Multiproduct Cost Passthrough: Edgeworth's Paradox Revisited

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Abstract

Edgeworth's paradox of taxation occurs when an increase in the unit cost of a product causes a multiproduct monopolist to *reduce* prices. We give simple illustrations of the paradox, including how it can arise with uniform pricing. We then give a general analysis of the case of linear marginal cost and demand conditions, and characterize which matrices of cost passthrough terms are consistent with profit maximization. When the firm supplies at least one pair of substitute products we show how Edgeworth's paradox always occurs with a suitable choice of cost function. We then establish a connection between Ramsey pricing and the paradox in a form relating to consumer surplus, and use it to find further examples where consumer surplus increases with cost.

JEL codes: D42, H22, L12

Keywords: Multiproduct pricing, Edgeworth's paradox of taxation, cost passthrough, Ramsey pricing.

1 Introduction

Recent analyses of the rate of passthrough from cost to price have focussed on single-product firms with market power: see, for example, Weyl and Fabinger (2013), and Miklos-Thal and Shaffer (2021). However, most firms supply a number of products, and it is by no means clear that cost passthrough results from the single-product case carry over to the multi-product setting. The economics of multiproduct cost passthrough has a long history.

In a remarkable article[1] on the pure theory of monopoly published in 1897, Edgeworth demonstrated his paradox of taxation—that a tax on (or cost increase of) one product supplied by a multiproduct monopolist could lead to a reduction in the prices charged by the monopolist, including the price of the more costly product. This finding, controversial at the time, was established in more detail by Hotelling (1932), who gave some illustrations of the phenomenon that were somewhat easier to comprehend than Edgeworth's.

In particular, Edgeworth (1925, pages 132-4) showed for the two-product case that the second-order condition for profit maximization was compatible with consumer surplus increasing with a tax on one product. He then provided a numerical example in which both prices decreased with a tax on the first product.[2] The example has zero costs but Edgeworth notes that the conclusion is strengthened when there are costs of production, "for then we have more functions at our disposal with which to manipulate a favourable example". He goes on to illustrate with rail fares: a tax on first-class tickets might lower both first- and second-class fares, though the number of first-class travellers will nonetheless decline, as the reduced second-class fare predominates.

An important application of Edgeworth's insight, and multiproduct cost-passthrough more generally, is to the price effects of vertical integration or upstream cartel behaviour. For instance, Salinger (1991) discussed a situation with a multiproduct retailer who sources its products from a number of upstream suppliers. If the retailer integrates with one of its suppliers, then the "double margin" for that product is eliminated and the retailer's cost for that product falls. If demand and cost conditions are such that the Edgeworth's paradox applies for the retailer, then the impact of this vertical integration, often considered to be beneficial for final consumers, would be to raise all retail prices.[3] Luco and Marshall (2020) investigate the empirical impact of partial vertical mergers in the carbonated drinks market, and find that integration with one supplier does decrease the associated price, but

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[1]In Italian, in the *Giornale degli Economisti*. The article appears in English with some modifications in Edgeworth (1925).

[2]Edgeworth's example had the prices p_1 and p_2 being related to to quantities x_1 and x_2 as $p_1 = 1.605\dot{3} - .2x_1 - \frac{2}{3}(x_1 - .96)^{\frac{3}{2}} - \frac{1}{2}x_2$ and $p_2 = 3.9\dot{1}\dot{8} - 2\sqrt{x_2 - .6975} - \frac{1}{2}x_1$. Production was assumed costless, and the profit-maximizing quantities were $x_1 = x_2 = 1$, before the introduction of a tax on product 1.

[3]A related effect can occur even with a single final product. Suppose multiple inputs are used in the production of a single final product. If a input is *inferior*, i.e., less of it is used when more of the final output is supplied, then a reduction in the price of that input causes the marginal cost (though not the total cost) of the final product to *rise*, which induces the firm to set a higher retail price. Hicks (1939, p. 93) noticed this possibility, observing that a decrease in the cost of one input will necessarily increase the demand for the input, but might also decrease the supply of the output.

raises the price for the non-integrated product (a cost passthrough effect from one product to another that they term the "Edgeworth-Salinger" effect).[4]

In this paper, the *Edgeworth paradox* is said to occur when a unit tax on a product (or uniform increase in that product's marginal cost of supply) induces the firm weakly to reduce all its prices, with some prices decreasing strictly. As a preliminary comment it is worth noting why this phenomenon cannot arise in the single-product case. If the cost of supplying product i increases, the monopolist will want to reduce the amount of product i that it supplies. (The revealed preference argument that confirms this is set out in section 2 below.) If i is the only product, and demand as a function of price slopes down, then the price of product i must go up, and consumer surplus decreases. But if i is not the only product, the monopolist will in general adjust its supply of other products. If products are substitutes, reduced supply of i will often induce the monopolist to supply more of product j. The latter effect will bear down on prices, including the price of i, offsetting at least partially the effect of reduced supply of i. The paradox arises when the price effect of expanded supply of j more than offsets the reduced supply of i – a phenomenon compatible with standard demand theory in the multiproduct case, just as Edgeworth observed.

Edgeworth's paradox is of interest not only for it own sake. It is part of the much wider question of which cost passthrough possibilities exist in the multi-product case. For an n-product firm there are n cost passthrough terms for each price, and therefore $(n \times n)$ cost passthrough terms altogether. What properties of the $(n \times n)$ matrix of cost passthrough terms are implied by standard theory of cost and demand? Our exploration of Edgeworth's paradox leads to an answer to this broader question. It also reveals a connection with Ramsey pricing—in particular how Ramsey quantities move as the welfare weight on consumer surplus varies.

The paper is organized as follows. We set out the model in section 2, and show how a tax on product i will induce the firm to reduce its supply of that product. This implies that in order to obtain Edgeworth's paradox for a tax on product i, that product i must have at least one substitute. We use this output reduction insight to generate two examples of Edgeworth's paradox: one with discrete choice where one valuation is known, and the other where the monopolist is restricted to uniform pricing of its outputs. In section 3 we provide a general analysis of the case with linear marginal cost and demand conditions,

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[4]See Asker and Nocke (2021, section 3.5.1) for further discussion of this topic.

and show how this analysis extends to general smooth demand and cost functions. We characterise the general range of possibilities for cost passthrough, and show that a matrix is a possible matrix of cost-passthrough terms if and only it is similar (in the technical sense) to a positive-definite matrix. We go on to show that, provided there is at least one pair of substitute products, there are always cost conditions that give rise to the paradox. In section 4 we turn to a weaker version of Edgeworth's paradox that we call the surplus paradox—i.e., that *consumer surplus* increases as the cost of one product rises. Arguably, this is the more relevant of the two paradoxes, as consumers only care about their surplus not the individual prices that generate this surplus. We show that there are always cost conditions that give rise to the surplus paradox—and indeed to a paradox relating to total welfare—even without products being substitutes. We also derive a connection between Ramsey pricing and the surplus paradox. This paradox cannot happen in the cost and demand conditions featured in section III of Armstrong and Vickers (2018), hereafter abbreviated to AV, but is quite possible more generally. An implication of the Ramsey connection is that the surplus paradox can be found whenever the profit-maximizing supply of some product exceeds its supply with marginal cost pricing. This insight gives a way to find further examples of the surplus paradox.

2 The model and output reduction result

A monopolist supplies $n \geq 2$ products. The price and quantity of product i are denoted by p_i and x_i respectively, and p and x denote the price and quantity vectors. Total output is $X \equiv \sum_{i=1}^{n} x_i$. As in AV, gross consumer utility $u(x)$ is assumed to be strictly concave; the inverse demand function is given by $p(x) = \nabla u(x)$, the vector of partial derivatives of u (where these exist); and revenue is given by $r(x) \equiv (p(x))^T x$, where 'T' stands for transpose and $(p(x))^T x = \sum_{i=1}^{n} p_i(x)x_i$ is the inner product. Consumer surplus as a function of quantities is given by the function $s(x) \equiv u(x) - r(x)$. Profit is $\pi(x) = r(x) - c(x)$, where the $c(x)$ is the cost function. In general the monopolist maximizes the weighted sum $\phi(x) = \pi(x) + \alpha s(x)$, with $\alpha \in [0,1]$. Profit maximization corresponds to $\alpha = 0$, efficient supply corresponds to $\alpha = 1$, while interior α correspond to more general Ramsey pricing. While for the most part we model the firm as choosing quantities x, where more convenient we sometimes consider it to choose prices, in which case quantities are determined by the (direct) demand function $x(p)$.

Suppose that the set of feasible quantity vectors lies in some set $\mathfrak{X} \subset \mathbb{R}^n_+$, and that initially $\phi(x)$ is maximized by $x^0 \in \mathfrak{X}$. (In many cases it makes sense that $\mathfrak{X} = \mathbb{R}^n_+$, but, as with the two illustrations of discrete choice and uniform pricing below, there are natural situations where the feasible set of quantities is restricted.) Compare the situation when product i has a per-unit cost increase (or tax) of $t_i > 0$, and let $x^* \in \mathfrak{X}$ then maximize $\phi(x) - t_i x_i$. By revealed preference we have

$$\phi(x^0) \geq \phi(x^*) \text{ and } \phi(x^*) - t_i x_i^* \geq \phi(x^0) - t_i x_i^0 \ .$$

Combining these inequalities we deduce that $t_i(x_i^0 - x_i^*) \geq 0$, confirming that the cost increase causes supply of product i to fall, at least weakly. Moreover, if $\phi(x)$ is smooth, $\mathfrak{X} = \mathbb{R}^n_+$ and $x_i^0 > 0$, then the supply of product i decreases *strictly*.[5] For if not, i.e., if $x_i^0 = x_i^*$, then x^* also maximizes ϕ, and we would have the contradiction that

$$\frac{\partial}{\partial x_i}\phi(x^*) = 0 \text{ and } \left. \frac{\partial}{\partial x_i}[\phi(x) - t_i x_i]\right|_{x=x^*} = 0 \Rightarrow \frac{\partial}{\partial x_i}\phi(x^*) = t_i \ .$$

Note that the Edgeworth paradox requires at least some product pairs to be substitutes, at least in the smooth case with a small cost increase. For if $\partial x_1/\partial t_1 < 0$ is the change in the supply of product 1 in response to the small cost increase for product 1, say, then we can decompose $\partial x_1/\partial t_1$ in terms of the direct demand function, $x_1(p)$, as

$$\frac{\partial x_1}{\partial t_1} = \frac{\partial x_1}{\partial p_1}\frac{\partial p_1}{\partial t_1} + ... + \frac{\partial x_1}{\partial p_n}\frac{\partial p_n}{\partial t_1} < 0 \ . \tag{1}$$

Since each $\partial p_i/\partial t_1$ is negative with the paradox, in order for x_1 to fall it is necessary that at least one $\partial x_1/\partial p_i$ be positive, i.e., for some product i to be a substitute in this direct demand sense for product 1.[6] Later, in section 3.2, we show that condition (1) is also sufficient for $(\partial p_1/\partial t_1, ..., \partial p_1/\partial t_1)$ to be a possible pattern of price-cost passthrough (for a given demand system), in the sense that we can find a cost function that induces such a pattern of passthrough.

A recurring theme in this paper, explored more systematically in section 4, is that situations in which consumers benefit from a cost increase are associated with *over*-provision of the relevant product by the monopolist, relative to efficient supply. Intuitively, if the

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[5] The following argument is in the spirit of Edlin and Shannon (1998).

[6] Two notions of substitutability are that x_i increases with p_j (which concerns the direct demand functions as here) or that p_i decreases with x_j (which concerns the inverse demand functions). With two products these notions coincide, but with more products they are distinct.

monopolist supplies too much of a product, a tax on the supply of that product—which as we have seen reduces that supply—may well lead to better outcomes for consumers. The fall in x_i will by itself tend to increase p_i but other x_j will adjust too. Edgeworth's paradox occurs when their adjustment both outweighs the upward effect on p_i of the fall in x_i, and causes other p_j to decrease too.

Before turning to a more systematic analysis, we present two simple illustrations of how the output reduction result can lead directly to Edgeworth's paradox. In both examples, it is more convenient to consider the firm as choosing prices rather than quantities.

Discrete choice where one product has known valuation: Consider a framework with discrete choice, where a consumer chooses to buy (one unit of) one of the $n \geq 2$ products, or nothing, where her valuation for product i is v_i. Thus, she buys the product with the highest $v_i - p_i$ provided this is non-negative. Suppose that the valuation for product 1 is *known* and for all consumers is equal to $\bar{v}_1$, while other v_i are continuously distributed in some fashion. The unit cost of product i is c_i, where $c_1 < \bar{v}_1$.

It is straightforward to see that the profit-maximizing firm will choose the price $p_1 = \bar{v}_1$.[7] For if $p_1 < \bar{v}_1$ then all consumers buy something (as they get positive surplus if they buy product 1) and if the firm increases each price p_i by the same small $\varepsilon > 0$ each consumer continues to buy the same product as before, and the firm gains revenue ε from each consumer. Therefore, it cannot be optimal to set $p_1 < \bar{v}_1$.[8] In particular, if c_1 increases (but does not exceed $\bar{v}_1$) the firm does not change p_1. Since $p_1 = \bar{v}_1$, the number of consumers who buy product 1, x_1, consists of those for whom $v_i < p_i$ for all $i \geq 2$. But since x_1 falls when c_1 rises, at least one other price p_i must fall. Therefore, if there are just two products (or if all products $i \geq 2$ are symmetric and the firm sets the same price for these products) then all prices weakly fall (and some strictly decrease) when c_1 increases, which illustrates the Edgeworth paradox.[9]

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[7]More generally, there are other situations in which the firm's price for one product is fixed. For example, aspects of the firm's service might be regulated, or the firm might face strong competition in one market. In these cases, the following analysis applies equally, so long as the firm's products are substitutes.

[8]It cannot be optimal to set $p_1 > \bar{v}_1$, since then when a consumer has $v_i > p_i$ for all $i > 1$ she buys nothing, and if the firm reduced p_1 down to $\bar{v}_1$ such consumers would instead buy product 1 which yields greater profit for the firm. In addition, it continues to be optimal to set $p_1 = \bar{v}_1$ in the Ramsey problem with $\alpha > 0$.

[9]For instance, with two products, $\bar{v}_1 = 1$, v_2 uniformly distributed on $[0, 1]$, $0 \leq c_1 \leq 1$, and $c_2 = 0$, one can check that profit is maximized with prices $p_1 = 1$ and $p_2 = 1 - \frac{1}{2}c_1$. More generally, suppose there are n products where $\bar{v}_1 = 1$ and the other other $n-1$ product valuations are continuously distributed in some fashion with full support on $[0, 1]^{n-1}$, and that production is initially costless. Then the firm initially

Since all consumers buy something from the firm, we have $X = \sum_{i=1}^{n} x_i = 1$ (if the number of consumers is normalized to 1), and so the feasible set of quantities is the simplex $\mathfrak{X} = \{x \mid \sum_{i=1}^{n} x_i = 1\}$. Since the firm chooses prices $p_i > c_i$ for $i \geq 2$, its supply x_1 *exceeds* the efficient supply of product 1, i.e., the supply corresponding to marginal-cost pricing. As we will discuss later, this implies that an increase to c_1 will benefit consumers, even in the more general case with n products where some prices p_i might rise in response to the cost increase.

Uniform pricing: Suppose now that the firm is constrained to set the same price for each of its products. For instance, regulation or social norms might require a restaurant to set the same price for dinner regardless of the day of the week, or a chain-store might set the same price across all of its retail outlets.[10] In such situations it is more convenient to work with direct demand $x(p)$ rather than inverse demand $p(x)$. Here, $\mathfrak{X}$ is the set of quantity vectors x traced out by the path $x(P, ..., P)$ as the scalar uniform price P varies.

Since an increase in product 1's cost causes the firm reduce its supply x_1, it follows that if $x_1(P, ..., P)$ is an increasing function of P then the cost increase will induce the firm to *reduce P*, which is a version of Edgeworth's paradox. In the differentiable case, the condition that x_1 rises with P is equivalent to total quantity X increasing with p_1. This is because by Slutsky symmetry

$$\frac{d}{dP} x_1(P, ..., P) = \sum_{i=1}^{n} \frac{\partial x_1}{\partial p_i} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial p_1} = \frac{\partial}{\partial p_1} X(P, ..., P) \ .$$

This condition cannot occur in a situation with standard discrete choice, since in that case the number of consumers who buy anything (which is X) weakly decreases when any price rises. However, more generally it is possible that total quantity rises when a price rises.[11]

In any demand system total output X falls with the uniform price P. In particular, in the two-product case if x_1 increases with P then x_2 must fall with P, in which case the set $\mathfrak{X}$ of feasible quantities with uniform pricing is a *downward*-sloping curve in $\mathbb{R}_+^2$. Revealed preference shows that the uniform price P that maximizes the Ramsey objective

chooses to supply product 1 to all consumers, but if c_1 increases from zero one can show that *all* other prices strictly fall, so that the Edgeworth paradox holds.

[10] Another reason why the firm might choose to offer uniform prices is if consumers regard its various products as perfect substitutes and if, for cost reasons, the firm wishes to supply a positive quantity of all products.

[11] For example, suppose demand functions are $x_1 = 1 - p_1 + \frac{3}{2}p_2$ and $x_2 = 3 - 4p_2 + \frac{3}{2}p_1$, and unit costs are c_1 (which varies) and $c_2 = 0$. Then x_1 increases with a uniform price P, and one can check that the profit-maximizing uniform price is $P = 1 - \frac{1}{8}c_1$ which decreases with c_1.

ϕ decreases with the weight on consumer surplus, α, and in particular that the profit-maximizing uniform price exceeds the welfare-maximizing uniform price. Therefore, as with the previous illustration, when x_1 increases with P the firm supplies too much product 1 relative to the efficient supply.

3 Patterns of cost passthrough and the Edgeworth paradox

In this section we aim to analyze the possibility of Edgeworth's paradox in general terms, by studying the feasible patterns of cost passthrough. To do this, we first study the simpler situation in which both marginal costs and demands vary linearly with quantities. While some of the insights from this analysis are special to the linear case, most will generalize in a straightforward manner to general smooth demand and cost systems.

3.1 Linear demand and marginal cost

Suppose for now that the n products have linear inverse demands

$$p(x) = a - Bx \tag{2}$$

where a is a vector of positive constants and B is a (symmetric) positive-definite matrix.[12] Thus the firm's revenue function is

$$r(x) = a^T x - x^T B x \; .$$

Suppose the firm's cost function is

$$c(x) = c^T x + x^T D x \; ,$$

where c is a vector of non-negative constants c_i, one per product, and D is symmetric (though not necessarily positive definite). Note that increasing c_1, say, corresponds to introducing a tax t_1 on the sale of product 1. Let $M \equiv 2(B + D)$ so that the firm's profit is

$$\pi = (a - c)^T x - \tfrac{1}{2} x^T M x \; . \tag{3}$$

To ensure that profit is strictly concave in quantities, suppose that M is positive definite, which requires that the cost function not be too concave relative to revenue. Assuming

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[12] We adopt the standard convention that a positive-definite matrix is symmetric.

an interior solution in which the firm supplies a positive quantity of each product, the first-order condition for profit-maximizing quantities is

$$Mx = a - c \Rightarrow x = M^{-1}(a - c) , \tag{4}$$

so that the matrix derivative dx/dc is equal to (minus) the positive-definite matrix M^{-1}. In particular, there is symmetry in the cross-cost effects on quantities in the sense that $\partial x_i/\partial c_j = \partial x_j/\partial c_i$. Since the only constraint on M^{-1} is that it is positive definite, expression (4) fully characterizes the feasible ways that quantities can vary with cost changes: the matrix derivative dx/dc is negative definite.

From (2), expression (4) implies optimal prices are given by

$$p = (I - \Gamma)a + \Gamma c , \tag{5}$$

where I is the identity matrix and

$$\Gamma \equiv BM^{-1} . \tag{6}$$

Thus, Γ is the matrix derivative dp/dc, and expression (6) is just an instance of the chain rule: the matrix M^{-1} describes how quantities x vary with costs c, and B describes how those quantity changes translate into price changes. While quantity responses to cost rises depend only on the profit function, price responses depend jointly on profit and demand functions. Even though the matrices B and M^{-1} are symmetric, their product Γ need not be and cross-cost effects on prices can be asymmetric. The Edgeworth paradox occurs for product i—that is, an increase in c_i (e.g., due to a new tax t_i on that product) will reduce all prices—if the ith column of Γ consists of negative entries.

If the cost function is linear (i.e., if $D = 0$), then $\Gamma = \frac{1}{2}I$ and (5) simplifies to

$$p = \tfrac{1}{2}(a + c) , \tag{7}$$

in which case optimal prices do not depend on the demand matrix B at all, there are no cross-cost effects on prices, and the paradox cannot occur.[13]

Turn next to the case with a quadratic cost function, and suppose for simplicity that there are two products, where we write $b = b_{12} = b_{21}$ and $m = m_{12} = m_{21}$. From (6), we

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[13]In the numerical example from footnote 11, if the uniform pricing constraint was removed then from (7) the firm would choose $p_1 = \frac{17}{7} + \frac{1}{2}c_1$ and $p_2 = \frac{9}{7}$, which are both higher than the firm's optimal uniform price $P = 1 - \frac{1}{8}c_1$.

have

$$\Gamma = \frac{2}{\det M} \begin{pmatrix} b_{11}m_{22} - bm & bm_{11} - b_{11}m \\ bm_{22} - b_{22}m & b_{22}m_{11} - bm \end{pmatrix} \tag{8}$$

where $\det M > 0$ is the determinant of M. Thus an increase in c_1 will reduce both prices if

$$b_{11}m_{22} - bm < 0 \tag{9}$$

and

$$bm_{22} - b_{22}m < 0 . \tag{10}$$

We know from section 2 that this pair of conditions can occur only if products are substitutes, i.e., if $b > 0$. Indeed, when $b > 0$ then condition (9) implies that (10) holds as well.[14] To see this, note that

$$b_{22}[b_{11}m_{22} - bm] - b[bm_{22} - b_{22}m] = (b_{11}b_{22} - b^2)m_{22} > 0 ,$$

where the inequality follows since both $b_{11}b_{22} > b^2$ and $m_{22} > 0$. Since $b_{22} > 0$ it follows that when (9) holds then $b[bm_{22} - b_{22}m] < 0$, as claimed. We summarise this discussion as follows:

Proposition 1 *Suppose there are two products and marginal cost and demand is linear. Then the Edgeworth paradox holds for an increase in the cost of product 1 if and only if products are substitutes and (9) is satisfied.*

To illustrate, consider the demand and profit matrices

$$B = \begin{pmatrix} 7/2 & 3 \\ 3 & 8/3 \end{pmatrix} , M = \begin{pmatrix} 10 & 8 \\ 8 & 20/3 \end{pmatrix} ,$$

both of which are positive definite and which satisfy (9).[15] These induce the passthrough matrix

$$\Gamma = BM^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} , \tag{11}$$

which exhibits the Edgeworth paradox for c_1.

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[14] Actually, when $b < 0$ then (9) implies that $bm_{11} - b_{11}m < 0$, in which case (8) shows that p_1 decreases with both costs c_1 and c_2.

[15] For instance, these matrices arise when inverse demand and cost are given by $p_1 = 18 - \frac{7}{2}x_1 - 3x_2$, $p_2 = \frac{44}{3} - \frac{8}{3}x_2 - 3x_1$, and $c(x) = \frac{1}{6}(3x_1 + 2x_2)^2$, in which case the firm chooses quantities $x_1 = x_2 = 1$.

Now consider the Ramsey problem rather than pure profit maximization. With a weight α on consumer surplus, which is $s(x) = \frac{1}{2}x^T B x$ with linear demand (2), the objective function becomes

$$\pi + \alpha s = (a - c)^T x - \tfrac{1}{2}x^T M x + \tfrac{1}{2}\alpha x^T B x \ ,$$

and so the preceding analysis carries over if the profit matrix M is replaced by $(M - \alpha B)$. Therefore, the cost-passthrough matrix for the Ramsey problem, denoted Γ_α, is $\Gamma_\alpha = B(M - \alpha B)^{-1}$. As with (6), it follows that

$$\Gamma_\alpha^{-1} = (B(M - \alpha B)^{-1})^{-1} = (M - \alpha B)B^{-1} = \Gamma^{-1} - \alpha I \ , \tag{12}$$

and the diagonal elements of the inverse passthrough matrix are shifted down as more weight is put on consumer surplus, while off-diagonal terms are unchanged. In the two-product case, expression (12) implies that Γ_α is proportional to $\Gamma - \det(\Gamma)\alpha I$. Here, increasing α does not affect the sign of the cross-cost effects, but can make the own-cost effects more negative. Thus, with linear demand and marginal cost the paradox becomes less difficult to achieve when more weight is placed on consumer surplus. In particular, if the paradox occurs under profit-maximization it occurs for all Ramsey weights as well.[16]

Before we turn to more general analysis, it is useful to derive one further result in the linear framework that will be useful as an ingredient for the general analysis. We know from section 2 that the firm must reduce its supply of product 1 if the cost of that product rises. The following result shows that this is the *only* restriction on the firm's pattern of quantity responses to a cost increase.

Lemma 1 *Let $x^* > 0$ be a vector of quantities and let $k = (k_1, ..., k_n)$ be a vector such that $k_1 < 0$. Then there exists a concave profit function $\pi(x)$ of the form (3) such that (i) π is maximized at $x = x^*$ and (ii) the firm's optimal response to an increase of the cost (or tax) of product 1 is given by $\partial x_i/\partial c_1 = k_i$ for each $1 \le i \le n$.*

Proof. Consider the quadratic profit function

$$\pi(x) = (x^*)^T M x - \tfrac{1}{2}x^T M x \ , \tag{13}$$

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[16]Hotelling (1932, pp. 602-3) presents a linear example in which the Edgeworth paradox arises with the Ramsey objective $\phi = \pi + \alpha s$ for all $\alpha \in (0, 1]$. But with pure profit maximization ($\alpha = 0$) the price of product 1 does not vary with its unit cost in his example.

where M is a positive-definite matrix. Then we have for $x \neq x^*$

$$
\begin{aligned}
2[\pi(x^*) - \pi(x)] &= (x^*)^T M x^* + x^T M x - 2(x^*)^T M x \\
&= (x^*)^T M x^* + x^T M x - (x^*)^T M x - x^T M x^* \\
&= (x^* - x)^T M (x^* - x) > 0 .
\end{aligned}
$$

Here, the second equality arises since $(x^*)^T M x$ is a scalar and so equal to its transpose, while the inequality follows since M is positive definite. Therefore, π in (13) is maximized at $x = x^*$ for any positive-definite matrix M.

If we introduce a vector of small taxes t to the profit function (13), the optimal quantities satisfy $\nabla \pi(x) = t$, so that

$$
x^* - x = M^{-1} t
$$

(where x^* are the optimal quantities with $t = 0$). Thus, as in (4), the matrix M^{-1} is (minus) the derivative dx/dt evaluated at $t = 0$. In particular, considering only a cost increase to product 1 we see that the vector of quantity responses $(\partial x_1/\partial c_1, ..., \partial x_n/\partial c_1)$ is equal to the first row or column of M^{-1}. Therefore, part (ii) is proved provided that we can construct a negative-definite matrix $-M^{-1}$ with first row or column equal to the vector $(k_1, ..., k_n)$.

Consider the symmetric matrix

$$
-M^{-1} = \begin{pmatrix}
k_1 & k_2 & k_3 & \cdots & k_n \\
k_2 & a_2 & 0 & \cdots & 0 \\
k_3 & 0 & a_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k_n & 0 & 0 & \cdots & a_n
\end{pmatrix}
$$

with $a_i < 0$, which we wish to be negative definite. Intuitively, if the diagonal terms a_i are large enough and negative then this matrix is negative definite. In more detail, for any vector z we have

$$
\begin{aligned}
z^T(-M^{-1})z &= k_1 z_1^2 + \sum_{i=2}^{n} z_i \left(2 k_i z_1 + a_i z_i \right) \\
&= k_1 z_1^2 + \sum_{i=2}^{n} \frac{1}{a_i} \left[(k_i z_1 + a_i z_i)^2 - k_i^2 z_1^2 \right] \\
&\leq k_1 z_1^2 - \sum_{i=2}^{n} \frac{k_i^2}{a_i} z_1^2 \\
&= \left(k_1 - \sum_{i=2}^{n} \frac{k_i^2}{a_i} \right) z_1^2 .
\end{aligned}
$$

If each a_i is large and negative enough, e.g., if $a_i \equiv a$ where $a < \frac{1}{k_1} \sum_{i=2}^{n} k_i^2$, then the above term $(\cdot)$ is negative, i.e., $z^T(-M^{-1})z < 0$ if $z \neq 0$, so M^{-1} is positive definite. $\blacksquare$

3.2   More general analysis

The linear analysis in the previous section carries over to more general situations. For suppose inverse demand is not necessarily linear, but is given by the smooth function $p(x)$, and that profit with a vector of cost-shifters (or tax rates) t is $\pi(x) - t^T x$. Then provided the profit function $\pi(x)$ is smooth and there is a unique profit-maximizing vector of quantities for each t (which is the case if π is strictly concave), the most profitable quantity vector x, if interior, satisfies $\nabla \pi(x) = t$, and this x varies smoothly with t. The matrix of quantity cost-passthrough terms is again given by $dx/dt = -M^{-1}$, where now M is (minus) the matrix of second derivatives of π, which is positive definite from the second-order condition for (and uniqueness of) the optimal x. Thus, a pattern of quantity responses to cost shocks is feasible provided that the matrix dx/dt is negative definite. The matrix of price cost-passthrough terms, evaluated at $t = 0$, is again given by (6), where B is (minus) the matrix of derivatives of inverse demand $p(x)$, which is positive definite from the assumption that utility is concave. Proposition 1 therefore extends to general demand and cost conditions as well, and with two products that are substitutes the Edgeworth paradox holds for product 1 if and only if expression (9) holds.

Not every insight from the linear case generalizes. One feature of the linear demand assumption was that the Edgeworth paradox is not possible when marginal costs are constant (see expression (7)). However, with other demand specifications it is perfectly possible to observe the paradox even with a constant unit cost per product. (Such an example was presented above in section 2, and others were provided in the early papers by Edgeworth and Hotelling.)

We showed in section 2 that to obtain the Edgeworth paradox it is necessary that product 1 has at least one substitute product, in the sense that direct demand for product 1 rises with another price. We now show that this condition is also sufficient for the paradox to be possible. That is, for any demand system where one product has a substitute we can find a well-behaved cost function which implements the paradox.

Proposition 2 *Consider a smooth demand system with revenue $r(x)$, and let x^* be a vector of positive quantities such that $\nabla r(x^*) \geq 0$. Suppose direct demand for product 1 is*

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increasing in the price of another product. Then there exists a cost function $c(x)$, where $\nabla c(x^*) \geq 0$ and $c(x^*) \geq 0$, such that $x = x^*$ maximizes profit $r(x) - c(x)$ and all prices fall when a small tax on product 1 is introduced.

Proof. Given $r(x)$ one can induce any desired profit function $\pi(x)$ by means of the cost function $c(x) = r(x) - \pi(x)$, and rather than choosing a cost function we work directly with the profit function. (We check at the end that the corresponding cost function is well behaved.) Specifically, consider a quadratic profit function $\pi(x)$ given by expression (13) in the proof of Lemma 1, where x^* is the quantity vector with $\nabla r(x^*) \geq 0$, and where the matrix M can be chosen to induce any pattern of quantity responses $\partial x_i / \partial t_1$ so long as $\partial x_1 / \partial t_1 < 0$. Since the change in quantity x_i in response to the tax t_1 can be expressed in terms of the direct demand functions $x_i(p)$ as

$$\frac{\partial x_i}{\partial t_1} = \frac{\partial x_i}{\partial p_1}\frac{\partial p_1}{\partial t_1} + ... + \frac{\partial x_i}{\partial p_n}\frac{\partial p_n}{\partial t_1} \ ,$$

and since the only constraint on the vector $\partial x / \partial t_1$ is that $\partial x_1 / \partial t_1 < 0$, we deduce that a vector of price responses $\partial p / \partial t_1$ can be implemented with a suitable cost function if and only if (1) holds, so that the combined price responses to the tax cause demand x_1 to fall.

Since by assumption direct demand $x_1(p)$ increases with some p_j, it is clear that one can find a feasible vector of price responses, $\partial p / \partial t_1$, that satisfies (1) and for which each $\partial p_i / \partial t_1 < 0$, and so the Edgeworth paradox holds. Specifically, since $\partial x_1 / \partial p_j > 0$, if one makes $\partial p_j / \partial t_1$ large and negative and all other $\partial p_i / \partial t_1$ small and negative, this vector will satisfy the condition (1). With this demand system and the profit function $\pi(x)$, introducing a small tax t_1 will induce the firm to reduce all prices. Note that if the profit function π works for this argument, then so does any profit function of form $\kappa \pi(x)$ where κ is a positive constant. (The introduction of the scaling factor κ does not affect the choice of optimal quantities x^*, and it scales down all the quantity responses $\partial x_i / \partial t_1$ by κ, which does not affect the feasibility constraint (1).) Therefore, without loss of generality we can choose a profit function such that $\pi(x^*) \leq r(x^*)$.

Finally, consider the associated cost function $c(x) = r(x) - \pi(x)$. Since $\pi(x)$ is maximized at $x = x^*$, it follows that $\nabla c(x^*) = \nabla r(x^*) \geq 0$ and so marginal costs at x^* are non-negative. Moreover, we have chosen a profit function such that $\pi(x^*) \leq r(x^*)$, and so $c(x^*) \geq 0$ as required. ∎

The cost function in the statement of the result is "well behaved" in the sense that it is positive and increasing at the relevant point $x = x^*$. The result shows that a very wide range of demand systems are compatible with the Edgeworth paradox. Indeed, the only demand systems incompatible with the paradox are those where all product pairs are complements. This validates Edgeworth's remark, quoted above, that a flexible choice of cost functions expands the scope for examples of the paradox.

Moving beyond the focus on the Edgeworth paradox, an important question is what patterns of price-cost passthrough are feasible in general. We have seen that a pattern of *quantity* responses to taxes is feasible provided that the matrix dx/dt is negative definite. Expression (6) shows that the matrix of price-cost passthrough terms under profit maximization is $dp/dt = \Gamma = BM^{-1}$. Since the only constraint on B and M^{-1} is that they are positive definite, the only constraint on Γ is that it be the product of two positive-definite matrices. It is not obvious from simply looking at a matrix to know whether it can be factored into positive-definite matrices. However, Ballantine (1968, Theorem 2) shows that a matrix Γ is the product of two positive-definite matrices if and only if it is similar to a positive-definite matrix.[17] As a positive-definite matrix is similar to a diagonal matrix with positive entries, Γ is the product of two positive-definite matrices if and only it is similar to a diagonal matrix with positive entries, i.e., if Γ is diagonalizable with positive eigenvalues. Thus, we have the following result:

Proposition 3 Γ *is a feasible price cost-passthrough matrix if and only if it is diagonalizable with positive eigenvalues.*

The condition that Γ be diagonalizable with positive eigenvalues is almost the same as requiring that all eigenvalues of Γ are real and positive. If Γ has *distinct* positive eigenvalues then it is diagonalizable. However, if Γ has some repeated eigenvalues, then it is not necessarily diagonalizable.

If Γ happens to be symmetric, then the condition for it to be a feasible cost-passthrough matrix is simply that it be positive definite. In particular, its diagonal entries are positive and a new tax on product i must induce the firm to raise its price p_i. Thus the Edgeworth paradox can only occur when there are asymmetries in price-cost passthrough.

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[17]Two matrices A and B are said to be similar if they are related as $A = Z^{-1}BZ$ for some invertible matrix Z. Similar matrices have the same determinant, trace, and eigenvalues. If a matrix A is similar to a diagonal matrix, it is said to be "diagonalizable", and the entries in the diagonal matrix are the eigenvalues of A.

Proposition 3 implies that the determinant of Γ is positive and that the trace of Γ is positive, where the latter observation says that on average the own-cost passthrough terms are positive. For instance, when there are two products the trace condition means that at most one of the own-cost terms can be negative. Therefore, if the Edgeworth paradox holds for product 1 (i.e., if γ_{11} and γ_{21} are negative) we have $\gamma_{11} < 0 < \gamma_{22}$, in which case the requirement that the determinant of Γ is positive requires that $\gamma_{12}\gamma_{21}$ be negative, so that γ_{21} is positive and an increase in product 2's cost must cause both prices to rise. More generally, if γ_{12} and γ_{21} have the same sign, then the determinant condition implies that γ_{11} and γ_{22} have the same sign and hence both are positive.

4   The surplus paradox and a Ramsey connection

For Edgeworth's paradox to occur, all prices must fall when the cost of one product rises. Consider instead the *surplus paradox* that he mentioned, where consumer surplus increases when the cost of one product rises. Clearly the surplus paradox occurs whenever Edgeworth's paradox does, but it can occur more generally and with less in the way of "manipulation" needed.

The next result modifies Proposition 2 to focus on surplus rather than individual prices, and shows that the surplus paradox can occur for essentially any demand system, if one chooses a suitable cost function. The result also demonstrates that a stronger "welfare paradox" can occur under the same demand conditions, so that the sum of profit and consumer surplus can rise when a tax is imposed.

Proposition 4 *Consider a smooth demand system with revenue $r(x)$, and let x^* be a vector of positive quantities such that $\nabla r(x^*) \geq 0$. Unless $s(x)$ depends only on quantity x_1, there exists a cost function $c(x)$, where $\nabla c(x^*) \geq 0$ and $c(x^*) \geq 0$, such that $x = x^*$ maximizes profit $r(x) - c(x)$ and consumer surplus $s(x)$ rises when a small tax on product 1 is introduced. Under the same conditions, there exists a cost function such that total welfare $s(x) + \pi(x)$ rises when a small tax of product 1 is introduced.*

Proof. The proof follows similar lines to that for Proposition 2. Consider a quadratic profit function $\pi(x)$ given by (13) in the proof of Lemma 1, where x^* is the quantity vector with $\nabla r(x^*) \geq 0$, and where the matrix M can be chosen to induce any pattern of quantity responses $\partial x_i / \partial t_1$ so long as $\partial x_1 / \partial t_1 < 0$. The change in consumer surplus in response to

the tax t_1 can be expressed in terms of the quantity responses as

$$\frac{\partial s}{\partial t_1} = \frac{\partial s(x^*)}{\partial x_1}\frac{\partial x_1}{\partial t_1} + ... + \frac{\partial s(x^*)}{\partial x_n}\frac{\partial x_n}{\partial t_1} \ . \tag{14}$$

As discussed in AV the function $s(x)$ might increase or decrease with x_i, although it cannot decrease with *all* x_i. (It increases with all x_i if all products are substitutes in the sense that each $p_i(x)$ decreases with each x_j.) But regardless of the signs of $\partial s/\partial x_i$ it is clear we can choose a feasible pattern of quantity responses to make (14) positive. For instance, if $\partial s/\partial x_1$ is positive, then we can set $\partial x_1/\partial t_1$ (which is necessarily negative) to be small in magnitude, and any or all of the other $\partial x_i/\partial t_1$ to be large and with the same sign as their corresponding $\partial s/\partial x_i$. The only situation in which this cannot be done is when $\partial s/\partial x_i = 0$ for all $i > 1$, which we rule out by assumption. (AV provide an example of a demand system where $s(x)$ depends only on x_1.)

As in the proof of Proposition 2, if $\pi(x)$ is a profit function maximized at $x = x^*$ and which makes (14) positive, then so is $\kappa\pi(x)$ for $\kappa > 0$. For smaller enough κ one can ensure that the corresponding cost function $c(x)$ is positive at $x = x^*$. (The associated cost function is necessarily increasing at $x = x^*$ given $\nabla r(x^*) \geq 0$.)

A similar argument shows that we can find a cost function such that total welfare $s(x) + \pi(x)$ rises when a tax t_1 is introduced. When a small tax dt_1 is introduced, an envelope argument shows that the firm's maximum post-tax profit falls by $x_1^* dt_1$, where x^* is the firm's optimal quantity vector without the tax. Therefore, from (14) the impact on total welfare is

$$\frac{\partial(s+\pi)}{\partial t_1} = \frac{\partial s(x^*)}{\partial x_1}\frac{\partial x_1}{\partial t_1} + ... + \frac{\partial s(x^*)}{\partial x_n}\frac{\partial x_n}{\partial t_1} - x_1^* \ . \tag{15}$$

But if $\pi(x)$ is a profit function maximized at $x = x^*$ and which makes (14) positive, then so does $\kappa\pi(x)$ for $\kappa > 0$. More precisely, the scaling factor κ multiplies (14) by $1/\kappa$, and so for small enough κ the expression (15) is positive too. For small κ we are also sure to have the corresponding cost function being increasing and positive. ∎

Thus, unlike the full Edgeworth paradox, the surplus paradox, and even a welfare paradox, can occur when the firm serves markets which are separate and independent in terms of consumer demand.[18]

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[18]Chen and Schwartz (2015) analyze the effect on consumer surplus and welfare of mean-preserving *spreads* of unit cost in a setting with separate single-product markets that have the same demand conditions.

The surplus paradox has a simple connection with Ramsey pricing. Recall that the monopolist is assumed to maximize $\phi(x) = \pi(x) + \alpha s(x)$, where α is the weight on consumer surplus relative to profit. A natural question is how the optimal quantity x_i of product i varies with α. As the next result records, the answer is that the rate that consumer surplus varies with the cost of product i is equal to minus the rate that x_i varies with α. Therefore, if x_i decreases with α then the surplus paradox occurs and consumer surplus increases with the introduction of a tax t_i.

Proposition 5 *Suppose that a small per-unit tax t_i is imposed on product i. Then*

$$\frac{\partial s}{\partial t_i} = -\frac{\partial x_i}{\partial \alpha} \ . \tag{16}$$

Proof. Define

$$\hat{\phi}(t_i, \alpha) \equiv \max_{x \geq 0} : \pi(x) - t_i x_i + \alpha s(x)$$

as maximum weighted welfare with Ramsey parameter α and tax t_i. By the envelope theorem

$$\frac{\partial \hat{\phi}}{\partial t_i} = -x_i \quad ; \quad \frac{\partial \hat{\phi}}{\partial \alpha} = s$$

and the symmetry of cross derivatives of $\hat{\phi}$ entails (16). ∎

A revealed preference argument shows that s necessarily increases with α, and so Proposition 5 can be interpreted as saying that the surplus paradox arises for a cost increase for product i if the most profitable quantity of product i needed to achieve a target consumer surplus s *decreases* with s.

For instance, suppose consumers view the products as *perfect* substitutes, so that they care only about total quantity X. Given the firm's cost function $c(x)$, suppose that the least-cost way to supply total quantity X involves quantity $x_1(X)$ of product 1. Then if $x_1(X)$ decreases with X, so that product 1 is akin to an inferior input, Proposition 5 implies that a tax on product 1 will increase consumer surplus. (The optimal choice of X in the Ramsey problem increases with α, and so the optimal choice of x_1 decreases with α if $x_1(X)$ falls with X.)

Using Proposition 5 we can apply Ramsey pricing results to understand better when the surplus paradox might arise beyond monopoly settings. For instance, AV connects Ramsey

pricing to Cournot competition. In particular, Section IIC in AV implies that Cournot competition between m symmetric multiproduct firms with cost function satisfying

$$c(x) \text{ is convex and homogeneous degree 1} \tag{17}$$

has the same outcome as the monopoly Ramsey problem with weight $\alpha = (m - 1)/m$. Proposition 5 then implies that, if entry into the Cournot market would have caused equilibrium supply of product i to fall, then an industry-wide tax on that product will cause consumer surplus to rise.

Another theme of AV is that under certain conditions optimal quantities move *equiproportionately* as the Ramsey weight α on consumer surplus varies, in which case the surplus paradox cannot occur. It is well known that the equiproportional property holds if α is close to 1 when (17) holds (see section IIB in AV). Thus when $\alpha \approx 1$ the only way to obtain either paradox is to have cost functions outside the class (17). For instance, as already noted, Hotelling (1932, section 7) gives an example with linear demand and a quadratic cost function in which Edgeworth's paradox arises with $\alpha = 1$. Section III in AV considers the situation where the cost function satisfies (17) and consumer surplus s is *homothetic* in x, i.e., s is an increasing function of the scalar "composite quantity" $q(x)$ where $q(\cdot)$ is homogenous degree 1 in outputs x. (Consumer surplus is homothetic in x when utility u is homothetic, and also when demands are linear or take a Logit form. More generally, Proposition 2 in AV shows that s is homothetic in x if the utility function u takes the form $u(x) = h(x) + g(q(x))$, where h and q are homogeneous degree 1 functions.) Proposition 3 in AV shows in this case that Ramsey quantities increase equiproportionately as α increases, so that neither paradox can occur. The reason is that the cost of producing composite quantity q increases if the cost of any component product rises, and this induces the firm to reduce q and so reduce consumer surplus.

An implication of Proposition 5 is that the surplus paradox can be found wherever the profit-maximizing quantity of some product i exceeds its efficient level, for in that case there must be a range of α over which x_i falls with α and hence a range of α for which consumer surplus increases with t_i. This phenomenon of excessive monopoly supply of one product can be viewed as the quantity analogue to a monopoly price being below marginal cost. A natural situation in which this occurs is when total quantity X does not vary as α varies, so (unless the profit-maximizing and efficient allocation of that quantity happened to coincide) one product must be in greater supply with profit-maximization than with

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marginal cost pricing. We have seen one such example already in section 2 when one product has a known valuation. Another example is the following:

Hotelling preferences: In the spirit of Hotelling (1929), consider a firm with two products located at each end of the unit interval $[0, 1]$ supplied by a single firm with unit costs c_1 and c_2 respectively. Consumers of mass 1 are uniformly distributed along the line and wish to buy or other product (or neither). Their willingness to pay for a product is $1 - \tau z$, where z is their distance travelled and τ is the transport cost. Assume $0 \leq c_1 - c_2 < \tau$, which ensures an interior solution with both profit maximization and with marginal-cost pricing. Assume also that $\frac{1}{2}(c_1 + c_2) < 1 - \tau$, which ensures that the firm will optimally choose to serve all consumers, and so total output does not depend on costs over this range. With marginal cost pricing the quantity of the high-cost product is

$$\tilde{x}_1 = \frac{1}{2}\left(1 - \frac{c_1 - c_2}{\tau}\right) \, ,$$

whereas with profit-maximization it is

$$\hat{x}_1 = \frac{1}{2}\left(1 - \frac{c_1 - c_2}{2\tau}\right) > \tilde{x}_1 \, .$$

So there is more asymmetry between x_1 and x_2 with marginal cost pricing than with profit-maximizing monopoly. In the latter case, increasing c_1 has the effect of increasing asymmetry, which is good for consumers and so we have the surplus paradox. The reason is that the consumer indifferent between products gets zero surplus, and the surplus of others is τ times their distance from the indifferent consumer. The average distance increases with asymmetry. In this example total quantity is at the efficient level with profit-maximization but is inefficiently allocated between products whenever cost levels differ. Thus increasing the cost of the more costly product 1 will benefit consumers in aggregate, and the surplus paradox always exists.

These discrete choice examples shared the feature that the firm's choice of total output X was unaffected by cost changes over the relevant range of costs. In effect, the firm's choice of quantities was taken from the constrained set $\mathfrak{X} = \{x \mid X = \sum_k x_k = 1\}$. Likewise, in the situation with uniform pricing in section 2, we saw that the uniform price decreases with the cost of a product when the constrained set $\mathfrak{X}$ took the form of a downward-sloping curve in $\mathbb{R}^2_+$. Indeed, it is generally the case that if the firm chooses its quantities from a

constrained set $\mathfrak{X}$ that has the property that an increase in one quantity necessarily causes another quantity to fall, then the surplus paradox will hold. Proposition 5 continues to hold when quantities are chosen from a (suitably smooth) constrained set $\mathfrak{X}$ rather than $\mathbb{R}_+^n$, and as long as increasing the Ramsey weight α above zero has *any* impact on its choice of x it must then cause one quantity to fall. For instance, if the firm operates under some form of average price regulatory constraint, that will usually entail this form of quantity constraint $\mathfrak{X}$, and so an increase in some product's cost will cause consumer surplus to rise.

In general, however, the fact that the firm supplies a greater quantity of product i than is efficient does not imply that the profit-maximizing firm will offer consumers greater surplus when the cost of product i rises. Proposition 5 implies that there is a range of α for which a rise in the cost of this product rises implies that consumers obtain more surplus, but this range need not include the profit-maximizing case $\alpha = 0$. An instance of this situation is described next.

Damaged goods: Deneckere and McAfee (1996) and a rich subsequent literature show how a firm sometimes has an incentive to introduce a "damaged" good (which no consumer prefers to the existing product, and which costs more to supply) in order to facilitate price discrimination. Such a product would not be supplied when the objective is to maximize welfare ($\alpha = 1$), and so trivially the firm supplies too much of the damaged good relative to efficient supply. Deneckere and McAfee (1996) show nevertheless that the introduction of the damaged good can boost both consumer surplus and profit, in which case raising the cost of the damaged good sufficiently will induce the profit-maximizing firm to offer less consumer surplus. However, the discussion in this section shows that there will a range of α (which then does not include $\alpha = 0$) for which an increase in the cost of the damaged good will boost consumer surplus.[19]

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[19] To illustrate, suppose there are two groups of consumers of equal size: high types who value the main product at 7 and the damaged variant at 3, and low types who value the main product at 3 and the damaged variant at 2. When the unit cost of the main product is zero and that of the damaged variant is sufficiently small, the profit-maximizing firm will supply low types with the damaged good and the high types with the main product, leaving the high types with positive surplus, while if c_2 rises sufficiently the firm will only supply the high types with the main product, and no consumer obtains any surplus. However, in the Ramsey problem with larger α, one can show that raising the cost of the damaged good will induce the firm to switch from supplying the damaged good to low types and the main product to high types to offering the main product to *all* consumers, which will indeed boost consumer surplus.

5 Conclusion

Edgeworth's paradox highlights that comparative statics in multi-product settings can be very different from what happens in the single-product case. They can also contrast with those familiar examples of multiproduct pricing often studied (such as with linear demand and constant unit cost). We described various simple examples in which all prices fell as a cost level increased, and have shown that this possibility always exists for some cost function provided at least one pair of products are substitutes. We then explored the milder consumer surplus paradox, and related it to Ramsey pricing. A common theme was that the paradox in either form involves the most profitable output of product i decreasing with consumer surplus. This is akin to product i being an inferior good in consumer theory—i.e., one for which demand decreases as income rises. Although Edgeworth's pricing paradox is rarer than the surplus paradox, examples of either kind are not hard to find once one considers situations outside the most familiar specifications for multiproduct cost and demand systems.

The paradox is but one aspect of the much wider question of what cost passthrough possibilities exist in the multi-product case. We established that with profit maximization a matrix is a possible cost-passthrough matrix if and only if it is similar to a positive-definite matrix. This result defines the range of multiproduct cost passthrough possibilities that are consistent with the basic economics of supply and demand.

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