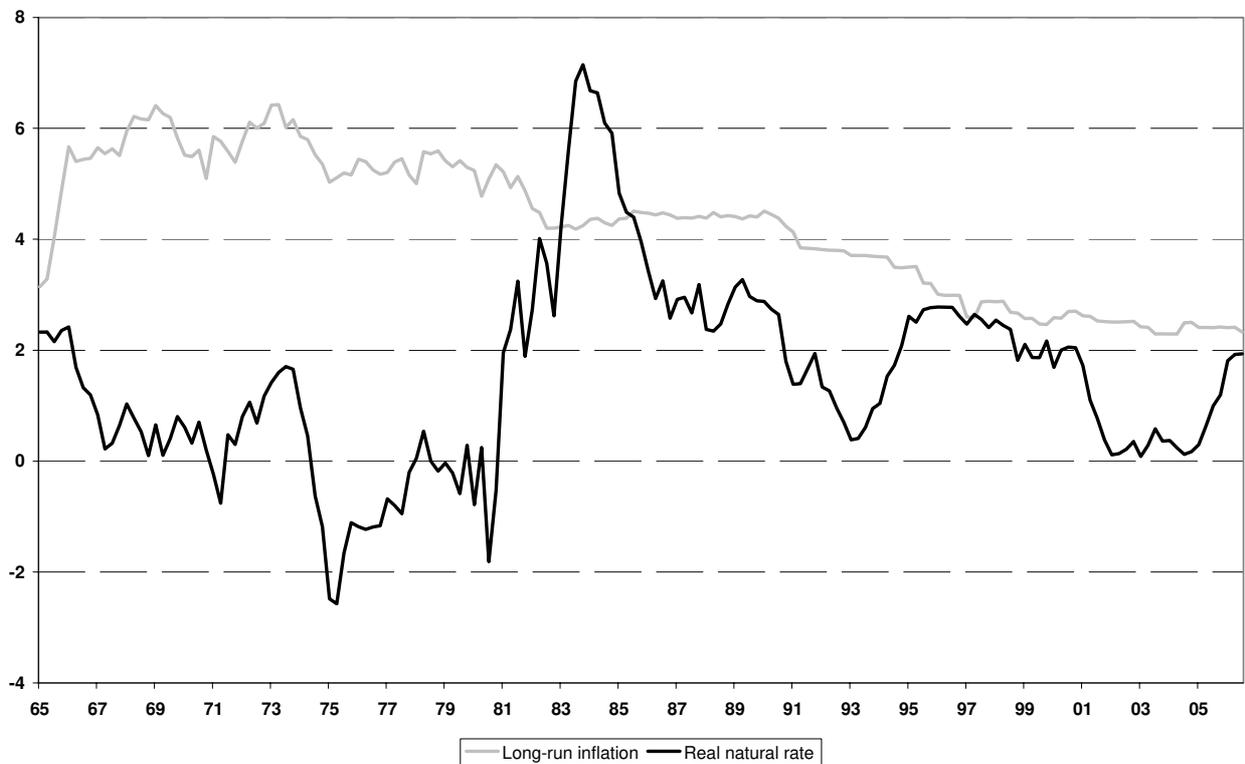
Figure 5 – Estimated forward premium – USA (65-06)
**Model with multiple equilibria and standard VAR**

Standard VAR — VAR with Multiple Equilibria
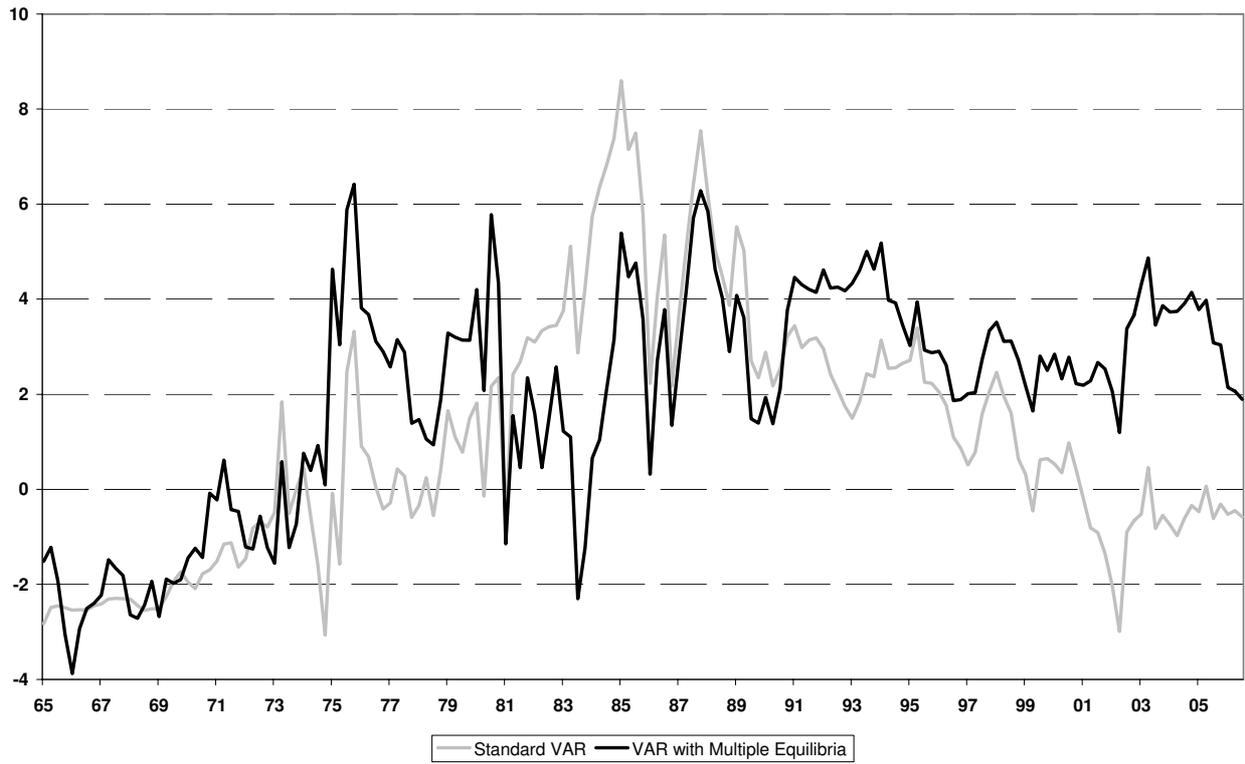
Figure 6 – Estimated forward premium – USA (65-06)
Model with multiple equilibria and no-arbitrage macro-VAR

No-Arbitrage VAR — VAR with Multiple Equilibria
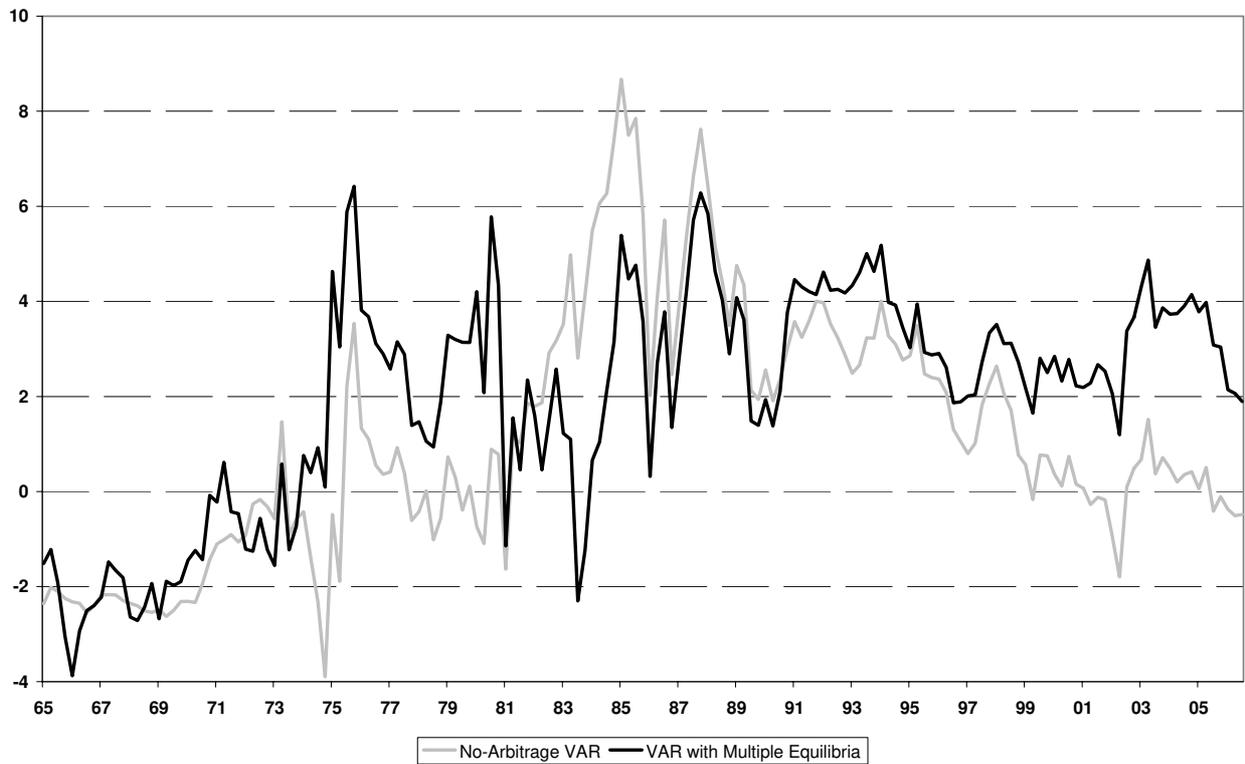
Figure 7 - Expected short rate 9-years ahead and forward rate – Euro Area (80-06)
**Standard Macro-VAR**
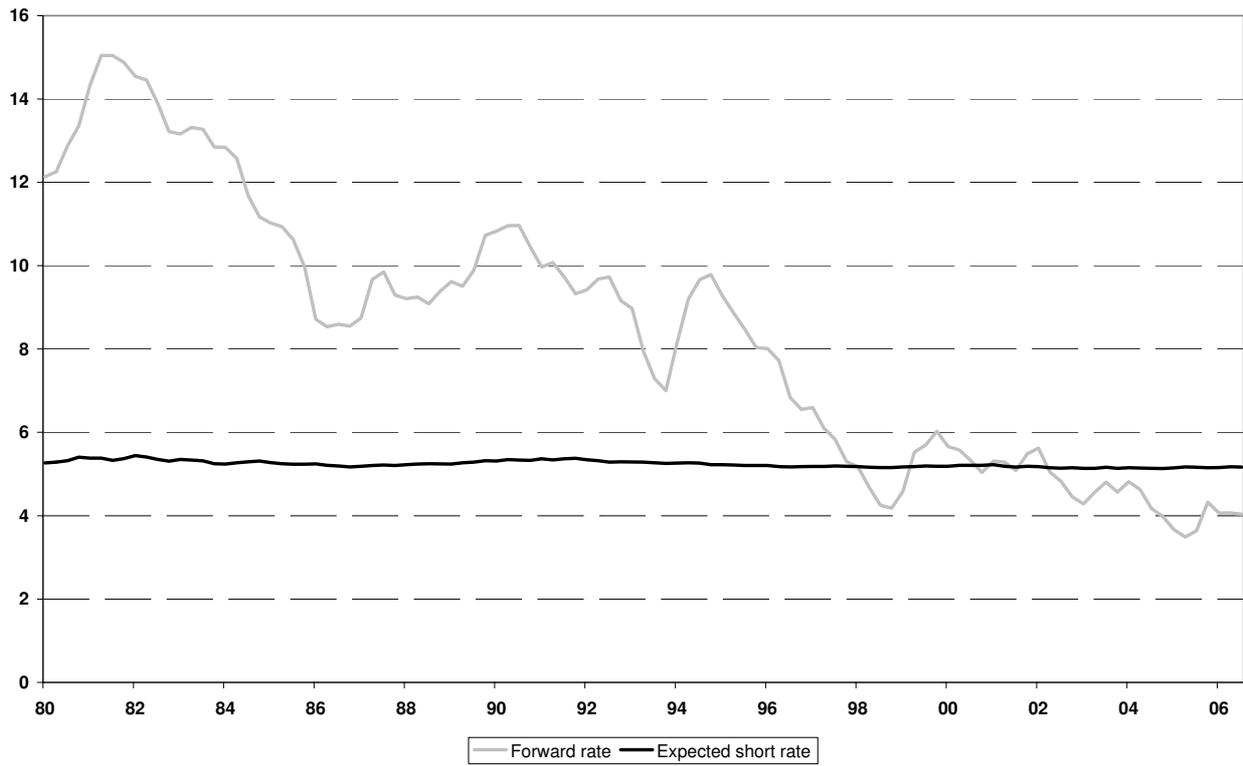
Figure 8 - Expected short rate 9-years ahead and forward rate – Euro Area (80-06)
**No-Arbitrage Macro-VAR with Latent Factors**
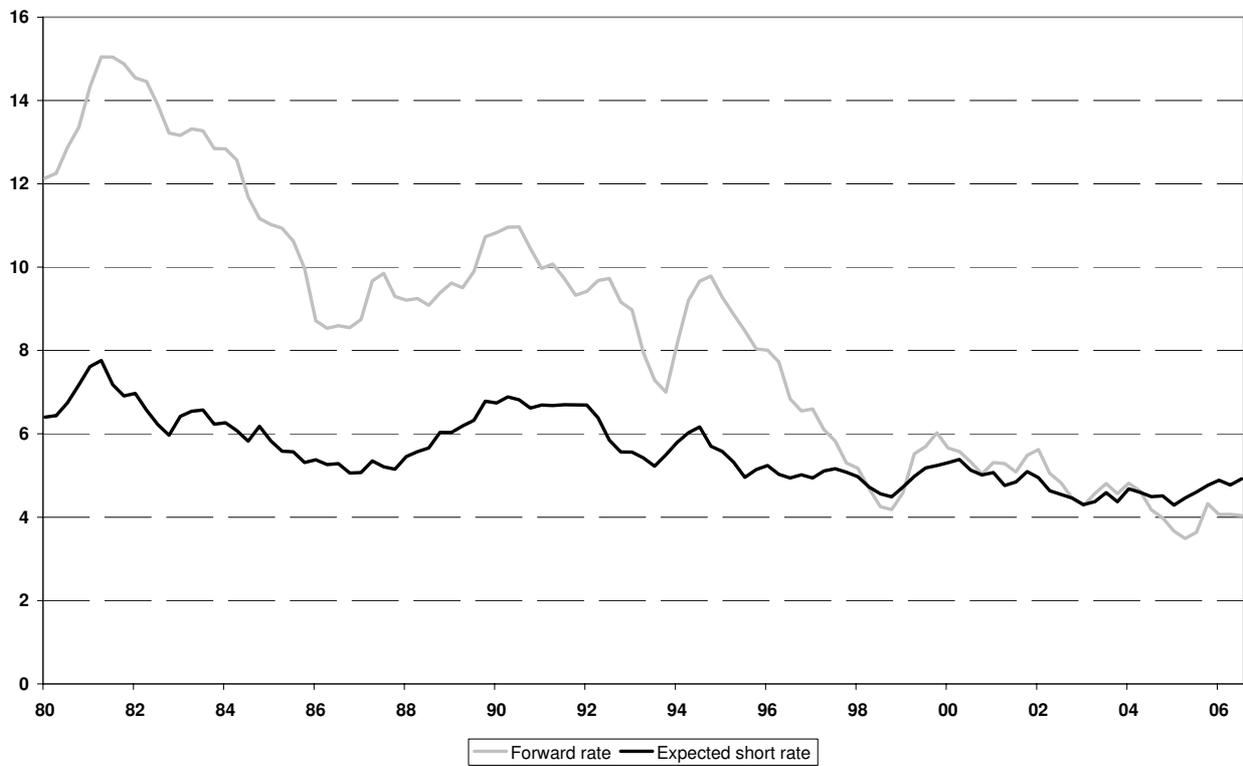
Figure 9 - Expected short rate 9-years ahead and forward rate – Euro Area (80-06)
Model with multiple equilibria
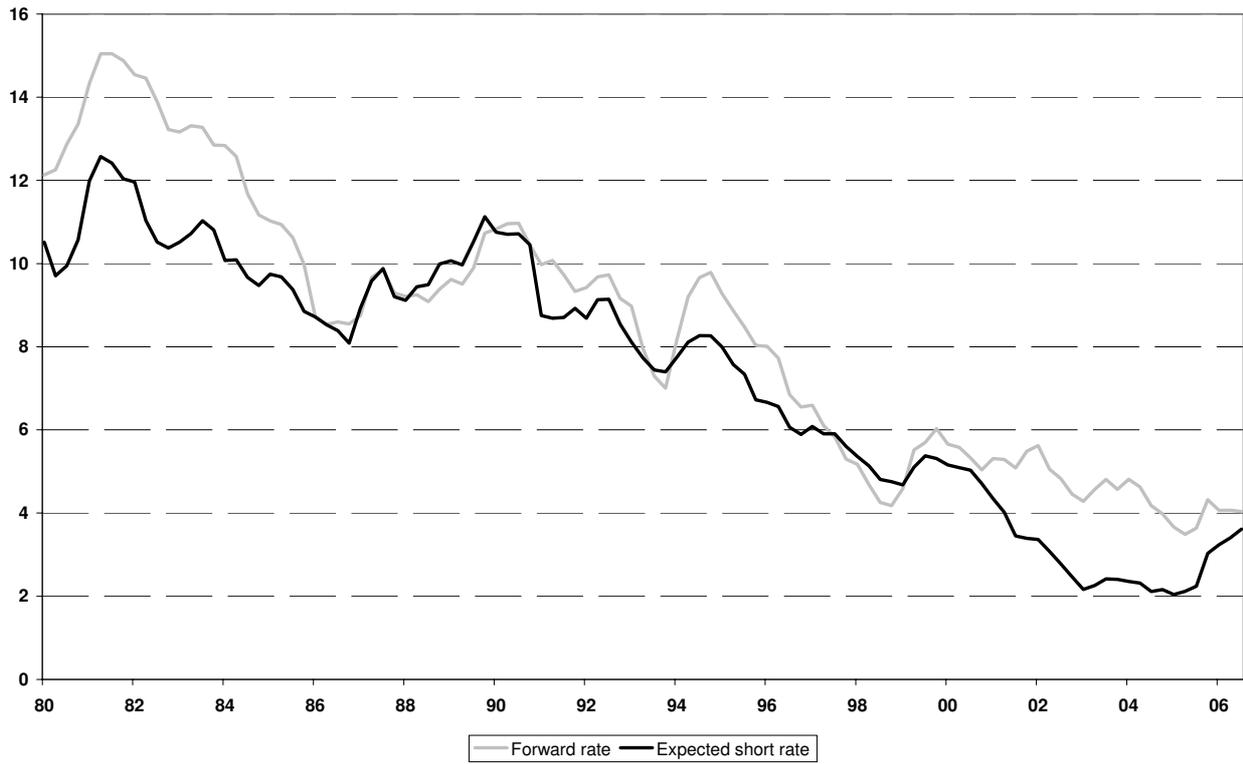
Figure 10 - Expected long-run inflation and the real natural rate – Euro Area (80-06)
Model with multiple equilibria
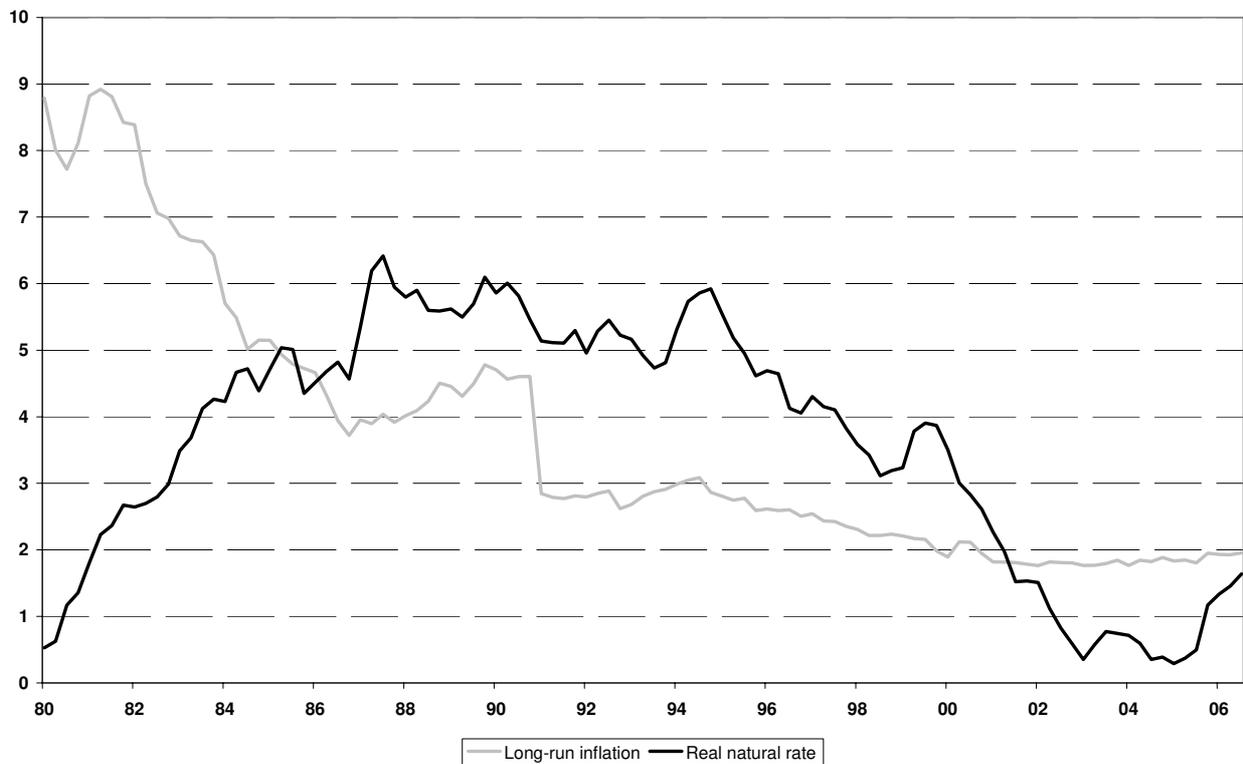
Figure 11 – Estimated forward premium – Euro Area (80-06)
Model with multiple equilibria and standard VAR
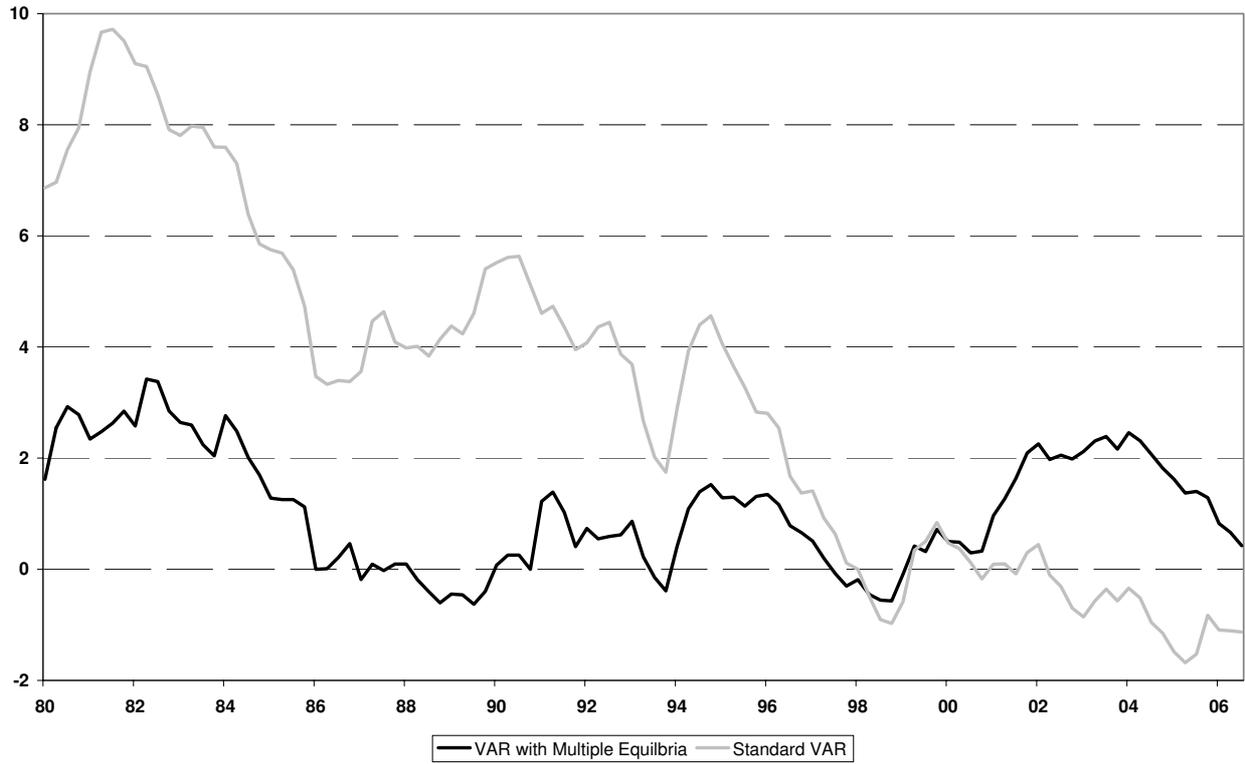
Figure 12 – Estimated forward premium – Euro Area (80-06)
Model with multiple equilibria and no-arbitrage macro-VAR