Sensitivity Analysis between Lagrange Multipliers and Consumer Coupon: Utility Maximization Perspective

Mohajan, Devajit and Mohajan, Haradhan

Department of Civil Engineering, Chittagong University of Engineering Technology, Chittagong, Bangladesh, Department of Mathematics, Premier University, Chittagong, Bangladesh

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Devajit Mohajan

Department of Civil Engineering, Chittagong University of Engineering & Technology, Chittagong, Bangladesh

Email: devajit1402@gmail.com

Mobile: +8801866207021

Haradhan Kumar Mohajan

Department of Mathematics, Premier University, Chittagong, Bangladesh

Email: haradhan1971@gmail.com

Mobile: +8801716397232

Abstract

Sensitivity analysis is a vital concept in optimization models. From this analysis an economist can predict on the outcome of an economic model by the use of certain range of input variables. Utility maximization policy is essential for the sustainability of the economic organizations. This study takes an attempt to discuss sensitivity analysis between Lagrange multipliers and consumer coupon, where utility maximization is analyzed with detail mathematical analysis. In the study method of Lagrange multipliers is used to investigate the utility function; subject to two constraints: budget constraint, and coupon constraint. Moreover, two Lagrange multipliers are used here with four commodity variables.

Keywords: Coupon, consumers, Lagrange multipliers, sensitivity analysis, utility maximization

1. Introduction

Mathematical modeling in economics is the application of mathematics in economics, which makes relationships among prices, production, employment, saving, investment, etc. It is a highly abstract discipline that covers many fields, such as economics, sociology, psychology, political science, etc. (Samuelson, 1947; Zheng & Liu, 2022). The concept of utility was developed in the late 18^{th} century by the English moral philosopher, jurist, and social reformer Jeremy Bentham (1748-1832) and English philosopher, political economist, Member of Parliament (MP) and civil servant John Stuart Mill (1806-1873) (Bentham, 1780). In modern economics, utility is a measure of a consumer's preferences on an alternative set of commodities or services (Coleman & Fararo, 1992, Islam et al., 2009a). Utility maximization policy is the best way for the sustainability of the organizations (Kirsh, 2017). It is a blessing both for individuals and the organizations (Eaton & Lipsey, 1975).

The method of Lagrange multiplier is a very useful and powerful technique in multivariable calculus, which transfers a constrained problem to a higher dimensional unconstrained problem (Islam et al., 2009a,b, 2010). The sensitivity analysis plays an important role to predict on future production of the commodities and about future profit of the organizations (Islam et al., 2010). In this study, we have included four commodity variables, the determinant of 6×6 Hessian matrix, and 6×10 Jacobian matrix to investigate the sensitivity analysis.

2. Literature Review

Two American researchers, John V. Baxley and John C. Moorhouse have discussed the utility maximization through the mathematical formulation by illustrating an explicit example (Baxley & Moorhouse, 1984). Well-known mathematician Jamal Nazrul Islam (1939-2013) and his coauthors have discussed utility maximization by considering reasonable interpretation of the Lagrange multipliers (Islam et al., 2010, 2011). Pahlaj Moolio and his coworkers have worked on the Cobb-Douglas production functions to determine maximum profit (Moolio et al., 2009). Novel and young researcher Lia Roy and her coauthors have applied necessary and sufficient conditions to make the economic model for the cost minimization problem of an industry for its sustainable development (Roy et al., 2021).

Sabo Nelson Pandi and his coauthors have used the Lagrange multiplier method to derive a mathematical formulation that works out an optimal solution for a 4-period overlapping generation model with autonomous consumption to maximize a lifetime utility for households subject to age-specific inter-temporal budget constraints (Pandi et al., 2022). Ying Hu and her coworkers consider the problem of utility maximization for small traders on incomplete financial markets. They have wanted to show that a small trader can maximize the utility from his/her final wealth measured by some utility function (Hu et al., 2005). Haradhan Kumar Mohajan has considered the utility maximization and cost minimization techniques (Mohajan, 2021a, 2022a). He and his coauthors also have investigated optimization problems for the social welfare (Mohajan et al., 2013).

Jannatul Ferdous and Haradhan Kumar Mohajan have tried to calculate a profit maximization problem from sale items of an industry. They have realized that profit function plays an important role in modern economics for the development of global financial structure, and to achieve maximum profit an industry must be careful in every step of its operation (Ferdous & Mohajan, 2022). Devajit Mohajan and Haradhan Kumar Mohajan have realized that the sensitivity analysis provides the economic predictions of future production of an industry (Mohajan & Mohajan, 2022a).

3. Research Methodology of the Study

Research is an essential and influential works to the academicians (Pandey & Pandey, 2015). Methodology is a system of explicit rules and procedures in which research is based (Ojo, 2003). Therefore, research methodology is a guideline to accomplish a good research (Kothari, 2008). To prepare this paper we have considered four commodity variables, two Lagrange multipliers λ_1 and λ_2, 6×6 Hessian, and 6×10 Jacobian. After using two Lagrange multipliers we have observed that 4-dimensional constrained problem has developed to a 6-dimensional unconstrained problem that maximizes utility function (Mohajan, 2017b, 2020). We have tried to preserve the reliability and validity of the research analysis

(Mohajan, 2017a, 2022b). Throughout the study we have shown mathematical calculations very clearly (Mohajan, 2018b, Mohajan & Mohajan, 2022a). In this study, we have used material from the secondary data sources of utility maximization. We have consulted the books and handbooks of famous authors, journal articles, internet, websites, etc. to enrich this article (Islam et al., 2012; Mohajan, 2012, 2014a,b, 2017c, 2018a, 2022c).

4. Objective of the Study

The chief objective of this study is to demonstrate sensitivity analysis between Lagrange multipliers and consumer coupon when utility maximization is analyzed. The other trivial objectives are as follows:

- to show the mathematical calculations elaborately, and
- to provide the economic results accurately.

5. An Economic Model of Utility

To study sensitivity analysis we consider four commodities: W_1, W_2, W_3, and W_4. Let the consumers in the society wants to purchase w_1, w_2, w_3, and w_4 amounts from these four commodities W_1, W_2, W_3, and W_4, respectively. The utility function for these four commodities can be written as (Islam et al., 2010; Mohajan & Mohajan, 2022b),

$$U(w_1, w_2, w_3, w_4) = w_1 w_2 w_3 w_4 . \qquad (1)$$

The budget constraint of the consumers is,

$$B(w_1, w_2, w_3, w_4) = p_1 w_1 + p_2 w_2 + p_3 w_3 + p_4 w_4 \qquad (2)$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities w_1, w_2, w_3, and w_4, respectively. Now the coupon constraint is,

$$K(w_1, w_2, w_3, w_4) = k_1 w_1 + k_2 w_2 + k_3 w_3 + k_4 w_4 , \qquad (3)$$

where k_1, k_2, k_3, and k_4 are the coupons necessary to purchase a unit of commodity of w_1, w_2, w_3, and w_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $V(w_1, w_2, w_3, w_4, \lambda_1, \lambda_2)$ as (Baxley & Moorhouse, 1984; Ferdous & Mohajan, 2022),

$$V(w_1, w_2, w_3, w_4, \lambda_1, \lambda_2) = w_1 w_2 w_3 w_4 + \lambda_1 (B - p_1 w_1 - p_2 w_2 - p_3 w_3 - p_4 w_4)$$
$$+ \lambda_2 (K - k_1 w_1 - k_2 w_2 - k_3 w_3 - k_4 w_4). \qquad (4)$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where λ_1 and λ_2 are two Lagrange multipliers.

Now taking first and second order and cross-partial derivatives in (4) we obtain (Islam et al. 2009a,b; Mohajan & Mohajan, 2022d);

$$B_1 = p_1, \quad B_2 = p_2, \quad B_3 = p_3, \quad B_4 = p_4.$$
$$K_1 = k_1, \quad K_2 = k_2, \quad K_3 = k_3, \quad K_4 = k_4. \qquad (5)$$
$$V_{11} = 0, \quad V_{12} = V_{21} = w_3 w_4, \quad V_{13} = V_{31} = w_2 w_4,$$
$$V_{14} = V_{41} = w_2 w_3, \quad V_{22} = 0, \quad V_{23} = V_{32} = w_1 w_4,$$
$$V_{24} = V_{42} = w_1 w_3, \quad V_{33} = 0, \quad V_{34} = V_{43} = w_1 w_2, \quad V_{44} = 0; \qquad (6)$$

where $\frac{\partial B}{\partial w_1} = B_1$, $\frac{\partial K}{\partial w_1} = K_1$, $\frac{\partial B}{\partial w_2} = B_2$, $\frac{\partial V}{\partial w_1} = V_1$, $\frac{\partial^2 V}{\partial w_1 \partial w_3} = V_{31}$, $\frac{\partial^2 V}{\partial w_2^2} = V_{22}$, etc. indicate partial derivatives of multivariate functions.

Now we consider the bordered Hessian (Mohajan, 2021a; Mohajan & Mohajan, 2022c),

$$|H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & V_{11} & V_{12} & V_{13} & V_{14} \\ -B_2 & -K_2 & V_{21} & V_{22} & V_{23} & V_{24} \\ -B_3 & -K_3 & V_{31} & V_{32} & V_{33} & V_{34} \\ -B_4 & -K_4 & V_{41} & V_{42} & V_{43} & V_{44} \end{vmatrix}. \qquad (7)$$

We use $p_3 = p_1$ and $p_4 = p_2$, i.e., amount of a pair of prices are same, and $k_3 = k_1$ and $k_4 = k_2$, i.e., a pair of coupon numbers are same. Now we consider that in the expansion of (7) every term contains $p_1 p_2 k_1 k_2$, then from (7) we can derive (Mohajan & Mohajan, 2022e);

$$|H| = -2 p_1 p_2 k_1 k_2 < 0. \qquad (8)$$

For w_1, w_2, w_3, w_4, λ_1, and λ_2 in terms of p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, *B*, and *K* we can calculate sixty partial derivatives, such as $\frac{\partial \lambda_1}{\partial p_1}$, $\frac{\partial \lambda_2}{\partial p_1}$, …, $\frac{\partial \lambda_1}{\partial k_1}$, $\frac{\partial \lambda_2}{\partial k_1}$, …, $\frac{\partial w_1}{\partial p_1}$, …, $\frac{\partial w_1}{\partial k_1}$, …,

$\frac{\partial \lambda_1}{\partial B}$, …, $\frac{\partial \lambda_1}{\partial K}$, etc., (Islam et al., 2011; Mohajan, 2021c). Now we consider 6×6 Hessian and Jacobian matrix as (Mohajan, 2021b; Mohajan & Mohajan, 2022a);

$$|J| = |H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & V_{11} & V_{12} & V_{13} & V_{14} \\ -B_2 & -K_2 & V_{21} & V_{22} & V_{23} & V_{24} \\ -B_3 & -K_3 & V_{31} & V_{32} & V_{33} & V_{34} \\ -B_4 & -K_4 & V_{41} & V_{42} & V_{43} & V_{44} \end{vmatrix} \tag{9}$$

which is non-singular at the optimum point $\left(w_1^*, w_2^*, w_3^*, w_4^*, \lambda_1^*, \lambda_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (9) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem. We have total 16 variables in our study, such as λ_1, λ_2, w_1, w_2, w_3, w_4 p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and K. By the implicit function theorem, we can write (Moolio et al., 2009; Islam et al., 2010);

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, K\right). \tag{10}$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by (Mohajan, 2021a; Mohajan & Mohajan, 2022a),

$$J_G = \begin{bmatrix} \frac{\partial \lambda_1}{\partial p_1} & \frac{\partial \lambda_1}{\partial p_2} & \frac{\partial \lambda_1}{\partial p_3} & \frac{\partial \lambda_1}{\partial p_4} & \frac{\partial \lambda_1}{\partial k_1} & \frac{\partial \lambda_1}{\partial k_2} & \frac{\partial \lambda_1}{\partial k_3} & \frac{\partial \lambda_1}{\partial k_4} & \frac{\partial \lambda_1}{\partial B} & \frac{\partial \lambda_1}{\partial K} \\ \frac{\partial \lambda_2}{\partial p_1} & \frac{\partial \lambda_2}{\partial p_2} & \frac{\partial \lambda_2}{\partial p_3} & \frac{\partial \lambda_2}{\partial p_4} & \frac{\partial \lambda_2}{\partial k_1} & \frac{\partial \lambda_2}{\partial k_2} & \frac{\partial \lambda_2}{\partial k_3} & \frac{\partial \lambda_2}{\partial k_4} & \frac{\partial \lambda_2}{\partial B} & \frac{\partial \lambda_2}{\partial K} \\ \frac{\partial w_1}{\partial p_1} & \frac{\partial w_1}{\partial p_2} & \frac{\partial w_1}{\partial p_3} & \frac{\partial w_1}{\partial p_4} & \frac{\partial w_1}{\partial k_1} & \frac{\partial w_1}{\partial k_2} & \frac{\partial w_1}{\partial k_3} & \frac{\partial w_1}{\partial k_4} & \frac{\partial w_1}{\partial B} & \frac{\partial w_1}{\partial K} \\ \frac{\partial w_2}{\partial p_1} & \frac{\partial w_2}{\partial p_2} & \frac{\partial w_2}{\partial p_3} & \frac{\partial w_2}{\partial p_4} & \frac{\partial w_2}{\partial k_1} & \frac{\partial w_2}{\partial k_2} & \frac{\partial w_2}{\partial k_3} & \frac{\partial w_2}{\partial k_4} & \frac{\partial w_2}{\partial B} & \frac{\partial w_2}{\partial K} \\ \frac{\partial w_3}{\partial p_1} & \frac{\partial w_3}{\partial p_2} & \frac{\partial w_3}{\partial p_3} & \frac{\partial w_3}{\partial p_4} & \frac{\partial w_3}{\partial k_1} & \frac{\partial w_3}{\partial k_2} & \frac{\partial w_3}{\partial k_3} & \frac{\partial w_3}{\partial k_4} & \frac{\partial w_3}{\partial B} & \frac{\partial w_3}{\partial K} \\ \frac{\partial w_4}{\partial p_1} & \frac{\partial w_4}{\partial p_2} & \frac{\partial w_4}{\partial p_3} & \frac{\partial w_4}{\partial p_4} & \frac{\partial w_4}{\partial k_1} & \frac{\partial w_4}{\partial k_2} & \frac{\partial w_4}{\partial k_3} & \frac{\partial w_4}{\partial k_4} & \frac{\partial w_4}{\partial B} & \frac{\partial w_4}{\partial K} \end{bmatrix}. \tag{11}$$

$$= -J^{-1}\begin{bmatrix} -w_1 & -w_2 & -w_3 & -w_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -w_1 & -w_2 & -w_3 & -w_4 & 0 & 1 \\ -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}. \tag{12}$$

The inverse of Jacobian matrix is, $J^{-1} = \frac{1}{|J|}C^T$, where $C = (C_{ij})$, the matrix of cofactors of J, and T indicates transpose, then (12) becomes (Mohajan, 2017a; Islam et al., 2009b, 2011),

$$J_G = -\frac{1}{|J|}C^T\begin{bmatrix} -w_1 & -w_2 & -w_3 & -w_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -w_1 & -w_2 & -w_3 & -w_4 & 0 & 1 \\ -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}. \tag{13}$$

Now 6×6 transpose matrix C^T can be represented by,

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \tag{14}$$

Using (14) we can write (11) as a 6×10 Jacobian matrix (Mohajan & Mohajan, 2022b);

$$J_G = -\frac{1}{|J|}\left[\begin{matrix} -w_1C_{11}-\lambda_1C_{31} & -w_2C_{11}-\lambda_1C_{41} & -w_3C_{11}-\lambda_1C_{51} & -w_4C_{11}-\lambda_1C_{61} & -w_1C_{21}-\lambda_2C_{31} \\ -w_1C_{12}-\lambda_1C_{32} & -w_2C_{12}-\lambda_1C_{42} & -w_3C_{12}-\lambda_1C_{52} & -w_4C_{12}-\lambda_1C_{62} & -w_1C_{22}-\lambda_2C_{32} \\ -w_1C_{13}-\lambda_1C_{33} & -w_2C_{13}-\lambda_1C_{43} & -w_3C_{13}-\lambda_1C_{53} & -w_4C_{13}-\lambda_1C_{63} & -w_1C_{23}-\lambda_2C_{33} \\ -w_1C_{14}-\lambda_1C_{34} & -w_2C_{14}-\lambda_1C_{44} & -w_3C_{14}-\lambda_1C_{54} & -w_4C_{14}-\lambda_1C_{64} & -w_1C_{24}-\lambda_2C_{34} \\ -w_1C_{15}-\lambda_1C_{35} & -w_2C_{15}-\lambda_1C_{45} & -w_3C_{15}-\lambda_1C_{55} & -w_4C_{15}-\lambda_1C_{65} & -w_1C_{25}-\lambda_2C_{35} \\ -w_1C_{16}-\lambda_1C_{36} & -w_2C_{16}-\lambda_1C_{46} & -w_3C_{16}-\lambda_1C_{56} & -w_4C_{16}-\lambda_1C_{66} & -w_1C_{26}-\lambda_2C_{36} \end{matrix}\right.$$

$$\left.\begin{matrix} -w_2C_{21}-\lambda_2C_{41} & -w_3C_{21}-\lambda_2C_{51} & -w_4C_{21}-\lambda_2C_{61} & C_{11} & C_{21} \\ -w_2C_{22}-\lambda_2C_{42} & -w_3C_{22}-\lambda_2C_{52} & -w_4C_{22}-\lambda_2C_{62} & C_{12} & C_{22} \\ -w_2C_{23}-\lambda_2C_{43} & -w_3C_{23}-\lambda_2C_{53} & -w_4C_{23}-\lambda_2C_{63} & C_{13} & C_{23} \\ -w_2C_{24}-\lambda_2C_{44} & -w_3C_{24}-\lambda_2C_{54} & -w_4C_{24}-\lambda_2C_{64} & C_{14} & C_{24} \\ -w_2C_{25}-\lambda_2C_{45} & -w_3C_{25}-\lambda_2C_{55} & -w_4C_{25}-\lambda_2C_{65} & C_{15} & C_{25} \\ -w_2C_{26}-\lambda_2C_{46} & -w_3C_{26}-\lambda_2C_{56} & -w_4C_{26}-\lambda_2C_{66} & C_{16} & C_{26} \end{matrix}\right]. \tag{15}$$

Now we analyze the nature of Lagrange multiplier λ_1 when total coupon K of the consumers increases. Taking $T_{1(10)}$, (i.e., term of 1st row and 10th column) from both sides of (15) we get (Islam et al., 2011; Mohajan & Mohajan, 2022e),

$$\frac{\partial \lambda_1}{\partial K} = -\frac{1}{|J|}\left[C_{21}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{21}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ -K_1 & V_{11} & V_{12} & V_{13} & V_{14} \\ -K_2 & V_{21} & V_{22} & V_{23} & V_{24} \\ -K_3 & V_{31} & V_{32} & V_{33} & V_{34} \\ -K_4 & V_{41} & V_{42} & V_{43} & V_{44} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{ B_1\begin{vmatrix} -K_1 & V_{12} & V_{13} & V_{14} \\ -K_2 & V_{22} & V_{23} & V_{24} \\ -K_3 & V_{32} & V_{33} & V_{34} \\ -K_4 & V_{42} & V_{43} & V_{44} \end{vmatrix} - B_2\begin{vmatrix} -K_1 & V_{11} & V_{13} & V_{14} \\ -K_2 & V_{21} & V_{23} & V_{24} \\ -K_3 & V_{31} & V_{33} & V_{34} \\ -K_4 & V_{41} & V_{43} & V_{44} \end{vmatrix} + B_3\begin{vmatrix} -K_1 & V_{11} & V_{12} & V_{14} \\ -K_2 & V_{21} & V_{22} & V_{24} \\ -K_3 & V_{31} & V_{32} & V_{34} \\ -K_4 & V_{41} & V_{42} & V_{44} \end{vmatrix} \right.$$
$$\left. - B_4\begin{vmatrix} -K_1 & V_{11} & V_{12} & V_{13} \\ -K_2 & V_{21} & V_{22} & V_{23} \\ -K_3 & V_{31} & V_{32} & V_{33} \\ -K_4 & V_{41} & V_{42} & V_{43} \end{vmatrix} \right\}$$

$$= \frac{1}{|J|}\left[B_1\left\{ -K_1\begin{vmatrix} V_{22} & V_{23} & V_{24} \\ V_{32} & V_{33} & V_{34} \\ V_{42} & V_{43} & V_{44} \end{vmatrix} - V_{12}\begin{vmatrix} -K_2 & V_{23} & V_{24} \\ -K_3 & V_{33} & V_{34} \\ -K_4 & V_{43} & V_{44} \end{vmatrix} + V_{13}\begin{vmatrix} -K_2 & V_{22} & V_{24} \\ -K_3 & V_{32} & V_{34} \\ -K_4 & V_{42} & V_{44} \end{vmatrix} - V_{14}\begin{vmatrix} -K_2 & V_{22} & V_{23} \\ -K_3 & V_{32} & V_{33} \\ -K_4 & V_{42} & V_{43} \end{vmatrix} \right\} \right.$$
$$- B_2\left\{ -K_1\begin{vmatrix} V_{21} & V_{23} & V_{24} \\ V_{31} & V_{33} & V_{34} \\ V_{41} & V_{43} & V_{44} \end{vmatrix} + V_{13}\begin{vmatrix} -K_2 & V_{21} & V_{24} \\ -K_3 & V_{31} & V_{34} \\ -K_4 & V_{41} & V_{44} \end{vmatrix} - V_{14}\begin{vmatrix} -K_2 & V_{21} & V_{23} \\ -K_3 & V_{31} & V_{33} \\ -K_4 & V_{41} & V_{43} \end{vmatrix} \right\} + B_3\left\{ -K_1\begin{vmatrix} V_{21} & V_{22} & V_{24} \\ V_{31} & V_{32} & V_{34} \\ V_{41} & V_{42} & V_{44} \end{vmatrix} \right.$$
$$\left. + V_{12}\begin{vmatrix} -K_2 & V_{21} & V_{24} \\ -K_3 & V_{31} & V_{34} \\ -K_4 & V_{41} & V_{44} \end{vmatrix} - V_{14}\begin{vmatrix} -K_2 & V_{21} & V_{22} \\ -K_3 & V_{31} & V_{32} \\ -K_4 & V_{41} & V_{42} \end{vmatrix} \right\} - B_4\left\{ -K_1\begin{vmatrix} V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \\ V_{41} & V_{42} & V_{43} \end{vmatrix} + V_{12}\begin{vmatrix} -K_2 & V_{21} & V_{23} \\ -K_3 & V_{31} & V_{33} \\ -K_4 & V_{41} & V_{43} \end{vmatrix} \right.$$
$$\left. \left. - V_{13}\begin{vmatrix} -K_2 & V_{21} & V_{22} \\ -K_3 & V_{31} & V_{32} \\ -K_4 & V_{41} & V_{42} \end{vmatrix} \right\} \right]$$

$$= \frac{1}{|J|}\{ -B_1K_1V_{23}V_{24}V_{34} - B_1K_1V_{23}V_{24}V_{34} - B_1K_2V_{12}V_{34}^2 + B_1K_4V_{12}V_{23}V_{34} + B_1K_3V_{12}V_{24}V_{34} + B_1K_2V_{13}V_{24}V_{34}$$
$$- B_1K_3V_{13}V_{24}^2 + B_1K_4V_{13}V_{23}V_{24} + B_1K_2V_{14}V_{23}V_{34} - B_1K_3V_{14}V_{23}V_{24} - B_1K_4V_{14}V_{23}^2 - B_2K_1V_{12}V_{14}V_{34}$$

$$+B_2K_1V_{14}V_{23}V_{34} + B_2K_1V_{13}V_{24}V_{34} - B_2K_2V_{13}V_{14}V_{34} + B_2K_4V_{12}V_{13}V_{34} + B_2K_3V_{13}V_{14}V_{24} - B_2K_4V_{24}V_{13}^2$$
$$- B_2K_2V_{13}V_{14}V_{34} + B_2K_3V_{12}V_{14}V_{34} - B_2K_3V_{23}V_{14}^2 + B_2K_4V_{13}V_{14}V_{23} + B_3K_1V_{12}V_{24}V_{34} - B_3K_1V_{13}V_{24}^2$$
$$+ B_3K_1V_{14}V_{23}V_{24} + B_3K_2V_{12}V_{14}V_{34} - K_3K_4V_{34}V_{12}^2 - B_3K_3V_{12}V_{14}V_{24} + B_3K_4V_{12}V_{14}V_{23} + B_4K_1V_{12}V_{23}V_{34}$$
$$+ B_4K_1V_{13}V_{23}V_{24} - B_4K_1V_{14}V_{23}^2 + B_4K_2V_{12}V_{13}V_{34} - B_4K_3V_{34}V_{12}^2 + B_4K_3V_{12}V_{14}V_{23} - B_4K_4V_{12}V_{13}V_{23}$$
$$- B_4K_2V_{24}V_{13}^2 + B_4K_2V_{13}V_{14}V_{23} + B_4K_3V_{12}V_{13}V_{24} - B_4K_4V_{12}V_{13}V_{23}\}$$

$$= \frac{1}{|J|}\{-2p_1k_1w_1^3w_2w_3w_4 - p_1k_2w_1^2w_2^2w_3w_4 + p_1k_4w_1^2w_2w_3w_4^2 + p_1k_3w_1^2w_2w_3^2w_4 - p_1k_2w_1^2w_2^2w_3w_4$$
$$- p_1k_3w_1^2w_2w_3^2w_4 + p_1k_4w_1^2w_2w_3w_4^2 + p_1k_2w_1^2w_2^2w_3w_4 - p_1k_3w_1^2w_2w_3^2w_4 - p_1k_4w_1^2w_2w_3w_4^2$$
$$- k_1p_2w_1w_2^2w_3^2w_4 + p_2k_1w_1^2w_2^2w_3w_4 + p_2k_1w_1^2w_2^2w_3w_4 - p_2k_2w_1w_2^3w_3w_4 + p_2k_4w_1w_2^2w_3w_4^2$$
$$+ p_2k_3w_1w_2^2w_3^2w_4 - p_2k_4w_1w_2^2w_3w_4^2 - p_2k_2w_1w_2^3w_3w_4 + p_2k_3w_1w_2^2w_3^2w_4 - p_2k_3w_1w_2^2w_3^2w_4$$
$$+ p_2k_4w_1w_2^2w_3w_4^2 + p_3k_1w_1^2w_2w_3^2w_4 - p_3k_1w_1^2w_2w_3^2w_4 + p_3k_1w_1^2w_2w_3^2w_4 + p_3k_2w_1w_2^2w_3^2w_4$$
$$- p_3k_4w_1w_2w_3^2w_4^2 - p_3k_3w_1w_2w_3^3w_4 + p_3k_4w_1w_2w_3^2w_4^2 + p_4k_1w_1^2w_2w_3w_4^2 + p_4k_1w_1^2w_2w_3w_4^2$$
$$- p_4k_1w_1^2w_2w_3w_4^2 + p_4k_2w_1w_2^2w_3w_4^2 - p_4k_3w_1w_2w_3^2w_4^2 + p_4k_3w_1w_2w_3^2w_4^2 - p_4k_4w_1w_2w_3w_4^3$$
$$- p_4k_2w_1w_2^2w_3w_4^2 + p_4k_2w_1w_2^2w_3w_4^2 + p_4k_3w_1w_2w_3^2w_4^2 - p_4k_4w_1w_2w_3w_4^3\}$$

$$= \frac{1}{|J|}\{-2p_1k_1w_1^3w_2w_3w_4 - 2p_2k_2w_1w_2^3w_3w_4 - p_3k_3w_1w_2w_3^3w_4 - 2p_4k_4w_1w_2w_3w_4^3 - 2p_1k_2w_1^2w_2^2w_3w_4$$
$$+ 2p_2k_1w_1^2w_2^2w_3w_4 + p_1k_4w_1^2w_2w_3w_4^2 + p_4k_1w_1^2w_2w_3w_4^2 - p_1k_3w_1^2w_2w_3^2w_4 + p_3k_1w_1^2w_2w_3^2w_4$$
$$+ p_2k_4w_1w_2^2w_3w_4^2 + p_4k_2w_1w_2^2w_3w_4^2 + p_2k_3w_1w_2^2w_3^2w_4 + p_3k_2w_1w_2^2w_3^2w_4 + p_4k_3w_1w_2w_3^2w_4^2\}$$

$$\frac{\partial\lambda_1}{\partial K} = \frac{w_1w_2w_3w_4}{|J|}\{-2p_1k_1w_1^2 - 2p_2k_2w_2^2 - p_3k_3w_3^2 - 2p_4k_4w_4^2 + 2(p_2k_1 - p_1k_2)w_1w_2 + (p_3k_1 - p_1k_3)w_1w_3$$
$$+ (p_1k_4 + p_4k_1)w_1w_4 + (p_2k_3 + p_3k_2)w_2w_3 + (p_2k_4 + p_4k_2)w_2w_4 + p_4k_3w_3w_4\}. \quad (16)$$

Using $w_1 = w_2 = w_3 = w_4 = 1$ in (16) we get,

$$\frac{\partial\lambda_1}{\partial K} = \frac{1}{|J|}\{-2p_1k_1 - 2p_2k_2 - p_3k_3 - 2p_4k_4 + 2p_2k_1 - 2p_1k_2 + p_3k_1 - p_1k_3 + p_1k_4 + p_4k_1 + p_2k_3 + p_3k_2$$
$$+ p_2k_4 + p_4k_2 + p_4k_3\}. \quad (17)$$

Using $k_3 = k_1$ and $k_4 = k_2$ in (17) we get,

$$\frac{\partial\lambda_1}{\partial K} = \frac{1}{|J|}(-3p_1k_1 - 2p_2k_2 + 5p_2k_1). \quad (18)$$

Let $k_1 = k_2 = k$, and $|J| = -2p_1p_2k^2$ then we get from (18),

$$\frac{\partial\lambda_1}{\partial K} = \frac{3(p_1 - p_2)}{2p_1p_2k} \quad (19)$$

where $p_1p_2k > 0$.

If $p_1 > p_2$ in (19) we get,

$$\frac{\partial\lambda_1}{\partial K} > 0. \quad (20)$$

Inequality (20) indicates that if the total coupon of the consumers' increases, the level of marginal utility will also increase. Therefore, in this situation the consumers will collect more coupons. Depending on the consumers' demand, the organization should take attempts to increase the production level.

If $p_2 > p_1$ in (19) we get,

$$\frac{\partial \lambda_1}{\partial K} < 0. \tag{21}$$

Inequality (21) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, in this situation the consumers will reduce the collection of coupons. Depending on the consumers' demand, the organization should take attempts to decrease the production level.

In this study we observe that, $\frac{\partial \lambda_1}{\partial K} \neq 0$. Therefore, from (19) we see that, $p_1 \neq p_2$, i.e., the prices of two commodities w_1 and w_2 are not equal and consequently, these are different goods.

Now we analyze the nature of Lagrange multiplier λ_2 when total coupon of the consumers increases. Taking $T_{2(10)}$, (i.e., term of 2nd row and 10th column) from both sides of (15) we get (Islam et al., 2010; Mohajan & Mohajan, 2022e,f),

$$\frac{\partial \lambda_2}{\partial K} = -\frac{1}{|J|}[C_{22}]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{22}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ -B_1 & V_{11} & V_{12} & V_{13} & V_{14} \\ -B_2 & V_{21} & V_{22} & V_{23} & V_{24} \\ -B_3 & V_{31} & V_{32} & V_{33} & V_{34} \\ -B_4 & V_{41} & V_{42} & V_{43} & V_{44} \end{vmatrix}$$

$$
= \frac{1}{|J|}\left\{ B_1 \begin{vmatrix} -B_1 & V_{12} & V_{13} & V_{14} \\ -B_2 & V_{22} & V_{23} & V_{24} \\ -B_3 & V_{32} & V_{33} & V_{34} \\ -B_4 & V_{42} & V_{43} & V_{44} \end{vmatrix} - B_2 \begin{vmatrix} -B_1 & V_{11} & V_{13} & V_{14} \\ -B_2 & V_{21} & V_{23} & V_{24} \\ -B_3 & V_{31} & V_{33} & V_{34} \\ -B_4 & V_{41} & V_{43} & V_{44} \end{vmatrix} + B_3 \begin{vmatrix} -B_1 & V_{11} & V_{12} & V_{14} \\ -B_2 & V_{21} & V_{22} & V_{24} \\ -B_3 & V_{31} & V_{32} & V_{34} \\ -B_4 & V_{41} & V_{42} & V_{44} \end{vmatrix} \right.
$$

$$
\left. - B_4 \begin{vmatrix} -B_1 & V_{11} & V_{12} & V_{13} \\ -B_2 & V_{21} & V_{22} & V_{23} \\ -B_3 & V_{31} & V_{32} & V_{33} \\ -B_4 & V_{41} & V_{42} & V_{43} \end{vmatrix} \right\}
$$

$$
= \frac{1}{|J|}\left[B_1 \left\{ -B_1 \begin{vmatrix} V_{22} & V_{23} & V_{24} \\ V_{32} & V_{33} & V_{34} \\ V_{42} & V_{43} & V_{44} \end{vmatrix} - V_{12} \begin{vmatrix} -B_2 & V_{23} & V_{24} \\ -B_3 & V_{33} & V_{34} \\ -B_4 & V_{43} & V_{44} \end{vmatrix} + V_{13} \begin{vmatrix} -B_2 & V_{22} & V_{24} \\ -B_3 & V_{32} & V_{34} \\ -B_4 & V_{42} & V_{44} \end{vmatrix} - V_{14} \begin{vmatrix} -B_2 & V_{22} & V_{23} \\ -B_3 & V_{32} & V_{33} \\ -B_4 & V_{42} & V_{43} \end{vmatrix} \right\} \right.
$$

$$
- B_2 \left\{ -B_1 \begin{vmatrix} V_{21} & V_{23} & V_{24} \\ V_{31} & V_{33} & V_{34} \\ V_{41} & V_{43} & V_{44} \end{vmatrix} + V_{13} \begin{vmatrix} -B_2 & V_{21} & V_{24} \\ -B_3 & V_{31} & V_{34} \\ -B_4 & V_{41} & V_{44} \end{vmatrix} - V_{14} \begin{vmatrix} -B_2 & V_{21} & V_{23} \\ -B_3 & V_{31} & V_{33} \\ -B_4 & V_{41} & V_{43} \end{vmatrix} \right\} + B_3 \left\{ -B_1 \begin{vmatrix} V_{21} & V_{22} & V_{24} \\ V_{31} & V_{32} & V_{34} \\ V_{41} & V_{42} & V_{44} \end{vmatrix} \right.
$$

$$
\left. \left. + V_{12} \begin{vmatrix} -B_2 & V_{21} & V_{24} \\ -B_3 & V_{31} & V_{34} \\ -B_4 & V_{41} & V_{44} \end{vmatrix} - V_{14} \begin{vmatrix} -B_2 & V_{21} & V_{22} \\ -B_3 & V_{31} & V_{32} \\ -B_4 & V_{41} & V_{42} \end{vmatrix} \right\} - B_4 \left\{ -B_1 \begin{vmatrix} V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \\ V_{41} & V_{42} & V_{43} \end{vmatrix} + V_{12} \begin{vmatrix} -B_2 & V_{21} & V_{23} \\ -B_3 & V_{31} & V_{33} \\ -B_4 & V_{41} & V_{43} \end{vmatrix} - V_{13} \begin{vmatrix} -B_2 & V_{21} & V_{22} \\ -B_3 & V_{31} & V_{32} \\ -B_4 & V_{41} & V_{42} \end{vmatrix} \right\} \right]
$$

$$
= \frac{1}{|J|}\left\{ -2B_1^2 V_{23}V_{24}V_{34} - B_1B_2V_{12}V_{34}^2 + B_1B_4V_{12}V_{23}V_{34} + B_1B_3V_{12}V_{24}V_{34} + B_1B_2V_{13}V_{24}V_{34} - B_1B_3V_{13}V_{24}^2 \right.
$$

$$
+ B_1B_4V_{13}V_{23}V_{24} + B_1B_2V_{14}V_{23}V_{34} + B_1B_3V_{14}V_{23}V_{24} - B_1B_4V_{14}V_{23}^2 - B_1B_2V_{12}V_{34}^2 + B_1B_2V_{14}V_{23}V_{34}
$$

$$
+ B_1B_2V_{13}V_{24}V_{34} - B_2^2V_{13}V_{14}V_{34} - B_2B_4V_{12}V_{13}V_{34} + B_2B_3V_{13}V_{14}V_{24} - B_2B_4V_{13}^2V_{24} - B_2^2V_{13}V_{14}V_{34}
$$

$$
+ B_2B_3V_{12}V_{14}V_{34} - B_2B_3V_{14}^2V_{23} + B_2B_4V_{13}V_{14}V_{23} - B_1B_3V_{12}V_{24}V_{34} - B_1B_3V_{13}V_{24}^2 + B_1B_3V_{14}V_{23}V_{24}
$$

$$
+ B_2B_3V_{12}V_{14}V_{34} - B_3B_4V_{12}^2V_{34} - B_3^2V_{12}V_{14}V_{24} + B_3B_4V_{12}V_{13}V_{24} + B_1B_4V_{12}V_{23}V_{34} + B_1B_4V_{13}V_{23}V_{24} - B_1B_4V_{14}V_{23}^2
$$

$$
+ B_2B_4V_{12}V_{13}V_{34} - B_3B_4V_{12}^2V_{34} + B_3B_4V_{12}V_{14}V_{23} - B_4^2V_{12}V_{13}V_{23} - B_2B_4V_{13}^2V_{24} + B_2B_4V_{13}V_{14}V_{23}
$$

$$
\left. + B_3B_4V_{12}V_{13}V_{24} - B_4^2V_{12}V_{13}V_{23} \right\}
$$

$$
\frac{\partial \lambda_2}{\partial K} = -\frac{1}{|J|}\left\{ -2p_1^2w_1^3w_2w_3w_4 - p_1p_2w_1^2w_2^2w_3w_4 + p_1p_4w_1^2w_2w_3w_4^2 + p_1p_3w_1^2w_2w_3^2w_4 + p_1p_3w_1^2w_2^2w_3w_4 \right.
$$

$$
- p_1p_3w_1^2w_2w_3^2w_4 + p_1p_4w_1^2w_2w_3w_4^2 + p_1p_2w_1^2w_2^2w_3w_4 + p_1p_3w_1^2w_2w_3^2w_4 - p_1p_4w_1^2w_2w_3w_4^2
$$

$$
- p_1p_2w_1^2w_2^2w_3w_4 + 2p_1p_2w_1^2w_2^2w_3w_4 - p_2^2w_1w_2^3w_3w_4 - p_2p_4w_1w_2^2w_3w_4^2 + p_2p_3w_1w_2^2w_3^2w_4
$$

$$
- p_2p_4w_1w_2^2w_3w_4^2 - p_2^2w_1w_2^3w_3w_4 + p_2p_3w_1w_2^2w_3^2w_4 - p_2p_3w_1w_2^2w_3^2w_4 + p_2p_4w_1w_2^2w_3w_4^2
$$

$$
- p_1p_3w_1^2w_2w_3^2w_4 - p_1p_3w_1^2w_2w_3^2w_4 + p_1p_3w_1^2w_2w_3^2w_4 + p_2p_3w_1w_2^2w_3^2w_4 - p_3p_4w_1w_2w_3^2w_4^2
$$

$$
- p_3^2w_1w_2w_3^3w_4 + p_3p_4w_1w_2w_3^2w_4^2 + p_1p_4w_1^2w_2w_3w_4^2 + p_1p_4w_1^2w_2w_3w_4^2 - p_1p_4w_1^2w_2w_3w_4^2
$$

$$
+ p_2p_4w_1w_2^2w_3w_4^2 - p_3p_4w_1w_2w_3^2w_4^2 + p_3p_4w_1w_2w_3^2w_4^2 - p_4^2w_1w_2w_3w_4^3 - p_2p_4w_1w_2^2w_3w_4^2
$$

$$
\left. + p_2p_4w_1w_2^2w_3w_4^2 + p_3p_4w_1w_2w_3^2w_4^2 - p_4^2w_1w_2w_3w_4^3 \right\}
$$

$$\frac{\partial \lambda_2}{\partial K} = -\frac{1}{|J|}\{-2p_1^2 w_1^3 w_2 w_3 w_4 \quad -2p_2^2 w_1 w_2^3 w_3 w_4 \quad - p_3^2 w_1 w_2 w_3^3 w_4 \quad -2p_4^2 w_1 w_2 w_3 w_4^3 \quad +2p_1 p_2 w_1^2 w_2^2 w_3 w_4$$
$$+2p_1 p_4 w_1^2 w_2 w_3 w_4^2 + 2p_2 p_3 w_1 w_2^2 w_3^2 w_4 + p_3 p_4 w_1 w_2 w_3^2 w_4^2\}. \tag{22}$$

$$\frac{\partial \lambda_2}{\partial K} = -\frac{w_1 w_2 w_3 w_4}{|J|}\{-2p_1^2 w_1^2 - 2p_2^2 w_2^2 - p_3^2 w_3^2 - 2p_4^2 w_4^2 + 2p_1 p_2 w_1 w_2 + 2p_1 p_4 w_1 w_4 + 2p_2 p_3 w_2 w_3$$
$$+ p_3 p_4 w_3 w_4\}. \tag{23}$$

Now we use $p_3 = p_1$, and $p_4 = p_2$ where pair of prices are same, a $|J| = |H| = -2p_1 p_2 k_1 k_2$. Now we use $w_3 = w_1$, and $w_4 = w_2$, then (23) becomes;

$$\frac{\partial \lambda_2}{\partial K} = \frac{w_1^2 w_2^2}{2p_1 p_2 k_1 k_2}(p_2 w_2 - p_1 w_1)(3p_1 w_1 - 4p_2 w_2), \tag{24}$$

where $p_1 p_2 k_1 k_2 > 0$ and $w_1^2 w_2^2 > 0$.

Now if $p_2 w_2 < p_1 w_1 < \frac{4}{3} p_2 w_2$ in (24) we get,

$$\frac{\partial \lambda_2}{\partial K} > 0. \tag{25}$$

Inequality (25) indicates that if the total coupon of the consumers' increases, the level of marginal utility will also increase. Therefore, the consumers will try to collect more coupons. The organization should take attempts to increase the production level, depending on the consumers' demand.

Now if $p_2 w_2 < \frac{3}{4} p_1 w_1$ in (24) we get,

$$\frac{\partial \lambda_2}{\partial K} < 0. \tag{26}$$

Inequality (26) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, the consumers will try to reduce the collection of more coupons. The organization should take attempts to decrease the production level, depending on the consumers' demand.

From (24) we see that, $\frac{\partial \lambda_2}{\partial K} \neq 0$, so that, $p_1 w_1 \neq p_2 w_2$ and also $3p_1 w_1 \neq 4p_2 w_2$, i.e., the prices of two commodities w_1 and w_2 are not equal, i.e., it seems that these goods are different.

6. Conclusions

In this study we have taken attempts to discuss sensitivity analysis between Lagrange multipliers and total coupon during the utility maximization investigation. To discuss utility maximization we have used two constraints: budget constraint and coupon constraint. In this article we have run the mathematical calculations with four commodity variables. We have observed that uses of the Lagrange multipliers are very fruitful both for the consumers and producers.

References

Baxley, J. V., & Moorhouse, J. C. (1984). Lagrange Multiplier Problems in Economics. *The American Mathematical Monthly*, 91(7), 404-412.

Bentham, J. (1780). *An Introduction to the Principles of Morals and Legislation*. CreateSpace Independent Publishing Platform.

Coleman, J. S. & Fararo, T. J. (1992). *Rational Choice Theory*. Nueva York: Sage.

Eaton, B., & Lipsey, R. (1975). The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition. *Review of Economic Studies*, 42(1), 27-49.

Ferdous, J., & Mohajan, H. K. (2022). Maximum Profit Ensured for Industry Sustainability. *Annals of Spiru Haret University. Economic Series*, 22(3), 317-337.

Hu, Y., Imkeller, P., & Müller, M. (2005). Utility Maximization in Incomplete Markets. *The Annals of Applied Probability*, 15(3), 1691-1712.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2009a). Preference of Social Choice in Mathematical Economics. *Indus Journal of Management & Social Sciences*, 3(1), 17-38.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2009b). Political Economy and Social Welfare with Voting Procedure. *KASBIT Business Journal*, 2(1), 42-66.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2010). Utility Maximization Subject to Multiple Constraints. *Indus Journal of Management & Social Sciences*, 4(1), 15-29.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2011). Output Maximization Subject to a Nonlinear Constraint. *KASBIT Business Journal*, 4(1), 116-128.

Islam, J. N., Mohajan, H. K., & Datta, R. (2012). Stress Management Policy Analysis: A Preventative Approach. *International Journal of Economics and Research*, 3(6), 1-17.

Kirsh, Y. (2017). Utility and Happiness in a Prosperous Society. Working Paper Series, No. 37-2017, Institute for Policy Analysis, The Open University of Israel.

Kothari, C. R. (2008). *Research Methodology: Methods and Techniques* (2nd Ed.). New Delhi: New Age International (P) Ltd.

Mohajan, D., & Mohajan, H. K. (2022a). Profit Maximization Strategy in an Industry: A Sustainable Procedure. *Law and Economy*, 1(3), 17-43. https://doi:10.56397/LE.2022.10.02

Mohajan, D., & Mohajan, H. K. (2022b). Utility Maximization Analysis of an Organization: A Mathematical Economic Procedure. *Law and Economy*, 2(1), 1-15.

Mohajan, D., & Mohajan, H. K. (2022c). Utility Maximization Investigation: A Bordered Hessian Method. *Annals of Spiru Haret University. Economic Series*, Manuscript Submitted.

Mohajan, D. & Mohajan, H. K. (2022d). Sensitivity Analysis among Commodities and Prices: Utility Maximization Perspectives (Unpublished Manuscript).

Mohajan, D. & Mohajan, H. K. (2022e). Sensitivity Analysis among Commodities and Coupons during Utility Maximization. *Frontiers in Management Science*, 1(3), 13-28.

Mohajan, D., & Mohajan, H. K. (2022f). Importance of Total Coupon in Utility Maximization: A Sensitivity Analysis. *Law and Economy*, 1(5), 65-67.

Mohajan, H. K. (2012). Green Marketing is a Sustainable Marketing System in the Twenty First Century. *International Journal of Management and Transformation*, 6(2), 23-39.

Mohajan, H. K. (2014a). Greenhouse Gas Emissions of China. *Journal of Environmental Treatment Techniques*, 1(4), 190-202.

Mohajan, H. K. (2014b). Improvement of Health Sector in Kenya. *American Journal of Public Health Research*, 2(4), 159-169.

Mohajan, H. K. (2017a). Optimization Models in Mathematical Economics. *Journal of Scientific Achievements*, 2(5), 30-42.

Mohajan, H. K. (2017b). Two Criteria for Good Measurements in Research: Validity and Reliability. *Annals of Spiru Haret University. Economic Series*, 17(3), 58-82.

Mohajan, H. K. (2017c). Roles of Communities of Practice for the Development of the Society. *Journal of Economic Development, Environment and People*, 6(3), 27-46.

Mohajan, H. K. (2018a). *Aspects of Mathematical Economics, Social Choice and Game Theory*. PhD Dissertation. University of Chittagong, Chittagong, Bangladesh.

Mohajan, H. K. (2018b). Qualitative Research Methodology in Social Sciences and Related Subjects. *Journal of Economic Development, Environment and People*, 7(1), 23-48.

Mohajan, H. K. (2020). Quantitative Research: A Successful Investigation in Natural and Social Sciences. *Journal of Economic Development, Environment and People*, 9(4), 52-79.

Mohajan, H. K. (2021a). Utility Maximization of Bangladeshi Consumers within Their Budget: A Mathematical Procedure. *Journal of Economic Development, Environment and People*, 10(3), 60-85.

Mohajan, H. K. (2021b). Product Maximization Techniques of a Factory of Bangladesh: A Sustainable Procedure. *American Journal of Economics, Finance and Management*, 5(2), 23-44.

Mohajan, H. K. (2021c). Estimation of Cost Minimization of Garments Sector by Cobb-Douglass Production Function: Bangladesh Perspective. *Annals of Spiru Haret University. Economic Series*, 21(2), 267-299.

Mohajan, H. K. (2022a). Cost Minimization Analysis of a Running Firm with Economic Policy. *Annals of Spiru Haret University. Economic Series*, 22(3), 171-181.

Mohajan, H. K. (2022b). An Overview on the Feminism and Its Categories. *Research and Advances in Education*, 1(3), 11-26. https://doi.org/10.56397/RAE.2022.09.02

Mohajan, H. K. (2022c). Four Waves of Feminism: A Blessing for Global Humanity. *Studies in Social Science & Humanities*, 1(2), 1-8. https://doi:10.56397/SSSH.2022.09.01

Mohajan, H. K., Islam, J. N., & Moolio, P. (2013). *Optimization and Social Welfare in Economics*. Lambert Academic Publishing, Germany.

Moolio, P., Islam, J. N., & Mohajan, H. K. (2009). Output Maximization of an Agency. *Indus Journal of Management and Social Sciences*, 3(1), 39-51.

Ojo, S. O. (2003). Productivity and Technical Efficiency of Poultry Egg Production in Nigeria. *International Journal of Poultry Science*, 2(6), 459-464.

Pandey, P., & Pandey, M. M. (2015). *Research Methodology: Tools and Techniques*. Bridge Center, Romania, European Union.

Pandi, S. N, Torsen, E., Martins, D., & Modibbo, U. M. (2022). Application of the Techniques of Determinants to the Utility Maximization for a 4-Period Age-specific Inter-temporal Budget Constraints: Determinants to the Utility Maximization. *Journal of Computational and Cognitive Engineering*. https://doi.org/10.47852/bonviewJCCE2202353

Roy, L., Molla, R., & Mohajan, H. K. (2021). Cost Minimization is Essential for the Sustainable Development of an Industry: A Mathematical Economic Model Approach. *Annals of Spiru Haret University. Economic Series*, 21(1), 37-69.

Samuelson, P. A. (1947). *Foundations of Economic Analysis*. Harvard University Press, Cambridge, MA.

Zheng, K., & Liu, Y. (2022). Application of Mathematical Models in Economic Variable Input and Output Models under the Scientific Visualization. *Computational Intelligence and Neuroscien*ce. Article ID 6269358.