A Simple Hypothesis Test for Heteroscedasticity

Guido Venier

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Guido Venier
Financial Risk Controller
guido-venier@freenet.de
http://www.myspace.com/guidovenierresearch

Abstract: The scope of this paper is the presentation of a simple hypothesis test that enables to discern heteroscedastic data from homoscedastic i.i.d. gaussian white noise. The main feature will be a test statistic that’s easy applicable and serves well in committing such a test. The power of the statistic will be underlined by examples where it is applied to stock market data and time series from deterministic diffusion a chaotic time series process. It will turn out that in those cases the statistic rejects with a high degree of confidence the random walk hypothesis and is therefore highly reliable. Furthermore it will be discussed, that the test in most cases also may serve as a test for independence and heteroscedasticity in general. This will be exemplified by independent and equally distributed random numbers.

1. Introduction
The history of this paper is basically, that it was originally part of [1] and was used to show that DAX and Model time series obey similar degrees of heteroscedasticity. However for the sake of economy it was left out of [1] since heteroscedasticity could also be easily demonstrated graphically. Nevertheless the used test procedure still has got a charm with respect to its simplicity and should therefore be presented in the following. It also contributes another measure in addition to the approaches shown in [2] and can be used complementary.

2. Heteroscedasticity
Heteroscedasticity is a common feature observed in certain time series e.g. financial time series like interest rates or stock returns. It happens to occur when a lot of large changes follow abruptly a series of moderate changes.

Definition 4.2.1(Heteroscedasticity)
Define the m-sample variance estimator at sample point k of a sample of N realizations of a variable \(x_1, x_2, \ldots, x_N\) as:

\[
\hat{\sigma}^2_{m,k} = \frac{1}{m-1} \sum_{i=k}^{m+k} (x_i - \hat{\mu}_{m,k})^2
\]

where \(\hat{\mu}_{m,k} = \frac{1}{m} \sum_{i=k}^{m+k} x_i\)

with \(1 < k < N\) and \(n+k < N\) \(\forall k\)

is defined as the m-sample mean estimator at sample point k. And define the sample variance estimator by:

\[
\hat{\sigma}^2_N = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \hat{\mu}_N)^2
\]

where \(\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} x_i\)

is defined as the sample mean estimator. In this context heteroscedasticity means that for N samples there exists significant many values k so that the m-sample estimators of non overlapping sample buckets differ significantly from the sample variance.

To have a clear discern between homoscedasticity and heteroscedasticity of course one needs a hypothesis test. By generating a test statistic, it can be defined what it means that there exist significantly many sample estimators that are distinct from the sample variance.

3. The Test Statistic
The hypothesis of heteroscedasticity will be tested against the Null Hypothesis of a homoscedastic Gaussian random process. To test for heteroscedasticity in time bucket \(\tau\) the following test statistic is suggested:

\[
T(m, \alpha) = \frac{\min(N/m) \sum_{k=1}^{m} I_{a,k}(\alpha)}{N}
\]

Where \(I_k\) is an indicator function for the non overlapping buckets \(k = 1,2,3,\ldots,N/m\) indicating:
The value $\beta$ will then be the level of confidence of the test.

4. Examples

Table 1 and 2 show the results of the test statistic $T$ for the DAX and Model time series of [1]. Test parameters chosen where $\alpha = 0.01$. The null hypothesis of a homoscedastic i.i.d Gaussian random process is rejected for every time bucket significantly at a confidence level of one percent.

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Table 2

Table 1, 2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points for the model time series and DAX time series respectively. Upper-T, Lower-T percentage of samples for which the m-sample bucket variance differs significantly from its expected value indicated by I at either the upside or the downside respectively. Sum-T total percentage of m-sample bucket variance being significantly different from its expected value. P-Value is the 1-$\beta$ Quantile of the Binomial Distribution $B(n,p)$ Multiplied by 1/N with $\beta = 0.01$
From the proof of general limit theorem in [3], one could conjecture, that the intermediate distribution of squares of i.i.d. random variables is a $\chi^2$ distributed random variable until also these sums of i.i.d. random variables converge to a normal distribution.

Therefore the test should also serve as a test for independence and a homoscedasticity in general. Finally table 3 shows the test results independent and equally on the interval [0,1] distributed random numbers. Please note that the test statistic only rejects the homoscedasticity and independence hypothesis for the smallest time bucket.

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Table 1, 2 Results for model time series of the test statistic $T$ for various time buckets by observing $N=2000$ sample date points of an i.i.d random variable equally distributed on the interval [0,1].

5. Summary and Conclusions

The presented method is straightforward and shows, when applied, significant results indicating its power. It can be used in addition to the methods of [2] but should also work stand alone. Furthermore the test seems to have power in general to verify homoscedasticity and independence at the same time.

6 References

