Boating Against the Current: The Advance-Retreat Analysis for Socio-Economic Process

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Abstract. Boating against the current is a kind of human behavior which can generalize many real socio-economic processes. Starting from the view of point, this paper suggests the problem for advance-retreat course and builds the general analytic models of advance-retreat. By the models, we could see the endogenous resistances, instead of exogenous resistances, causes the periodic fluctuation in socio-economic process, and get the critical condition under which the periodic fluctuation is occured. A series of results for developing motivity and investing strategies are obtained. Finally, the conclusions and strategies are illuminated to be rational and maneuverable by two examples.

Key wards. socio-economic development, advancing motivity, environment resistance, advance-retreat course

1 Introduction

There are many of the important studies for the economic growth, such as the business cycle (Lucas, 1981), the real business cycle (Kydland and Prescott, 1982, and Long and Plosser, 1983), and new growth theory (Romer, 1986), etc. These theories has availably propeled forward the economic development of world. In recent years, the economists pay their more attention to make the efficiency of economic development be higher (Collier and Dollar, 2004), evaluate the government's economic aid to nation or district and their actual results (Guillaumont and Chauvet, 2001), insure the economic growth by establishing and choosing the policies and perfecting the system of economic education (Graziella, 2001 and 2004), etc.

Also, the economists study the economic growth in many of other ways, such as the economic dynamics from the points of view of overlapping generations, bionomics or homogeneous (Croix and Michel, 2002; Gandolfo. 2003; Barro and Sala-i-Martin, 1995), the rational choice on the economic policies (Colman and Hines, 1998; Brams, 2004), game theory for analyzing economic behaviour (Fudenberg and Tirole, 1991; Osborne, 2003; Brams, 2004), etc.

But, there is an importment problem to which should be paid great attention, that is the problem of relation between the sapiential human being and non-sapiential environment, for examples, the economic developing and the polluted environment, firm operating and market environment, policies making and environments in which the policies will be put into practice, knowledges learning and their applied environment, making use of energy sources and the lack of energy sources, traffic operation and congestion, and so on. In these problems, human being, as a sapiential subject, can make a choice in strategies independently and actualize them in order to forward economy, and the the objective environments, as a non-sapiential object, can not make the choice independently and have to admit the human's choices. But, the objective environments can bring their resistance to human's choice and behaviours. The quicker the socio-economic development is and the higher the socio-economic level is, the larger the environment resistances are. This kinds of environment resistances include the endogenous and exogenous.

The relation between the human and environment is called the advance or retreat problem in this paper. Here, the author try to study the advance or retreat problem, describe its basic concept, generalize its basic features, build its general model, and give the basic rules and an analytic model. Further more, author will, based on the analytic advance or retreat model, illustrate that the endogenous resistances, instead of exogenous resistances, results in the periodic fluctuation in socio-economic process, and give the corresponding investment strategies for avoiding the economic fluctuation.

Human's socio-economic behaviours are various, and also the environmental resistances are various, so the advance or retreat problem should not be neglected. The advance or retreat problem could be visually described as the boating against the current, then we start from the boating against the current as follows.

2 Boating Against The Current And Its Concept Model

In this chapter, the basic description and cases about boating against the current are given, and its basic features and basic model are researched.

2.1 The description of boating against the current

There is a current in a river, and someone is boating against the current. It is difficult for the boater to go against the current, and easier to go with the current. The environment of boating is complicated, and the resistances (or the pressure) from the current exist at any places and any time. The boater will be success and gain the bonus income if he advances despite difficulties, and will be failure and eliminated if he always retreat. The far the boater goes ahead, the more his income is, and the larger the resistance he will face is.

The boater could be a person or an organization. In the progress of boating against the current, the resistances are called to be endogenous if they are caused by the factors in internal environment, such as the boater is tired, the boating way is not very good, the boating equipment is not very advanced, the organizing for boating is in confusion, the income is lower, etc. And the resistances are called to be exogenous if they are caused by the factors from external environment , such as the current is rapid, the river has a winding course, submerged rock are densely covered, the weather is changing and unfathomable, the gradient of runway is undulate, etc. The boater could use his income to invest in order to increase his force to go forward, to reduce the endogenous and exogenous resistances, and to gain more income. And the investment can be in the way to make the distribution in his strength being more rational, his way of boating being better, the organizing for boating being more efficient, the equipments of boating being more advanced, or the runway being more wider and flat, etc. In the analysis for boater going, it is worth to pay attention the questions as follow:

- How far the boater will go from now?
- What is the time at which the income reaches its maximum, and what is the time at which the income will reduce?
- How is the relation between the forward force and resistance?
- What is the way in which forward process will change along with different resistances?
- What is largest resistance below which the boater can go forward against the current?
- What is smallest resistance upon which the boater have to invest if he want to go forward stably against the current?
- Whether the investment should be applied to increase the boater's forward force or to reduce the resistances if its quantity is limited?
- Whether the endogenous or the exogenous resistances should be reduced if the resistances must be reduced?

These questions would be significant if we take many activities in socio-economic development as the “boating against the current”.

2.2 The cases of boating against the current

The following cases will tell us that the boating against the current, as a model, will generalize many of activities in socio-economic field.

2.2.1 Learning the knowledge

In modern society, everybody is always facing many kinds of pressures because there are many problems in the environment he or she lived in. In order to live and to succeed, a person, an organization or a nation needs to study many of necessary knowledges constantly. If abandoning effort and stopping studying new knowledges, they would not overcome the new difficulties, would fall behind gradually, and would be eliminated. Studying new knowledge continuously and making use of them, one can hold the advantages to exist and keep on progressing.

In the process of learning new knowledge, one will face many kinds of difficulties, such as the perplexities in understanding the knowledges, how is the best way to learn knowledge and to apply them, the time and fee for learning knowledge, etc. These difficulties could be regard as the endogenous resistances in learning knowledge. There are some of other difficulties in learning knowledge, such as, the cultural difference in background of science and technology, the lagged education system, lack of input in education, the imperfect education policies, etc. These difficulties could be regard as the exogenous resistances.

2.2.2 Constituting policies and laws

In the process of society progressing and economy developing, many kinds of problems always occur, humankind need to constitute and use all kinds of policies and laws continuously, in order to keep society and economy in order. Many of socio-economic problems would come forth, and society or economy would come into confusion if new policies and laws stop being constituted at some time. So it is necessary for humankind to constitute and use all kinds of policies and laws continuously.

In the policies and laws constituting and being applied, government will face many kinds of difficulties, such as the choice in concept and diction for a concrete item, the validity and scope of application of a concrete item, the way how to let civilians know and accept the items, and the consistence between different policies and laws, etc. These difficulties could be regard as the endogenous resistances in constituting policies and laws. There are some of other difficulties in constituting policies and laws, such as, the difference between the new and inhered systems of policies and laws, new systems of policies and laws is corresponding with socio-economic development or not, the change in socio-economic environment makes the policies and laws be lagged, etc. These difficulties could be regard as the exogenous resistances.

2.2.3 Operating a firm

In the process of operating a firm, many kinds of problems need to be solved. For examples, making the decision for the product plan, organizing the productions, knowing the market and the possible changes in socio-economic environment, establishing and perfecting the bylaws of firm, appointing the managers and employees, etc. Once stop doing the things like those, the income of firm would decrease, and some of other costs must be paied. So it is necessary for a director to operate the firm continuously if he or she wants to develop and expand the firm.

In the operating a firm, many kinds of problems always occur, such as, what kind of product the firm should produce, how to price the products, what scale the firm should be in its production, what kinds of equipments the firm should purchase, which bylaws should be constituted for managing firm, etc. These difficulties could be regard as the endogenous resistances in operating a firm. In addition, there are some exogenous resistances in operating a firm, such as, the difference in developing trends between firm and macroeconomy, the imperfect market system, lack of outlay for developing firm, the products being behind the needs of market, etc.

2.2.4 Developing transportation

A region (nation or city) can not get away from the transportation facilities, and will face the amount of vehicles increase continuously. If a region stops increasing its traffic capacity at some time, the traffic jam would occur, all kinds of its developments in society and economy would be restricted seriously, and this region would suffer the loss in economy. So, it is only the continuous developments in transportation facilities that can resolve continuously the new problems occurred in transportation and keep on developing region's economy.

In the developing transportation, many kinds of problems always occur, for example, the roads are narrow, vehicles increase rapidly and the total amount of them are larger, and the width and length of highway can not meet the need of traffic, the transportation management are imperfect, etc. These could be regard as the endogenous resistances in transportation developing. In addition, the contradictions between transportation developing and land using, the law systems of transportation need to be renewed, the differences between transportation developing and macroeconomic building, the waste gas brought from the vehicles will result in the environmental pollution, etc. These could be taken as the exogenous resistances in developing transportation.

2.2.5 Developing energy sources

The human life and all kinds of productions can not be got away from the energies. Large amount of energy is consumed continuously along with the production level becomes higher and production scale extends continuously so that the demand and lack of energy becomes obvious more and more. We need to produce various energies continuously to. Once the current energies stop being produced, not only all kinds of production are hard to keep on, but also the existence of humankind would be threatened seriously. So it is necessary to produce the current energies and to developthe new energies econtinuously.

In energy developing and energy system constructing, human will face many problems, such as, the life quantity raising may cause the household-use energy to increase, the lower efficiency and waste in using energy, energies researching and developing are lag behind the needs for them, the prices of main energy (such as crude oil) are rising, and so on. These can be seen as the endogenous resistances in developing energy. The developments in transportation, real estate, communication, catering outlet cause the needs in energy make many of unrenewable energies reduce, the energy crisis may break out at any time, developing the new energy is more and more urgent, the strategies for energy development is difficult to decide, etc. All of these could be taken as the exogenous resistances in developing energy.

2.2.6 Controlling pollutions

The industrial productions bring much of pollution inevitably, and these pollutions have been threatening the environments. So human has been paying attentions, in many of past years, to control pollutions and protect environment. Once these efforts stop, the environment will be quickly worsened. For this reason, mankind must overcome many kinds of difficulties, control diligently and reduce effectively the pollutions brought from industry productions and life garbage, to present a good environment for everybody.

There are many things will happen in pollutions controlling, such as, large quantity of abandoned packs of products, the daily life and industrial production bring on the physical and chemical pollutions, the waste gas will be caused by various vehicles, communication will cause the electromagnetism pollution. The difficulties in controlling these pollutions can be taken as the the endogenous resistances. And the differences in science and technique between the different countries or regions result in the imbalance in pollutions, the ability and attitude in controlling pollution are different among the different countries and regions, both the policies and the understanding are differed in controlling pollutions in various countries or regions, etc. All of these will become the exogenous resistances in controlling pollutions.

All kinds of the processes mentioned above can be sum up to boating against the current, i.e., human must face various resistances or pressures, and overcome them by means of working hard or investing to reduce them, then get the liveing and existing environments which is higher quality. In contrary, human will be punished by environment if doing nothing. Of course, there are many other examples which we do not list here. If we regard the boating against the current as the game between humankind and natural environments, thus, it has a wide background in society and economy.

2.3 The rules, assumptions and characters of boating against the current

According to the cases above, author can induce the basic rules, assumptions and characters of boating against the current as follow.

2.3.1 The basic rules

- The attendee of game will gain the imcome if he overcome the difficulties and run against the current.
- The attendee must pay the cost or fine if he do not run against the current.
- The attendee will be bowled out if having no the dint to pay a fine.
- The attendee have some of charge at the beginning of his game.

2.3.2 The general assumptions

- The attendee will do his best to gain the bonus income for living and success.
- The distance attendee to go foward can be measured by his income.
- The far the distance of attendee going ahead is, the larger the resistances or pressures are.
- Only keeping on going ahead, attendee would make his income increase continuously.
- The kind and intensity of resistance can not be known beforehand.
- The food and time needed in boating process, etc. are accounted into cost, and do not calculate in other way.
- Attendees would have no any fees when he bowls out, including his original charge. The original charge is limited.

2.3.3 The main characters

From the rules and assumptions above, we could see there are three main characters in the game of boating against the current as follow:

- The deterministic structure. There is a motivity for attendee to live and to develop, and this motivity, though it is large at some time or small at another time, impels attendee to gain more income all the time. So, any of attendees prefers to decide boating against the current, or has to decide drifting with the current when he is too tired. Therefore, the basic trend and direction of the game is going ahead, especially to socio-economic development, and the total income will increase.
- The stochastic environment. In the progressing, some of resistances may occur stochasticly or unexpectedly, those will influence the motivity to go ahead, and progressing motivity will change along with environment, so the attendees have the stochastic choices to go ahead or back in some degree.
- The uncertain outcome. Attendee needs to overcome various resistances and many of the difficulties in order to acquire income. Not all of attendees will be successful and most of them will bowl out, the attendees would become less and less along with the game progressing, so it is hard to determine who will be the victor.

In addition, attendee would lose his income if slackening a little bit and some opportunities would be lost if he or she can not hold them on time.

2.4 The description model of boating against the current

The basic model of boating against the current will be established based on defining the attendees of game and describing the strategies of subjective attendee.

2.4.1 The attendees

There is one attendee and a latent attendee in the game of boating against the current, the subjective attendee (subject for short) and the objective attendee (object for short).

Subject: the initiative doer of the game (boater). Subject could be a person, an organization or a government. The basic characters of subject are

- Rationality. Subject will not lose the opportunities to gain the income though he has to overcome many resistances and difficulties, and will regard the existence and development in the rational way.
- Activity. Subject can analyze and judge actively the state and situation himself, make a choice and decision on various strategies, and put his decision into effect.
- Wisdom. Subject has the enough courage and is able to do his best in making a correct decision for rising the progressing motivity and reducing the various resistances.
- Stratergy. The results of subject strategies are finite, include advance, stop and retreat, but the ways and method to carry out the stratergy are infinite.

Object: The passive doer of the game (boating environment). Object is composed of all the factors which will embarrass the subject as progressing. The basic characters of object are

- Existence. Object exists always, this is impersonal and inevitable.
- Memory. Object can remember the distance of subject going ahead, this is represented in that the resistance become larger and larger along with the subject going ahead far and far, include both of factors and intensity of resistance are increase.
- Passivity. Object can not make a decision, but it can react to any choice of subject by the resistance or pressure.
- Behavior. The object's behavior is only the natural and non-intellectual reaction to subject, and its reactions present stronger or weaker according to the different behaviours of subject and at the different timeas.

2.4.2 The description for subject strategy

Denote:

u : The subject motivity to go ahead, $u>0$

$\overline{u}$: The object resistance to retard subject progressing, $\overline{u}\geq 0$.

Then, the boating against the current has the three games as follow:

- When $u-\overline{u}>0$, the advancing force exists. Subject will determine to go ahead continuously and his income is increasing.
- When $u-\overline{u}=0$, the motivity not exists. Subject can adopt measures actively to raise his motivity and go ahead continuously, or choose stop or retreat according to current situation.
- When $u-\overline{u}<0$, the retreating force exists. Subject can choose retreaing if the resistance is larger. Subject can adopt measures actively to raise his motivity and go ahead if the resistance is smaller.

3 The Basic Model for Advance-Retreat Analysis

The development processes of anything in socio-economic field include the two basic activities behaviors, growth and recession. From this view of point and the above discussion, if we take the subject as the society, economy, firm or person in economy, the object as the socio-economic environment and the resistances as the force from the factors in socio-economic environment which retard subject to progress, thus, the “boating against the current” can describe widely many things in socio-economic field. At this time, we call “boating against the current” the advance or retreat problem, and call the related researches on advance or retreat problems the advance or retreat analysis or process analysis.

3.1 The concept model of advance-retreat analysis

The basic concepts, assumptions, characters and models about advance-retreat problem are given as follow.

3.1.1 The basic components of advance or retreat problem

The advance or retreat problem includes two pair of basic components, the subject and the object, and the motivity and the resistance.

Definition 1 (subject and object) Subject is the things that can choose one or more strategies and carry out them actively. The basic behavior of subject is going in the direction of progressing and developing. Object is the things existed in the environment in which subject is going forward, and their basic behavior is passively retarding subject to progress or let the progressing subject retreat in the reverse direction.

The subjects include all the doers who are able to do anything actively. As saying above, subjects include the society, economy, firm or person, etc. The objects include all the the environment factors which may influence subjects to go forward or make the trouble to them, such as, the problems in policy, management, science and thchnology, production, market, energy, transportation, service and other problems in socio-economic field; the storm, tsunami, drought and waterlogging, temperature and other problems in natural environment; disease, pollution, lack of energy, species reduced, natural disaster and other problems in existent environment.

Definition 2 (subject power) Subject power is the motivity to impel subject to go forward. The subject power is also called the subject energy, and motivity for short.

The motivity includes the physical force, vigor, wisdom, ability, time, investment, etc. The subject incomes include the asset, cash, resource, service, etc. The subject motivity is brought from the needs to exist and to develop, the expectation to gain the income and the corresponding investment.

Definition 3 (object resistance) Object resistance is the force which is come from object and it try to retard subject to progress. Object resistance is also called the environment resistance, and resistance for short. The resistances include the endogenous one and exogenous one. Endogenous resistance is the one come from the environment factors inside subject system, and exogenous resistance is the one come from the environment factors outside the subject system.

The resistance is generally brought from the problems in environment, like the unsuitable policies, confused management, backward technology, short energy, obstructed transportation, prevalent disease, serious pollution, natural disasters, etc.

As we see, advance or retreat problem is the universal induction on boating against the current, and the latter is a concrete example of the former. So, the rules, assumptions and characters about boating against the current are also suitable for advance or retreat analysis.

3.1.2 The basic assumotions and characters about advance or retreat analysis

The additional assumotions and characters about advance or retreat analysis are given as follow besides the rules, assumptions and characters of boating against the current.

- The assumption on goal. The goal of subject is the maximum in its integrated income. The integrated income includes economic income, social benefit and environment efficiency.
- The assumptions on condition.

1) The initial capital (or initial income) of subject is larger than zero. The subject needs some capital for

startup payout.

2) The subject income is unable to be overdraft. Subject is able to get forward force when his stock income (the capitals in holding) is larger than zero. Subject will be bowled out or perished if his stock income is equal to or less than zero. If acquiring the loan, it is regard as subject stock income is larger than zero.

3) There is no branch in progressing route. In general, there may be branch in progressing route. In the following discussion, it is assumed no branch in progressing route.

- The behavior characters. In the progressing, subject and object have the different characters in their status and behavior.

1) Subject's go-aheadism. Subject can adopt any strategies and do in any ways independently to gain the income by going ahead.

2) Subject's mobility. Subject can apply his stock income to invest in order to raise the progressing force or reduce the object resistance, and choose retreat by losing his income to avoid the larger resistance.

3) Object's passivity. Object does not apply any strategy actively, and react passively to any choice of subject by the resistance (or pressure) in order to decrease the subject income or reduce the progressing speed of subject.

4) Object's memory. The object memory will appears as its resistance to subject becomes larger and larger when the subject goes forward far and far and gains the income more and more, i.e. it is more and more difficult for subject to go forward. Generally, this memory of object makes it get a lag in releasing resistance.

5) Controllability about resistance. The endogenous resistances could be controlled in a larger degree because they are produced in the inside environment of subject system, and the exogenous resistances could be controlled in a smaller degree or even not be controlled because they are brought from outside environment around subject system. The main way to control resistance is the investment.

3.1.3 The frame model

In order to establish the frame model of advance or retreat process, the following notations and their explanations are presented.

u	:	The value of power, power for short also, u>0.
$\bar{u}=\bar{u}(u,g)$	:	The value of resistance, resistance for short also. g is the value of exogenous resistance. $\bar{u}\geq 0$, g>0. Here, the memory of object is presented in the resistance related to the subject motivity.
$U=U(u,\bar{u})$	:	The rate of variationin power. $u>\bar{u}$, $U(u,\bar{u})>0$; $u=\bar{u}$, $U(u,\bar{u})=0$; $u<\bar{u}$, $U(u,\bar{u})<0$
$L(U)=L(u,\bar{u})$	:	The stock income of subject at the time t. $L(U)=L(u,\bar{u})$The initial capital (or initial income) of subject is $L_0=L(u(0),\bar{u}(0))>0$. In general, the income increase when $u>\bar{u}$, and the income decrease when $u>\bar{u}$.

Definition 4 (advance-retreat course) if existing time $\bar{t}>0$ (or $\bar{t}=+\infty$), make $L[U(\bar{t})]=0$ and $L[U(t)]>0$ ($t\in[0,\bar{t}$)), then $\{L[U(t)],\ t\in[0,\bar{t}]\}$ is called a simple advance-retreat course. If time $\bar{t}$ (>0) exists, and make $U(\bar{t})<0$ and $L[U(t)]>L[U(\bar{t})]$ when $t>\bar{t}$, then call $\{L[U(t)],\ t\in[0,\bar{t}]\}$ a complex advance-retreat course. Both the simple and complex advance-retreat course are called the advance-retreat course, and course for short. If having $t^+\in[0,\bar{t})$ and $L[U(t^+)]=\max\limits_{0<t<\bar{t}}\{L[U(t)]\}$, then $L[U(t^+)]$ is called a solution of $L[U(t)]$.

In the following discussion, the course $L[U(t)]$ is as $L(U)$ for short.

Definition 5 (the course series) Let n be a positive integer, and $L_i(U_i)$ be a advance-retreat course ($0\leq\bar{t}_{i-1}<t_i\leq\bar{t}_i$) for integer $i\in[1, n]$, then $\{L_i(U_i),\ i\in[1, n]\}$is called a finite course series, $L_{i\leq n}(U_i)$ for short. If the positive integer n in finite course series $L_{i\leq n}(U_i)$ could be large arbitrarily, this course series is called a infinite course series, and $L_{i<\infty}(U_i)$ for short. Both infinite course series and finite series are called the course series. If time t_i^+ (>0) exists, $t_i^+\in[\bar{t}_{i-1}, \bar{t}_i)$, and make $L_i[U_i(t_i^+)]=\max\limits_{\bar{t}_{i-1}<t<\bar{t}_i}\{L_i[U_i(t_i)]\}$, then

$\boldsymbol{U}=(U_1^+,\cdots,U_n^+)^{\mathrm{T}}$ is called the solution of course series $L_{i\le n}(U_i)$.

Theorem 1 If subject course $L[U(t)]$ ($t\in[0,\ \bar{t}\ ]$) is continuous about t, then its solution is existent.

Proof. Because $L[U(0)]>0$ (there is the initial asset for subject to go forward)and $L[U(\bar{t})]=0$, we have t^+ and $L[U(t^+)]=\max\limits_{0<t<\bar{t}}\{L[U(t)]\}$ according to the property of continue function, then the result as follows.

Theorem 2 If course series $L_{i\le n}(U_i(t))$ are continuous about t, $L_0=\lim\limits_{n\to\infty}L_n(U_n)$ is bounded above, thus the solution of infinite course series $L_{i<\infty}(U_i)$ is existing.

$L=\lim\limits_{n\to\infty}L_n(U_n)$ is finite means the stock income is bounded above finally, so all the advance-retreat course $L_i(U_i)$ in this series are continue and bounded above, then the solution of $L_{i<\infty}(U_i)$ is existent.

It is worth to explain that, the advance-retreat problem is not the zero sum game if regarding it as the game between subject and object, and there is no steady equilibrium point, because

■ If $u-\bar{u}\neq 0$, subject should be going forward or retreating. There might be equilibrium between subject and object only when $u-\bar{u}=0$, and subject should stay at the current place. If the subject stays at the current place, he must pay the cost for related consumptions. So the subject will not do like that in a long term.

■ Because of the activity and mobility, i.e., adopting any strategies and doing in any ways independently, subject will try to go forward again instead of staying the current place. Otherwise, he will be bowled out or perished finally.

■ Due to the lagging effect of object, the subject will bear larger resistance after the possible equilibrium point has occurred. This means the lagging effect will destroy the possible equilibrium and urge subject to adopt more active measures to go forward.

To sum up, the advance-retreat is not a game problem in the traditional meaning.

3.2 The analytic models for advance-retreat problem

The analytic models will be given here for single subject and multi-subject advance-retreat problem. Of course, there shoule be the analytic models in other kinds of form.

3.2.1 *AR*(1,1/1) -the analytic model for single subject advance-retreat.

If both subject motivity $u(t)$ and object resistance $\bar{u}=\bar{u}(u,g)$ in section 3.1.3 are the differentiable, then we could express the $F(u,\bar{u})$ and $\bar{u}=\bar{u}(u,g)$ as the analytic model in expression (1).

$$\begin{cases}\dfrac{du}{dt}=b(u-\bar{u})\\[2ex]\dfrac{d\bar{u}}{dt}=\delta u+\psi g\end{cases}\tag{1}$$

where, b is the coefficient of motivity , δ is the coefficient of endogenous resistance, ψ is the coefficient of exogenous resistance, g is the factor of endogenous resistance, constant $b>0$, $\delta>0$, $\psi>0$. There is one subject and one object, and the object includes one factor of exogenous resistance and one factor of exogenous resistance in model (1), so it is called the single subject model, and denote as $AR_b(1,1/1)$. If $b=1$, the model $AR_1(1,1/1)$ is denoted as $AR(1,1/1)$. We will emphasize to discuss the model $AR(1,1/1)$.

Considering the subject income $L[\,u(t),\bar{u}(t)\,]$ and combining model (1), we could express the frame model in section 3.1.3 as

$$\begin{cases}\dfrac{du}{dt}=b(u-\bar{u})\\[2ex]\dfrac{d\bar{u}}{dt}=\delta u+\psi g\\[2ex]\dfrac{dL}{dt}=auL\end{cases}\tag{2}$$

where, constant $a>0$ is the growth coefficient of income.

3.2.1 *AR*(*n*, *n*/*m*) - the analytic model for multi-subjects advance-retreat.

Similarly to single subject model (1), we could express the model for multi-subjects advance-retreat as

$$\begin{cases} \dfrac{dU}{dt} = B(U - \overline{U}) \\ \dfrac{d\overline{U}}{dt} = KU + \Psi G \end{cases} \tag{3}$$

Where, $\boldsymbol{U}=(u_1, \cdots, u_n)^{\mathrm{T}}$ is the vector of subject motivity, $\overline{U} = (\overline{u}_1, \cdots, \overline{u}_n)^{\mathrm{T}}$ is the vector of object resistance, $\boldsymbol{G}=(g_1, \cdots, g_m)^{\mathrm{T}}$ is the vector of exogenous resistance ; $\boldsymbol{B}=(b_{ij})_{n\times n}$ is the coefficient matrix of motivity, $b_{ij}\geq 0$, $\boldsymbol{K}=(\delta_{ij})_{n\times n}$ is the coefficient matrix of endogenous resistance, $\delta_{ij}\geq 0$, $\boldsymbol{\Psi}=(\psi_{ij})_{n\times m}$ is the coefficient matrix of exogenous resistance, $\psi_{ij}\geq 0$. There are n subjects, n objects and m factors of exogenous resistance in model (3). If $b_{ii}=1$, the model (3) is expressed as $AR(n, n/m)$.

Also, we could have similarly the models of one subject and multi-objects, multi-subjects and one objects, etc.

3.3 The general solution of model *AR*(1,1/1)

The model *AR*(1,1/1) can be converted to

$$\frac{d^2u}{dt^2} = \frac{du}{dt} - (\delta u + \psi g)$$

and denoting $\varphi(t)=-\psi g$, we have the nonhomogeneous differential equation

$$u'' - u' + \delta u = \varphi(t) \tag{4}$$

The homogeneous one of differential equation (4) is

$$u'' - u' + \delta u = 0 \tag{5}$$

The characteristic equation of model (5) is $\lambda^2 - \lambda + \delta = 0$, and its solution is $\lambda_{1,2} = \dfrac{1 \pm \sqrt{1-4\delta}}{2}$.

Denoting $w=\sqrt{|1-4\delta|}$ and $\Delta=1-4\delta$. According to appendix A, we get the general solution of model *AR*(1,1/1) as follows

$$u(t)=\begin{cases} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dfrac{1}{w}\left(e^{\lambda_1 t}\displaystyle\int_0^t \varphi(\tau)e^{-\lambda_1\tau}d\tau - e^{\lambda_2 t}\int_0^t \varphi(\tau)e^{-\lambda_2\tau}d\tau\right) & if \quad \Delta>0 \\ \qquad c_1 = \dfrac{u'(0)-\lambda_2 u(0)}{w}, \quad c_2 = \dfrac{\lambda_1 u(0) - u'(0)}{w} & \\ e^{\frac{t}{2}}\left(c_1 + c_2 t + t\displaystyle\int_0^t \varphi(\tau)e^{-\frac{\tau}{2}}d\tau - \int_0^t \varphi(\tau)\tau e^{-\frac{\tau}{2}}d\tau\right) & if \quad \Delta=0 \\ \qquad c_1 = u(0), c_2 = u'(0) - \dfrac{1}{2}u(0) & \\ e^{\frac{t}{2}}\left(c_1\cos\left(\dfrac{wt}{2}\right) + c_2\sin\left(\dfrac{wt}{2}\right)\right) + & if \quad \Delta<0 \\ +\dfrac{2}{w}e^{\frac{t}{2}}\left(\sin\left(\dfrac{wt}{2}\right)\displaystyle\int_0^t \varphi(t)e^{\frac{-\tau}{2}}\cos\left(\frac{w}{2}\tau\right)d\tau - \cos\left(\frac{wt}{2}\right)\int_0^t \varphi(\tau)e^{\frac{-\tau}{2}}\sin\left(\frac{w}{2}\tau\right)d\tau\right) & \\ \qquad c_1 = u(0), c_2 = \dfrac{1}{w}[2u'(0) - u(0)] & \end{cases} \tag{6}$$

4 The Related Analysis on *AR*(1,1/1)

If letting $g(t) = e^{\psi t}$ in the general solution (6) of model *AR*(1,1/1), i.e. $\varphi(t) = -\psi e^{\psi t}$, and denoting $\Delta=1-4\delta$ and $w=\sqrt{|1-4\delta|}$, we obtain the analysis results about the general solution (6) of model *R*(1,1/1).

4.1 The analysis for subject motivity under the condition of Δ>0

If Δ=1-4δ>0, i.e., δ<0.25, then $w=\sqrt{1-4\delta}$ >0. And let $u(0)=1, u'(0)=1$. According to the solution (6) of model AR(1,1/1), we have $c_1=\frac{\lambda_1}{w}$, $c_2=-\frac{\lambda_2}{w}$, thus, the model of subject motivity to progress can be expressed as

$$u(t)=\begin{cases}\left(\frac{\lambda_1}{w}+\frac{\psi}{w(\psi-\lambda_1)}\right)e^{\lambda_1 t}-\left(\frac{\lambda_2}{w}+\frac{\psi}{w(\psi-\lambda_2)}\right)e^{\lambda_2 t}-\frac{\psi e^{\psi t}}{(\psi-\lambda_1)(\psi-\lambda_2)} & \psi\neq\lambda_1,\psi\neq\lambda_2\\ \frac{\psi}{w}\left(1-t+\frac{1}{w}\right)e^{\psi t}-\frac{1}{w}\left(\lambda_2+\frac{\psi}{w}\right)e^{(\psi-w)t} & \psi=\lambda_1\\ \frac{1}{w}\left(\lambda_1-\frac{\psi}{w}\right)e^{(\psi+w)t}+\frac{\psi}{w}\left(-1+t+\frac{1}{w}\right)e^{\psi t} & \psi=\lambda_2\end{cases}\tag{7}$$

4.1.1 The basic results for subject motivity under the condition Δ>0

As we see, from model AR(1,1/1) and its solution (6), the subject income will increase continuously if the motivity rises continuously. But in reality, the motivity can not rise always, and it will decrease under the special conditions, this is implied in expression (7). Now, we discuss the conditions and give the related analysis.

Theorem 3 let $\psi\neq\lambda_1$ and $\psi\neq\lambda_2$, i.e., $\psi^2-\psi+\delta\neq0$,

1) If $\psi>\lambda_1>\lambda_2$, then having the time T, and the motivity $u(t)<0$ when $t>T$.

2) If $\lambda_1>\psi>\lambda_2$ or $\lambda_1>\lambda_2>\psi$, then having the time T, the motivity $u(t)<0$ when $t>T$ if and only if the following inequation (8) is tenable

$$\psi+\sqrt{\psi^2+4\psi}-\sqrt{1-4\delta}>1\tag{8}$$

Proof.

1) If $\psi>\lambda_1>\lambda_2$, the coefficient of e^{wt} in model (7) is $-\frac{\psi e^{\psi t}}{(\psi-\lambda_1)(\psi-\lambda_2)}$<0, so $u(t)<0$ will be tenable if the time t is larger sufficiently.

2) According to expression (7), we have $0<\lambda_1<\frac{\psi+\sqrt{\psi^2+4\psi}}{2}$ when $\lambda_1>\psi>\lambda_2$ or $\lambda_1>\lambda_2>\psi$ and the inequation (8) is tenable, i.e., $\lambda_1^2-\psi\lambda_1-\psi<0$, then we obtain $\lambda_1+\frac{\psi}{(\psi-\lambda_1)}$<0, namely, the coefficient of item $e^{\lambda_1 t}$ in expression (7) is smaller than zero. Therefore, having the time T which is larger sufficiently, and the motivity $u(t)<0$ when $t>T$. Whereas, if having $T>0$, and $u(t)<0$ when $t>T$, then we will obtain the inequation (8).

Theorem 4 If $\psi=\lambda_1>\lambda_2$, we have the certain time T, and the motivity $u(t)<0$ when $t>T$.

Proof. If $\psi=\lambda_1>\lambda_2$, we have $\frac{\lambda_1}{w}-\frac{\psi t}{w}+\frac{\psi}{w^2}$ <0 when $t>1+\frac{1}{w}=1+\frac{1}{\sqrt{1-4\delta}}$, namely, the coefficient of item $e^{\psi\cdot t}$ in expression (7) is smaller than zero.

Theorem 5 When $\psi=\lambda_2<\lambda_1$, then the time T is existent, and $u(t)<0$ if $\delta>\frac{\sqrt{2}-1}{2}$ and $t>T$.

Proof. From $\delta>\frac{\sqrt{2}-1}{2}$, we obtain $1+w<\sqrt{2}$, i.e., $w(1+w)<1-w$, also $w\lambda_1<\lambda_2=\psi$. Therefore, the coefficient of item $e^{(\psi+w)t}$ in expression (7) is smaller than zero.

According to the theorems above, we have the basic results as following

■ If Δ=1-4δ>0, then δ<0.25, this means the subject motivity will change in the way of a single-peak if the endogenous resistance is smaller than the 25% of subject motivity. At this time, subject motivity does not change in the cycle way, namely, the subject income does not change in the cycle way.

- ■ Both the 1) in theorem 3 and theorem 4 incate that the subject motivity will fall when the exogenous resistance increases to a special degree
- ■ The 2) in theorem 3 indicate that the subject motivity mjght be rising always if the coefficient of exogenous resistance ψ is smaller than λ_1 and the inequation $\psi+\sqrt{\psi^2+4\psi}$ - $\sqrt{1-4\delta}$ <1 is tenable.
- ■ Theorem 5 indicate that that the subject motivity mjght be rising always if the coefficient of exogenous resistance $\psi=\frac{1-\sqrt{1-4\delta}}{2}$ and the coefficient of exogenous resistance $\delta \le \frac{\sqrt{2}-1}{2}$.

4.1.2 The methods to compute the zero point of subject motivity under the condition Δ>0

As we know, the motivity $u(t)$ will be smaller than zero if the resistances are larger to a special degree. When motivity $u(t)$ is equal to zero, subject needs to make a important decision, that is, whether he retreats now or goes forward by means of investing. Therefore, it is necessary for subject to compute the zero point of his motivity.

1) If $\psi\neq\lambda_1$ and $\psi\neq\lambda_2$, let $u(T)=0$ in the expression (7), we have the following iterative algorithm:

$$T_{i+1}=\frac{1}{\lambda_1-\psi}\ln\left[\frac{w\psi+[\lambda_2(\psi-\lambda_2)+\psi](\psi-\lambda_1)e^{(\lambda_2-\psi)T_i}}{[\lambda_1(\psi-\lambda_1)+\psi](\psi-\lambda_2)}\right]$$

Letting $T_0=0$ and applying repeatedly the algorithm above, we will get the time T.

2) If $\psi=\lambda_1>\lambda_2$, let $u(T)=0$ in the expression (7), we have the following iterative algorithm:

$$T_{i+1}=\left(1+\frac{1}{w}\right)-\left(\frac{\lambda_2}{\psi}+\frac{1}{w}\right)e^{-wT_i}$$

Letting $T_0=0$ and applying repeatedly the algorithm above, we will get the time T.

3) When $\lambda_1>\psi=\lambda_2$, the zero point of motivity can not be calculated by a simple iterative algorithm. At this time, let

$$X(t)=u(t)=\frac{1}{w}\left(\lambda_1-\frac{\psi}{w}\right)e^{(\psi+w)t}+\frac{\psi}{w}\left(-1+t+\frac{1}{w}\right)e^{\psi t}$$

and by means of the algorithm in appendix B, we will get the T to make $u(T)=0$.

4.1.3 The methods to compute the maximum value of motivity under the condition Δ>0

Because the subject motivity $u(t)$ is a single peak function when Δ>0, so the maximum value of motivity $\max\limits_{0<t<+\infty}\{u(t)\}=u(\hat{T})$ if $\hat{T}$ make $u'(t)\big|_{t=\hat{T}}=0$.

1) If $\psi\neq\lambda_1$, $\psi\neq\lambda_2$, $(2\psi-1)\sqrt{1-4\delta}+4(\psi+\delta)>1$ is tenable, and letting $u'(\hat{T})=0$ on the expression (7), we obtain the following iterative algorithm:

$$\hat{T}_{i+1}=\frac{1}{\lambda_1-\psi}\ln\left[\frac{w\psi^2+\lambda_2[\lambda_2(\psi-\lambda_2)+\psi](\psi-\lambda_1)e^{(\lambda_2-\psi)\hat{T}_i}}{\lambda_1[\lambda_1(\psi-\lambda_1)+\psi](\psi-\lambda_2)}\right]$$

Letting $\hat{T}_0=0$ and applying repeatedly the algorithm above, we will get the time $\hat{T}$.

2) If $\psi=\lambda_1>\lambda_2$, and letting $u'(\hat{T})=0$ on the expression (7), we obtain the following iterative algorithm:

$$\hat{T}_{i+1}=1-\frac{1}{\psi}+\frac{1}{w}-\frac{(\psi-w)}{\psi^2}\left(\lambda_2+\frac{\psi}{w}\right)e^{-w\hat{T}_i}$$

Letting $\hat{T}_0=0$ and applying repeatedly the algorithm above, we will get the time $\hat{T}$.

3) When $\lambda_1>\psi=\lambda_2$, the maximum value of motivity can not be calculated by a simple iterative algorithm. At this time, let

$$X(t)=u'(t)=\frac{1}{w}\left(\lambda_1-\frac{\psi}{w}\right)e^{(\psi+w)t}+\frac{\psi}{w}\left(-1+t+\frac{1}{w}\right)e^{\psi t}$$

and by means of the algorithm in appendix B, we will obtain the $\hat{T}$ to make $u'(\hat{T})=0$.

4.1.4 A computing example under Δ>0

Here, we give a computing example under Δ>0 to analyze the properties of subject motivity movement. Let the initial motivity $u(0)=1$(unit of motivity), and the initial rate (growth rate) $u'(0)=1$. if the coefficient of endogenous resistance δ=0.18 and the coefficient of exogenous resistance ψ=0.35, by means of $\lambda_{1,2}=\frac{1\pm\sqrt{1-4\delta}}{2}$, we have got the λ_1=0.7645751311, λ_2=0.2354248689, $\psi+\sqrt{\psi^2+4\psi}-\sqrt{1-4\delta}$ =1.0547461>1. It is obvious that δ=0.18<0.25 and $\lambda_1>\psi>\lambda_2$. And by the algorithms in section 4.1.2 and 4.1.3, the times of zero motivity and maximum motivity are computed separately as T=8.453179155 and $\hat{T}$ = 6.769242661. The curve of subject motivity is drawn in figure 1.

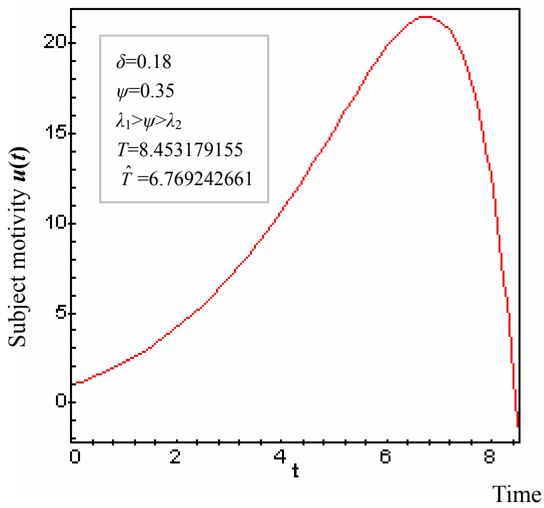

Figure 1 motivity changing process under Δ=1-4δ>0

At the beginning of motivity movement, the motivity rises slowly and subject goes forward hardly. In middle stage, the motivity rises quickly, this means subject progresses very smoothly. At the end of this process, the motivity decreases rapidly.

We see, from figure 1, there are three basic stages in the process of subject motivity moving. The first stage, motivity rises slowly and it is hard for subject to go forward. The second stage, motivity presents to rise quickly and subject comes into progressing in a high speed. The third stage, the subject motivity is almost exhausted, the object resistances are large, and the retreating force is so large that the motivity declines rapidly.

This kind of process is represented in practice as the development process of many industries in socio-economic field. At the bigainning, there is a period of time for suiting and groping, and many of difficulties will be faced in this period. If unremitting, a regular and rapidly developing stage will come. All the works will be systematically done in this stage. Finally, many of new problems, which can not be solved by the current methods, will appear, and the enormous pressure causes the progressing motivity to exhaust. In this stage, if not adopting the positive measures, making the innovations and looking for the new developing way, the end may be occurring now.

The above process is suitable to any industries in socio-economic field almost.

4.2 The analysis for subject motivity under the condition of Δ=0

If Δ=1-4δ=0, i.e., δ=0.25, then w=0. Let $u(0)=1$, $u'(0)=1$. According to the solution (6) of model $AR(1,1/1)$, we have $c_1=1$, $c_2=\frac{1}{2}$. At this time, the model of subject motivity to progress can be expressed as

$$u(t)=\begin{cases}\dfrac{8\psi^2+2+(2\psi-1)(6\psi-1)t}{2(2\psi-1)^2}e^{\frac{t}{2}}-\dfrac{4\psi}{(2\psi-1)^2}e^{\psi t} & \delta=\dfrac{1}{4},\psi\neq\dfrac{1}{2}\\ e^{\frac{t}{2}}\left(1+\dfrac{t}{2}-\dfrac{t^2}{4}\right) & \delta=\dfrac{1}{4},\psi=\dfrac{1}{2}\end{cases}\tag{9}$$

4.2.1 The basic results for subject motivity under the condition Δ=0

If Δ=1-4δ=0, we could obtain some results on expression (9) as follow.

Theorem 6 when $\delta=\frac{1}{4}$, we have

1) If $\psi\neq\frac{1}{2}$, and $\psi>\frac{1}{2}$ or $\frac{1}{2}>\psi>\frac{1}{6}$, then the time T is existent, and $u(t)<0$ when $t>T$.

2) If $\psi=\frac{1}{2}$, then the time T is existent, and $u(t)<0$ when $t>T$.

Proof.

1) If $\psi>\frac{1}{2}$, the coefficient of item $e^{\psi\cdot t}$ in expression (9) is smaller than zero, i.e. $-\frac{4\psi}{(2\psi-1)^2}<0$. So, the $u(t)<0$ will be smaller then zero as long as the time t is large sufficiently. And if $\psi<\frac{1}{2}$ and $\psi-\frac{1}{6}>0$ (i.e., $\frac{6\psi-1}{2(2\psi-1)}<0$), the coefficient of item $te^{\frac{t}{2}}$ in expression (9) is smaller than zero as long as the time t is large sufficiently, this means $u(t)<0$ as long as the time t is large sufficiently.

2) We always have $1+\frac{t}{2}-\frac{t^2}{4}<0$ as long as the time t is large sufficiently.

According to models (9) and theorem 6, we have the two basic results as follows:

- If $\Delta=1-4\delta=0$, i.e. $\delta=0.25$, this means the endogenous resistance is equal to 25% of subject motivity. At this time, subject motivity does not change in the cycle way.
- If the coefficient of endogenous resistance δ is equal to $\frac{1}{4}$ and the coefficient of exogenous resistance ψ is smaller than $\frac{1}{6}$, then subject motivity will rise continuously up to infinitude.

4.2.2 The methods to compute the zero point of subject motivity under the condition Δ=0

Based on the models (9), the zero point of subject motivity can be computed by the following approaches.

1) If $\psi>\frac{1}{2}$, let $u(T)=0$ in the expression (9), we have the following iterative algorithm:

$$T_{i+1}=\frac{2}{2\psi-1}\ln\left(\frac{1}{8\psi}[8\psi^2+2+(2\psi-1)(6\psi-1)T_i]\right)$$

Letting $T_0=0$ and applying repeatedly the algorithm above, we will get the time T.

2) If $\frac{1}{2}>\psi>\frac{1}{6}$, let $u(T)=0$ in the expression (9), we have the following iterative algorithm:

$$T_{i+1}=\frac{2}{(2\psi-1)(6\psi-1)}\left(-(4\psi^2+1)+4\psi e^{\left(\frac{2\psi-1}{2}\right)T_i}\right),\ T_0=0.$$

Letting $T_0=0$ and applying repeatedly the algorithm above, we will get the time T.

What we need to indicate is that the convergence speed of two iterative algorithms above will slower when ψ is close to $\frac{1}{2}$. At this time, the algorithm in appendix B can be applied.

3) If $\psi=\frac{1}{2}$, let $u(T)=0$ in the expression (9), we obtain $T=1+\sqrt{5}$.

4.2.3 The methods to compute the maximum value of motivity under the condition Δ=0

Because the subject motivity $u(t)$ is a single peak function when $\Delta=0$, so the maximum value of motivity $\max\limits_{0<t<+\infty}\{u(t)\}=u(\hat{T})$ if $\hat{T}$ make $u'(t)\big|_{t=\hat{T}}=0$.

1) If $\psi>\frac{1}{2}$, let $u'(\hat{T})=0$ on the expression (9), we have the following iterative algorithm:

$$\hat{T}_{i+1}=\frac{2}{2\psi-1}\ln\left(\frac{4(8\psi^2-4\psi+1)+(2\psi-1)(6\psi-1)\hat{T}_i}{16\psi^2}\right)$$

Letting $\hat{T}_0$ =0 and applying repeatedly the algorithm above, we will get the time $\hat{T}$.

2) If $\frac{1}{2}>\psi>\frac{1}{6}$, let $u'(\hat{T})=0$ on the expression (9), we have the following iterative algorithm:

$$\hat{T}_{i+1}=\frac{-4(8\psi^2-4\psi+1)+16\psi^2 e^{\frac{2\psi-1}{2}\hat{T}_i}}{(2\psi-1)(6\psi-1)}$$

Letting $\hat{T}_0$ =0 and applying repeatedly the algorithm above, we will get the time $\hat{T}$.

What we need to indicate is that the convergence speed of two iterative algorithms above will slower when ψ is close to $\frac{1}{2}$. At this time, the algorithm in appendix B can be applied to get $\hat{T}$, the root of $u'(t)=0$.

3) If $\psi=\frac{1}{2}$, let $u'(\hat{T})=0$ on the expression (9), we can obtain $\hat{T}=2$.

4.3 The analysis for subject motivity under the condition of Δ<0

When $\Delta=1-4\delta<0$ and denoting $w=\sqrt{4\delta-1}$, let $u(0)=1$, $u'(0)=u(0)$. From the general solution (6) of model $AR(1,1/1)$, we get $c_1=1$, $c_2=\frac{1}{w}$, and

$$u(t)=e^{\frac{t}{2}}\left[\cos\left(\frac{wt}{2}\right)+\frac{1}{w}\sin\left(\frac{wt}{2}\right)\right]+\frac{4\psi}{(2\psi-1)^2+w^2}\left[e^{\frac{t}{2}}\left(\cos\left(\frac{wt}{2}\right)+\frac{(2\psi-1)}{w}\sin\left(\frac{wt}{2}\right)\right)-e^{\psi t}\right]$$

According to the appendix C, $u(t)$ can be expressed as

$$u(t)=\frac{Me^{\frac{t}{2}}\cos\left(\frac{wt}{2}+\theta\right)-4\psi we^{\psi t}}{w[(2\psi-1)^2+w^2]} \tag{10}$$

where, θ is determined by $\cos\theta=\frac{w(4\psi^2+1+w^2)}{M}$, $M=\sqrt{\left[w(4\psi^2+1+w^2)\right]^2+\left[(6\psi-1)(2\psi-1)+w^2\right]^2}$.

We see, from expression (10), the $\cos\left(\frac{wt}{2}+\theta\right)$ will cause subject motivity $u(t)$ to change in the cycle way. This is different essentially from the cases $\Delta=1-4\delta>0$ and $\Delta=1-4\delta=0$. And we have the following conclusions:

- The subject motivity fluctuates in a cycle way as long as $\Delta=1-4\delta<0$, i.e., $\delta>\frac{1}{4}$.This means the periodic phenomenon in subject motivity is caused by the larger endogenous resistance.
- The larger the endogenous resistance is, i.e., the larger the coefficient δ is, the more frequent the periodic fluctuation in subject motivity is. Because both the periods of triangle functions $\cos\left(\frac{wt}{2}\right)$ and $\sin\left(\frac{wt}{2}\right)$ is $\frac{wt}{2}=2\pi$, the interval time $t=\frac{4\pi}{w}$. So, the periodic fluctuation of subject motivity is more frequent as long as δ is larger, i.e, $w=\sqrt{4\delta-1}$ is larger and the interval time t is smaller.

These two results are shown in figure 2 and figure3.

4.4 The motivities comparison analysis nearby Δ=0

Now, we illustrate the transition from model (7) to (9) and to (10) is continuous, i.e., all these models are consistent. Letting $u(0)=1$, $u'(0)=1$, $\psi=0.5$, and separately $\delta=0.2499(\Delta>0)$, $\delta=0.25(\Delta=0)$ and $\delta=0.2501$ $(\Delta<0)$, we compute the times of zero motivity, T $(u(T)=0)$, on model (7), (9) and (10) separately. The times of zero motivity are listed in table 1. At the same time, the motivity changing process is shown in figure 4. We can see, from figure 4, though the motivity models of (7), (9) and (10) are different in their form, their curves are very similar nearby $\Delta=0$. Further more, whether the model (7), (9) or (10), the motivity $u(t)$ will fall to smaller as long as the resistance becones larger, i.e., δ or/and ψ become larger. And $u(t)$ will rise to larger as long as δ or/and ψ become smaller. All of these indicate the model (7), (9) and (10) are reasonable.

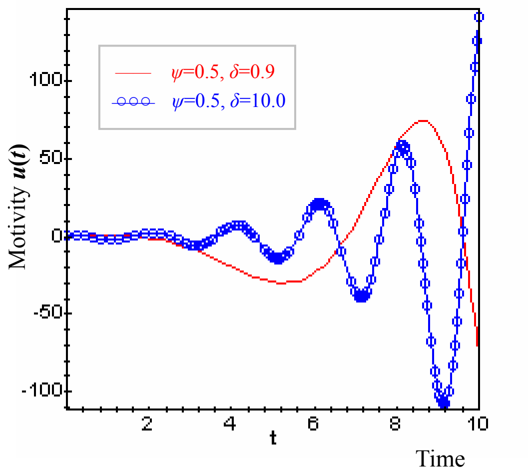

Figure 2 Motivity fluctuation on different endogenous resistances

When endogenous resistance is larger, i.e., $\Delta=1-4\delta<0$, the motivity will fluctuate to diffuse in periodic way. The scope of fluctuation becomes larger and larger at $\psi=0.5$. when $\delta=0.9$, fluctuating frequency is lower; when $\delta=10$, fluctuating frequency is lower.

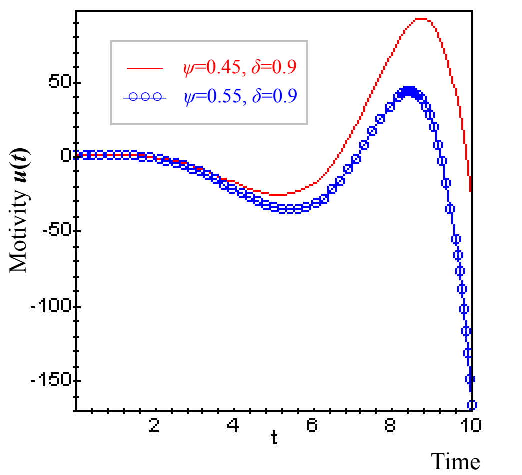

Figure 3 Motivity fluctuation on different exogenous resistances

$\delta=0.9$, subject motivity fluctuate to diffuse in periodic way. When $\psi=0.45$, exogenous resistance is smaller, then motivity level is higher;when $\psi=0.55$, exogenous resistance is larger, then motivity level islower.

Table 1 The times of zero motivity nearby $\Delta=0$

The coefficient of endogenous resistance	The times of zero motivity
$\delta=0.2499$ $(\Delta>0)$	$T=3.236414030$
$\delta=0.2500(\Delta=0)$	$T=3.236067978$
$\delta=0.2501(\Delta<0)$	$T=3.235551430$
Initial values $u(0)=1$, $u'(0)=1$	The coefficient of exogenous resistance $\psi=0.5$

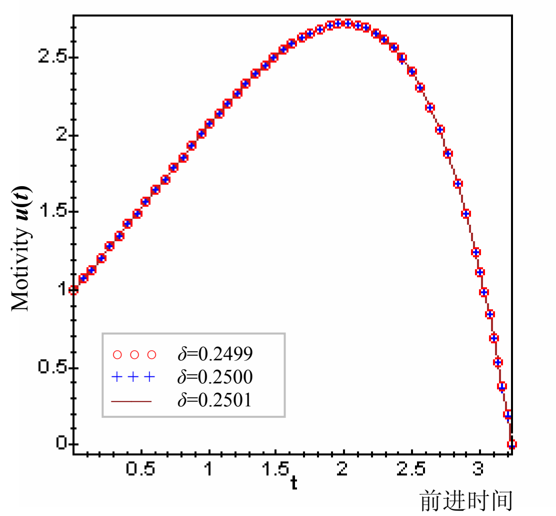

Figure 4 The motivities comparison nearby $\Delta=0$

The subject motivity models (7), (9) and (10) are different formally from one to another, however, their curves are very similar nearby $\Delta=0$, i.e., $\delta=0.25$.

4.5 The subject income and the solution of advance-retreat

Here, we shall give the computing model of subject income and the expression of solution of advance- retreat.

4.5.1 The computing model of subject income

In model (2), we suppose subject income follows $\frac{dL}{dt}=auL$. Then

$$L(u,t)=L_0e^{\beta(t)t} \tag{11}$$

where, $\beta(t)=a\hat{u}(t)$ is the average yield of subject in movement, $\hat{u}(t)=\frac{1}{t}\int_0^t u(\tau)d\tau$. L_0 is the initial income (i.e., initial capital) of subject.

The subject motivity $u(t)$ changes in a non-periodic way when $\Delta\geq 0$, but, in a periodic way when $\Delta<0$. So we discuss the computing models of subject income separately under the different conditions.

4.5.2 The solution of an advance-retreat under $\Delta\geq 0$

When $\Delta\geq 0$, from the expression (11), we have

$$\frac{dL}{dt}=auL\ ,\quad \frac{d^2L}{dt^2}=aL\left(\frac{du}{dt}+u\right)$$

Letting $\frac{dL}{dt}=0$, obtain $u(t)$=0. According to the previous discussions in section 4, the unique T existed make $u(T)$=0, and

$$\left.\frac{d^2L}{dt^2}\right|_{t=T}=aL\left.\frac{du}{dt}\right|_{t=T}=aL\frac{du(T)}{dt}<0$$

Thus, $u(T)$=0 make $L(u(T),T)=L_0e^{\beta(T)T}=\max_{0<t<+\infty}L(u(t),t)$.

If the conditions in theorem 3, theorem 4, theorem 5 and theorem 6 are satisfied, $\lim_{t\to\infty}u(t)$ =-∞, i.e., $\lim_{t\to\infty}L(u(t),t)$ =0.

We see, from definition 4, $u(t)$ is the solution of the complete advance-retreat $L(u)=L_0e^{\beta(t)t}$ when t=T.

4.5.3 The solution of the advance-retreat series under $\Delta<0$

When $\Delta<0$, the subject motivity is as the expression (10), and his stock income is

$$L(u,t)=L_0e^{\beta(t)t}$$

Because the motivity $u(t)$ fluctuates to diffuse in periodic way at this time, $L(u)=L_0e^{\beta(t)t}$ is a infinite half advance-retreat series from definition 5. According to model (10), we have

$$u(t)=\frac{e^{\frac{t}{2}}}{w^2}\left[(2+w^2)\cos\left(\frac{wt}{2}\right)-2\right],\quad u'(t)=\frac{1}{2}u(t)-\frac{(2+w^2)}{2w}e^{\frac{t}{2}}\sin\left(\frac{wt}{2}\right)\quad \text{if } \psi=0.5$$

Letting $u(t)$=0, obtain $t=\frac{2}{w}\left[2i\pi\pm\arccos\left(\frac{2}{2+w^2}\right)\right]$, i=0,1, ···.

From $u'(t)=\mp\frac{1}{2w}e^{\frac{t}{2}}\sin\alpha$, $\alpha=\arccos\left(\frac{2}{2+w^2}\right)$, we know that $u(T_i)$=0 (i=0,1, ···) is the solution of advance-retreat series $L(u)=L_0e^{\beta(t)t}$ when $T_i=\frac{2}{w}\left[2i\pi+\arccos\left(\frac{2}{2+w^2}\right)\right]$.

If $\psi\neq 0.5$, the solution may be more complicated, or there is not the closed form in solution of advance-retreat series $L(u)=L_0e^{\beta(t)t}$. However, we could confirm the solution is existent because the exogenous resistance the does not influence the motivity in its essential structure.

4.4 The summary on previous conclusiones

Taking a synthesis for previous researches in section 4, we have the basic conclusions as follow:

■ The strong exogenous resistances may cause the sudden recession in society or economy, like the large scale of war, disaster, etc. However, they do not cause the periodic fluctuation in socio-economic level, i.e., the ultimate reason that society and economy fluctuate periodicly is not the exogenous resistances.

■ The periodic fluctuation in socio-economic level is brought by the endogenous resistances at the time when the coefficient of endogenous resistances δ is larger than 0.25 (the critical value) if b=1 in expression (1). So we could think the essential reasons, the periodic fluctuations are existed in socio-economic fields like economic production, consumed market, national development and so on, are the endogenous resistances in system being larger than their special critical values.

■ If being larger than the the critical value, the larger the endogenous resistances is, the lager the subject motivity aroused may be, and the larger the risk to subject is. In the view of economy, the temporary lose may bring more return. Of course, the larger risk will be chaperonaged with.

■ In order to avoid the periodic fluctuation in motivity, the subject need to take measures to decrease its endogenous resistance. In the socio-economic fields, government need to formulate the related policies and take some investment measures in order to raise the development motivity and reduce the loss brought from periodic fluctuated risk.

■ The endogenous resistances are small at the beginning of some socio-economic subjects developing and promoting. Afterwards, the endogenous resistances become larger and larger. The periodic fluctuations appear as long as the endogenous resistance is larger than the special critical values. This may be the real and intrinsic reason socio-economic fluctuations always in the periodic way.

To a summary, the larger endogenous resistances in socio-economic system are prone to cause its periodic turbulences. These kinds of turbulences may result in widening gap between rich and poor nations, and then threaten the world peace, the security in society and the stable growth in economy.

5 *ARI*(1,1/1)—The Subject Motivity Model with Investment

As we know, from analyzing on model *AR*(1,1/1), the subject will be more difficult to go ahead and its motivity will fluctuate in a periodic way if endogenous resistances is larger than a special value, i.e., $\Delta = 1 - 4\delta < 0$. This means, in socio-economic fields, the developing foreground is more uncertain. At the same time, new possible opportunities may present along with the new possible risks. In order to decrease the endogenous resistance, then reduce the risks and make it stable to develop in society and economy, the concerned policies needed to be formulated with some investment measures.

5.1 The advance-retreat model with investment

Denoting:

r_1 : The investment rate to strengthen the subject motivity for going forward, $r_1 \geq 0$.

r_2 : The investment rate to decrease the endogenous resistance, $r_2 \geq 0$.

r_3 : The investment rate to decrease the exogenous resistance, $r_3 \geq 0$.

r : The total investment rate, $r = r_1 + r_2 + r_3 > 0$.

Then, the corresponding to model *AR*(1,1/1), the analytic model for advance-retreat with investment can be expressed as

$$\begin{cases} \dfrac{du}{dt} = (1 + r_1)u - \overline{u} \\ \dfrac{d\overline{u}}{dt} = (\delta - r_2)u + (\psi - r_3)g \end{cases} \tag{12}$$

There is one subject and one object, and the object includes one factor of endogenous resistance and one factor of exogenous resistance in model (12), so it is called the single subject model with investment. The model (12) is denoted as *ARI*(1,1/1).

If there are multi-subjects in advance-retreat problem, and suppose *n* subjects and *n* objects, and *n* objects include *n* factors of endogenous resistances and *m* factors of exogenous resistances, we could express the model for multi-subjects advance-retreat as

$$\begin{cases} \dfrac{dU}{dt} = B_R U - B\overline{U} \\ \dfrac{d\overline{U}}{dt} = K_R U + \Psi_R G \end{cases} \tag{13}$$

where, $\boldsymbol{B_R}$=$\boldsymbol{B}$+$\boldsymbol{R}_1$, $\boldsymbol{K_R}$= $\boldsymbol{K}$–$\boldsymbol{R}_2$, $\boldsymbol{\Psi_R}$=$\boldsymbol{\Psi}$-$\boldsymbol{R}_3$, $\boldsymbol{U}$, $\overline{U}$, $\boldsymbol{B}$, $\boldsymbol{K}$, $\boldsymbol{\Psi}$, $\boldsymbol{G}$ are the sames as in model (3), and $\boldsymbol{R}_1$=$(r_{ij}^{(1)})_{n\times n}$ is the matrix of investment rate to strengthen the subject motivity for going forward, $r_{ij}^{(1)}\geq 0$, $\boldsymbol{R}_2$=$(r_{ij}^{(2)})_{n\times n}$ is the matrix of investment rate to decrease the endogenous resistance, $r_{ij}^{(2)}\geq 0$, i, j=1, ···, n. $\boldsymbol{R}_3$=$(r_{ij}^{(3)})_{n\times m}$ is the matrix of investment rate to decrease the exogenous resistance, $r_{ij}^{(3)}\geq 0$, i =1, ···, n ; j=1, ···, m. The model (13) is denoted as ARI (n, n/m).

5.2 The general solution of model *ARI*(1,1/1)

Denoting $w=\sqrt{|(1+r_1)^2-4(\delta-r_2)|}$ and $\lambda_{1,2}=\dfrac{(1+r_1)\pm\sqrt{(1+r_1)^2-4(\delta-r_2)}}{2}$, and letting $g(t)=e^{(\psi-r_3)t}$, i.e., $\varphi(t)$ =-(ψ-r_3)g. Similarly to expression (6), the general solution of AR (1,1/1), we have

$$u(t)=\begin{cases} c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}+\dfrac{1}{w}\left(e^{\lambda_1 t}\displaystyle\int_0^t\varphi(\tau)e^{-\lambda_1\tau}dt-e^{\lambda_2 t}\int_0^t\varphi(\tau)e^{-\lambda_2\tau}dt\right) \\ \qquad c_1=\dfrac{u'(0)-\lambda_2u(0)}{w},\quad c_2=\dfrac{\lambda_1u(0)-u'(0)}{w},\ if\quad \Delta=(1+r_1)^2-4(\delta-r_2)>0 \\ e^{\frac{1+r_1}{2}t}\left(c_1+c_2t+t\displaystyle\int_0^t\varphi(\tau)e^{-\frac{1+r_1}{2}\tau}d\tau-\int_0^t\varphi(\tau)\tau e^{-\frac{1+r_1}{2}t}d\tau\right) \\ \qquad c_1=u(0), c_2=u'(0)-\dfrac{1}{2}u(0),\qquad if\quad \Delta=(1+r_1)^2-4(\delta-r_2)=0 \\ e^{\frac{1+r_1}{2}t}\left(c_1\cos\left(\dfrac{wt}{2}\right)+c_2\sin\left(\dfrac{wt}{2}\right)\right)+ \\ +\dfrac{2}{w}e^{\frac{1+r_1}{2}t}\left[\sin\left(\dfrac{wt}{2}\right)\displaystyle\int_0^t\varphi(t)e^{\frac{-(1+r_1)}{2}\tau}\cos\left(\dfrac{w}{2}\tau\right)d\tau-\cos\left(\dfrac{wt}{2}\right)\int_0^t\varphi(\tau)e^{\frac{-(1+r_1)}{2}\tau}\sin\left(\dfrac{w}{2}\tau\right)d\tau\right] \\ \qquad c_1=u(0), c_2=\dfrac{1}{w}[2u'(0)-u(0)]\qquad if\quad \Delta=(1+r_1)^2-4(\delta-r_2)<0 \end{cases} \tag{14}$$

5.3 The investing strategies analysis based on *ARI*(1,1/1)

when Δ=1-4δ<0, we can increase the subject motivity or decrease the endogenous resistance by investment, eliminate the periodic fluctuation in process of subject going forward, and reduce the uncertainty and risk. At this time, the following investment strategies are available:

- Investing only to subject. r_1 is enlarged to increase the subject motivity till $(1+r_1)^2$-4δ>0, i.e. $r_1>2\sqrt{\delta}-1$.
- Investing only to endogenous resistance. r_2 is enlarged to decrease the endogenous resistance till 1-4(δ-r_2)>0, i.e. $r_2>\delta-\dfrac{1}{4}$. Because the end to decrease the endogenous resistance should be it is entirely eliminated, so we should have $r_2\leq\delta$.
- Investing to subject and to endogenous resistance at the same time. r_1 is enlarged to increase the subject motivity and r_2 is enlarged to decrease the endogenous resistance till $(1+r_1)^2$-4(δ-r_2)>0, i.e. $(1+r_1)^2$+4r_2)>4δ.

Obviously, if hoping to let $(1+r_1)^2$-4(δ-r_2)≥0 when 1-4δ<0, we could not only enlarge r_1, but also enlarge r_2. Namely, at this time, the r_1 and r_2 can be replaced one with another. The previous investment strategies can be supplemented as the following propositions.

Proposition 1 If Δ=1-4δ<0 and r_1+r_2≤r≤1, the investing efficiency of maximizing r_2 is higher than that of maximizing r_1.

Proof. Denoting $s=r_1+r_2$. $\Delta=(1+s-r_2)^2-4(\delta-r_2)$ is a increasing function about r_2 when $s\le r\le 1$, so, Δ reaches its maximum if r_2 reaches its maximum.

When $\Delta=(1+r_1)^2-4(\delta-r_2)>0$, let $u(0)=1, u'(0)=1$, and according to expression (7), the model (14) can be expressed as

$$u(t)=\begin{cases}\left(\frac{\lambda_1}{w}+\frac{\psi-r_3}{w(\psi-r_3-\lambda_1)}\right)e^{\lambda_1 t}-\left(\frac{\lambda_2}{w}+\frac{\psi-r_3}{w(\psi-r_3-\lambda_2)}\right)e^{\lambda_2 t}-\frac{(\psi-r_3)e^{(\psi-r_3)t}}{(\psi-r_3-\lambda_1)(\psi-r_3-\lambda_2)} \\ \qquad where, \psi-r_3\neq\lambda_1, \psi-r_3\neq\lambda_2 \\ \frac{\psi-r_3}{w}\left(1-t+\frac{1}{w}\right)e^{(\psi-r_3)t}-\frac{1}{w}\left(\lambda_2+\frac{\psi-r_3}{w}\right)e^{(\psi-r_3-w)t} & \psi-r_3=\lambda_1 \\ \frac{1}{w}\left(\lambda_1-\frac{\psi-r_3}{w}\right)e^{(\psi-r_3+w)t}+\frac{\psi-r_3}{w}\left(-1+t+\frac{1}{w}\right)e^{(\psi-r_3)t} & \psi-r_3=\lambda_2\end{cases} \tag{15}$$

where $\lambda_{1,2}=\frac{(1+r_1)\pm\sqrt{(1+r_1)^2-4(\delta-r_2)}}{2}$.

At the same time, the condition (8) in theorem 1 should be expressed as

$$\psi\text{-}r_3+\sqrt{(\psi-r_3)^2+4(\psi-r_3)}-\sqrt{(1+r_1)^2-4(\delta-r_2)}>1+r_1 \tag{16}$$

We have the following three propositions under the condition $\Delta=(1+r_1)^2-4(\delta-r_2)>0$ and inequation (16).

Proposition 2 When $\lambda_2<\psi-r_3<\lambda_1$ or $\psi-r_3<\lambda_2<\lambda_1$, the investing efficiency of maximizing r_1 is higher than that of maximizing r_2.

Proof. We know, from model (15), the item $\left(\frac{\lambda_1}{w}+\frac{\psi-r_3}{w(\psi-r_3-\lambda_1)}\right)e^{\lambda_1 t}$ play a main part in increasing motivity along with the time being larger. At this time, enlarging λ_1 is most beneficial to prolong the subject to go forward and to gain more income. Because the function

$$\lambda_1=\frac{(1+r_1)+\sqrt{(1+r_1)^2-4[\delta-(r-r_1-r_3)]}}{2}$$

is a increasing function about r_1, so, λ_1 reaches its maximum if r_1 reaches its maximum.

Proposition 3 When $\lambda_2<\lambda_1<\psi-r_3$, the investing efficiency of maximizing r_3 is higher than others.

Proof. We can see, from model (15), $e^{(\psi-r_3)t}>e^{\lambda_1 t}$ when $\lambda_1<\psi-r_3$. And the item of exogenous resistance, $-\frac{(\psi-r_3)e^{(\psi-r_3)t}}{(\psi-r_3-\lambda_1)(\psi-r_3-\lambda_2)}<0$, is the main part in resistance. So, maximizing the investment rate r_3 and minimizing $\psi-r_3$ will reduce in maximum the resistance which obstructs subject to go forward and let subject gain more income.

Proposition 4 If the total investment rate r is invariable and $\psi-r_3=\lambda_1$ or $\psi-r_3=\lambda_2$, then maximizing r_2 and minimizing r_3 will let subject have a higher investing efficiency if investing to reduce resistance.

Proof. Based on the model (15) and when $\psi-r_3=\lambda_1$, the item $e^{(\psi-r_3)t}$, which plays a main part in increasing subject motivity, will rises if decreasing r_3. And at the same time, increasing r_2 will make w increase, and then $e^{(\psi-r_3-w)t}$ becomes smaller. All of these are most beneficial for subject to gain more income. Having a similar proof with proposition 1, we could know increasing r_2 is higher than increasing r_1 in efficiency for maximizing w. It is the similar way for $\psi-r_3=\lambda_2$.

We could see, from the model (15), the longer the time when the subject goes forward is if his motivity will decrease finally, the more difficult the subject will face. So the investment will be larger and larger to maintain the motivity for subject to go forward. All these are represented, in socio-economic fields, as the investments concentrate to the larger project, the individual investment is trend to larger and larger, and scale of firm or enterprise becomes larger and largr.

Applying the theory as we say to analyze the macroeconomy or production fields, we could know the effective investments will promote to increase the developing motivity, to decrease the developing resistance, and to raise the stability and endurance in socio-economic development and gaining income. The actual ways to investment are as follows:

- Investing to increase subject motivity. These kinds of ways include investing to optimize the organization structure and policies environments, to reduce the running cost, to improve the job facilities, to raise the the scientific and technological content in products, to exploit and develop the new products and services, to train manpowers, to open new building and developing projects, etc.
- Investing to reduce the endogenous resistance. These kinds of ways include investing to reduce the unnecessary spendings, to repeal the rules and regulations which are not beneficial to increase produced efficiency and develop continuously, to make the operation process standardization and specialization, to recoup for the rised cost, improve confused managements and lag backward techniques, etc. In fact, increasing the motivity can promote to decrease the endogenous resistance, and vice versa.
- Investing to exogenous resistance. These kinds of ways include investing to constitute or improve macroeconomic policies and statutes, to reasarch and develop new energy sources, to develop effectively the communication and transportation, to perfect social security system, to suppress the disease popularity, to control the pollution sufficiently, to improve the ecological environment, to establish the nature and man-made disasters defensing system, to develop new markets, etc.

5.4 The approaches to compute the income on *ARI*(1,1/1)

If considering only the investment, we let δ=0 and ψ=0 in *ARI*(1,1/1), namelythe model (12).Then we have

$$\begin{cases} \dfrac{du}{dt} = r_1 u - \overline{u} \\ \dfrac{d\overline{u}}{dt} = -r_2 u - r_3 g \end{cases} \tag{17}$$

Denote $s=\sqrt{r_1^2+4r_2}$, $\kappa_{1,2}=\dfrac{r_1 \pm \sqrt{r_1^2+4r_2}}{2}$ and $g(t)=e^{-r_3 t}$, i.e. $\varphi(t)=r_3 g$, and let $u(0)=1, u'(0)=1$.

Similarly to expression (6), we have the general solution of model (17) as

$$u_r(t) = \left(\frac{\kappa_1}{s} + \frac{r_3}{s(r_3+\kappa_1)}\right)e^{\kappa_1 t} - \left(\frac{\kappa_2}{s} + \frac{r_3}{s(r_3+\kappa_2)}\right)e^{\kappa_2 t} + \frac{r_3 e^{-r_3 t}}{(r_3+\kappa_1)(r_3+\kappa_2)} \tag{18}$$

Thus, $u_r(t)$ is the net motivity by investment, from model (2), the net income of subject follow

$$\frac{dL}{dt} = a(u-u_r)L$$

where $u(t)$ is determined by expression (15). Then the net income of subject is

$$L(u,t) = L_0 e^{a\int_0^t (u-u_r)dt} \tag{19}$$

where L_0 is the initial value of captital of subject, $v=u-u_r$ is the net motivity of subject at the time t, $V=\int_0^t [u(\tau)-u_r(\tau)]d\tau$ is the total of net motivity in the field of [0, t], $\beta(t)=\dfrac{a}{t}\int_0^t [u(\tau)-u_r(\tau)]d\tau$ means the average of net income ratio of subject in movement.

6 The Analytical Examples For Investing Strategies on *ARI*(1,1/1)

Two examples are given here based on the total of net motivity $V=\int_0^t[u(\tau)-u_r(\tau)]d\tau$, in which investment strategies in section 5.3 are compared analytically. Then we indicate that the different investment strategies will bring the different incomes (19) to subject if total investment rate r is invariable. Suppose the developing process of a economic industry follows the advance-retreat model $AR(1,1/1)$, where the total investment rate r =0.3.

6.1 The analysis on investment strategies when the exogenous resistance is smaller

If the coefficient of endogenous resistance is δ=0.3, and the coefficient of exogenous resistance ψ= 0.51. Where, ψ= 0.51 is the smallest value among those satisfying the condition (16).

Because Δ=1-4δ=-0.20<0, this industry will fluctuate periodically in the future. We need to frame the strategy for investment to subject or endogenous resistance to reduce the risk and the uncertainty.

According to proposition 1, should reduce the endogenous resistance first, increasing r_2, as long as $\Delta=1-4(\delta-r_2)>0$. When $\Delta=1-4(\delta-r_2)>0$, according to proposition 2 or proposition 3, should increase r_1 to strengthen the subject motivity or increase r_3 to reduce the exogenous resistance. We analyze here three following strategies to illustrate the results above to be correct. The contents of three strategies and the concerned computations are listed in table 2.

Table 2 The contents of strategies and the concerned computations

Strategy	Content	The concerned computation
1	r_1=0.00 r_2=0.30 r_3=0.00	Δ=1, λ_1=1, λ_2=0 ψ-r_3 =0.51 $\lambda_1>\psi-r_3>\lambda_2$
2	r_1=0.30 r_2=0.00 r_3=0.00	Δ=0.49, λ_1=1 λ_2=0.3,ψ-r_3 =0.51 $\lambda_1>\psi-r_3>\lambda_2$
3	r_1=0.0, r_2=0.051 r_3=0.249	Δ=0.004, λ_1=0.5316227766 λ_2=0.4683772234 ψ-r_3 =0.261 $\lambda_1>\lambda_2>\psi-r_3$

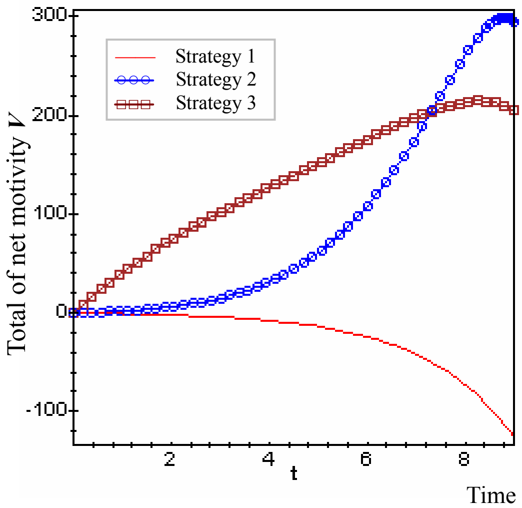

Figure 5 Comparison between investment strategies when ψ is smaller

This industry will not fluctuate periodically in the future when Δ=1-4δ>0. The condition of proposition 2 is satisfied because ψ is smaller. So the total of net motivity of strategy 2 is higher than those of strategy 1 and strategy 3.

We see, from table 2, strategy 1 invests the entire fund to reduce the endogenous resistance. And it ensures the Δ>0 and $\lambda_1>\psi-r_3>\lambda_2$ at the same time. But, when $\lambda_1>\psi-r_3>\lambda_2$, we should choose strategy 2 according to proposition 2. Strategy 2 invests the entire fund to raise the subject motivity, and not only ensures Δ>0, but also increases the efficiency of investment. At this time and according to proposition 2, it should be still applied to strengthen the subject motivity if the investment is added because $\lambda_1>\psi-r_3>\lambda_2$ is tenable. Strategy 3, under Δ>0, invests the entire fund to reduce the exogenous resistance. But, according to proposition 2, it is more efficient to invest the entire fund to raise the subject motivity because $\lambda_1>\lambda_2>\psi-r_3$.

The results discussed above can be tested in figure 5. In figure 5, Strategy 2 has a highest efficiency of investment among these three strategies, so the proposition 2 is correct.

6.2 The analysis on investment strategies when the exogenous resistance is larger

We have illustrated the proposition 2 is correct previously. Here we try to illustrate the proposition 3 is correct. If the coefficient of endogenous resistance is δ=0.3, and the coefficient of exogenous resistance ψ= 1.01. Where, ψ= 1.01 will make the condition of proposition 3 be satisfied.

We also analyze the three strategiesin section 6.1 under δ=0.3 andψ=1.01. The concerned computations are listed in table 3.

Table 3 The contents of strategies and the concerned computations

Strategy	Content	The concerned computation
1	r_1=0.00 r_2=0.30 r_3=0.00	Δ=1, λ_1=1, λ_2=0 ψ-r_3 =1.01 ψ-r_3>λ_1>λ_2
2	r_1=0.30 r_2=0.00 r_3=0.00	Δ=0.49, λ_1=1, λ_2=0.3 ψ-r_3 =1.01 ψ-r_3>λ_1>λ_2
3	r_1=0.0, r_2=0.051 r_3=0.249	Δ=0.004 λ_1=0.5316227766 λ_2=0.4683772234 ψ-r_3 =0.761 ψ-r_3>λ_1>λ_2

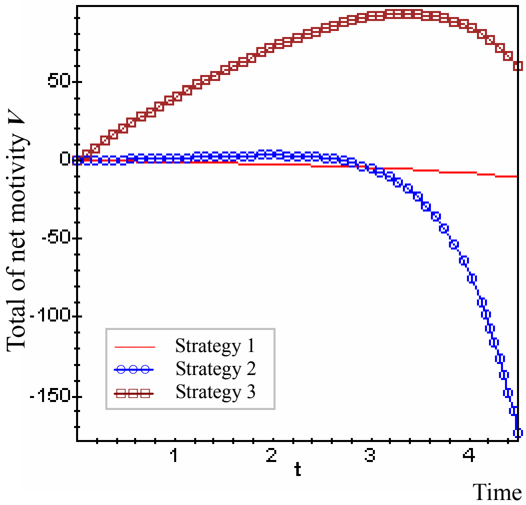

Figure 6 Comparison between investment strategies when ψ is larger

This industry will not fluctuate periodically in the future when Δ=1-4δ>0. The condition of proposition 3 is satisfied because ψ is larger. So the total of net motivity of strategy 3 is higher than those of strategy 1 and strategy 2.

We see, from table 3, strategy 1 invests the entire fund to reduce the endogenous resistance and ensures the Δ>0. Strategy 2 invests the entire fund to raise the subject motivity, and not only ensures Δ>0, but also make its efficiency of investment higher than that of strategy 1. But, there is ψ-r_3>λ_1>λ_2 in the cases of strategy 1 and strategy 2, at the same time, we need to choose the strategy 3 according to proposition 3. All the results given here can be tested in figure 6. In figure 6, Strategy 3 has a highest efficiency of investment among these three strategies, so the proposition 3 is correct.

7 The Conclusions

Based on summing up many of real problems in socio-economic fields, this paper proposes visually the “boating against the current”. After analyzing the problem of “boating against the current”, the problem of advance-retreat and their models are given. The core of advance-retreat is the relation between the subject of going forward and object of creating resistance. The subject, which is sapiential and active, can analyze and judge actively the state and situation himself, make a choice and decision on various strategies, and put his decision into effect. The object, which is non-sapiential and passive, can react to any choice of subject by the resistance or pressure. Subject and object are always in the dynamic rivalry one with another.

In essential, the analysis on advance-retreat is the researches to advance or retreat between subject and object. Based on the basic frame of analysis on advance-retreat, this paper designs the kinds of analytic models. From the number of subject and object, this kinds of analytic models are includes AR(1,1/1) and AR(n,n/m).

By the analysis a nd computation on the general solution of model AR(1,1/1), expression (6), we have seen that the endogenous resistances, instead of exogenous resistances, causes the periodic fluctuation in socio-economic process, and obtained the critical condition beyond which the periodic fluctuation is occured. Further more, a series of results on the motivity for subject going forward are found (theorem 3—theorem 6 and concerned explanations).

Combining with real problems in socio-economic fields, this paper also puts forward the advance-retreat models with investment, *ARI*(1,1/1) and *ARI*(*n*,*n*/*m*). Based on the model *ARI*(1,1/1), we have got some available strategies to invest (proposition 1—proposition 4), which will be different along with the different conditions. Finally, we have illustrated, by two examples, those strategies are correct and effective to the real economic investment and industial investment.

Of course, there are many problems needed to be solved in the future, such as

- The concept model of advance-retreat should be perfected further.
- The empirical researches for the analytic model of advance-retreat (AR(1,1/1), ARI(1,1/1), etc.) need to be unfolded.
- It might be worth to discuss the advance-retreat model with the case that there are the branches in the process of going forward.
- The basic characters of model *AR*(*n*,*n*/*m*) and *ARI*(*n*,*n*/*m*).
- The stochastic model for advance-retreat problem.
- More analytic models are built for application.
- Combing the advance-retreat theory with game theory, we could have the composing problem which may be valuable.

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Appendix A: The approach to solve the model *AR*(1,1/1).

According to the *Handbook of Exact Solutions for Ordinary Differential Equations* (Polianin et al., 2002), we could convert model *AR*(1,1/1), the group of differential equations

$$\begin{cases} \dfrac{du}{dt} = u - \overline{u} \\ \dfrac{d\overline{u}}{dt} = \delta u + \psi g \end{cases}$$

to

$$u'' - u' + \delta u = \varphi(t) \text{ (where } \varphi(t) = -\psi g) \qquad (A_1)$$

The homogeneous one of differential equation (A_1) is

$$u'' - u' + \delta u = 0 \qquad (A_2)$$

The characteristic equation of (A_2) is $\lambda^2 - r_1\lambda + \delta = 0$, its solutions are $\lambda_{1,2} = \dfrac{1 \pm \sqrt{1-4\delta}}{2}$.

Then we could obtain the special integral of (A_2) are $u_1 = u_1(t)$ and $u_2 = u_2(t)$. Let a special integral of (A_1) is

$$u^*(t) = c_1(t)u_1(t) + c_2(t)u_2(t)$$

where $c_1(t)$ and $c_2(t)$ satisfy the following group of equations

$$\begin{cases} c_1'(t)u_1 + c_2'(t)u_2 = 0 \\ c_1'(t)u_1' + c_2'(t)u_2' = \varphi(t) \end{cases}$$

i.e.

$$m(t) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}, \quad c_1'(t) = -\frac{\varphi(t)u_2}{m(t)}, \quad c_2'(t) = \frac{\varphi(t)u_1}{m(t)}.$$

Thus, the general solution of (A_1) is

$$u(t) = c_1u_1(t) + c_2u_2(t) - u_1(t)\int_0^t \frac{\varphi(\tau)u_2(\tau)}{m(\tau)}d\tau + u_2(t)\int_0^t \frac{\varphi(\tau)u_1(\tau)}{m(\tau)}d\tau$$

Denoting $w = \sqrt{|1-4\delta|}$.

1) If $\Delta = 1-4\delta > 0$, the special integrals of (A_2) are $u_1 = e^{\lambda_1 t}$ and $u_2 = e^{\lambda_2 t}$, thus

$$m(t) = -we^{r_1 t}, \quad c_1'(t) = \frac{\varphi(t)e^{-\lambda_1 t}}{w}, \quad c_2'(t) = -\frac{\varphi(t)e^{-\lambda_2 t}}{w}.$$

Because $c_1(t) = c_1 + \dfrac{1}{w}\displaystyle\int_0^t \varphi(\tau)e^{-\lambda_1\tau}dt$, $c_2(t) = c_2 + -\dfrac{1}{w}\displaystyle\int_0^t \varphi(\tau)e^{-\lambda_2\tau}dt$, the general integral of (A_1) is

$$u(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \frac{1}{w}\left(e^{\lambda_1 t}\int_0^t \varphi(\tau)e^{-\lambda_1\tau}dt - e^{\lambda_2 t}\int_0^t \varphi(\tau)e^{-\lambda_2\tau}dt\right)$$

And the initial values are $u(0) = c_1 + c_2$, $u'(0) = \lambda_1 c_1 + \lambda_2 c_2$. The initial values in the general integral of (A_1) satisfy

$$\begin{cases} c_1 = \dfrac{u'(0) - \lambda_2 u(0)}{w} \\ c_2 = \dfrac{\lambda_1 u(0) - u'(0)}{w} \end{cases}.$$

2) If $\Delta = 1-4\delta = 0$, the special integrals of (A_2) are $u_1 = e^{\frac{t}{2}}$ and $u_2 = te^{\frac{t}{2}}$, thus

$m(t)= e^{t}$, $c_1'(t) = -\varphi(t)te^{-\frac{t}{2}}$, $c_2'(t) = \varphi(t)e^{-\frac{t}{2}}$

and the general integral of (A_1) is

$$u(t)= c_1 e^{\frac{t}{2}} + c_2 te^{\frac{t}{2}} + te^{\frac{t}{2}} \int_0^t \varphi(\tau)e^{-\frac{\tau}{2}} dt - e^{\frac{t}{2}} \int_0^t \varphi(\tau)\tau e^{-\frac{\tau}{2}} dt$$

where $c_1 = u(0)$, $c_2 = u'(0) - \frac{1}{2}u(0)$.

3) If $\Delta = 1-4\delta < 0$, the special integrals of (A_2) are $u_1 = e^{\frac{t}{2}} \cos\left(\frac{wt}{2}\right)$, $u_2 = e^{\frac{t}{2}} \sin\left(\frac{wt}{2}\right)$, thus

$$m(t)= \frac{w}{2}e^{t},\quad c_1'(t) = -\frac{2\varphi(t)}{w} e^{\frac{-t}{2}} \sin\left(\frac{w}{2}t\right),\quad c_2'(t) = \frac{2\varphi(t)}{w} e^{\frac{-t}{2}} \cos\left(\frac{w}{2}t\right)$$

and the general integral of (A_1) is

$$u(t)= c_1 e^{\frac{t}{2}} \cos\left(\frac{wt}{2}\right) + c_2 e^{\frac{t}{2}} \sin\left(\frac{wt}{2}\right) + $$
$$+ \frac{2}{w}\left(e^{\frac{t}{2}} \sin\left(\frac{wt}{2}\right) \int_0^t \varphi(t)e^{\frac{-\tau}{2}} \cos\left(\frac{w}{2}\tau\right) d\tau - e^{\frac{t}{2}} \cos\left(\frac{wt}{2}\right) \int_0^t \varphi(t)e^{\frac{-\tau}{2}} \sin\left(\frac{w}{2}\tau\right) d\tau \right)$$

where $c_1=u(0)$, $c_2=\frac{1}{w}[2u'(0)-u(0)]$.

Appendix B: The algorithm for computing the root of equation $X(t)=0$.

Let $t_0=0$, choose a larger $t_1>0$ to make $X(t_1)<0$, and give a smaller $\varepsilon>0$.

1° Evaluating: $T \leftarrow t_1$

2° computing: $X(T)=0$

3° If $|X(T)-0|<\varepsilon$, end. Otherwise, let $t_1=T$ if $X(T)<0$ or let $t_0=T$ if $X(T)>0$.

4° Evaluating: $T \leftarrow \frac{t_0+t_1}{2}$, goto 2°.

Appendix C: The process of obtaining the expression $u(t)$ when $\Delta<0$.

$$u(t) = e^{\frac{t}{2}}\left[\cos\left(\frac{wt}{2}\right) + \frac{1}{w}\sin\left(\frac{wt}{2}\right)\right] + \frac{4\psi}{(2\psi-1)^2 + w^2}\left[e^{\frac{t}{2}}\left(\cos\left(\frac{wt}{2}\right) + \frac{(2\psi-1)}{w}\sin\left(\frac{wt}{2}\right)\right) - e^{\psi t}\right],\text{ i.e.,}$$

$$u(t) = \frac{1}{w[(2\psi-1)^2 + w^2]}\left\{e^{\frac{t}{2}}\left[w(4\psi^2+1+w^2)\cos\left(\frac{wt}{2}\right) + ((6\psi-1)(2\psi-1)+w^2)\sin\left(\frac{wt}{2}\right)\right] - 4\psi w e^{\psi t}\right\}$$

Denoting $\cos\theta = \dfrac{w(4\psi^2+1+w^2)}{\sqrt{\left[w(4\psi^2+1+w^2)\right]^2 + \left[(6\psi-1)(2\psi-1)+w^2\right]^2}}$

and $\sin\theta = \dfrac{(6\psi-1)(2\psi-1)+w^2}{\sqrt{\left[w(4\psi^2+1+w^2)\right]^2 + \left[(6\psi-1)(2\psi-1)+w^2\right]^2}}$,

then $u(t) = \dfrac{Me^{\frac{t}{2}}\cos\left(\frac{wt}{2}+\theta\right) - 4\psi w e^{\psi t}}{w[(2\psi-1)^2 + w^2]}$.

where $M=\sqrt{\left[w(4\psi^2+1+w^2)\right]^2 + \left[(6\psi-1)(2\psi-1)+w^2\right]^2}$.