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**Abstract**

Coordination games admit two types of equilibria: coordinated pure equilibria in which everyone plays the same action, and inefficient mixed equilibria with miscoordination. The existing literature shows that populations will converge to one of the pure coordinated equilibria from almost any initial state. By contrast, we show that plausible learning dynamics, in which agents sample the aggregate behavior of the opponent's population and best reply to their samples, can induce stable miscoordination if there is heterogeneity in the sample sizes: some agents base their choices on noisy small samples (anecdotal evidence), while others rely on large samples.

**Keywords**: sampling best Response dynamics, action-sampling dynamics, coordination games, hawk-dove games, evolutionary stability, logit dyanmics. **JEL Classification**: C72, C73.

## 1 Introduction

Many real-life situations can be modeled as (two-player) coordination games, in which the best reply against each opponent's action is to play the same action (possibly, after

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Table 1: Standard Representation of Two-Action Coordination Game

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $u_1$, $u_2$ | 0, 0 |
| $b_1$ | 0, 0 | 1, 1 |

relabeling the actions). Two-action coordination games admit two types of equilibria: two coordinated pure equilibria in which everyone plays the same action, and an inefficient mixed equilibrium with miscoordination. A key result in evolutionary game theory is that under a broad set of learning dynamics the mixed equilibrium is unstable, and populations in which players are randomly matched to play coordination games must converge to everyone playing one of the pure coordinated equilibria (as surveyed in Section 2). By contrast, in this paper we show that the mixed equilibrium with miscoordination can be stable if the populations are heterogeneous in the sense that some (but not all) of the agents rely on anecdotal evidence induced by small samples.

**Highlights of the model** Consider a setup in which pairs of agents from two infinite populations are repeatedly randomly matched to play a (one-shot) coordination game. Agents occasionally die and are replaced by new agents (or, alternatively, agents occasionally receive opportunities to revise their actions). The new agents do not have precise information about the aggregate behavior in the opponent's population, and estimate this from sampling the opponent's population. Specifically, each population $i$ is characterized by a distribution of sample sizes $\theta_i$, such that $\theta_i(k)$ is the frequency of agents with sample size $k$. Each such agent observes the behavior of $k$ random opponents, and then adopts the action that is a best reply to her sample (with an arbitrary tie-breaking rule). This learning dynamics, which seems plausible in various setups, is called *sampling best-response dynamics* (Sandholm, 2001; Osborne and Rubinstein, 2003; Oyama et al., 2015, henceforth abbreviated as *sampling dynamics*).

As explained in Section 3, any two-action coordination game can be represented WLOG by the payoff matrix presented in Table 1 that has two positive parameters $(u_1, u_2)$ that represent the players' payoffs when both playing the first action; the payoffs when coordinating on the second action are normalized to 1, and the payoffs for miscooridnating are normalized to 0.

The following definitions will be helpful for the presentation of our results. An action is $q$-dominant (Morris et al., 1995) to player $i$ if it is the player's best reply against any opponent's mixed action that assigns mass of at-least $q$ to the opponent playing the same action. An action is risk dominant for player $i$ if it is $q$-dominant for $q = \frac{1}{2}$. Observe that action $a_i$ (resp., $b_1$) is risk dominant for player $i$ iff $u_i > 1$ (resp., $u_i < 1$). A pure equilibrium is risk dominant if each equilibrium action is risk dominant for each population (e.g., $(a_1, a_2)$ is risk-dominant equilibrium iff $u_1, u_2 > 1$).

**Global Convergence to Miscoordination** Our first main result (Theorem 1) presents a full characterization for environments in which the populations converge to states with miscoordination from almost all initial states. This happens if (and essentially only if): (1) each population has a different risk dominant action (i.e., $u_2 < 1 < u_1$), (2) the product of the mass of agents with sample size 1 times the expected sample size is larger than one (i.e., $\forall i,\ \theta_i(1) \cdot \mathbb{E}(\theta_{-i}) > 1$ ), and (3) the risk-dominant action of each population is $q$-dominant for a sufficiently low $q$ (which is satisfied iff each $u_i$ is sufficiently far from 1).

The proof idea is as follows. Global convergence to miscoordination occurs iff both pure equilibria are unstable. Assume that $u_2 < 1 < u_1$. Consider a slightly perturbed state near the equilibrium $(a_1, a_2)$, in which $\epsilon_i << 1$ of the agents in population $i$ plays $b_i$. Events in which a new agent observes multiple occurrences of the rare opponent action $b_{-i}$ in her sample are negligible ($O\left(\epsilon_{-i}^2\right)$). Neglecting these very rare events imply that new agents of population 1 will adopt the risk-dominated action $b_1$ only when they have sample size 1, and they have observed the rare action $b_2$ (the probability of this is $\theta_1(1) \cdot \epsilon_2$). By contrast, if $u_2 < 1$ is sufficiently small, then a single occurrence of the rare action $b_1$ in a sample of size $k$ (which occurs with probability of $k \cdot \epsilon_1$) is sufficient to induce a new agent of population 2 to play her risk-dominant action $b_2$. This implies that the total share of new agents of population 2 who plays action $b_2$ is $\mathbb{E}(\theta_2) \cdot \epsilon_2$. This, in turn, implies that the product of the number of agents playing the rare action in each population increases iff $\theta_1(1) \cdot \mathbb{E}(\theta_2) > 1$.

**Heterogeneity and Stability of Miscoordination** Our second result shows that heterogeneity of the sample sizes is necessary for stable miscoordination. Specifically, we show that if all agents in each population $i$ have the same sample size $k_i \geq 2$, then all states with miscoordination are unstable (the case in which everyone has sample size 1 is discussed in Remark 1). Our final result (Theorem 3) shows that many heterogeneous distributions of sample sizes in which some agents have relatively small samples, while others have sufficiently large samples induce locally-stable states with miscoordination in coordination games in which $u_1$and $u_2$ are sufficiently far from 1 (both for games with or without risk-dominant equilibria).

The intuition why heterogeneity of sample sizes in important for the stability of states with miscoordination is as follows. Consider homogeneous populations with a fixed sample size $k$ in a stationary state with miscoordination. In such a state the random sample of size $k$ frequently yields both outcomes for which action $a_i$ is a best reply and outcomes for which $b_i$ is a best reply. We show that in such situations the probability of each action being a best reply is sensitive to small perturbations in the opponent's distribution of actions. That is, if $\epsilon$ more of the opponent's population play $a_{-i}$, it increases the probability for which $a_i$ is the best reply to the random sample by more than $\epsilon$.

Next consider a heterogeneous population in which some agents have relatively small samples, while other agent have large samples. The stationary interior state with miscoordination does not coincide with the Nash equilibrium, which implies that almost all agents with large samples play the same action (the unqiue best reply to the the true distribution of the opponents' actions), and that their play is insensitive to small perturbations of the opponents' behavior. This allows the overall sensitivity of the entire population to small perturbations to be sufficiently small to allow stable miscoordination.

Theorem 2's proof relies on deriving a property of binomial distributions (Proposition 2), which may be of independent interest. This new property states that a composition of two binomial cumulative distributions have a unique interior fixed point.

**Main Insights** Heterogeneous populations in which some agents base their decisions on noisy anecdotal evidence, while others have more accurate data are plausible in various applications. For example, bargaining situations between buyers and sellers in the housing market can be modeled as hawk-dove coordination-games, in which each agent chooses either a soft or a tough bargaining approach (see Example 2). Such markets often involve professional real-estate investors (who can rely on large samples) and agents who only buy or sell houses once or twice in their lives, and often rely on anecdotal evidence.

In Section 7 we demonstrate that our main insight of heterogeneity inducing stable miscoordination does not depend on the specific details of the sampling dynamics. Specifically, we numerically study the commonly-used logit dynamics, in which agents play noisy best reply to the opponent's aggregate behavior, with $\eta_i$ describing the noise level in population $i$. We first demonstrate that if the noise level in each population is homogeneous, then one can induce stable miscoordination only with implausibly high levels of noise. By contrast, when we introduce an extension of logit dynamics that allows heterogeneity in the level of noise in each population, we show that stable miscoordination can be supported by moderate heterogeneous levels of noise.

Taken together, our results show that the conventional wisdom that miscoordination is unstable in learning dynamics is not accurate. Miscoordination can be stable in heterogeneous populations in which some agents rely on anecdotal evidence or noisy data, while other agents have access to more accurate data. This might explain why in various real-life situations, such as bargaining in the housing market (Example 2), states with miscoordination and frequent bargaining failures might persist. The experimentally testable implications of our results are discussed in Section 2.

**Structure** Section 2 presents the related literature. Our model is described in Section 3. Section 4 presents an "complete" characterization for global convergence to states with miscoordination. In Section 5 we show that homogeneous populations always converge to one of the pure coordinated states. Section 6 shows that many heterogeneous distributions of sample sizes can lead to locally stable states with miscoordination. In Section 7 we numerically analyze the logit dynamics, and demonstrate that our main insights hold in this setup as well. We conclude in Section 8. Formal proofs are presented in the appendix. Appendix 4.1 describes the proof an interesting general result on binomial distributions.

# 2 Related Literature

**Instability of Miscoordination** It is well-known that the pure coordinated equilibria satisfy strong stability refinements, while mixed equilibria with miscoordination do not satisfy even weak stability refinements. In particular, coordinated pure strict equilibria are evolutionary stable (Maynard Smith and Price, 1973), while the mixed equilibrium with miscoordination does not satisfy neutral stability (Maynard-Smith, 1982), or even the mild refinement of weak stability (Heller, 2017). Moreover, it is well known that interior stationary states in multiple-population games cannot be asymptotically stable under the commonly-used replicator dynamics (see, e.g., Sandholm, 2010, Theorem 9.1.6), and this is true for all underlying games. Instability of interior states is further shown for various classes of learning dynamics in Crawford (1989).

Moreover, various papers in the literature have proven results for various dynamics that populations playing coordination games would converge to pure coordinated states from almost all initial states. Kaniovski and Young (1995) has proven global convergence to one of the pure coordinated equilibria for sampling dynamics with sufficiently large samples. Oprea et al. (2011) has proven global stability to one of the pure coordinated equilibria monotone dynamics in which an action becomes more frequent iff it yields a higher payoff than the alternative action.[1] Recently, Oyama et al. (2015) has proven global stability to the risk-dominant pure coordinated equilibrium in symmetric coordination games for distributions of sample sizes in which sufficiently many agents have sufficiently small samples. The stochastic evolutionary literature (pioneered in Kandori et al., 1993; Young, 1993; see also the recent application to hawk-dove coordination games in Bilancini et al., 2022) shows that only pure coordinated equilibria can be stochastically stable in dynamics in large finite populations in which agents most of the time best reply to a large sample, but occasionally they mistakenly play the other action.

Thus, taken together, the various existing literature suggest that states with miscoordination are unstable. Our contribution is showing that is not true in plausible learning dynamics in which agents base their behavior on sampling the opponent's population, and there is substantial heterogeneity in the sample sizes in each population.

**Sampling (Best-Response) Dynamics** The sampling (best-response) dynamics were pioneered by Sandholm (2001) and Osborne and Rubinstein (2003). As argued by Oyama et al. (2015), the deterministic nature of the sampling dynamics implies that when there is convergence to a stable state the convergence is fast.[2] As we show in Proposition 1 that the populations always converge to stable states in our setup. Recently, Heller and Mohlin (2018) studied the conditions on the expected sample size that implies global

[1] Oprea et al. (2011) has shown it to hawk-dove coordination games, but the proof can be extended to all coordination games.

[2] Conditions in which also stochastic dynamics induce fast convergence are studies in Kreindler and Young (2013); Arieli et al. (2020).

convergence for all payoff functions and all sampling dynamics.[3]

Salant and Cherry (2020) (see also Sawa and Wu, 2021) generalized the sampling dynamics by allowing new agents to use various procedures to infer from their samples the aggregate behavior of the opponents (in addition to allowing for payoff heterogeneity in the population). Salant and Cherry pay special attention to *unbiased* inference procedures in which the agent's expected belief about the share of opponents who play hawk coincides with the sample mean. Examples of unbiased procedures are maximum likelihood estimation, beta estimation with a prior representing complete ignorance, and a truncated normal posterior around the sample mean. In our setup, the payoffs are linear in the share of agents who play hawk, which implies that the agent's perceived best reply depends only on the expectation of her posterior belief. This implies that our results hold for any unbiased inference procedure.

**Experimental Literature and Testable Predictions** Our model yields a novel testable prediction: miscoordination can persist when there is heterogeneity in the amount of information that the agents have about the opponents' behavior, and if for each population one of the actions is $q$-dominant for a sufficiently low $q$.

Selten and Chmura (2008) applied an experimental design that can help in testing our prediction. In their design: (1) agents are randomly and repeatedly matched to anonymous pairs within a relatively large matching groups, (2) each agent gets explicit feedback only about her most recent opponent's play (and can rely on her memory of her previous feedback abut past opponents). Selten and Chmura applied this design for two-action, two-player games with a unique completely mixed Nash equilibrium. They showed that sampling dynamics explain well the aggregate experimental behavior (much better than the predictions of quantal response equilibria and Nash equilibria).

Assuming that all agents use the same sample size, Selten and Chmura (2008) showed that a sample size of 7 fitted the data best (which is roughly in line with the typical estimates of people short-run memory captivity). The prediction of global convergence to miscoordination (Theorem 1) requires that a substantial part of the population to have a much smaller sample size of 1. Possibly, the this can be induced by having the subjects

[3] Hauert and Miekisz (2018) use the term "sampling dynamics" to refer to a variant of the replicator dynamics, in which when an agent samples another agent and mimics the other agent's behavior, it is more likely that these two agents will be matched with each other. This is less related to our use of the notion of "Sampling dynamics", which is in line with the literature cited in the main text.

playing in different rounds different underlying games. For example, in each round, one out of several underlying games will be played. Each subject will be informed about the payoff matrix of the current game, and will be reminded about her past opponent's behavior in the previous time in which the same game was played. It seems plausible that many subjects will rely only on this feedback when deciding how to play (i.e., essentially have a sample of size 1), while only few subjects will exert efforts to remember relevant past feedback from less recent times in which the same game was played.

Recently, Lyu et al. (2022) has applied a more elaborate experimental setup that aims to implement sampling dynamics. In their implementation the subjects are endowed with a specific "default" action to play in the first round. This novel design component allows to test dynamic predictions with respect to specific initial states. At the end of each round each player observes the $k$ most recent actions ($k = 2$ and $k = 7$ were applied in their treatments) of random opponents among the 14 players in their matching group. Their underlying games were symmetric coordination games with Pareto ranked pure equilibria. Lyu et al. show that action sampling dynamics (with respect to the most recent $k$ observed actions) fits about 80% of the subjects' behavior.

A central reason why some subjects deviated from the predictions of sampling dynamics (specially in early rounds) was that they played the action that is part of the Pareto-dominant equilibrium, in order to "teach" the other subjects in the matching group to move from the Pareto-dominated equilibrium to the Pareto-better equilibrium. These "teaching" incentives were justified, as they often helped groups starting in the Pareto-dominated equilibrium to shift to the Pareto-dominant equilibrium. We think that "teaching"incentives would be substantially reduced, and the fit of sampling dynamics would much improve, if either (1) the strict equilibria of the underlying coordination games will not be Pareto-ranked, such as the case in hawk-dove games (see Example 1), or (2) the matching groups would be substantially larger.

The examples demonstrating the predictions of Theorem 3 (local stability of miscoordination) require both an appropriate initial state (which can be implemented à la Lyu et al., 2022), and that some agents will have accurate information about the opponents' aggregate behavior. This can be implemented by providing some agents with a full feedback about the behavior of all opponents in the previous round (while the remaining agents will only get feedback about the behavior of their own matched opponents).

Table 2: Standard Representation of Two-Action Coordination Game

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $u_1$ , $u_2$ | 0 , 0 |
| $b_1$ | 0 , 0 | 1 , 1 |

$u_1 \geq 1$, $u_2 > 0$

# 3 Model

## 3.1 Coordination Games

**Standard Representation** Let $G = \{A_1 \times A_2, u\}$ denote a normal-form two-action two-player coordination game. Let $i \in \{1, 2\}$ be an index denoting one of the players ("she"), and let $-i$ denote her opponent ("he"). For each $i \in \{1, 2\}$, let $A_i = \{a_i, b_i\}$ denote the two feasible actions of player $i$. Let $c_i \in A_i$ denote one of these two actions (and $c_{-i}$ its counterpart action for the opponent). The standard 2-parameter payoff matrix of a coordination game is given in Table 2: The players get a low payoff (normalized to zero) if they miscoordinate (i.e., one player plays $a_i$ and the opponent plays $b_{-i}$), they get a high payoff (normalized to 1) if they coordinate on both playing on $(b_1, b_2)$, and they get a payoff of $(u_1, u_2)$, where $u_1 \geq 1$ and $u_2 > 0$ if they coordinate on $(a_1, a_2)$. We say the the coordination game is *symmetric* if $u_1 = u_2$. Although, this standard representation may look as a specific subset of coordination games, as we explain below, this standard representation does capture all coordination games WLOG.

We extend the game to mixed actions in the standard linear way. We identify each mixed action with the probability it assigns to the first action ($a_i$), and we denote it by $p_i \in [0, 1]$. We identify the degenerate mixed action 1 with the pure action $a_i$, and the degenerate mixed action 0 with $b_i$. Observe that the coordination game admits 3 Nash equilibria: two pure (*coordinated*) equilibria $(a_1, a_2)$ and $(b_1, b_2)$, and one mixed equilibrium with miscoordination $\boldsymbol{p}^{NE} = \left(\frac{1}{1+u_2}, \frac{1}{1+u_1}\right)$ with expected payoff of $\left(\frac{u_1}{1+u_1}, \frac{u_2}{1+u_2}\right)$.

**General Coordination Games** We next explain why the above 2-parameter representation indeed captures WLOG any 2-action coordination game. The most general definition of a 2-action coordination game is a game that admits two strict Nash equi-

Table 3: Normalization of General Two-Action Coordination Games

Original Representation

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $u_{11}$, $v_{11}$ | $u_{12}$, $v_{12}$ |
| $b_1$ | $u_{21}$, $v_{21}$ | $u_{22}$, $v_{22}$ |

$\Rightarrow$

Standard Representation

| | $a_2$ | $b_2$ |
|---|---|---|
| $a_1$ | $\frac{u_{11}-u_{21}}{u_{22}-u_{12}}$, $\frac{v_{11}-v_{12}}{v_{22}-v_{21}}$ | 0, 0 |
| $b_1$ | 0, 0 | 1, 1 |

$u_{11} > u_{21}, u_{22} > u_{12}, \; v_{11} > v_{12}, v_{22} > v_{21}$

libria. By relabeling the actions of player 1, we can assume WLOG that these two pure equilibria are $(a_1, a_2)$ and $(b_1, b_2)$ (i.e., if the two strict equilibria where $(a_1, b_2)$ and $(b_1, a_2)$, then we switch the labels of player 1's actions: $a_1 \leftrightarrow b_1$). This implies that the left panel of Table 3 shows a $2 \cdot 4$-parameter representation of all coordination games.

Sampling dynamics depends only on the differences between the payoffs a player can get by playing different actions (the same property holds for best-reply dynamics and logit dynamics, which implies that the sets of Nash equilibria, quantal response equilibria and evolutionary stable strategies depend only on these differences).[4] These differences are invariant to subtracting a constant payoff from all the payoffs of a player when fixing the opponent's action (e.g., subtracting $u_{21}$ from all of player's 1 first column payoffs). Moreover, sampling dynamics (as well as all the other dynamics and solution concepts mentioned above) are also invariant to dividing all of a player's payoff by a positive constant (which preserves the vN-M utility). The left matrix in Table 3 is reduced to the right matrix by the following steps (each of which does not affect the sampling dynamics):

1. Three changes to player 1's payoffs: (I) subtracting $u_{21}$ from player's 1 payoffs in her first column, (II) subtracting $u_{12}$ from player's 1 payoffs in her second column, (III) dividing player 1's payoff by $u_{22}-u_{12}$; and

2. Three changes to player 2's payoffs: (I) subtracting $v_{12}$ from player's 1 payoffs in her first row, (II) subtracting $v_{21}$ from player's 1 payoffs in her first row, (III) dividing player 1's payoff by $v_{22} - v_{21}$.

Observe that the assumption that $u_1 = \frac{u_{11}-u_{21}}{u_{22}-u_{12}} \geq 1$ in the standard representation of

[4]A notable exception is the best-experienced payoff dynamics (see, e.g., Osborne and Rubinstein, 1998; Cárdenas et al., 2015; Mantilla et al., 2018; Sandholm et al., 2019, 2020; Sethi, 2021), where the dynamics depend directly on the payoffs, and not only on payoff differences.

Table 4: Normalization of Hawk-Dove Games ($g, l \in (0, 1)$)

Original Representation

| | $h_2$ | $d_2$ |
|---|---|---|
| $h_1$ | $0$, $0$ | $1+g$, $1-l$ |
| $d_1$ | $1-l$, $1+g$ | $1$, $1$ |

$\Rightarrow$

Standard Representation

| | $a_2 = h_2$ | $b_2 = d_2$ |
|---|---|---|
| $a_1 = d_1$ | $\frac{1-l}{g}$, $\frac{g}{1-l}$ | $0$, $0$ |
| $b_1 = h_1$ | $0$, $0$ | $1$, $1$ |

Table 2 is WLOG. If $\frac{u_{11}-u_{21}}{u_{22}-u_{12}} < 1$, then we can multiply all of player 1's payoffs by $\frac{u_{22}-u_{12}}{u_{11}-u_{21}}$ and all of player 2's payoffs by $\frac{v_{22}-v_{21}}{v_{11}-v_{12}}$, relabel the actions $a_i \leftrightarrow b_i$ for both players, and obtain a standard representation as in Table 2 in which $u_1 \geq 1$.

**Example 1** (Motivating Example – Hawk-Dove Games)**.** Consider a hawk-dove (AKA, Chicken) game (as described in the left panel of Table 4), which can be interpreted as a bargaining over a price of an asset (e.g., house) between a buyer and a seller. Each player can either insist on a more favorable price ("hawk") or agree to a less favorable price in order to close the deal ( "dove"). A hawkish The left panel of Table 4 shows the original payoffs of a hawk-dove game. Two doves agree on an equally favorable price. A hawk obtains a favorable price when being matched with a dove, but has a substantial probability of bargaining failure against another hawk.

Observe that a hawk-dove game can be transformed to our standard representation of a coordination game (the right panel of 4) as follows: (1) relabeling the payoffs of player 1 such that $a_1 = d_1$ and $b_1 = h_1$(while $a_2 = h_2$ and $b_2 = d_2$), (2) subtract a payoff of 1 from player 1's payoffs in her second column and from player 2's payoffs in her first column, and (3) divide all the payoffs of player 1 by $g$, and all payoffs of player 2 by $1 - l$. Observe that the induced standard representation has the special property that $u_1 = \frac{1-l}{g} = \frac{1}{u_2}$. Thus, we say that a coordination game in its standard representation is hawk-dove game if $u_1 = \frac{1}{u_2}$.

**Risk-Dominance** Fix $q \in [0, 1]$. We say that action $c_i \in A_i$ is *q-dominant* (Morris et al., 1995; Oyama et al., 2015) for player $i$ if it is a strict best reply to any opponent's mixed action that assigns mass of at-least $q$ to the counterpart action $c_{-i}$. Observe that both actions are 1-dominant (which is equivalent to being part of a strict equilibrium). Further observe that the lower the $q$, the more demanding the $q$-dominance requirement becomes (i.e., if an action is $q$-dominant, it is also $r$-dominant for any $q \leq r \leq 1$). Finally, observe that action $a_i$ is $\frac{1}{1+u_i}$-dominant and action $b_i$ is $\frac{u_i}{1+u_i}$-dominant.

We say that action $c_i \in A_i$ is *risk dominant* for player $i$ if it is $\frac{1}{2}$-*dominant*. We say that the pure equilibrium $c$ is *risk dominant* if each action $c_i$ is *risk* dominant; in this case we say the remaining equilibrium to be risk dominated. Observe that a coordination game admits a risk-dominant equilibrium iff $u_1, u_2 > 1$. By contrast, if $u_1 > 1 > u_2$, then $a_1$ is risk dominant to player 1 and $b_2$ is risk dominant to player 2.

Observe that $q$-dominance depends only on the differences between the payoffs a player can get by playing the different actions. This implies that $q$-dominance (and risk-dominance) is invariant to all the transformations described above; i.e., an action is $q$-dominant in the standard representation (right panel of Table 3) iff it is $q$-dominant in the original representation (left panel). By contrast, payoff dominance is not invariant to changing all of a player's payoff when fixing the opponent's action. For example the payoff for $(d_1, d_2) = (a_1, b_2)$ strictly Pareto dominates the payoff of $(h_1, h_2) = (b_1, a_2)$ in the original representation of hawk-dove games in the left panel of Table 4, but the two payoff profiles coincide in the standard representation in the right panel.

## 3.2 Evolutionary Dynamics

We assume that there are two unit-mass continuums of agents and that agents in population 1 are randomly matched with agents in population 2. Aggregate behavior at time $t \in \mathbb{R}^+$ is described by a *state* $\mathbf{p}(t) = (p_1(t), p_2(t)) \in [0,1]^2$, which is equivalent to a mixed action profile (i.e., $p_i(t)$ represents the share of agents playing action $a_i$ at time $t$ in population $i$). A state $\mathbf{p} = (p_1, p_2)$ is *interior* (or mixed) if $p_1, p_2 \in (0,1)$.

Agents die at a constant rate of 1, and are replaced by new agents (or, equivalently, agents get opportunities to revise their actions). The evolutionary process is represented by a function $\mathbf{w} : [0,1]^2 \to [0,1]^2$, which describes the frequency of new agents in each population who play action $a_i$as a function of the current state. Thus, the instantaneous change in the share of agents of population $i$ that play $a_i$ is given by $\dot{p}_i = w_i(\mathbf{p}) - p_i$.

**Sample sizes** We allow heterogeneity in the sample sizes used by new agents. Let $\theta_i \in \Delta(\mathbb{Z}_+)$ denote the distribution of sample sizes of new agents of population $i$. We assume that $\theta_i$ has a finite support. A share of $\theta_i(k)$ of the new agents have a sample of size $k$. Let $\operatorname{supp}(\theta_i)$ denote the support of $\theta_i$. If there exists some $k$, for which $\theta_i(k) = 1$, then we use $k$ to denote the degenerate (homogeneous) distribution $\theta_i \equiv k$.

*Remark* 1. We assume that some agents have sample sizes larger than 1 (i.e., $\forall i$ $\max(\text{supp}(\theta_i)) > 1$). This rules our the trivial case in which all agents have sample size 1. In this case the entire diagonal $\{(p_1, p_1) \,|p_1 \in [0,1]\}$ is Lyapunov stable, and no state is asymptotically stable (as analyzed in Sethi, 2000, Example 7 in a related setup).

**Definition 1.** An *environment* is a tuple $E = (\boldsymbol{u}, \boldsymbol{\theta}) = ((u_1, u_2), (\theta_1, \theta_2))$ where $(u_1, u_2)$ describe the payoffs of the underlying coordination game, and $(\theta_1, \theta_2)$ describes the distributions of sample sizes in each population.

**Sampling best-response dynamics** The sampling best-response dynamics (Sandholm, 2001; Oyama et al., 2015, henceforth abbreviated as *sampling dynamics*) fit situations in which agents do not know the exact distribution of actions in the opponent's population. New agents estimate this unknown distribution by sampling opponents' actions. Specifically, each new agent with sample size $k$ (henceforth, a $k$-agent) samples $k$ randomly drawn agents from the opponent's population and then plays the action that is the best reply against the sample. *To simplify notation, we assume that in case of a tie, the new agent plays $a_i$.* Our results are essentially the same for any tie-breaking rule.

Let $X(k, p_{-i}) \sim Bin\,(k, p_{-i})$ denote a random variable with binomial distribution with parameters $k$ (number of trials) and $p_{-i}$ (probability of success in each trial), which is interpreted as the number of $a_{-i}$-s in the sample. Observe that the sum of payoffs of playing action $a_i$ against the sample is $u_i \cdot X(k, p_{-i})$ and the sum of payoffs of playing action $b_i$ against the sample is $k - X(k, p_{-i})$.

This implies that action $a_i$ is a best reply to a sample of size $k$ iff $u_i \cdot X(k, p_{-i}) \geq k - X(k, p_{-i}) \Leftrightarrow X(k, p_{-i}) \geq \frac{k}{u_i+1}$. This, in turn, implies that the sampling dynamics in environment $(\boldsymbol{u}, \boldsymbol{\theta})$ is given by

$$w_i\,(p_{-i}) \equiv w_i\,(\mathbf{p}) = \sum_{k \in \text{supp}(\theta_i)} \theta_i(k) \cdot \Pr\left(X\,(k, p_{-i}) \geq \frac{k}{u_i + 1}\right). \tag{3.1}$$

Observe that $\Pr\left(X\,(k, p_{-i}) \geq \frac{k}{u_i+1}\right) = \sum_{l=\left\lceil \frac{k}{u_i+1} \right\rceil}^{k_i} \begin{pmatrix} k \\ l \end{pmatrix} p_{-i}^{l}\,(1 - p_{-i})^{k-l}$, which is a polynomial of $p_{-i}$ of degree $k$. This implies that $w_i\,(p_{-i})$ is a polynomial with a finite degree of $\max \text{supp}\,(\theta_i) > 1$.

*Remark* 2 (Symmetric games played within a single population). Our formal model deals with coordination games played between two different populations (or, equivalently, by games played within a single population where an agent can condition her play on the role she was allocated to in the game: player 1 or player 2). When dealing with symmetric coordination games ($u_1 = u_2$), one can also consider a variant of our dynamics in which the game is played by agents from a single population, and a player cannot condition her play on allocated role: player 1 or player 2 (e.g. the setup analyzed in Oyama et al., 2015).[5] It turns out that both types of dynamics (two-population or one-population) yield exactly the same results regarding the characterization of stationary states and asymptotically stable states. This has been shown formally in the related setup of best-experienced payoff-sampling dynamics (Sethi, 2000, Thoerem 3; Arigapudi et al., 2021, Corollary 2), and these proofs can be adapted to the current setup. Thus all of our results remain valid for symmetric coordination games played within a single population.

# 4 Global Convergence to Miscoordination

As discussed in Section 2, various existing papers have shown that populations playing coordination games will always converge to pure coordinated states under various dynamics. In this section, we fully characterize the conditions for which the opposite result holds under sampling dynamics; i.e., the populations converge from almost any initial state to one of the interior states with miscoordination.

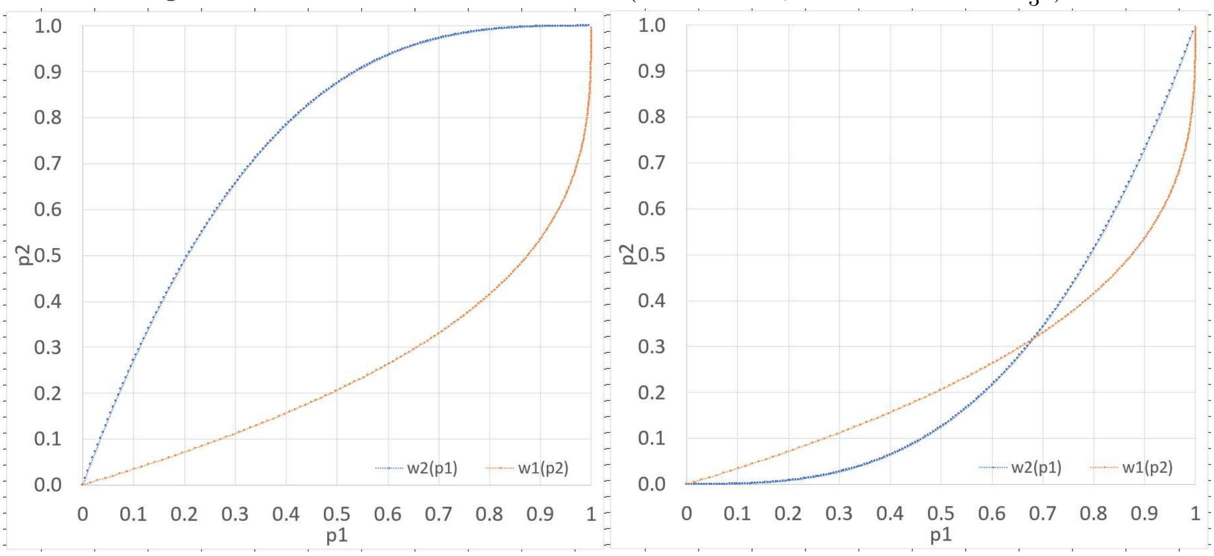
## 4.1 Analysis of $w_i(\boldsymbol{p})$ and Preliminary Results

The characteristics of sampling dynamics are closely related to the properties of the polynomials $w_1(p_2)$ and $w_2(p_1)$ and their intersection points, which are analyzed in this subsection.

Figure 4.1 illustrates the phase plots of the sampling dynamics and the properties of the polynomials $w_2(p_1)$ and $w_1(p_2)$. We refer to the latter polynomial also as $w_1^{-1}(p_1)$. The left panel illustrates a symmetric coordination game with $u_1 = u_2 = 3$, and the right

[5] In one-population dynamics it is important that the strict equilibria are obtained by both agents playing the same action (i.e., being on the main diagonal of the payoff matrix), as one cannot relabel the actions of only one of the roles in the game without break the symmetry. Thus, in one-population environments, hawk-dove games are no longer considered to be coordination games.

Figure 4.1: Illustrative Phase Plots ($\theta_i \equiv 3 = u_1$; $u_2$ is either 3 or $\frac{1}{3}$ )




The figure illustrates the phase plots of the sampling dynamics for two environments: (1) $\forall i\ \theta_i \equiv 3 = u_i$ (left panel, a symmetric coordination game), and (2) $\forall i\ \theta_i \equiv 3 = u_1 = \frac{1}{u_2}$ (right panel, a hawk-dove game). The blue solid (resp., orange dashed) curve shows the states for which $\dot{p}_1 = 0$ (resp., $\dot{p}_2 = 0$ ). The intersection points of these curves are the stationary states. A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) stationary state.

panel a hawk-dove game with $u_1 = 3$ and $u_2 = \frac{1}{3}$. In both panels all agents have sample size of 3 ($\theta_i \equiv 3$). The blue solid curve is the polynomial $p_2 = w_2(p_1)$, which describes the states in which $\dot{p}_2 = 0$. In all states above (resp., below) this curve $\dot{p}_2 < 0$ (resp., $\dot{p}_2 > 0$). The orange dashed curve is the polynomial $p_1 = w_1(p_2)$, which describes the states in which $\dot{p}_1 = 0$. In all states to the right (resp., left) of this curve $\dot{p}_1 < 0$ (resp., $\dot{p}_1 > 0$). Observe that on both panels all non-stationary initial states converge to a pure state. In the left panel there is global convergence to $(1, 1)$, while in the right panel some states converge to $(1, 1)$ and others to $(0, 0)$.

The following fact is immediate from basic properties of binomial random variables.

**Fact 1.** *$w_i(p_{-i})$ is a strictly increasing polynomial function that satisfies $w_i(0) = 0$ and $w_i(1) = 1$. This implies that the inverse function $w_i^{-1} : [0, 1] \to [0, 1]$ exists, is continuously differentiable, and that $w_i^{-1}(0) = 0$ and $w_i^{-1}(1) = 1$.*

Fact 1 implies that the two curves intersect at $(0, 0)$ and $(1, 1)$.

Appendix A.2 presents the standard definitions of stationary states, asymptotically stable states, and unstable states. Observe that a state is stationary (i.e., it is a fixed point of the dynamics) iff it is an intersection point of the two curves $w_1$ and $w_2$.

**Fact 2.** *State $\boldsymbol{p}$ is stationary iff $p_2 = w_2(p_1)$ and $p_1 = w_1(p_2)$.*

Further observe that $p_1 \in [0,1]$ is part of a stationary state iff it is a rest point of the composition of the two $w_i$-s.

**Fact 3.** *State* $(p_1, w_2(p_1))$ *is stationary iff* $p_1 = w_1(w_2(p_1)) = (w_1 \circ w_2)(p_1)$.

We begin by showing that the dynamics admit a finite number of stationary states.

*Claim* 1. The curves $w_2(p_1)$ and $w_1(p_2)$ intersect in a finite number of points.

*Proof.* As explained in the end of Section 3, $w_i(p_{-i})$ is a polynomial of degree $\max \operatorname{supp}(\theta_i) > 1$. This implies that $w_1(w_2(p_1))$ is a polynomial of a finite degree strictly larger than 1. This, in turn, implies that the equation characterizing intersection points (and stationary states) $w_1(w_2(p_1)) = p_1$ has a finite number of solutions. □

Next, we show that the dynamics always takes the populations to a state that is below one of the curves, and above the other curve.

*Claim* 2. There exists $t < \infty$ such that either $w_1^{-1}(p_1(t)) \leq p_2(t) \leq w_2(p_1(t))$ or $w_2(p_1(t)) \leq p_1(t) \leq w_1^{-1}(p_1(t))$.

*Sketch of Proof.* Observe that $\dot{p}_1 > 0 > \dot{p}_2$ (resp., $\dot{p}_1 < 0 < \dot{p}_2$) in any trajectory that begins above (resp., below) both curves, which implies that the trajectory moves downward and to the right (resp., upward and to the left) until it intersects with one of the curves. See Appendix A.3 for the formal proof. □

Our next result shows that if a trajectory reaches a state between the two curves, then it must converge to one of the neighboring stationary states: the stationary state in the right side if the curve $w_2(p_1)$ is above the curve $w_1(p_2)$ , and the stationary state in the left side otherwise.

*Claim* 3. Let $\boldsymbol{p}(t)$ be an interior state. Let $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}}$ be the neighboring stationary states, i.e.,: (1) $\underline{p}_i < p_i(t) < \overline{p}_i$ and (2) there does not exist any stationary $\hat{\boldsymbol{p}}$ satisfying $\underline{p}_i < \hat{p}_i < \overline{p}_i$.

1. If $p_i(t) \in \left[w_1^{-1}(p_1(t)), w_2(p_1(t))\right]$, then $\lim_{t\to\infty} \mathbf{p}(t) = \overline{\boldsymbol{p}}$, and
2. If $p_i(t) \in \left[w_2(p_1(t)), w_1^{-1}(p_1(t))\right]$, then $\lim_{t\to\infty} \mathbf{p}(t) = \underline{\boldsymbol{p}}$.

*Sketch of Proof for case (1).* The fact that $w_1^{-1}(p_1(t)) < p_i(t) < w_2(p_1(t))$ implies that $\dot{p}_1(t), \dot{p}_2(t) > 0$, and thus the trajectory moves upward and to the right until meeting one of the curves. This meeting point must be the intersection point $\overline{\boldsymbol{p}}$, because if the meeting point were only with $w_2(p_1)$ (resp., $w_1(p_2)$), the trajectory there would have been horizontal to the right (resp., vertical upward), which implies that the trajectory would come from the left side of the curve (resp., from below the curve), leading to a contradiction. See Appendix A.4 for a formal proof. □

The claims immediately imply that any trajectory converges to a stationary state.[6]

**Corollary 1.** $\lim_{t\to\infty} \mathbf{p}(t)$ *exists for any* $\boldsymbol{p}(0)$, *and it is a stationary state.*

Next we show that if any initial state converges to one of the pure equilibria, then this equilibrium must be asymptotically stable (as defined in Appendix A.2).

*Claim* 4. Assume that $\mathbf{p}(0) \neq (0,0)$ and $\lim_{t\to\infty} \mathbf{p}(t) = (0,0)$; then $(0,0)$ is asymptotically stable. The same result holds when replacing $(1,1)$ with $(0,0)$.

*Sketch of Proof.* Due to Claim 1 we can assume WLOG that $\mathbf{p}(0)$ is between the two curves. Claim 2 implies that the trajectory converges to $(0,1)$ iff $w_2(p_1)$ is below $w_1(p_2)$, and the closest intersection point of the curves to the left of $\mathbf{p}(0)$ is (0,0). These conditions imply that any initial state sufficiently close to (0,0) converges to (0,0), which, in turn, implies that (0,0) is asymptotically stable. See Appendix A.5 for the formal proof. □

## 4.2 Asymptotic Stability of Pure Equilibria

In order to state our next results, it will be helpful to consider the condition in which a single appearance of a rare action can change the behavior of a new agent. Specifically, Consider a new agent in population $i$ with a sample size of $k$. Observe that:

1. Action $a_i$ induces a weakly higher payoff against a sample with a single opponent's action $a_{-i}$ (and $k-1$ opponent actions $b_{-i}$) iff $u_i \geq k-1 \Leftrightarrow k \leq u_i + 1$.

2. Action $b_i$ induces a strictly[7] higher payoff against a sample with a single opponent's action $b_{-i}$ (and $k-1$ opponent actions $a_{-i}$) iff $1 > (k-1)\,u_i \Leftrightarrow k < \frac{1}{u_i} + 1$.

Next we define *m-bounded expectation* as the expected value of a probability distribution when taking into account only values smaller than $m$. Formally,

[6] An alternative way to prove Corollary 1 is to rely on the Bendixson–Dulac theorem (see Theorem 9.A.6 of Sandholm, 2010).

[7] We require strictly higher payoffs for action $b_i$ and weakly higher payoffs for action $a_i$ due to our tie-breaking rule in favor of action $a_i$.

**Definition 2.** The *m-bounded expectation* $\mathbb{E}_{\le m}$ (resp., $\mathbb{E}_{<m}$) of distribution $\theta_i$ with support on integers is[8] $\mathbb{E}_{\le m}(\theta_i) = \sum_{1\le k\le m} \theta_i(k)\cdot k$ (resp., $\mathbb{E}_{<m}(\theta_i) = \sum_{1\le k<m} \theta_i(k)\cdot k$).

Our next result characterizes the asymptotic stability of the pure states. It shows that the asymptotic stability depends only on whether the product of the bounded expectations of the distributions of sample sizes is larger or smaller than one, where the bound of each distribution is the maximal sample size for which a single appearance of a rare action can change the behavior of a new agent. Formally (where replacing the $a_i$-favorable tie-breaking rule with a $b_i$-favorable one would replace the "$<$"-s and the "$\le$"-s in the bounded expectations in the statement):

**Proposition 1.**

1. $\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)\cdot\mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) = \theta_1(1)\cdot\mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) > 1 \Rightarrow \boldsymbol{a}{=}(a_1,a_2)$ *is unstable;*
2. $\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)\cdot\mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) = \theta_1(1)\cdot\mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) < 1 \Rightarrow \boldsymbol{a}$ *is asymptotically stable;*
3. $\mathbb{E}_{\le u_1+1}(\theta_1)\cdot\mathbb{E}_{\le u_2+1}(\theta_2) > 1 \Rightarrow \boldsymbol{b}{=}(b_1,b_2)$ *is unstable; and*
4. $\mathbb{E}_{\le u_1+1}(\theta_1)\cdot\mathbb{E}_{\le u_2+1}(\theta_2) < 1 \Rightarrow \boldsymbol{b}{=}\ (b_1,b_2)$ *is asymptotically stable;*

*Sketch of Proof.* Consider a slightly perturbed state $(1-\epsilon_1, 1-\epsilon_2)$ near $\boldsymbol{a}=(1,1)$ (the argument for $\boldsymbol{b}$ is analogous) in which almost all agents play action $a_i$. The event of two rare actions ($b_i$-s) appearing in a sample of a new agent has a negligible probability of $O(\epsilon_i^2)$. If a new agent has a sample size of $k$, then the probability of a rare action appearing in the sample is approximately $k\cdot\epsilon_{-i}$. This rare appearance changes the perceived best reply of a new agent of population $i$ iff $k$ is smaller than $\frac{1}{u_i}+1$. Thus, the total probability that a new agent of population $i$ adopts the rare action $b_i$ is equal to $\mathbb{E}_{<\frac{1}{u_i}+1}(\theta_i)$ . This implies that the product of the share of new agents adopting a rare action in each population is $\epsilon_1\cdot\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)\cdot\epsilon_2\cdot\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_2)$. This shows that the share of agents playing rare actions gradually increases (resp., decreases) if $\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)\cdot\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_2) > 1$ (resp., $\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)\cdot\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_2) < 1$), which implies instability (resp., asymptotic stability). Finally, observe that our assumption that $u_1\ge 1$ implies that $\frac{1}{u_1}+1\le 2$, which, in turn, implies that $\theta_1(1) = \mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1)$. See Appendix A.6 for a formal proof. □

[8] Observe that in our notation the parameter $k$ takes only (positive) integer values (although we allow the upper bound $m$ to be a non-integer).

An interesting implication of Proposition 1 is the substantial difference in the stability of a risk dominated equilibrium and a risk-dominant equilibrium (related results fro symmetric coordination games are derived in Oyama et al., 2015). A risk-dominated equilibrium (say, $\boldsymbol{b}$) is unstable as long as sufficiently many agents have not too-large samples (i.e., samples below $u_i + 1$). The process inducing instability is as follows. A small perturbation of few players who play $\boldsymbol{a}_i$ induces a slightly larger number of new agents to observe $\boldsymbol{a}_i$ at-least one in their samples, which induces them to play $a_i$ as well, which allows the small perturbations to gradually increase, until, in the end of the process, everyone playing the risk-dominant action $\boldsymbol{a}_i$.

By contrast, risk-dominant equilibria are always asymptotically stable. To see this, observe that if $\boldsymbol{a}$ is a weakly risk-dominant equilibrium (i.e., if $u_2 \geq 1$)[9] then $\frac{1}{u_2} + 1 \leq 2$, which implies that $\mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) = \theta_2(1) < 1$. This implies that $\mathbb{E}_{<\frac{1}{u_1}+1}(\theta_1) \cdot \mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) = \theta_1(1) \cdot \theta_2(1) < 1$, and, thus $\boldsymbol{a}$ is asymptotically stable. This implies that:

**Corollary.** *A risk-dominant equilibrium is always asymptotically stable.*

Thus, only in games in which neither equilibrium is risk-dominant (i.e., those in which $u_2 < 1 \leq u_1$), it might be possible for both pure equilibria to be unstable.

## 4.3 Global Convergence Result

Combining the previous results yields the main result of this section. It shows that the population converges from almost any initial state to an interior stationary state if (and essentially only if) in each population the product of the bounded expectations of the distribution of sample sizes and the opponent population's share of agents with sample size 1 is larger than 1.

**Theorem 1.** *Assume that $u_2 < 1 \leq u_1$ (no risk-dominant equilibrium). Then*

1. ***Global convergence to miscoordination**: Assume that*

$$\theta_1(1) \cdot \mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) > 1 \quad AND \quad \theta_2(1) \cdot \mathbb{E}_{\leq u_1+1}(\theta_2) > 1.$$

*If $\boldsymbol{p}(0) \notin \{(0,0),(1,1)\}$, then* $\lim_{t\to\infty} \mathbf{p}(t) \notin \{(0,0),(1,1)\}$.

[9] The result holds also for $u_2 = 1$ due to the tie-breaking rule in favor of $a_i$.

2. ***Local convergence to coordination**: Assume that*

$$\theta_1(1) \cdot \mathbb{E}_{<\frac{1}{u_2}+1}(\theta_2) < 1 \quad OR \quad \theta_2(1) \cdot \mathbb{E}_{\leq u_1+1}(\theta_2) < 1.$$

*Then there exist* $\boldsymbol{p}(0) \notin \{(0,0),(1,1)\}$ *such that* $\lim_{t\to\infty} \mathbf{p}(t) \in \{(0,0),(1,1)\}$.

*Proof.*

1. Proposition 1 implies that both pure stationary states are unstable. Combining Propositions 1–4 implies that from almost any initial state, the population converges to an interior stationary state.

2. Proposition 1 implies that at-least one of the pure equilibria is asymptotically stable, which implies that some interior initial states converge to a pure equilibrium. □

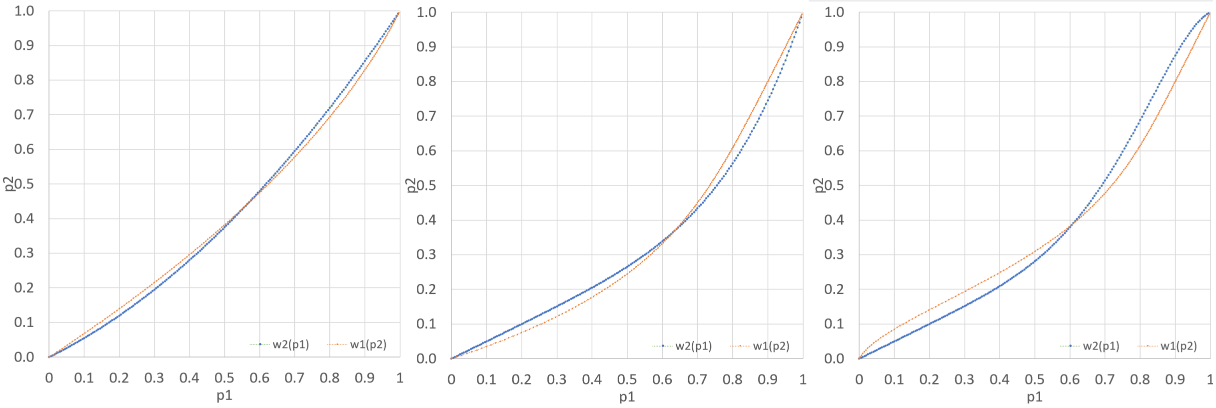
Theorem 1 shows that global converge to miscoordination requires heterogeneity in the sample sizes in each population that includes both agents with a small sample of a single action, and agents with larger samples (but not too large, such that are still below the bound for which a single observation of a rare action can influence behavior). Specifically, in each population it is required the product of (1) share of agents with a sample size of 1 and (2) the bounded expected sample size should be sufficiently large.

Observe that the bound in the expression for the bounded expected value is higher (and, thus, less restrictive), the farther are the $u_i$-s are from 1. That is, games in which in each population one of the actions is much riskier than the other (i.e., action $a_2$ is very risky for population 2, and action $a_1$ is very risky for population 1) are more likely to have stable miscoordination.

The following example demonstrates global convergence to miscoordination, and the fact that the stability of pure states is non-monotone in the sample sizes.

**Example 2** (Global convergence to miscoordination and non-monotone Impact of Sample Size)**.** Consider a hawk-dove game in which $u_1 = 5$ and $u_2 = 0.2$ and both populations have the same distribution of sample sizes. Consider 3 distributions of sample sizes, in each of which half of the population has sample size 1. In the first distribution (left panel of Figure 4.2) the remaining half have sample size 3, in the second distribution (middle panel) they have sample size 5, and in the third distribution (right panel of Figure 4.2) they have sample size 7. Observe that the second distribution satisfies the condition for

Figure 4.2: Illustration for Theorem 1 and Example 2




The figure illustrates the phase plots for three environments. In each environment the underlying game is hawk-dove with $u_1 = 5$ and $u_2 = 0.2$, and 50% of the agents in each population have sample size 1. In the environment illustrated in the left (resp., middle, right) panel the remaining half of the population have sample size 2 (resp., 5, 7). The middle panel shows global convergence to the interior state with miscoordination $(0.37, .0.63)$, while the other two panels show global convergence to one of of pure coordinated equilibria.

global convergence to miscoordination

$$\theta_1 (1) \cdot \mathbb{E}_{<\frac{1}{u_2}+1} (\theta_2) = \theta_1 (1) \cdot \mathbb{E}_{<6} (\theta_2) = \theta_1 (1) \cdot \mathbb{E} (\theta_2) = 0.5 \cdot 3 > 1$$

(and the same holds for population 2). Indeed, the middle phase plot shows that the populations converge from any interior state to the state $(0.37, 0.63)$ with substantial miscoordination (specifically, the players miscoordinate and get a payoff of zero in $46\% \approx 2 \cdot 0.37 \cdot 0.63$ of the interactions). By contrast, either decreasing the larger sample size from 5 to 2, or increasing it to 7, yields a product $\theta_1 (1) \cdot \mathbb{E}_{<\frac{1}{u_2}+1} (\theta_2)$ that is strictly smaller than 1 ($\theta_1 (1) \cdot \mathbb{E} (\theta_2) = 0.5 \cdot 1.5 < 1$ in the first case, and $\theta_1 (1) \cdot \mathbb{E}_{<6} (\theta_2) = \theta_1 (1) \cdot \theta_1 (1) = 0.5 \cdot 0.5 < 1$ in the second case). The left and right panels of Figure 4.2 illustrate that in both cases almost all initial states converge to a pure coordinated state. Thus, changing sample sizes of agents have non-monotone impact on the stability of miscoordination.

# 5 Homogeneity and Unstable Miscoordination

In this section, we show that heterogeneity is necessary for stable miscoordination. Specifically, we show that any environment in which all agents in each population have the same sample size admits at most one interior stationary state, and that this state is unstable.

**Auxiliary Results** We begin by showing that that a stationary state $\hat{\boldsymbol{p}}$ is asymptotically stable iff the curve $w_2$ is above the curve $w_1^{-1}$ in a left neighborhood of $\hat{\boldsymbol{p}}$, and it is below the curve $w_1^{-1}$ in a right neighborhood of $\hat{\boldsymbol{p}}$.

*Claim* 5. Let $\hat{\boldsymbol{p}}$ be a stationary state. $\hat{\boldsymbol{p}}$ is asymptotically stable if both conditions hold:

1. **Left neighborhood**: If $\hat{p}_1 > 0$,then there exists $\underline{p}_1 \in (0, \hat{p}_1)$ such that $w_2(p_1) > w_1^{-1}(p_1)$ for any $p_1 \in \left(\underline{p}_1, \hat{p}_1\right)$, and

2. **Right neighborhood**: If $\hat{p}_1 < 1$, then there exists $\overline{p}_1 \in (\hat{p}_1, 1)$ such that $w_2(p_1) < w_1^{-1}(p_1)$ for any $p_1 \in (\hat{p}_1, \overline{p}_1)$.

Moreover, if any of the above two conditions is not satisfied, then $\hat{\boldsymbol{p}}$ is unstable.

*Proof.* Assume that conditions (1) and (2) hold. Let $\boldsymbol{p} \neq \hat{\boldsymbol{p}}$ be any sufficiently close state. By Claim 2 any trajectory beginning at $\boldsymbol{p}$ will enter one of the two areas between the two curves in either side of $\hat{\boldsymbol{p}}$. Due to Claim 3, Condition (1) (resp., (2)) implies convergence to $\hat{\boldsymbol{p}}$ if the trajectory has entered the area between the curves to the left (resp., right) of $\hat{\boldsymbol{p}}$. This implies that any trajectory that starts sufficiently close to $\hat{\boldsymbol{p}}$ must converge to $\hat{\boldsymbol{p}}$, and, thus, $\hat{\boldsymbol{p}}$is asymptotically stable.

Next assume that $\hat{p}_1 > 0$ and condition (1) (resp., $\hat{p}_1 < 1$, and condition (2)) is not satisfied. This implies that $w_2(p_1) < w_1^{-1}(p_1)$ (resp., $w_2(p_1) > w_1^{-1}(p_1)$ for any $p_1$ that is sufficiently close from the left (resp., right) to $\hat{p}_1$. Due to Claim 3, this implies that a trajectory starting at $(p_1, p_2)$ with $p_1$ sufficiently close to $\hat{p}_1$ from the left (resp., right) and with $p_2 \in \left(w_2(p_1), w_1^{-1}(p_1)\right)$ (resp., $p_2 \in \left(w_1^{-1}(p_1), w_2(p_1)\right)$) must converge to the neighboring stationary point from the left (right) of $\hat{\boldsymbol{p}}$. Thus, $\hat{\boldsymbol{p}}$ is unstable. □

Claim 5 implies that the neighbor of an asymptotically stable state must be unstable.

**Corollary 2.** *Let $\underline{\mathbf{p}} \neq \overline{\mathbf{p}}$ be two neighboring stationary states (i.e., there does not exist any stationary state $\hat{\mathbf{p}}$ satisfying either $\hat{p}_1 \in \left(\underline{p}_1, \overline{p}_1\right)$ or $\hat{p}_1 \in \left(\overline{p}_1, \underline{p}_1\right)$). If $\underline{\mathbf{p}}$ is asymptotically stable, then $\overline{\mathbf{p}}$ is unstable.*

*Proof.* Assume that $\underline{p}_1 < \overline{p}_1$ (resp., $\overline{p}_1 < \underline{p}_1$). Due to Claim 5 the fact that $\overline{\boldsymbol{p}}$ is asymptotically stable implies that $w_2(p_1) > w_1^{-1}(p_1)$ for any $p_1 \in \left(\underline{p}_1, \overline{p}_1\right)$ (resp., $w_2(p_1) < w_1^{-1}(p_1)$ for any $p_1 \in \left(\overline{p}_1, \underline{p}_1\right)$), which, in turn, implies that $\overline{\boldsymbol{p}}$ is unstable. □

**New Property of Binomial Distributions** The main result is implied by deriving a property of binomial distributions (which might be of independent interest). Recall our notation of $X(k,p) \sim Bin\,(k,p)$ denoting a random variable with binomial distribution. Define $F_m^k\,(p) \equiv \Pr\,(X\,(k,p) \geq m)$ as the probability of having at-least $m$ success in $k$ trials when the probability of success in each trial is $p$. Observe that $F_m^k\,(0) = 0$, $F_m^k\,(1) = 1$ and $\left(F_m^k\right)' > 0$. It is known that $F_m^k$ has at most one interior fixed point, i.e.,

**Fact 4** (Green, 1983, Theorem 1)**.** *Fix arbitrary integers $0 < m \leq k$. Then there is at most one $p \in (0,1)$ such that $F_m^k\,(p) = p \Leftrightarrow \Pr\,(X\,(k,p) \geq m) = p$.*

Our new result shows that the same is true also for a composition of any two cumulative binomial distributions, i.e., that $F_{m_2}^{k_2} \circ F_{m_1}^{k_1}$ has at most one fixed point.

**Proposition 2.** *Fix arbitrary integers satisfying $0 < m_1 \leq k_1$ and $0 < m_2 \leq k_2$. There exists at most one probability $p \in (0,1)$ such that $\left(F_{m_1}^{k_1} \circ F_{m_2}^{k_2}\right)(p) = p$.*

*The proof, which is detailed in Appendix A.1, shows that $F_{m_2}^{k_2} \circ F_{m_1}^{k_1}$ has at most one inflection point, and that this implies having at most one interior fixed points.*

**Main Result** Next we show the main result of this section: any environment with homogeneous sample sizes admits at most one interior stationary state, which is unstable. This implies that almost all initial states converge to one of the coordinated equilibria.

**Theorem 2.** *Assume that $\theta_i \equiv k_i > 1$ for each $i \in \{1,2\}$. There exists at most one interior stationary state, and this state (if exists) is unstable.*

*Proof.* The fact that $\theta_i \equiv k_i > 1$ implies that $w_i\,(p_{-i}) = F_{m_i}^{k_i}\,(p)$ for some $1 \leq m_i \leq k_i$. This implies that any stationary state $\hat{\boldsymbol{p}}$ must satisfy $\left(F_{m_1}^{k_1} \circ F_{m_2}^{k_2}\right)(p_1) = p_1$. Proposition 2 implies that this holds for at most one interior state $\hat{\boldsymbol{p}}$. This implies that the stationary state $\hat{\boldsymbol{p}}$ (if exists) is a neighbor of both pure stationary states $(0,0)$ and $(1,1)$. Proposition 1 and the fact that no agents have sample size of 1 implies that at-least one of these pure states is asymptotically stable. Finally, Corollary 2 implies that $\hat{\boldsymbol{p}}$ is unstable. □

# 6 Heterogeneity and Stable Miscoordination

The conditions presented for global convergence to miscoordination in Section 4 are somewhat narrow (e.g., they require sufficiently many agents with sample size one). In this section we show that a much broader set of heterogeneous distributions of sample sizes can induce asymptotically stable states with miscoordination. Specifically, we show that any distribution of sample sizes can be combined with a group of agent with sufficiently large samples to induce locally stable miscoordination if for each population one of the actions is $q$-dominant for a sufficiently low $q$. As demonstrated below, this type of heterogeneity in the sample sizes is plausible in various setups.

**Example** (Example 1 revisited)**.** In housing markets (in which the bargaining situations which can be modeled as hawk-dove coordination games), it is often the case that each population includes two types of agents: (1) professional traders who buy/sell houses for investment, and (2) the remaining agents who buy/sell a house only a couple of times during their life. It seems plausible the the professional traders have reliable information on the aggregate behavior of the opposing population (captured in our model by having large samples), while the remaining agents are likely to have limited information about the aggregate behavior of the opposing population , which is influenced by anecdotal evidence from friends and relatives (captured in our model by small samples).

**Auxiliary Result and Definition** Our auxiliary results characterize asymptotically stable states in terms of the ratio of the slopes of the two curves $\frac{w_2'(\hat{p}_1)}{\left(w_1^{-1}\right)'(\hat{p}_1)}$; we show that must be at-most 1 in an asymptotically stable state. Formally,

*Claim* 6. Let $\hat{\boldsymbol{p}}$ be a stationary state.

1. If $\hat{\boldsymbol{p}}$ is asymptotically stable, then $\frac{w_2'(\hat{p}_1)}{\left(w_1^{-1}\right)'(\hat{p}_1)} \leq 1$.
2. If $\frac{w_2'(\hat{p}_1)}{\left(w_1^{-1}\right)'(\hat{p}_1)} < 1$, then $\hat{\boldsymbol{p}}$ is asymptotically stable.

*Proof.* The Claim is immediately implied by either Claim 5, or by the observation that the Jacobian of the dynamics is

$$J_{\boldsymbol{w}}(\boldsymbol{p}) = \begin{pmatrix} -1 & w_1'(p_2) \\ w_2'(p_1) & -1 \end{pmatrix},$$

which implies that the eigenvalues are $-1 \pm \sqrt{w_1'(p_2)\, w_2'(p_1)}$, and that both eigenvalues are negative (resp., one eigenvalue is positive) if $\frac{w_2'(\hat{p}_1)}{\left(w_1^{-1}\right)'(\hat{p}_1)} < 1$ (resp., $\frac{w_2'(\hat{p}_1)}{\left(w_1^{-1}\right)'(\hat{p}_1)} > 1$), which implies asymptotic stability (resp., instability). □

In order to state our result we define a *weighted average of two distributions*, as a population in which $\alpha$ of the agents have sample sizes according to the first distribution, and $1-\alpha$ have sample sizes according to the second distribution. Formally, given two distributions of sample sizes $\theta_i, \hat{\theta}_i$ and $\alpha_i \in [0,1]$, let $\theta_i \alpha_i \hat{\theta}_i$ be the weighted average of the two distributions: $\theta_i \alpha_i \hat{\theta}_i (k_i) = \alpha_i \cdot \theta_i (k_i) + (1-\alpha_i) \cdot \hat{\theta}_i (k_i)$.

**Main Result** Our final result shows that any pair of distributions can be combined with distributions of players having sufficiently large samples, such that the environment admits a locally-stable interior state with miscoordination, provided that the game admits a sufficiently risk-dominant action for each population.

**Theorem 3** (Asymptotic stability of miscoordination)**.** *For any distribution profile* $\boldsymbol{\theta}$*, there exists* $q \in (0,1)$ *such that if the game* $u$ *admits a* $q$*-dominant action for each population, then there exists* $\bar{k} > 1$ *and an open interval of* $\alpha_i$*-s, such that the environment* $\left(u, \left(\theta_1 \alpha_1 \overline{\theta}_1, \theta_2 \alpha_2 \overline{\theta}_2\right)\right)$ *admits a locally stable interior stationary state for any* $\alpha$ *in the interval if* $\min\left(supp\left(\overline{\theta}_i\right)\right) > \bar{k}$ *.*

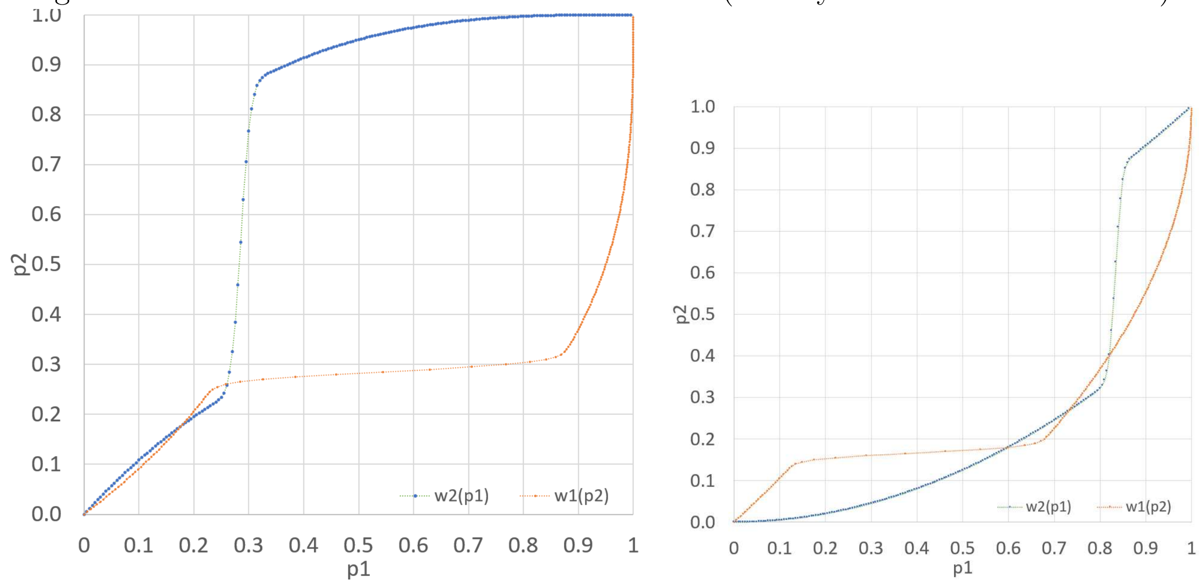
*Proof idea (see, Appendix A.7 for a formal proof).* Fix a distribution profile $\boldsymbol{\theta}$. We present two constructive proofs for games with / without a risk-dominated equilibrium. The arguments are illustrated in the two panels of Figure 6.1.

1. The risk-dominated equilibrium $\boldsymbol{b}$ is unstable for the distribution profile $\theta$, if it is $q$-dominated for a sufficiently low $q$. Observe that combining $\theta_i$ with agents with very large samples decrease the average probability of a new agent playing $a_i$ in state $(\epsilon_1, \epsilon_2)$ near $\boldsymbol{b}$. We choose $\alpha_i$, such that the share of new agents playing action $a_i$ (1) is above $\epsilon_{-i}$ in state $(\epsilon_1, \epsilon_2)$, and (2) it is below $p_{-i}$ in an interior point $\hat{\boldsymbol{p}}$ (this is possible because $\frac{w_i(p_{-i})}{p_{-i}}$ is decreasing for small $p_{-i}$-s). This, in turn, implies that there is a stable interior state between $\boldsymbol{b}$ and $\hat{\boldsymbol{p}}$.

2. Next consider games with different risk-dominant actions (i.e., $u_2 < 1 < u_1$). Observe that $w_i^{\theta}(p_{-i})$ remains the same for all values of $u_i$ that are sufficiently far from one, while $p_{NE}^1$ (resp., $p_{NE}^2$) converge to 0 (resp., 1) as $u_1$ converges to infinity (resp., $u_2$ converges to zero). Thus, for $u_i$-s sufficiently far from 1, $p_2^{NE} < \frac{w_2^{\theta_2}\left(\frac{1}{2}\right)}{2}$. We show that this implies that there must be an unstable stationary state with $p_1 < \frac{1}{2}$ and $p_2 \simeq p_2^{NE} < \frac{1}{2}$. By an analogous argument there is an unstable stationary state

with $p_1 \simeq p_1^{NE} > \frac{1}{2}$ and $p_2 > \frac{1}{2}$. This implies that there must be an asymptotically stable interior state between these two unstable states. □

Figure 6.1: Illustrative Phase Plots for Theorem 3 (Locally Stable Miscoordination)




The left panel illustrates the game $u_1 = u_2 = 2.5$, where 40% (resp., 60%) of agents in each population have sample size 3 (resp., 1,000). The pure risk-dominated equilibrium $(0,0)$ is unstable, and its neighbor the interior stationary state with miscoordination $(0.17, 0.17)$ is asymptotically stable. The right panel illustrates the game $u_1 = \frac{1}{u_2} = 5$, where 50% (resp., 50%) of the agents have sample size 2 (resp., 1,000). The mixed Nash equilibrium of this game is $\left(\frac{5}{6}, \frac{1}{6}\right)$. The environment admits 3 interior stationary states: 2 unstable interior states where one of the coordinates is close to the Nash equilibrium: $(0.6, 0.18)$ and $(0.82, 0.3)$, and a stable equilibrium at $(0.73, 0.27)$.

# 7 Comparison with Logit Dynamics

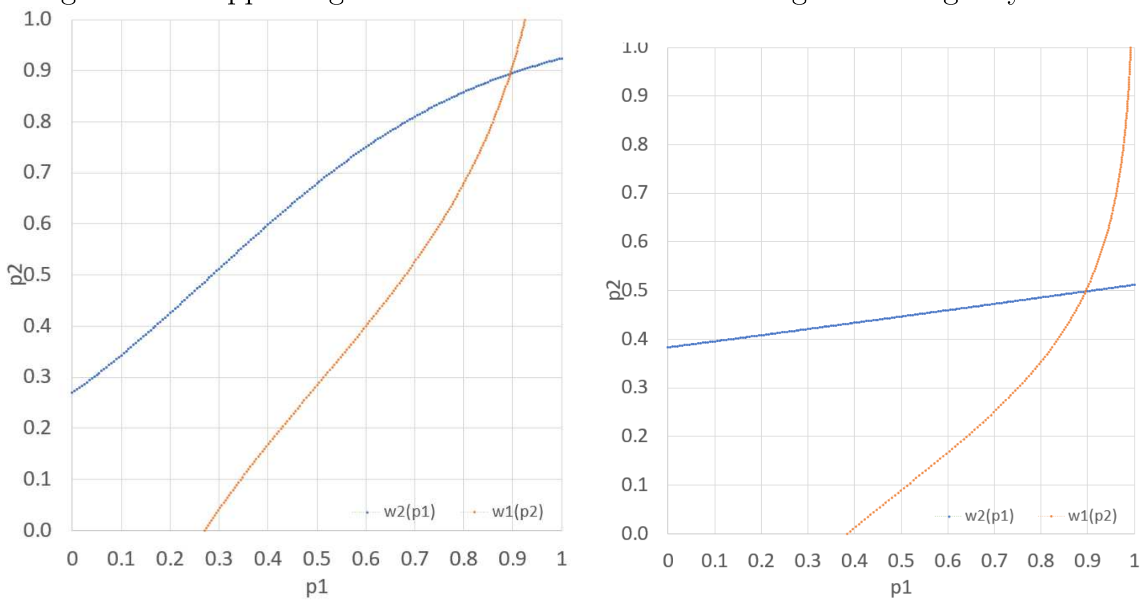
Another candidate to induce stable miscoordination in coordination games is logit dynamics. In this section we numerically demonstrate that (1) the standard logit dynamics with homogeneous level of noise in each population can induce stable miscoordination only with high levels of noise that seem implausible, (2) a variant of logit dynamics with heterogeneity in the noise level can induce stable miscoordination with substantially lower levels of noise. Thus, *our main insight of stable miscoordination induced by heterogeneous plausible levels of noises remains the same with other specifications of noise.*

**Standard (Homogeneous) Logit dynamics** The logit dynamics (see Sandholm, 2010, Section 6.2.3 for a textbook exposition) are characterized by a single parameter for each population $i$ - the *noise level* $\eta_i$. If player $i$ plays action $a_i$, she will get payoff of $p_{-i} \cdot u_{-i}$. If she plays action $b_i$ she will get payoff of $(1 - p_{-i}) \cdot 1$. The logit dynamics assume that the probability in which revising agents play action $a_i$ is proportional to $e^{\frac{\text{Payoff of } a_i}{\eta}}$. Specifically, the logit dynamics is given by:

$$w_i(p_{-i}) \equiv w_i(\mathbf{p}) = \frac{e^{\frac{p_{-i} \cdot u_{-i}}{\eta_i}}}{e^{\frac{1-p_{-i}}{\eta_i}} + e^{\frac{p_{-i} \cdot u_{-i}}{\eta_i}}}. \tag{7.1}$$

Trivially, logit dynamics can induce substantial coordination by having high values of noise. In what follows we demonstrate that this is indeed the case. One can support stable miscoordination in the examples studied in Section 6, but this requires a high level of noise. As illustrated in Figure 7.1 the minimal level of $\eta$ (which are assumed, for simplicity, to be the same on both populations) that is required to sustain asymptotically stable equilibrium with miscoordination in which each action is played with a probability of at-least 10% is $\eta = 1$ in both the environments of $u_1 = u_2 = 2.5$(left panel) and $u_1 = \frac{1}{u_2} = 5$ (right panel).

Figure 7.1: Supporting Stable Coordination with Homogeneous Logit dynamics




The figure revisits the examples presented in Figure 6.1: a game with a risk-dominant equilibrium with $u_1 = u_2 = 2.5$ in the left panel, and a game without risk-dominant equilibria with $u_1 = \frac{1}{u_2} = 5$. It turns out that in both cases the minimal homogeneous level of noise that sustains an asymptotically stable equilibrium with miscoordination in which each action is played with a probability of at least 10% in each population is $\eta = 1$.

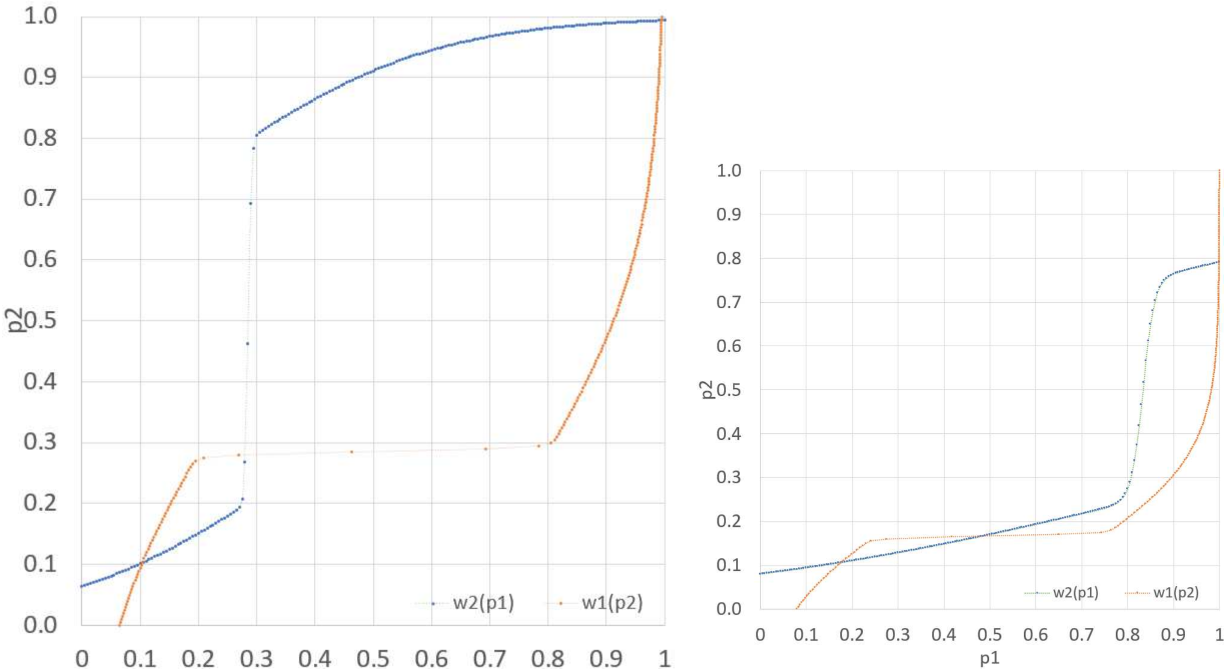
Such high noise levels implies that 27% of the revising agents do the mistake of playing $a_i$ when facing an opponent population in which almost everyone plays $b_{-i}$; by contrast, with sampling dynamics all agents play the payoff-maximizing action whenever all opponents play the same action. Moreover, the expected payoff obtained by revising agents who follow the logit dynamics against opponent populations in which the share of agents playing action $a_i$ is distributed uniformly is 85% (resp., 71%) of the maximal payoff that can be obtained by payoff-maximizing revising agents in the left (resp., right) panel, while it is 98% (resp., 95%) of the maximal payoff with sampling dynamics. Thus, stable cooperation can be supported by the standard (homogeneous) logit dynamics, only when assuming the agents have high levels of noise. Numeric calculations show that this is true for coordination games in general, and not only to the two examples above.

**Heterogeneous Logit dynamics** Consider a variant of logit dynamics in which there is heterogeneity in the level of noise of agent in each population. For example, in a population in which there are $n$ groups, the size $j$-th group is $\mu_i^j$ and its members have noise level of $\eta_i^j$ the heterogeneous logit dynamics is:

$$w_i(p_{-i}) \equiv w_i(\mathbf{p}) = \sum_j \mu_i^j \cdot \frac{e^{\frac{p_{-i}\cdot u_{-i}}{\eta_i^j}}}{e^{\frac{1-p_{-i}}{\eta_i^j}} + e^{\frac{p_{-i}\cdot u_{-i}}{\eta_i^j}}}. \tag{7.2}$$

Figure 7.2 demonstrates numerically that heterogeneity in the noise levels can induce asymptotically stable miscoordination with levels of noise that are substantially smaller than in homogeneous populations. Specifically in both panels (corresponding to $u_1 = u_2 = 2.5$ in the left panel, and to $u_1 = \frac{1}{u_2} = 5$ in the right panel) populations in which 50% of the the agents have moderate level of noise of $\eta = 0.6$ and 50% have a negligible level of $\eta = 0.01$ induce asymptotically stable equilibrium with miscoordination8% pl ((0.27, 0.27)in the left panel and (0.11, 0.18) in the right panel). Given these heterogeneous levels of noise, only 8% of the agents make the mistake of playing action $a_i$ when facing an opponent population in which everyone plays $a_{-i}$, and the expected payoff obtained by against opponent populations in which the share of agents playing action $a_i$ is distributed uniformly is 96% (resp., 89%) of the maximal payoff that can be obtained by payoff-maximizing revising agents in the left (resp., right) panel.

Figure 7.2: Supporting Stable Coordination with Homogeneous Logit dynamics




The figure revisits the examples presented in Figure 7.1: a game with $u_1 = u_2 = 2.5$ in the left panel, and a game with $u_1 = \frac{1}{u_2} = 5$ in the right panel. In both panels a heterogeneous variant of logit dynamics in which 50% of the the agents in each population have moderate level of noise $\eta = 0.6$ and 50% have a negligible level of $\eta = 0.01$ induce an asymptotically stable state with miscoordination in which each action is played with a probability of at-least 10%.

# 8 Conclusion

The conventional wisdom, which is supported by key results in evolutionary game theory, is that only coordinated outcomes can predict long-run behavior in coordination games. By contrast, we show that plausible learning dynamics, in which new agents rely on samples to estimate the behavior of the opponent's population, can induce stable miscoordination. This happens if there is heterogeneity in the sample sizes: some agents have accurate information about the opponents' aggregate behavior, while other agents rely on anecdotal evidence induced by small samples. We further show that stable miscoordination holds under a broader set of heterogeneous distributions, for coordination games in which one of the action is sufficiently more risk-dominant than the other action.

Our numeric analysis of logit dynamics suggest that the insight that heterogeneity in the noise level can induce stable miscoordination is relevant to various learning dynamics. As such heterogeneity is plausible in many applications (e.g., bargaining involving both professional traders, and inexperienced buyers/sellers), we think that this provides an interesting testable new explanation for the persistence of miscoordination in some interactions (see, Section 2, for our suggestion of how to experimentally test our predictions).

# A Appendix

## A.1 General Result for Binomial Distributions

Recall our notation of $X(k,p) \sim Bin\,(k,p)$ denoting a random variable with binomial distribution, and of the function $F_m^k\,(p) \equiv \Pr\,(X\,(k,p) \geq m)$.

**Proposition.** 2 *Fix arbitrary integers satisfying* $0 < m_1 \leq k_1$ *and* $0 < m_2 \leq k_2$. *There exists at most one probability* $p \in (0,1)$ *such that* $\left(F_{m_1}^{k_1} \circ F_{m_2}^{k_2}\right)(p) = p$.

*Proof.* Let $w_i\,(p) \equiv F_{m_1}^{k_1}\,(p)$ for each $i \in \{1,2\}$, $F\,(p) \equiv \left(F_{m_1}^{k_1} \circ F_{m_2}^{k_2}\right)(p) \equiv (w_1 \circ w_2)\,(p)$ ,and $G\,(p) = F\,(p) - p$. Proposition is implied by showing that there is at most one $p \in (0,1)$ such that $G\,(p) = 0$. Observe that $G\,(0) = G\,(1) = 0$. Assume to the contrary that there exists two different interior points $0 < \underline{p} < \overline{p} < 1$ such that $G\left(\underline{p}\right) = G\,(\overline{p})$. Thus $G$ is equal to zero in 4 points in the interval $[0,1]$. By Rolle's Theorem, this implies that $G'$ is equal to zero in at-least 3 points in the interval $(0,1)$, and, this, in turn, implies that $G''$ is equal to zero in at-least 2 interior points in the interval $(0,1)$. Observe that $G'' \equiv F''$. Thus, in order to obtain a contradiction we have to show that $F''\,(p) = 0$ in at most one interior point. Recall that (see, e.g., Green, 1983, Eq. (5)):

$$w_i'\,(p) = m_i \begin{pmatrix} k_i \\ m_i \end{pmatrix} p^{m_i - 1}\,(1-p)^{k_i - m_i},$$

which immediately implies that

$$\frac{w_i''\,(p_{-i})}{w_i'\,(p_{-i})} = \frac{m_i - 1}{p} - \frac{k_i - m_i}{1-p}.$$

Observe that $F'\,(p) = w_1'\,(w_2\,(p))\,w_2'\,(p)$, which implies that

$$F''\,(p) = w_1''\,(w_2\,(p))\,(w_2'\,(p))^2 + w_1'\,(w_2\,(p))\,w_2''\,(p) =$$

$$w_1'\,(w_2\,(p))\,w_2'\,(p)\left[\left(\frac{m_1-1}{w_2\,(p)} - \frac{k_1 - m_1}{1 - w_2\,(p)}\right) w_2'\,(p) + \frac{m_2 - 1}{p} - \frac{k_2 - m_2}{1-p}\right]$$

The fact that each $w_i(p)$ is strictly increasing implies that $F''(p) = 0$ iff

$$\left(\frac{m_1 - 1}{w_2(p)} - \frac{k_1 - m_1}{1 - w_2(p)}\right) w_2'(p) = \frac{k_2 - m_2}{1 - p} - \frac{m_2 - 1}{p} \Leftrightarrow$$

$$m_2 \begin{pmatrix} k_2 \\ m_2 \end{pmatrix} \left(\frac{m_1 - 1}{w_2(p)} - \frac{k_1 - m_1}{1 - w_2(p)}\right) p^{m_2 - 1} (1 - p)^{k_2 - m_2} = \frac{k_2 - m_2}{1 - p} - \frac{m_2 - 1}{p} \Leftrightarrow$$

$$m_2 \begin{pmatrix} k_2 \\ m_2 \end{pmatrix} \left(\frac{(m_1 - 1) p^{m_2} (1 - p)^{k_2 - m_2}}{\sum_{l=m_2}^{k_2} \begin{pmatrix} k_2 \\ l \end{pmatrix} p^l (1 - p)^{k_2 - l}} - \frac{(k_1 - m_1) p^{m_2} (1 - p)^{k_2 - m_2}}{\sum_{l=0}^{m_2 - 1} \begin{pmatrix} k_2 \\ l \end{pmatrix} p^l (1 - p)^{k_2 - l}}\right) \frac{1}{p} = \frac{k_2 - m_2}{1 - p} - \frac{m_2 - 1}{p} \Leftrightarrow$$

$$m_2 \begin{pmatrix} k_2 \\ m_2 \end{pmatrix} \left(\frac{(m_1 - 1)}{\sum_{l=m_2}^{k_2} \begin{pmatrix} k_2 \\ l \end{pmatrix} \left(\frac{p}{1-p}\right)^{l - m_2}} - \frac{k_1 - m_1}{\sum_{l=0}^{m_2 - 1} \begin{pmatrix} k_2 \\ l \end{pmatrix} \left(\frac{1-p}{p}\right)^{m_2 - l}}\right) \frac{1}{p} = \frac{k_2 - m_2}{1 - p} - \frac{m_2 - 1}{p}$$

Observe that the rhs (resp., lhs) is strictly increasing (resp., decreasing) in $p$, which implies that they are equal in at most one point. □

Observe that in the symmetric case when $k_1 = k_2$ and $m_1 = m_2$, the fact that there is at most one fixed point to $F_m^k \circ F_m^k$ implies that any fixed point of $F_m^k \circ F_m^k$ must be a fixed point of $F_m^k$. This implies the following corrolary on random binomial variables.

**Corollary 3.** *If* $\Pr(X(k,p) \geq m) = q$*,* $\Pr(X(k,q) \geq m) = p$ *for* $p \in (0,1)$*, then* $p = q$*.*

## A.2 Standard Definitions of Dynamic Stability

For completeness, we present in this appendix the standard definitions of dynamic stability that are used in the paper (see, e.g., Weibull, 1997, Chapter 5).

A state is said to be stationary if it is a rest point of the dynamics.

**Definition 3.** State $\mathbf{p}^* \in [0,1]^2$ is a *stationary state* if $w_i(\mathbf{p}^*) = p_i^*$ for each $i \in \{1,2\}$.

Let $\mathcal{E}(w)$ denote the set of stationary states of $w$, i.e., $\mathcal{E}(w) = \{\mathbf{p}^* | w_i(\mathbf{p}^*) = p_i^*\}$. Under monotone dynamics, an interior (mixed) state $\mathbf{p}^* \in (0,1)^2$ is a stationary state iff it is a Nash equilibrium (Weibull, 1997, Prop. 4.7). By contrast, under nonmonotone dynamics (such as the sampling dynamics analyzed below) the two notions differ.

A state is Lyapunov stable if a population beginning near it remains close, and it is asymptotically stable if, in addition, it eventually converges to it. A state is unstable if it is not Lyapunov stable. It is well known (see, e.g., Weibull, 1997, Section 6.4) that every Lyapunov stable state must be a stationary state. Formally:

**Definition 4.** A stationary state $\mathbf{p}^* \in [0,1]^2$ is *Lyapunov stable* if for every neighborhood $U$ of $\mathbf{p}^*$ there is a neighborhood $V \subseteq U$ of $\mathbf{p}^*$ such that if the initial state $p(0) \in V$, then $\mathbf{p}(t) \in U$ for all $t > 0$. A state is *unstable* if it is not Lyapunov stable.

**Definition 5.** A stationary state $\mathbf{p}^* \in [0,1]^2$ is *asymptotically stable* (or *locally stable*) if it is Lyapunov stable and there is some neighborhood $U$ of $\mathbf{p}^*$ such that all trajectories initially in $U$ converge to $\mathbf{p}^*$, i.e., $\mathbf{p}(0) \in U$ implies $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}^*$.

## A.3 Proof of Claim 2 (Reaching Area Between the Curves)

We say that state $\mathbf{p}$ is *above* (resp., *below*) curve $w_2(p_1)$ if $p_2 > w_2(p_1)$ (resp., ($p_2 < w_2(p_1)$). Similarly, we say that state $\mathbf{p}$ is to the *right* (resp., *left*) of the curve $w_1(p_2)$ if $p_1 > w_1(p_2)$ (resp., $p_1 < w_1(p_2)$). We say that the state $\mathbf{p}$ is on the curve $w_i(p_{-i})$ if $p_i = w_i(p_{-i})$. Due to the fact that the two curves are strictly increasing, we identify the notion of being above a curve and being to the left of the curve, and similarly we identify the notion of being below a curve and being to the right of the curve. The states on the curve $w_2(p_1)$ (resp., $w_1(p_2)$) are characterized by having $\dot{p_2} = 0$ (resp., $\dot{p_1} = 0$). Observe that $\dot{p_2} > 0$ (resp., $\dot{p_2} < 0$) in any state $\mathbf{p}$ above and to the left (resp., below and to the right) of the curve $w_2(p_1)$. Similarly, $\dot{p_1} > 0$ (resp., $\dot{p_1} < 0$) in any state $\mathbf{p}$ above and to the left (resp., below and to the right) of the curve $w_1(p_2)$.

Any state $\mathbf{p} \in [0,1]$ can be classified in one of $9 = 3 \cdot 3$ classes, depending on its relative location with respect to the two curves, i.e., whether $\mathbf{p}$ is below, above or on each of the two curve $w_i(p_{-i})$. If state $\mathbf{p}$ is on (resp., above, below) the curve $w_2(p_1)$, then $\dot{p_2}$ is zero (resp., negative, positive). Similarly, if state $\mathbf{p}$ is on (resp., above, below) the curve $w_1(p_2)$, then $\dot{p_1}$ is zero (resp., positive, negative).

In particular, any state $\mathbf{p}$ that is above (resp., below) both curves must satisfy that $\dot{p_1} > 0 > \dot{p_2}$. This implies that any trajectory that begins above (resp., below) both curves must always move downward and to the right. This (together with the fact that all stationary states are in the intersections of the two curves) implies that the trajectory must cross one of the curves, and reach a state $\boldsymbol{p}(t)$ that satisfies either $w_1^{-1}(p_1(t)) \leq p_2(t) \leq w_2(p_1(t))$ or $w_2(p_1(t)) \leq p_1(t) \leq w_1^{-1}(p_1(t))$.

## A.4 Proof of Claim 3 (Convergence to Stationary States)

We first show why we can assume WLOG that $\boldsymbol{p}(t)$ is strictly between the two curves. If $\boldsymbol{p}(t)$ crosses one of the curves and is strictly above (resp., below) the remaining curve, then the dynamics must take the populations to a state that is strictly below one of the curves and strictly above the remaining curve. This is so because on the crossing point one of the $\dot{p_i}$ is zero and the remaining derivative $\dot{p} - i$ is negative (resp., positive), which implies that the dynamics take the trajectory below and to the right (resp., above and to the left) of the curve that was crossed.

Next assume that $p_i(t) \in \left(w_1^{-1}(p_1(t)), w_2(p_1(t))\right)$ (resp., $p_i(t) \in \left(w_2(p_1(t)), w_1^{-1}(p_1(t))\right)$. By the classification presented in the proof of claim 2, the trajectory must move upward and to the right; i.e., $\dot{p_1}, \dot{p_2} > 0$ (resp., downward and to the left; i.e., $\dot{p_1}, \dot{p_2} < 0$). This implies that the trajectory must cross one of the curves. The crossing point cannot be only on the curve of $w_2(p_1)$ (resp., $w_1(p_2)$), because at such a point the trajectory moves horizontally to the left (vertically upward), which implies that it must cross the lower (resp., higher) curve $w_2(p_1)$ (resp.,$w_1(p_2)$) from the left side (resp., from below) and we get a contradiction. This implies that the crossing point must the closest intersection points of the two curves to the right (resp., left) of $p_i(t)$, namely, $\overline{\boldsymbol{p}}$ (resp., $\underline{\boldsymbol{p}}$).

## A.5 Proof of Claim 4 (Convergence to Pure States)

If $\mathbf{p}$ is below (resp., above) both curves, then by the classification presented in the proof of Claim 2 it must that $\dot{p_2} > 0$ (resp., $\dot{p_1} > 0$), which implies that convergence to (0,0) is possible only if the trajectory passes through a state that is strictly between the two curves, and that the closest intersection point of the two curves to the left of this state is (0,0). By the classification presented in the proof of Proposition 1 it must be that the curve of $w_2(p_1)$ is strictly below the curve of $w_1(p_2)$ in a right neighborhood of (0,0), which implies that (0,0) is asymptotically stable because any sufficiently close initial state would converge to (0,0).

## A.6 Proof of Proposition 1 (Stability of Pure States)

We are interested in deriving conditions for the stability of the pure stationary states. In what follows, we compute the Jacobian of the sampling dynamics in the pure state $\boldsymbol{a} = (0,0)$ (resp., $\boldsymbol{b} = (1,1)$). For this, we consider a slightly perturbed state with a "very small" $\epsilon_i$ share of agents playing $b_i$ (resp., $a_i$) in population $i$. By "very small," we mean that higher-order terms of $\epsilon_i$ and $\epsilon_j$ are neglected.

Consider a new agent of population $i$ with a sample size of $k_i$. Action $b_i$ (resp., $a_i$) has a weakly (resp., strictly) higher mean payoff against a sample size of $k_i$ iff (neglecting rare events of having multiple $b_{-i}$-s (resp., $a_{-i} - s$) in the sample): (1) the sample includes the single opponent action $b_{-i}$ (resp., $a_{-i}$), and (2) $k_i < \frac{1}{u_i} + 1$ (resp., $k_i \leq u_i + 1$). The probability of (1) is $k_i \cdot \epsilon_{-i} + o(\epsilon_{-i})$, where $o(\epsilon_{-i})$ denotes terms that are sublinear in $\epsilon_{-i}$, and, thus, it will not affect the Jacobian as $\epsilon_{-i} \to 0$. This implies that the probability that a new agent of population $i$ (with a random sample size distributed according to $\theta_i$) has a higher mean payoff for action $b_i$ (resp., $a_i$) against her sample is $w_i(1-\epsilon_{-i}) = \epsilon_{-i} \cdot \mathbb{E}_{<\frac{1}{u_i}+1}(\theta_i) + o(\epsilon_{-i})$ (resp., $w_i(\epsilon_{-i}) = \epsilon_{-i} \cdot \mathbb{E}_{\leq u_i+1}(\theta_i) + o(\epsilon_{-i})$ ). Therefore, the sampling dynamics at $(\epsilon_1, \epsilon_2)$ (resp., $(1-\epsilon_1, 1-\epsilon_2)$) can be written as follows (ignoring the higher-order terms of $\epsilon_1$ and $\epsilon_2$):

$$\dot{\epsilon}_i = \epsilon_{-i} \cdot \mathbb{E}_{<\frac{1}{u_i}+1}(\theta_i) - \epsilon_i \;\; (\text{resp.}, \dot{\epsilon}_i = \epsilon_i - \epsilon_{-i} \cdot \mathbb{E}_{\leq u_i+1}(\theta_i)). \tag{A.1}$$

Define: $a_{\theta_i} = \mathbb{E}_{<\frac{1}{u_i}+1}(\theta_i)$ (resp., $b_{\theta_i} = \mathbb{E}_{\leq u_i+1}(\theta_i)$). The Jacobian of the above system of Equations (A.1) is then given by $J_a = \begin{pmatrix} -1 & a_{\theta_1} \\ a_{\theta_2} & -1 \end{pmatrix}$ (resp., $J_a = \begin{pmatrix} 1 & -b_{\theta_1} \\ -b_{\theta_2} & 1 \end{pmatrix}$). The eigenvalues of $J_a$ (resp., $J_b$) are $-1-\sqrt{a_{\theta_1}a_{\theta_2}}$ and $-1+\sqrt{a_{\theta_1}a_{\theta_2}}$ (resp., $-1-\sqrt{b_{\theta_1}b_{\theta_2}}$ and $-1+\sqrt{b_{\theta_1}b_{\theta_2}}$), Observe that: (1) if $a_{\theta_1}a_{\theta_2} < 1$ (resp., $b_{\theta_1}b_{\theta_2}>1$) then both eigenvalues are negative, which implies that the pure state $\boldsymbol{a}$ (resp., $\boldsymbol{b}$) is asymptotically stable, and (2) if $a_{\theta_1}a_{\theta_2} > 1$ (resp., $b_{\theta_1}b_{\theta_2}>1$) then one of the eigenvalues is positive, which implies that this state is unstable (see, e.g., Perko, 2013, Theorems 1–2 in Section 2.9).

## A.7 Proof of Theorem 3 (Stability of Miscoordination)

Fix a distribution profile $\boldsymbol{\theta}$. Let $w_i^k(p_{-i})$ (resp., $w_i^{\theta_i}(p_{-i})$) denote the sampling dynamics of an agent with sample size $k$ (resp., of agents with a distribution of sample sizes $\theta_i$). We present two constructive proofs, one for games with a risk-dominant equilibrium, and the games in which each population has a different risk-dominant action (the arguments are illustrated in Figure 6.1).

Consider first games that admit a risk-dominant equilibrium (i.e., $u_1, u_2 > 1$). Fix any $q < \frac{1}{\max(\text{supp}(\theta_1), \text{supp}(\theta_2))}$, and any $u_1, u_2 > \frac{1}{q}$. Observe that for any $k \in \text{supp}(\theta_i)$

$$w_i^k(p_{-i}) = \Pr(X(k, p_{-i}) \geq 1) = 1 - (1 - p_{-i})^k = kp_{-i} - \begin{pmatrix} k \\ 2 \end{pmatrix} (p_{-i})^2 + O\left((p_{-i})^3\right).$$

This implies that $w_i^{\theta_i}(p_{-i}) = \mathbb{E}(\theta_i) p_{-i} - A_{\theta_i}(p_{-i})^2 + O\left((p_{-i})^3\right)$,where $A_{\theta_i} > 0$. Fix a sufficiently small $\epsilon > 0$. Let $0 < \hat{p}_1 < p_1^{NE} - \epsilon$ be sufficiently small such that the term $O\left((p_{-i})^3\right) < \epsilon$ is negligible. For each $i \in \{1, 2\}$, let $\alpha_i \in (0, 1)$ be such that: (1) $\alpha_i \mathbb{E}(\theta_i) > 1$ and (2) $\alpha_i \mathbb{E}(\theta_i) - A_{\theta_i} p_{-i} < 1 - 2\epsilon$. Observe that there is an open interval of $\alpha_i$-s that satisfy these inequalities. This implies that $w_i^{\theta_i}(p_{-i}) > \frac{p_{-i}}{\alpha_i}$ in a right neighborhood of zero, and $w_i^{\theta_i}(p_{-i}) < \frac{p_{-i}}{\alpha_i}$ in a left neighborhood of $\hat{p}_1$.

Observe that $\lim_{k \to \infty} w_2^k(\hat{p}_1) = 0$ and $\lim_{k \to \infty} \left(w_1^k\right)^{-1}(\hat{p}_1) = p_2^{NE}$. This implies that there exists $\bar{k}$ sufficiently large such that $w_2^k(\hat{p}_1) < \epsilon$ and $\left(w_1^k\right)^{-1}(\hat{p}_1) > p_2^{NE} - \epsilon > \hat{p}_1$ for any $k \geq \bar{k}$. Let $\bar{\theta}_i$ be any distribution of types satisfying $\min\left(\text{supp}\left(\bar{\theta}_i\right)\right) > \bar{k}$. Observe that $w_i^{\theta_i \alpha \bar{\theta}_i}(p_{-i}) > p_{-i}$ in a right neighborhood of zero, and $w_i^{\theta_i \alpha \bar{\theta}_i}(p_{-i}) < p_{-i}$ in a left neighborhood of $\hat{p}_1$. This, in turn, implies that $w_2^{\theta_i \alpha \bar{\theta}_i}(p_1) > p_1 > \left(w_1^{\theta_i \alpha \bar{\theta}_i}\right)^{-1}(p_1)$ in a right neighborhood of zero, and $w_2^{\theta_i \alpha \bar{\theta}_i}(p_1) < p_1 < \left(w_1^{\theta_i \alpha \bar{\theta}_i}\right)^{-1}(p_1)$ in a left neighborhood of $\hat{p}_1$. Thus, there exist a stationary state $\tilde{\boldsymbol{p}}$ that satisfies $0 < \tilde{p}_1 < \hat{p}_1 < 1$, and that $w_2^{\theta_i \alpha \bar{\theta}_i}(p_1) > \left(w_1^{\theta_i \alpha \bar{\theta}_i}\right)^{-1}(p_1)$ (resp., $w_2^{\theta_i \alpha \bar{\theta}_i}(p_1) < \left(w_1^{\theta_i \alpha \bar{\theta}_i}\right)^{-1}(p_1)$) in a left (resp., right) neighborhood of $\tilde{p}_1$. This implies, due to Claim 5, that $\tilde{\boldsymbol{p}}$is asymptotically stable. The argument in this case is illustrated in the left panel of Figure 6.1.

Next we analyze games that do not admit a risk-dominant equilibrium (i.e., $u_2 < 1 < u_1$). Observe that $w_1^{\theta}(p_2) = \Pr(\text{having at-least one } a_2 \text{ in the sample})$ is the same for all values of $u_1 > \max(\text{supp}(\theta_1), \text{supp}(\theta_2))$, and, similarly, that $w_2^{\theta}(p_1) = \Pr(\text{having no } b_1\text{-s in the sample})$ is the same for all values of $u_2 < \frac{1}{\max(\text{supp}(\theta_1), \text{supp}(\theta_2))}$. Fix a sufficiently small $\epsilon > 0$. Let $\bar{u}_1, \bar{u}_2$ be payoffs satisfying $p_2^{NE} = \frac{1}{1+\bar{u}_1} < \frac{w_2^{\theta_2}\left(\frac{1}{2}\right)}{2} - \epsilon$ and $1 - p_1^{NE} = \frac{\bar{u}_2}{1+\bar{u}_2} < \frac{1-w_1^{\theta_1}\left(\frac{1}{2}\right)}{2} - \epsilon$. Let $q = \min\left(\frac{1}{1+u_1}, \frac{u_2}{1+u_2}\right)$. Let $\overline{k}$ be sufficiently large such that $w_i^k\left(p_{-i}^{NE} - \epsilon\right) < \epsilon$, $w_i^k\left(p_{-i}^{NE} + \epsilon\right) > 1 - \epsilon$ for any $k \geq \overline{k}$. Let $\bar{\theta}_i$ be any distribution of types satisfying $\min\left(\text{supp}\left(\bar{\theta}_i\right)\right) > \overline{k}$.

In what follows any environment with $u_1 > \frac{1}{q}$, $u_2 < q$ and distribution $\theta_i \frac{1}{2} \bar{\theta}_i$ with $\min\left(\text{supp}\left(\bar{\theta}_i\right)\right) \geq \overline{k}$ admits an interior asymptotically stable state. By Proposition 1, the two pure equilibria are asymptotically stable and $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) < \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$ (resp., $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) > \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$) in a right (resp., left) neighborhood of zero (resp., one). Next observe that $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}\left(\frac{1}{2}\right) > \frac{1}{2} w_2^{\theta_2}\left(\frac{1}{2}\right) > p_{NE}^2 + \epsilon > \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}\left(\frac{1}{2}\right)$, which implies that there an (unstable) interior stationary state $\hat{\boldsymbol{p}}$ that satisfies $\hat{p}_1 < 0.5$ and $p_{NE}^2 - \epsilon < \hat{p}_2 < p_{NE}^2 + \epsilon < 0.5$, and that $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) > \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$ in a right neighborhood of $\hat{p}_1$. By an analogous argument there is also an (unstable) interior stationary state $\tilde{\boldsymbol{p}}$ that satisfies $\tilde{p}_2 > 0.5$ and $p_1^{NE} - \epsilon < \tilde{p}_1 < p_1^{NE} + \epsilon < 0.5$, and that $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) < \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$ in a left neighborhood of $\tilde{p}_1$. This implies that must be an interior stationary state $\grave{\boldsymbol{p}}$ between $\hat{\boldsymbol{p}}$ such that $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) > \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$ (resp., $w_2^{\theta_2 \frac{1}{2} \bar{\theta}_2}(p_1) > \left(w_1^{\theta_1 \frac{1}{2} \bar{\theta}_1}\right)^{-1}(p_1)$) in a left (resp., right) neighborhood of $\grave{\boldsymbol{p}}$. Finally, Claim 5 implies that $\grave{\boldsymbol{p}}$ is asymptotically stable.

# References


Arieli, I., Y. Babichenko, R. Peretz, and H. P. Young (2020). The speed of innovation diffusion in social networks. *Econometrica 88*(2), 569–594.

Arigapudi, S., Y. Heller, and I. Milchtaich (2021). Instability of defection in the prisoner's dilemma under best experienced payoff dynamics. *Journal of Economic Theory 197*, 105174.

Bilancini, E., L. Boncinelli, S. Ille, and E. Vicario (2022). Memory retrieval and harshness of conflict in the hawk–dove game. *Economic Theory Bulletin 10*(2), 333–351.

Cárdenas, J., C. Mantilla, and R. Sethi (2015). Stable sampling equilibrium in common pool resource games. *Games 6*(3), 299–317.

Crawford, V. P. (1989). Learning and mixed-strategy equilibria in evolutionary games. *Journal of Theoretical Biology 140*(4), 537–550.

Green, D. N. (1983). Fixed point properties of the binomial function. *Journal of the Franklin Institute 316*(1), 51–62.

Hauert, C. and J. Miekisz (2018). Effects of sampling interaction partners and competitors in evolutionary games. *Physical Review E 98*(5), 052301.

Heller, Y. (2017). Instability of belief-free equilibria. *Journal of Economic Theory 168*, 261–286.

Heller, Y. and E. Mohlin (2018). Social learning and the shadow of the past. *Journal of Economic Theory 177*, 426–460.

Kandori, M., G. J. Mailath, and R. Rob (1993). Learning, mutation, and long run equilibria in games. *Econometrica: Journal of the Econometric Society*, 29–56.

Kaniovski, Y. M. and H. P. Young (1995). Learning dynamics in games with stochastic perturbations. *Games and economic behavior 11*(2), 330–363.

Kreindler, G. E. and H. P. Young (2013). Fast convergence in evolutionary equilibrium selection. *Games and Economic Behavior 80*, 39–67.

Lyu, J., Z. Li, and Y. Xu (2022). Inexact information, strategic sophistication and equilibrium transition: A quasi-continuous-time experiment. *Mimeo (SSRN 4216572).*

Mantilla, C., R. Sethi, and J. C. Cárdenas (2018). Efficiency and stability of sampling equilibrium in public goods games. *Journal of Public Economic Theory 22*(2), 355–370.

Maynard-Smith, J. (1982). *Evolution and the Theory of Games.* Cambridge University Press.

Maynard Smith, J. and G. R. Price (1973). The logic of animal conflict. *Nature 246*(5427), 15–18.

Morris, S., R. Rob, and H. S. Shin (1995). p-dominance and belief potential. *Econometrica: Journal of the Econometric Society*, 145–157.

Oprea, R., K. Henwood, and D. Friedman (2011). Separating the hawks from the doves: Evidence from continuous time laboratory games. *Journal of Economic Theory 146*(6), 2206–2225.

Osborne, M. J. and A. Rubinstein (1998). Games with procedurally rational players. *American Economic Review 88*(4), 834–847.

Osborne, M. J. and A. Rubinstein (2003). Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior 45*(2), 434–441.

Oyama, D., W. H. Sandholm, and O. Tercieux (2015). Sampling best response dynamics and deterministic equilibrium selection. *Theoretical Economics 10*(1), 243–281.

Perko, L. (2013). *Differential Equations and Dynamical Systems*, Volume 7. Berlin: Springer Science & Business Media.

Salant, Y. and J. Cherry (2020). Statistical inference in games. *Econometrica 88*(4), 1725–52.

Sandholm, W. H. (2001). Almost global convergence to $p$-dominant equilibrium. *International Journal of Game Theory 30*(1), 107–116.

Sandholm, W. H. (2010). *Population Games and Evolutionary Dynamics.* Cambridge, MA: MIT Press.

Sandholm, W. H., S. S. Izquierdo, and L. R. Izquierdo (2019). Best experienced payoff dynamics and cooperation in the centipede game. *Theoretical Economics 14*(4), 1347–85.

Sandholm, W. H., S. S. Izquierdo, and L. R. Izquierdo (2020). Stability for best experienced payoff dynamics. *Journal of Economic Theory 185*, 104957.

Sawa, R. and J. Wu (2021). Statistical inference in evolutionary dynamics. *SSRN 3767635*.

Selten, R. and T. Chmura (2008). Stationary concepts for experimental 2x2-games. *American Economic Review 98*(3), 938–66.

Sethi, R. (2000). Stability of equilibria in games with procedurally rational players. *Games and Economic Behavior 32*(1), 85–104.

Sethi, R. (2021). Stable sampling in repeated games. *Journal of Economic Theory 197*, 105343.

Weibull, J. W. (1997). *Evolutionary Game Theory.* Cambridge, MA:MIT press.

Young, H. P. (1993). The evolution of conventions. *Econometrica 61*(1), 57–84.