Returns to scale with a Cobb-Douglas production function for four small Northern Italian firms

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Samenvatting

With this piece of evidence, I try to shed light upon the effects of fixed and variable costs on revenues for four firms operating in the sectors of lathing and milling, packaging machine construction, mechanical component production and shoe parts building, all four in the vicinity of Bologna, Italy, through the estimation of a linear bivariate simultaneous equation model where variable and fixed costs explain revenues; with a sample of eleven/twelve years of annual data for each firm, and find that a marginal increase in variable costs lead to more than proportional increases in revenues; similarly for fixed costs; I consider both contemporaneous regressions and distributed lags ones. I further estimate a Cobb-Douglas production function, in order to find out whether the returns to scale are increasing, constant or decreasing comparing various estimation methods: OLS, instrumental variable method, dynamic panel methods, as well as the Levinsohn and Petrin 2003 method, first separately for each single firm and then pooling the individual firms' samples in a panel; I find support for the hypothesis of slightly increasing returns to scale with the baseline Cobb-Douglas transformed in logarithms with capital, labour and materials as inputs.

Key words: production functions, returns to scale, cobb - douglas, stochastic frontier model, non linear least squares, production sets

JEL codes: C01, C51, C80, C81, C87, D01

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1 Literature on Cobb-Douglas' estimation

The literature on the empirical estimation of the Cobb-Douglas, 1928 production function began to appear in economics in the early 1950s; one of the most significant contributions is De Min Wu 1975, which appeared in Econometrica: there, the author deals with the exact distribution of the indirect least squares estimator of the coefficients of the Cobb-Douglas production function in the context of a stochastic frontier production model of the Marschak-Andrews 1944 type and derives finite sample tests of hypotheses on the coefficients of the production function with the underlying assumption of profit maximization.

In the realm of stochastic frontier production function models, Aigner, Knox-Lovell and Schmidt 1977 developed a whole theory, based on the theoretical production frontier of the firm, and constructed maximum likelihood estimates of such a stochastic frontier - see, for example, the exposition in Greene 2012, pp. 501-505 - the efficient production frontier is an *ideal concept*, and any positive stochastic error that leads the firm to deviate from such an ideal, should be interpreted as a form of inefficiency. Aigner, Knox-Lovell and Schmidt 1977 include in the error term also any measurement error. As such, inefficiency can be derived from 1. productive inefficiency; and/or 2. firm specific idiosynchratic effects, which can enter the model with both signs. The Cobb-Douglas production function would then take the form $\ln y = \beta_1 + \sum_k \beta_k \ln x_k + u$, $u \geq 0$, with u being the error term related with the inefficiency; the stochastic frontier would then be $\ln y = \beta_1 + \sum_k \beta_k \ln x_k - u + v$, with $v \sim N(0, \sigma^2)$ therefore $\ln y = \beta_1 + \sum_k \beta_k \ln x_k + \epsilon$; the frontier of every single firm is $h(\mathbf{x}, \beta) + v$; the inefficiency term is u, a random variable $\rightarrow$ u is a percentage measure of the extent to which the particular observation does not reach the frontier $\rightarrow$ the ideal rate of production. The authors posit two likely distributions for the inefficiency term u, the absolute value of a normally distributed random variable and of an exponentially distributed one; $\epsilon = v - u$; $\lambda = \frac{\sigma_u}{\sigma_v}$; $\sigma = (\sigma_u^2 + \sigma_v^2)^{1/2}$; $\Phi(z)$ is the probability of being to the left of z in a standard normal distribution probability density function.

S. N. Afriat 1972 also considers the efficiency estimation of production functions including a disturbance term in the model; as does Phoebus Dhrymes 1962, who previously studied the issue of devising unbiased estimation for the parameters of the Cobb-Douglas production function, referring first to investigations on the geometric mean of observed factor shares as an estimate of the factor exponents in a Cobb-Douglas; then referring to the work of Klein 1953. Dhrymes 1962 also claims, in note 2 of the paper, that the name Cobb-Douglas is inappropriately attributed, since Wicksell 1916 already defined a production function with analogous features as those of the so called Cobb-Douglas type. Overall, Dhrymes 1962 derives a estimators for the exponents of the Cobb-Douglas production function that are unbiased, sufficient, efficient and consistent.

Mundlak and Hoch 1965 studied the consequences of alternative specifications in the estimation of the Cobb-Douglas production functions; they claimed that the usual Cobb-Douglas production function is based on the assumptions of **perfect competition** between firms and **profit maximization** of each single firm, and that the **consistency** of the **estimators** depends on whether the stochastic error term is directly transmitted to the inputs of production: the least squares estimates would be unbiased only if the disturbance is not transmitted to the inputs; alternatively, the estimates would be unbiased only if some restrictions are imposed on the second moments of the disturbances of the system; most often, in empirical applications, the disturbance is partially transmitted to the imputs of productions, as such none of estimators would be definitely consistent.

A. A. Walters 1963, instead considers **both production functions and cost functions** in order to evaluate the efficiency and productivity of a single firm, starting from set theoretical definitions of production sets and their convexity in order to represent technological possibilities;

he draws up an analogue among decisions of allocations *within* the firm and those *between* firms and industries, claiming the relevance of studying the decisions of allocations *internal* to the firm. He also considers both *production* functions and *cost* functions, which I attempt to do as well in the present contribution[1].

Dennis J. Aigner 1976 considered the estimation of production frontiers; Houa, Zhaob, and Khumbakar 2023, develop the GMM estimation of semi-parametric spatial stochastic frontier models, especially as an application of productivity differentials across space, specifying spatial autoregressive models; they consider inefficiency and controlled random noise in a production frontier context $\rightarrow$ random productivity shocks; and undertake an estimation of the production technology, accouting for geographic or economic distances among units (firms, states, or countries); they relate their study to the spatial frontier literature $\rightarrow$ spatial auto-regressive models or spatial error models $\rightarrow$ with no closed form expression for the likelihood function; the errors in the model follow a spatial moving average process and have a sparse spatial weight matrix; the degree of a country spatial dependence upon other countries increases with net trade openness.

Farrell 1957 studied average labour productivity, analyzing the French census of manufacturing industries, considering different sectors of activity among which were machine construction and mechanical tools, textile, electrical engineering, paper, industrial chemicals, vehicles and cycles, footwear, milk products, sugar works, distilleries and beverages, glass products; he considered some likelihood functions of the form $\log L(\mathbf{w} \mid \theta) = \sum_i \log f_i(w_i \mid \theta)$, where $\theta = \begin{pmatrix} \lambda & \sigma & \log A & \beta_1 & \beta_2 \end{pmatrix}$, where β_1 and β_2 are the production elasticities of capital and labour respectively, and it is retained a purely statical definition of efficiency - as we do in our Cobb-Douglas estimation[2].

Aigner and Chu 1968 estimate Cobb-Douglas production functions for entire industries in the US, while Z. Griliches and V. Ringstad 1971 study the economies of scale and the form of the production function, while J. Mairesse 1976 compared production function estimates on the French and Norwegian censuses of manufacturing industries in order to achieve a measure of factor productivities.

Timmers 1971 applied a probabilistic frontier production function to measure technical efficiency, for US agriculture from 1960 to 1967, using the “average farm” in each state as an observation, adopting ordinary leasts squares and analysis of covariance estimates of the production function, finding low technical inefficiency across states, when the inputs are, beyond capital and labour, also intermediate factors of productions (such as raw materials)[3].

In the context of operational research, operational efficiency is the crucial point of enquiry while studying production functions; production and/or cost efficiency are related with performance, depending on the productivity and competitiveness of operational units, which, in our case are a single plant, located in the Northern Italian Appennines between the two regions of Emilia-Romagna and Toscana.

Finally, Zellner Kmenta and Drèze 1966 considered very broadly the specification and estimation of Cobb-Douglas production function models, assuming deterministic profit maximization; in such a context, output, inputs and profit of a firm are determined by a production function; the definition of profit, and the conditions of profit maximization are therefore:

$X = AL^{\alpha_1}K^{\alpha_2} \rightarrow$ production function;

$\pi = pX - wL - rK \rightarrow$ profit definition;

[1] Starting to study the connections between fixed and variable costs and revenues, and afterwards proceeding towards the simple estimation of a production function of the Cobb-Douglas form, starting first with a non linear least squares estimation of the Cobb-Douglas with the variables in levels, with capital and labour as the only inputs of production, and further onwards considering a (linear) logarithmic transformation of the production function which we estimate via ordinary least squares; in an extension, we consider also purchases of raw materials as a third factor or input of production - Borowski and Borwein 1989, p. 163.

[2] Where we consider contemporaneous values of inputs and output.

[3] As I will show in the subsequent sections of the paper.

$\frac{\partial \pi}{\partial L} = 0$ and $\frac{\partial \pi}{\partial K} = 0 \rightarrow$ maximizing conditions;

in a production model of the firm, Zellner, Kmenta and Drèze 1966 developed a production model of the firm where X, L and K are the quantities of output, labour and capital; p, w and r are their respective prices; firm i thus solves:

$X_i = AL_i^{\alpha_1} K_i^{\alpha_2} e^{u_{0i}}$

namely, a stochastic production function, where u_{0i} is the random term, which, if equal to zero implies that the firms maximize profits; if different than zero (namely with weather shocks, variations in machine or labour performance, etc.) implies that the production process is affected not instantaneously but with a delay of one or two periods by the stochastic shocks; the entrepreneurs therefore maximize *anticipated* profits, facing expectations of prices p_i^+, w_i^+, $r_i^+ \rightarrow$ expectations of factor prices $\rightarrow$ profit maximizing conditions become thus:

$\frac{\partial \mathbb{E}(\pi)}{\partial L} = 0$ and $\frac{\partial \mathbb{E}(\pi)}{\partial K} = 0$; and $\mathbb{E}(\pi) = p^+(X) - w^+L - r^+K$;

$[X] = AL^{\alpha_1} K^{\alpha_2} e^{\frac{1}{2}\sigma_{00}}$;

$\sigma_{00} = [u_{0i}]$.

1.1 Recent methodological contributions

Zvi Griliches and Jacques Mairesse 1995 noticed that the econometric production functions are a tool for testing hypotheses about the workings of marginal productivity theory originally started with the analysis of macro data, more recently dealing with micro data instead. At first the analysis and estimation of production functions was done for agriculture and afterwards the interest shifted towards the industrial sector of the economy; they stress the existence of the dual literature on cost functions and factor demand systems, emphasizing the importance of a correct measure of outputs and inputs; other relevant issues in the estimation of production function are the presence of economies of scale in production, as well as the market structure and markups of a given industry.

The simultaneous equations methodology originated with the study of Marschak and Andrews 1944 in agricultural economics as a system of functional relationships of the type $y = \alpha z + \beta x + u$ where labour was a treated as a variable input, while capital as a fixed input, and u as embodying the left out factors, and the functional form discrepancies, as well as the errors of measurement. $y = x + w + v - \ln \beta$; $x = (1-\beta)^{-1}[\alpha z - (w+v) + u]$; $y = (1-\beta)^{-1} \cdot [\alpha z - \beta(w+v) + u]$ $\longrightarrow$ reduced form system which leads to no structural interpretation of OLS estimates; the micro data analyzed came from farms, and there was no transmission, and thus no simultaneity bias, with weather and land quality as stochastic factors; the objective of the estimates was the estimation of the factor shares for the unkown parameters.

Specifically, the estimation methods were the within estimator and first-differences, in order to solve the issue of simultaneity; the error in the production function was

$$u_{it} = \underbrace{a_{it} + e_{it}}_{\text{known by the economic agent}} + \underbrace{\varepsilon_{it}}_{\text{data collection, measurement error}}$$

where ε_{it} is the net error, embodying the capital components unmeasured by the econometrician. The errors transmitted to the choice of x and z constitute the simultaneity problem in production functions estimation. In the error terms there are also factors connected with risk aversion, as well as functional form's approximation errors, and actual errors in optimization.

In the past thirty years there has been an increasing availability of panel data, which allowed researchers to estimate models of the type $y_{it} = \alpha z_{it} + \beta x_{it} + a_i + \lambda_t + e_{it}$ or $(y_{it} - y_i) = \alpha(z_{it} - z_i) + \beta(x_{it} - x_i) + (e_{it} - e_i)$.

Industrial micro data led to an empirical and theoretical problem connected with the strict exogeneity of the x's; furthermore, the typical heteroskedasticity of micro data led to the necessity of specification testing; a square diagonal matrix leading to $\mathbb{E}[a_i|x_1, x_2, x_3] = \delta_1 x_{1i} + \delta_2 x_{2i} + \delta_3 x_{3i}$; other estimation options related with the fixed effects panel data models with terms such as $\zeta_t = a_t - a_{t-1}$; the log - differences are very weakly correlated and covariance analysis becomes a relevant tool of enquiry $\longrightarrow$ ancova, which included quasi - fixed labour and land inputs.

Between and within estimators are used in the literature of panel data estimation of production functions in the context of micro data of entire sectors in a given industry.

$u_{it} = a_{it} + \lambda_{it} + \varepsilon_{it} + e_{it}$
with $\lambda_{it} = \lambda_t + g_{it}$
$a_{it} = \zeta_{it} + a_{it-1}$
$du_{it} = \zeta_{it} + c_t + g_i + de_{it} + d\varepsilon_{it}$
$\frac{\sum dy}{T-1} = \frac{y_T - y_1}{T-1}$
time and industry dummies are adopted.
$i_t = i_t(\underbrace{a_t}_{\text{random product. shocks}}, \underbrace{z_t}_{\text{fixed invest.}})$
$a_t = h_t(i_t, z_t)$
$y_t = \beta x_t + \phi_t(i_t, z_t) + e_t$
$\phi_t = \alpha z_t + h_i(i_t, z_t)$
$y_t = \alpha z_t + \beta x_t + a_t + e_t$

The difficulty in estimating production function lies mainly in the identification of pure production function parameters, which are characterized by the individual level heterogeneity, which occurs due to the existence of micro data on single firms; such an estimation is certainly connected with stochastic frontiers for different sectors. It seems that the individual firm heterogeneity might somehow be connected with frailty in survival models. Often, it might occur that the production functions are misspecified, also because census data are too general to be synthesized by single equation models. To achieve the aim of consistent estimation of production functions parameters, data quality and model specification are essential prerequisite of any analysis.

To achieve credible identification, detailed financial data is needed as well as differential cost-of-capital variables, possibly instrumental variables; also the product mix of production is required; it is likely that a large number of products, labour types, machines, technologies exists within each given firm. In order to reduce aggregation biases and to reduce heterogeneity, it is necessary to distinguish between these various categories of factors of production; output, labour, capital are indeed quite vague concepts; issues such as the quality of the labour force, technologies used, and organizational structures, markets served, all contribute to reduce multicollinearity.

Other relevant issues in estimation of production functions, always according to Griliches and Mairesse 1996, are economies of scale, rates of technological change, rates of return to R and D; and new data and appropriate theoretical and econometric models are needed to account for real heterogeneity.

Ackerberg, Caves and Frazer 2015 study the functional dependence problem, an econometric problem due to unobserved heterogeneity, unobserved by the econometrician; LAD estimates of a Cobb-Douglas of Greene 2012; a very relevant contribution in production function estimation is Olley and Pakes 1996; Ackerberg et al. 2015 invert optimal input decisions to control for the unobserved productivity shocks. An intermediate input demand function is thus derived.

The data generating process is crucial, though unobserved by the researcher as well as by the economic agent; a Cobb-Douglas production function in logarithms (with lower case letters denoting logged variables) might be $y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon_{it}$; ω_{it} and ε_{it} are two econometric unobservables; ε_{it} are completely unobservable, while ω_{it} are observable productivity shocks, such as managerial ability of a firm or down time due to a machine breakdown; the three most relevant variables variables in estimation are thus (k_{it}, l_{it}) and ω_{it}.

ω_{it} is a time-invariant shock absorbing all the endogeneity, which leads to low estimates of β_k; the first order conditions of the firm's maximization problem carry over information; ε_{it} expresses the deviations from the expected breakdown or measurement error in a variable; in panel data models with fixed effects, $\omega_{it} = \omega_i \longrightarrow$ fixed effects observed by the firm prior to choosing the inputs.

Estimating cost functions requires a theoretical and statistical model accounting for the cost shares of each input $\longrightarrow$ to derive the input elasticities $\longrightarrow$ flexible function and stochastic specification; dynamic issues are not to be neglected, such as adjustment costs or wedges between purchase and release prices; dynamic implications of the static first order conditions are to be taken into account as well.

An option for estimation is to consider wages rate and interest rate as instruments for labour and capital $\longrightarrow$ input prices are exogenous; $\rightarrow$ uncorrelated with $\omega_{it}+\varepsilon_{it} \rightarrow$ a source of exogenous variation $\rightarrow k_{it}$ and l_{it} are identified via input prices based IV methods.

A subset of inputs $\longrightarrow$ dynamic in nature, while there are no dynamic first order conditions; an econometric problem is the input endogeneity in production functions' estimation; therefore, a discrete time model of dinamically optimizing firms is needed $\longrightarrow \{\omega_{i\tau}\}_{\tau=0}^{+\infty} \rightarrow$ past productivity shocks; $\{\omega_\tau\}_{\tau=t+1}^{+\infty}$, with the assumption that $\mathbb{E}[\varepsilon_{it}|I_{it}] = 0$, where I_{it} denotes the information available to the econometrician up to time t for firm i.

$p(\omega_{it+1}|I_{it}) = p(\omega_{it+1}|\omega_{it}) \rightarrow$ law of motion of productivity shocks.

A new set of statistical and theoretical restrictions are necessary to identify the model $\longrightarrow$ functional dependence issue; timing assumptions; scalar unobservable assumptions anf monotonicity.

$k_{it} = \kappa(k_{it-1}, i_{it-1}) \rightarrow$ law of motion of capital; the identifying assumption is that investment is chosen one period in advance with respect to the production time.

The firm knows something about the distribution of future productivity shocks and labour choice is non - dynamic; capital is the dynamic input subject to an investment process $\rightarrow k_{it} \in I_{it-1}$, i_t is the policy function resulting from a dynamic optimization problem.

$i_{it} = f_t(k_{it}, \omega_{it}) \rightarrow$ stricly $\nearrow$ in ω_{it}; $k_{it} \rightarrow$ state variable; $l_{it} \rightarrow$ non - dynamic variable;

firm - specific unobservables $\rightarrow$ differ across firms on cost of capital as well as the demand or labour market conditions;

differences in these variables across time $\longrightarrow p(\omega_{it+1}|\omega_{it})$, stochastically $\nearrow$ in ω_{it};

$\omega_{it} = f_t^{-1}(k_{it}, i_{it})$

$\phi(k_{it}, i_{it})$, where what matters is the estimation of $\hat{\beta}_i$ and $\hat{\phi}(k_{it}, i_{it})$;

$K_{t+1} = (1-\delta)K_t + \psi\big(\frac{I_t}{K_t}\big) \longrightarrow$ convex adjustment costs, Wang and Wen, 2011;

$\phi_t(k_{it}, i_{it}) = \beta_k k_{it} + f_t^{-1}(k_{it}, i_{it})$

a complex dynamic programming problem dealing with the evolution of industry - wide prices and treat f_t^{-1} non - parametrically.

$$\begin{aligned}\omega_{it} &= \mathbb{E}[\omega_{it}|I_{it-1}] + \xi_{it}\\ &= \mathbb{E}[\omega_{it}|\omega_{it-1}] + \xi_{it} = g(\omega_{it-1} + \xi_{it})\end{aligned}$$

numerical, non - linear search; polynomials or kernels;

β_l and ϕ_{t-1}

Demand function for an intermediate input $\rightarrow y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \beta_m m_{it} + \omega_{it} + \varepsilon_{it}$

$m_{it} = f_t(k_{it}, \omega_{it})$

$\omega_{it} = f_t^{-1}(k_{it}, m_{it})$

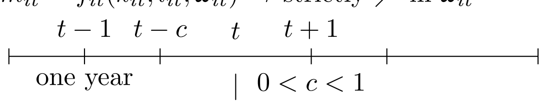
l_{it} and $m_{it} \rightarrow$ non - dynamic inputs;

ω_{it} is the only unobservable entering the intermediate input demand function; the intermediate input can be a heterogeneous good or differentiated product;

firm specific shocks investment prices;

intermediate input $\rightarrow$ non dynamic inputs; lumpy investment data.

$m_{it} = f_t(k_{it,}, \omega_{it}, p^m_{it}, p^i_{it})$
$\varepsilon_{it} \rightarrow$ pure measurement error
$y_{it} = \ln\left(\frac{1}{\beta_m}\right)\ln\left(\frac{p_m}{p_y}\right) + m_{it} + \varepsilon_{it}$
inverted first order conditions $\rightarrow$ treat them non - parametrically; partially linear model;
$\mathbb{E}[\{l_{it} - \mathbb{E}[l_{it}|k_{it}, m_{it}, t]\}'\{l_{it} - \mathbb{E}[l_{it}|k_{it}, m_{it}, t]\}]$
this object is positive definite
$l_{it} = h_t(k_{it}, \omega_{it})$
labour has no dynamic implications
$m_{it} = f_t(k_{it}, \omega_{it})$
$m_{it}|k_{it}, l_{it}, \omega_{it}$
$\iota_{it} = g_t(k_{it}, \underbrace{f_t^{-1}(k_{it}, m_{it})}_{\omega_{it}})$
$\phi_t(k_{it}, m_{it})$
firm specific input prices of labour and materials affect optimization error; the firm chooses optimal levels of labour + noise and we should consistently estimate β;
sick days, union issues, unobserved union shocks are amoung the classical measurement errors in $l_t \longrightarrow$ noise in observed labour;
information set;
$y_{it} + \Phi(k_{it}, l_{it}, m_{it}) + \varepsilon_{it}$
an investment function is useful to control for differentials in productivity;
$p(\omega_{it-b}|I_{it-1}) = p(\omega_{it-b}|\omega_{it-1})$, with $0 < b < 1$
$p(\omega_{it}|I_{it-1}) = p(\omega_{it}|\omega_{it-b})$
$i_{it} = f_t(k_{it}, \omega_{it})$
$\iota_{it} = g_t(\omega_{it-b}, k_{it})$
$f_t^{-1}(k_{it}, i_{it}) \rightarrow$ non - parametric function
labour is non - dynamic;
firm specific adjustment costs, unobserved to the labour input;
$k_{it} = \kappa(k_{it-1}, i_{it-1})$
$y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon_{it}$
$m_{it} = \tilde{f}_{it}(k_{it}, l_{it}, \omega_{it}) \rightarrow$ strictly $\nearrow$ in ω_{it}



conditional input demand function; input function of m_{it} conditional on l_{it}; value added production function; leontief production function in the intermediate input; mixed systems and their dual variables; revenue share equations + foc
m_{it} and l_{it} can be chosen simultaneously in presence of multiple structural observables
$\omega_{it} = f_t^{-1}(k_{it}, l_{it}, m_{it})$
first stage moment condition
$\Phi_t(k_{it}, l_{it}, m_{it}) = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it}$
like an IV or GMM procedure; one first difference
$\omega_{it} = \rho\omega_{it-1} + \xi_{it} \rightarrow AR(1)$
$\max_{l_{it}, m_{it}}\{\cdot\} = \max_{l_{it}}\{\max_{m_{it}|l_{it}}\{\cdot\}\}$
$m_{it} = f_t(k_{it}, \omega_{it})$
$\hat{\Phi}_t(k_{it}, l_{it}, m_{it})$
current state (k_{it}, ω_{it})
estimate β_0, β_k, and β_l
$l_{it} \in I_{it-1} \rightarrow$ in some industries;
$\omega_{it} \rightarrow$ unoebserved by the firm in $t + 1$

search, non-linear, in the second step

conditional intermediate input demand $\rightarrow$ avoiding functional dependence issue; i.i.d. firm specific shocks on output and wage prices; labour is a dynamic input $\rightarrow$ adjustment costs;

wage adjustment costs;
serially correlated, exogenous, unobserved shocks to the k_{it}, l_{it}, m_{it};
$i_{it} = f_t(k_{it}, l_{it}, \omega_{it})$
$y_{it} = \phi_t(k_{it}, l_{it}, i_{it}) + \varepsilon_{it}$
adjustment costs of capital;
non-linear search on:
$\beta_0, \beta_k, \beta_l \rightarrow$ OLS estimation
serially correlated unobserved wage shocks;
dynamic labour with adjustment costs;
dynamic panel data methods;

$y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon_{it}$
$\omega_{it} = \rho\omega_{it-1} + \xi_{it}$
endogenous exit from a sample

variables measured in physical units
price taker firms
KLETTE AND GRILICHES 1996, JAE

quality $\neq$ across firm $\rightarrow$ same
menu of prices
chilean data
convex capital adjustment costs
$K_{it} = (1-\delta)K_{it-1} + I_{it-1}$

$Y_{it} = \min\{\beta_0 K_{it}^{\beta_k} L_{it}^{\beta_l} \exp\{\omega_{it}, \beta_m M_{it}\}\} \cdot \exp\{\varepsilon_{it}\}$
$\rightarrow$ leontief production function

$y_{it} = \beta_0 + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon it$

intermediate input variable approach
$\searrow$ unobserved variation across firms
$\omega_{it} = \rho\omega_{it-1} + \varepsilon_{it}$

labour $\rightarrow$ dynamic implications
sensitivity to measurement error $\rightarrow$ Monte Carlo experiments;
robustness to misspecification;
across firm variation;

$K = \alpha + \beta L + \varepsilon$

$\omega_{it} = \rho\omega_{it-1} + \xi_{it}$

labour $\rightarrow$ dynamic input, $\xi_{it} \sim N(0, \sigma_\xi^2)$

firm specific wage shocks ω_{it}
knowledge of ω_{it}

$\omega_{it-b} = \rho^{1-b}\omega_{it-1} + \xi_{it}^A$
$\omega_{it} = \rho^b\omega_{it-1} + \xi_{it}^B$
firms $\rightarrow$ less than perfect information about ω_{it}

$(\rho^b)\xi_{it}^A + \xi_{it}^B = (\xi_{it})$
$(\xi_{it}^A) = \sigma_{\xi^A}^2$
$(\xi_{it}^B) = \sigma_{\xi^B}^2$

within firms across time
$Y_{it} = \min\{\beta_0 K_{it}^{\beta_k} L_{it}^{\beta_l} e^{\omega_{it}} m(M_{it})\} e^{\varepsilon it}$
$m(M_{it}) = \beta_0 K_{it}^{\beta_k} L_{it}^{\beta_l} e^{\omega_{it}}$
commitment to planned optimal L_{it} and M_{it} $\rightarrow$ try a non parametric estimation;
$Y_{it} = \beta_0 K_{it}^{\beta_k} L_{it}^{\beta_l} \underbrace{e^{\omega_{it}} e^{\varepsilon_{it}}}_{\text{unobserved}}$

steady state $\rightarrow$ capital $\equiv$ dynamic input
$K_{it} = (1-\delta)K_{it-1} + I_{it-1}$
shock to the price of material input;
between firm wage variation

convex adjustment costs
$\frac{1}{\phi_i} \sim \log N \rightarrow$ unobserved heterogeneity

$C_i(I_{it}) = \frac{\phi_i}{2} \cdot I_{it}^2$
within firm versus across firm variation

ω_{it}, ϕ_i, and $\ln(W_{it})$

K_{i0} and $(\omega_{i0}, \phi_i, ln(W_{i0}))$
leontief - derived value added production function

$Y_{it} = \beta_0 K_{it}^{\beta_k} L_{it}^{\beta_l} e^{\omega_{it}} e^{\varepsilon_{it}}$
higher order polynomial in the explanatory variables

$m_{it} \rightarrow$ a linear function of k_{it}, l_{it}, and ω_{it}

$\tilde{\beta}_0 + \omega_{it}(\beta_k, \beta_l)$

$\rightarrow \tilde{\beta}_0 + \omega_{it-1}(\beta_k, \beta_l)$

Petrin, Poi and Levinsohn 2004 implement in stata the Levinsohn and Petrin 2003 method to estimate production function, which is based on the idea of using inputs to control for unobservables, specifically unobservable productivity shocks and input levels;

lumpy investment at the firm level; plant level research
manufacturing surveys
cobb-douglas in logarithms

$y_t = \beta_0 + \beta_l l_t + \beta_k k_t + \beta_m m_t + \omega_t + \eta_t$
the error has two components $\rightarrow$ transmitted productivity components $\rightarrow$ state variables + an error term uncorrelated with input choices $\rightarrow$ simultaneity problem in production function estimation
$m_t = m_t(k_t, \omega_t)$
$\omega_t = \omega_t(k_t, m_t)$
first - order Markov process $\rightarrow \omega_t = \mathbb{E}[\omega_t | \omega_{t-1}] + \xi_t$

trans-log production functions
$v_t = \beta_0 + \beta_l l_t + \beta_k k_t + \omega_t + \eta_t$
$= \beta_{lt} + \phi(k_t, m_t)$
where $\underbrace{\phi(k_t, m_t)}_{\text{third order polynomial}} = \beta_0 + \beta_k k_t + \omega_t(k_t, m_t)$

first stage: estimate β_l and ϕ_t;
second stage

$\hat{\phi}_t = \hat{v}_t - \hat{\beta}_l l_t$

$= \hat{\delta}_0 + \sum_{i=0}^{3} \sum_{i=0}^{3-i} \hat{\delta}_{ij} k_t^i m_t^j - \hat{\beta}_l l_t$

$\hat{\omega}_t = \hat{\phi}_t - \beta_k^* k_t$

$\hat{\omega}_t = \gamma_0 + \gamma_1 \omega_{t-1} + \gamma_2 \omega_{t-1}^2 + \gamma_3 \omega_{t-1}^3 + \varepsilon_t$

$\mathbb{E}[\widehat{\omega_t|\omega_{t-1}}]$
bootstrap
$\hat{\beta}_t$, β_k^* and $\mathbb{E}[\widehat{\omega_t|\omega_{t-1}}]$
$\widehat{\eta_t+\xi_t} = v_t - \hat{\beta}_l l_t - \beta_k^* k_t - \mathbb{E}[\widehat{\omega_t|\omega_{t-1}}]$
$\hat{\beta}_k, \beta_k = \min_{\beta_k^*} \sum_t (v_t - \hat{\beta}_l l_t - \beta_k^* k_t - [\widehat{\omega_t|\omega_{t-1}}])^2$

$$\begin{aligned} y_t &= \beta_0 + \beta_l l_t + \beta_k k_t \beta_m m_t + \omega_t + \eta_t \\ &= \beta_l l_t + \phi_t(k_t, m_t) + \eta_t \end{aligned}$$

$\phi_t(k_t, m_t) = \beta_0 + \beta_k k_t + \beta_m m_t + \omega_t(k_t, m_t)$
moment conditions on the conditional distribution of the errors on each of the three factors
$k_t, m_{t-1}, l_{t-1}, m_{t-2}, k_{t.1}$
$\mathbb{E}[\eta_t + \xi_t | l_{t-1}] = 0;$
$\mathbb{E}[\eta_t + \xi_t | m_{t-2}] = 0;$
$\mathbb{E}[\eta_t + \xi_t | k_{t-1}] = 0;$
$\min_{\{\beta_k^*, \beta_m^*\}} \sum_h \left\{ \sum_t (\eta_t + \hat{\xi}_t | Z_{ht}) \right\}$
recenter the moment conditions
a series of global macros
panel data
ricavi lordi
$\hat{\omega}_t = \exp\{y_t - \hat{\beta}_l l_t - \hat{\beta}_k k_t - \hat{\beta}_m m_t\}$
1987 - 1996 → chilean data
OLS, FE, LP comparison

Yasar, Raciborski, and Poi 2008 implement operationally in STATA the Olley and Pakes 1996 semi-parametric method; the idea is approximating the sum of the inputs from the estimation of the Cobb-Douglas production function in order to overcome the drawbacks of simultaneity and selection bias of OLS; the method allows to control for these biases → leading to reliable productivity estimates; productivity is known to the profit maximizing firms but not to the econometrician; unobserved productivity shocks are thus neglected by the OLS method.

`levpet` → controls for simultaneity bias, but not for selection bias; an issue is the existence of productivity shocks and their connection with the probability of exit from the market of a firm; in particular, the amount of capital stock is linked with the $\mathbb{P}r[exit]$; unobserved time-varying productivity shocks affect survival probabilities of firms to remain in the sample.

$\underbrace{\Omega_{it}}_{\text{product. indicator or shock}}, \underbrace{K_{it}}_{\text{capital stock}}, \underbrace{a_{it}}_{\text{age of the firm}} \rightarrow$ state variables

intermediate inputs to control for correlation between inputs and the unobserved productivity shocks;

$\mathbb{E}[\Omega_{it} | \Omega_{it, k_{it}}] \rightarrow$ expected productivity;

the dynamic optimization problem of the representative firm is:
$\max_{u_0, u_1} \sum_{t=0}^{+\infty} \beta^t F(x_t, u_t)$ s.t. $x_{t+1} = f(x_t, u_t)$, x_0 given, $u_t \in \mathcal{U}$
$I(x) = \max_{u \in \mathcal{U}} F(x, u) + \beta I(f(x, u))$
the bellman equation looks like:
$V_{it}(k_{it}, a_{it}, \Omega_{it}) = \max[\Phi, \sup_{I_{it} \geq 0} \Pi_{it}(k_{it}, a_{it}, \Omega_{it}) - C_{it} + \rho \mathbb{E}\{V_{i,t+1}(k_{i,t+1}, a_{i,t+1}, \Omega_{it}) | J_{it}\}]$
where J_{it} is the information set of the firm and C_{it} is a cost function;

$$\chi_{it} \begin{cases} = 1 & \text{if} \quad \Omega_{it} \geq \underline{\Omega}_{it}(k_{it}, a_{it}) \\ = 0 & \quad \text{otherwise} \end{cases}$$

$\underline{\Omega}_{it}(k_{it}, a_{it}) \rightarrow$ lower bound of productivity to stay in the market;
$I_{it} = I(\Omega_{it}, k_{it}, a_{it})$

the inputs of production here are:
labour
materials
energy
capital
age
five inputs

$Y_{it} = F(L_{it}, E_{it}, K_{it}, a_{it}, \omega_{it})$
$y_{it} = \beta_0 + \beta_l l_{it} + \beta_m m_{it} + \beta_e e_{it} + \beta_k k_{it} + \beta_a a_{it} + u_{it}$
$u_{it} = \underbrace{\Omega_{it}}_{\text{unobs. by econometrician}} + \underbrace{\eta_{it}}_{\text{unobs. by both firm and econ.}}$

a first order markov process
$\Omega_{it} = I^{-1}(I_{it}, k_{it}, a_{it}) = h(I_{it}, k_{it}, a_{it}) \rightarrow$ inverse function for the unobserved shock;
$y_{it} = \beta_l l_{it} + \beta_m m_{it} + \beta_e e_{it} + \phi(i_{it}, k_{it}, a_{it}) + \eta_{it}$
$phi(i_{it}, k_{it}, a_{it}) = \beta_0 + \beta_k k_{it} + \beta_a a_{it} + h(i_{it}, k_{it}, a_{it})$

first step;

second step to control for selection bias and estimate survival probabilities;
probability of survival $\rightarrow$ in period t depends on $\Omega_{i,t-1}$ and $\underbrace{\Omega}_{i,t-1}$ and on age, capital, and investment at time $t-1$ $\rightarrow$ probit model of χ_{it} on $I_{i,t-1}, K_{i,t-1}$, and $a_{i,t-1}$;
second step, thid step

$y_{it} - \hat{\beta}_l l_{it} - \hat{\beta}_m m_{it} - \hat{\beta}_e e_{it} =$
$= \beta_k k_{it} + \beta_a a_{it} + g(\hat{\phi}_{t-1} - \beta_k K_{i,t-1} - \beta_a a_{i,t-1}, \hat{P}_{it}) + \xi_{it} + \eta_{it}$
kernel estimator for the second stage;
for the STATA implementation, a panel variable and a time variable must be specified;
$\phi(\cdot)$ and $g(\cdot) \rightarrow$ polynomial expansion;
3,772 firms, $t = 1995, \ldots, 2002$
coeffs. associated w/variable inputs $\rightarrow$ biased upward;
and coeffs. for capital input $\rightarrow$ biased downward.

Arthur S. Goldberger 1968 points out that, to carry out a Cobb-Douglas production function estimation, it is customary to append a multiplicative log-normal disturbance and fit a linear regression in the logarithmic variables; in addition to that, the attention is shifted from the conditional median to the conditional mean, which is ordinarily the primary object of study; the typical procedure may be varied in order to assure minimum variance unbiased estimation of the conditional median or the conditional mean.

Marjorie B. McElroy 1987 studies input demand or share systems in the context of the stochastic frontier literature; specifically, she deals with a cost function setup, input demand, and share functions in the framework of general error models; the nature of the problem is of primal or dual optimization based on a standard duality relationship. a stochastic share system may be based on a dirilichet distribution; the model is a stochastic frontier one with technical errors compared with a standard neoclassical model;

$y = q(x_1 - \varepsilon_1, \ldots, x_n - \varepsilon_n; \theta) = q(\mathbf{x} - \varepsilon; \theta)$

the known parameters are θ and ε
$\varepsilon \sim (\mathbf{0}, \mathbf{\Sigma})$
$c(y, \mathbf{w}) \rightarrow$ cost function
$\varepsilon \rightarrow$ known to the firm but not to the researcher;
θ are the same for all firms $\rightarrow$ suppressed;
minimum cost of producing output level y under the above production function, given output prices $\mathbf{w}$ when $\varepsilon = \mathbf{0}_n$

additive general error model
$c(y, \varepsilon, \mathbf{w}) \rightarrow$ firm specific cost function

$$\underbrace{C = C(y, \mathbf{w}, \varepsilon)}_{\text{firm specific cost funct.}} \quad = \quad \underbrace{c(y, \mathbf{w})}_{\text{deterministic cost funct.}} \quad + \underbrace{\sum_{j=1}^{n} w_j \varepsilon_j}_{\text{stochastic part}}$$

the stochastic part is price weighted;
the sum of the ε_j is $\mathbb{E}[C(y, \mathbf{w}, \varepsilon)] = c(y, \mathbf{w})$

$C(\cdot)$ and $c(\cdot)$ are both concave and linear homogeneous in $\mathbf{w}$, and dual to $q(\mathbf{x} - \varepsilon)$ and $q(\mathbf{x})$, respectively; errors in cost - minimizing behaviour; demand system:
$\frac{q_i(\mathbf{x}-\varepsilon)}{q_j(\mathbf{x}-\varepsilon)} = \frac{w_i}{w_j}$, for $i, j = 1, \ldots, n$; $i \neq j$
$x_i = c_i(y, \mathbf{w}, \varepsilon) = c_i(y, \mathbf{w}) + \varepsilon_i$, $i = 1, \ldots, n$

$C_i(\cdot)$ and $c(\cdot)$ are homogeneous of degree zero in $\mathbf{w}$; $\mathbb{E}[C_i(y, \mathbf{w}, \varepsilon)] = c_i(y, \mathbf{w}) \rightarrow$ deterministic or average demand function of the population of firms; $\varepsilon_i > 0 \rightarrow$ the firm uses ε_i more of x_i to produce y than does the average firm;

$\sum_i w_i x_i = \sum_i w_i c_i(y, \mathbf{w}) + \sum_i w_i \varepsilon_i = c(y, \mathbf{w}) + \sum_i w_i \varepsilon_i \equiv C(y, \mathbf{w}, \varepsilon)$
the cost function is the dual of the production function
an n-input demand system and a dost-cum-share system,
with conditional firm - specific input demand functions;

$S_i = s_i(y, \mathbf{w}) + v_i$, $i = 1, \ldots, n$
$S_i \equiv \frac{w_i x_i}{\sum_j w_j x_j}$; $\sum_i S_i \equiv 1$
$s_i(y, \mathbf{w}) \equiv \frac{\partial \ln c(y, \mathbf{w})}{\partial \ln w_i} \equiv \frac{w_i c_i(y, \mathbf{w})}{c(y, \mathbf{w})}$, $i = 1, \ldots, n$

$v_i = \nu_i(y, \mathbf{w}, \varepsilon) = \frac{1}{c(y, \mathbf{w}) + \sum_j w_j \varepsilon_j} \cdot [w_{ii} - s_i(y, \mathbf{w}) \sum_j w_j \varepsilon_j]$, $i = 1, \ldots, n$
$\sum_i w_i c_i(y, \mathbf{w}) = c(y, \mathbf{w})$

$$\frac{w_i x_i}{\sum_j w_j x_j} = w_i \cdot \frac{c_i(y, \mathbf{w})}{c(y, \mathbf{w}) + \sum_j w_j \varepsilon_j} + \frac{w_i \varepsilon_i}{c(y, \mathbf{w}) + \sum_j w_j \varepsilon_j},\ i = 1, \ldots, n;$$

$C = c(y, \mathbf{w}) + v_c$
$S_i = s_i(y, \mathbf{w}) + v_i$, $i = 1, \ldots, n-1$
$v_c \rightarrow \sum_{j=1}^{n} w_j \varepsilon_j$
$d(y, \mathbf{w}) = (c_1(y, \mathbf{w}), \ldots, c_n(y, \mathbf{w}))$
$= (\varepsilon_1, \ldots, \varepsilon_n) \rightarrow$ deterministic demand + noise
$S_i = s_i^*(d(y, \mathbf{w}), \mathbf{w}; \varepsilon)$

$x_i = x_i^*(\bar{s}(y, \mathbf{w}), \mathbf{w}, \tilde{v}) = \frac{1}{w_i} s_i(y, \mathbf{w}) c(y, \mathbf{w}) + \frac{1}{w_i}\{[c(y, \mathbf{w}) + v_c] v_i + s_i(y, \mathbf{w}) w_c\}\}$ for $i = 1, \ldots, n$
$e(\tilde{s}(y, \mathbf{w}), \mathbf{w}; \tilde{v}) \equiv \varepsilon_i$, $\forall i = 1, \ldots, n$
$\exists$ a specific firm production function
$y = q^*(\mathbf{x}, \mathbf{w}, \varepsilon)$
$q(\mathbf{x} - e(y, \mathbf{w}, \tilde{v}))$

$\ln C = \theta(y, \mathbf{w}) + \sum_i \eta_i \ln w_i + \eta_0$

$S_i = \frac{\partial \theta(y,\mathbf{w})}{\partial \ln w_i} + \eta_i,\ i = 1,\ldots,n$

$\sum_{i=1}^{n} \eta_i = 0$

stochastic frontier model with technical errors;
panel data vs a single cross - section or a time series;

$\frac{\partial v_i}{\varepsilon_i} > 0$; $\frac{\partial v_i}{\partial \varepsilon_k} < 0$ for $k \neq i$

$\frac{\partial S_i}{\partial \varepsilon_i} \cdot \frac{\varepsilon_i}{S_i} = \frac{w_i \varepsilon_i}{C(y,\mathbf{w},\varepsilon)} \cdot \frac{1 - S_i(y,\mathbf{w},\varepsilon)}{S_i(y,\mathbf{w},\varepsilon)}$

$$\begin{aligned} h(y,\mathbf{w}) = \ln \alpha_0 + \textstyle\sum_i \alpha_i \ln w_i + \\ \textstyle\sum_i \sum_j 1/2 \gamma_{ij} \ln w_i \ln w_j + \\ \textstyle\sum_i \mu_i \ln w_i \ln y + \\ \mu \ln y + \tfrac{\theta}{2} (\ln y)^2 \end{aligned}$$

$$\begin{aligned} c(y,\mathbf{w},\varepsilon) = \exp\{h(y+\mathbf{w})\} + \textstyle\sum_j w_j \varepsilon_j \\ \textstyle\sum_i \alpha_i = 1;\ \gamma_{ij} = \gamma_{ji} \\ \textstyle\sum_i \gamma_{ij} = \sum_j \gamma_{ji} = 0 \\ \textstyle\sum_i \mu_i = 0 \end{aligned}$$

isomorphic demand representation

$\hat{s}_i(y,\mathbf{w}) = w_i \hat{c}_i(y,\mathbf{w}) / \sum_j w_j \hat{c}_j(y,\mathbf{w})$

$\varepsilon(t) \sim (\mathbf{0}_n, \mathbf{\Sigma})$

cost and share system $\rightarrow$ two components of the error term

$d_i = \varepsilon_i + e_i(y, \mathbf{w}, \tilde{w}),\ \imath = 1,,n$

$\varepsilon \sim N(0_n, \mathbf{\Sigma})$

$\tilde{v} \sim N(0_n, \mathbf{\Omega})$

are positive definite matrices

$d_i = \varepsilon_i$

$H_A : \mathbb{E}[d_i(t) d_j(t)] = \sigma_{ij} + z'_{ij}(t) \alpha_{ij}(t),$
for $i,j = 1,\ldots,n$ and $t = 1,\ldots,T$

$\sigma_{ij} \rightarrow ij^{th}$

$0 + \Sigma_{n \times n}$

$H_0 : \mathbb{E}[d_i(t) d_j(t)] = \sigma_{ij},\ i = 1,\ldots,n$ and $t = 1,\ldots,T$

$\varepsilon_i(t) \sim$ jointly normal and iid;

$LM_{ij} = \frac{\sum_{t=1}^{T} [z'_{ij}(t_t, w_t) \hat{\alpha}_{ij}]^2}{2}$
$i,j = 1,\ldots,n,\ i \leq j$

1 to n commodities (i and j)

$\frac{n^2+n}{2} = \frac{n(n+1)}{2}$ hyps.

non - linear iterated seemingly unrelated regression

$\mu_i = \theta = 0;\ \mu_i = 1$

annual aggergates for U.S. manufacturing
capital, labour, energy, and materials
iterated three stages least squares;
constant returns to scale tanslog models;
$(\varepsilon_1, \ldots, \varepsilon_n) \sim iidN(\mathbf{0}_n, \Sigma)$

$S_i = \alpha_i + \sum_j \gamma_{ij} \ln w_j + v_i$, $i = 1, \ldots, n$
$\ln C = h(y, \mathbf{w}) + v_c$

full information maximum likelihood;
elasticities are often of interest $\rightarrow$ allen - uzawa partial elasticities
partial elasticity of substitution between capital and labour $\rightarrow$ small cost shares of capital and energy; ten cases;

ith and jth input demand equations

error specifications $\rightarrow$ fundamental to empirical work on general error model; theory - based research strategy; static neoclassical model; production theory or approximation theory

$\mathbb{E}[\varepsilon_i(t)\varepsilon_j(t)]$, $i, j = 1, 2, 3, 4$, $i \neq j$

GEMs and stochastic frontier production function.

Steven Olley and Ariel Pakes 1996 estimated a production function for the telecommunication equipment industry in the U.S. tackling the selection and simultaneity biases; their data was taken from the U.S. bureau of census and it allowed to deal with liquidation and simultaneity bias $\rightarrow$ endogeneity; they studied the production function residuals within a reduced form sales equation; they constructed a balanced panel data set, where output share was the weighted average of productivity of all active plants; they thus pursued a decomposition of industry productivity; capital, age and productivity were the main independent variables in the study.

switching and transmission signals $\rightarrow$ equipment industry; 24 years of data; industrial outlook is linked to the entry process in that market of telecommunication equipment, where AT&T held a form of monopoly with bell as main supplier; $1982, \ldots, 1986$; exit and input demand decisions; the production function developed by Olley and Pakes 1996 are accounting for entry and exit of firms from a market; the exit action is connected with selection; each firm $\max\{\mathbb{E}[\text{discounted value of future cash flow}]\}$;

the data analyzed are balanced panels with fixed effects $\rightarrow$ unobserved productivity realizations; selection bias; first order markov process; markov perfect nash equilibrium; they developed a dynamic model of firm behaviour $\rightarrow$ firm - specific efficiency differences and idiosyncratic changes over time info available when input decisions are made $\rightarrow$ simultaneity;

input demand and liquidation are the two relevant phenomena which the paper models; state variables and factor prices $\rightarrow a_t \rightarrow$ age; $k_t \rightarrow$ capital; $\omega_t \rightarrow$ efficiency; the combination of the these three state variables gives a market structure;

dynamics and industry structure

$k_{t+1} = (1 - \delta)k_t + i_t$ and $a_{t+1} = a_t + 1$

$F_\omega = \{F(\cdot|\omega), \omega \in \Omega\}$

restricted profit function conditional on the vector of state variables

$$\chi_t \begin{cases} = 1 & \text{if} \quad \omega_t \geq \underline{\omega}_t(a_t, k_t) \\ = 0 & \quad\quad \text{otherwise} \end{cases}$$

$i_t = i_t(\omega_t, a_t, k_t)$

markov perfect nash equilibrium

$y_0 = \beta_0 + \beta_a a_{it} + \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \eta_{it}$
ω_{it} is a state variable, η_{it} isn't input survival;
$i_t(a_t, \omega_t, k_t) \geq 0$
$\omega_t = h_t(i, a_t, k_t)$
$\rightarrow$ productivity as a function of observables;
$\mathbb{E}[\omega_t | a_t, k_t, \omega_{t-1}, \chi_t = 1]$

value function is $\nearrow$ in ω_t;
age effects on productivity are small $\rightarrow$ continue operations or sell off;

$\omega \rightarrow$ unobserved, firm specific state $\rightarrow$ simple estimation algorithm
$\omega_{t+1} \rightarrow$ reservation productivity;
β_a and β_k are age and capital, identify them separately through β_l and $\phi(\cdot)$
$\omega_t = h_t(i_t, a_t, k_t)$
$y_{t+1} - \beta_l l_{t+1}$
selection equation

$$\begin{aligned}\mathbb{P}r\{\chi_{t+1} = 1 | \underline{\omega}_{t+1}(k_{t+1}, a_{t+1}), J_t\} \\ &= \mathbb{P}r\{\omega_{t+1} \geq \underline{\omega}_{t+1}(k_{t+1}, a_{t+1}) | \underline{\omega}_{t+1}(k_{t+1}, a_{t+1}), \omega_t\} \\ &= p_t\{\underline{\omega}_{t+1}(k_{t+1}, a_{t+1}), \omega_t\} \\ &= p_t(i_t, a_t, k_t) \\ &= P_t\end{aligned}$$

$$\phi_t = \beta_0 + \beta_a a_t + \beta_k k_t + \omega_t$$

$$y_{it} = \beta_t l_{it} + \phi_t(i_{it}, a_{it}, k_{it}) + \eta_{it}$$

$$\phi_t(i_{it}, a_{it}, k_{it}) = \beta_0 + \beta_a a_{it} + \beta_k k_{it} + h_t(i_{it}, a_{it}, k_{it})$$

semi-parametric regression of ω_t and ω_{t+1}

$$g(\underline{\omega}_{t+1}, \omega_t) = \beta_0 + \int_{\underline{\omega}_{t+1}} \omega_{t+1} \frac{F(d\omega_{t+1}|\omega_t)}{\int_{\underline{\omega}_{t+1}} F(d\omega_{t+1}|\omega_t)}$$

partially linear model;
kernel and series estimators;
polynomial series estimators;
probit estimation $\rightarrow$ dynamic discrete choice model; a series of approximating functions; triple $(a_t, i_t, k_t) \rightarrow$ age, investment, capital $\rightarrow$ fourth order polynomial; stopping rule and investment equation; form of the survival probability; unit of analysis $\rightarrow$ plant; β_t, ϕ_t, and P_t.

$$\hat{h}_t = \hat{\phi}_t - \beta_a a_t - \beta_k k_t$$

$$y_{t+1} = b_l l_{t+1} - \beta_a a_{t+1} - \beta_k k_{t+1}$$

bias $\searrow$; bandwidth selection; smoothness conditions; capital, age, and time coefficients; age is negligible;
three step estimation procedure;
selecting on survival; annual survey of manufacturers: OLS and within estimates; single index selection model; a polynomial expansion in the triple (i_t, a_t, k_t) for ω_t
$K_t = (1-\delta)K_{t-1} + I_t$

$\mathbb{E}[\omega_{t+1}|\omega_t, \chi_{t+1} = 1] = \mathbb{E}[\omega_{t+1}|\omega_t]$
total and within estimates; probit model and mill's ratio; micro estimates of capital and labour coefficients; std. non - linear search routines; low density regions of the data $\omega_{t+1} = \rho\omega_t + \xi_{t+1}$; $\xi_t \sim N(0, \sigma^2)$;

polynomial estimates of a survival probability; testing investment assumption;
$y_{t+1} - b_l l_{t+1} = \beta_a a_{t+1} + \beta_k k_{t+1} + g(P_t, \phi_t - \beta_a a_t - \beta_k k_t) + \gamma_L L_t + \xi_{t+1} + \eta_{t+1}$

$\xi_{t+1} + \eta_{t+1} \to$ error term
$\xi_{t+1} \equiv \omega_{t+1} - \mathbb{E}[\omega_{t+1}|J_t, \chi_{t+1} = 1]$

investment demand $= \phi$(capital, age, productivity) $\to$ assmpt.
mills ratio
$m(x) := \frac{\bar{F}(x)}{f(x)}$; $f(x)$ is the probability density function;
$\bar{F}(x) := \mathbb{P}r[X > x] = \int_x^{+\infty} f(u) \to$ survival rate;
$h(x) := \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}r[x < X \leq x + \delta | X > x]$, $m(x) = \frac{1}{h(x)} \to$ hazard rate;
$\mathbb{E}[X|X > \alpha] = \mu + \sigma \cdot \frac{\phi\left(\frac{\alpha-\mu}{\sigma}\right)}{1-\Phi\left(\frac{\alpha-\mu}{\sigma}\right)}$[4]

$\mathbb{E}[X|X < \alpha] = \mu - \sigma \cdot \frac{\phi\left(\frac{\alpha-\mu}{\sigma}\right)}{\Phi\left(\frac{\alpha-\mu}{\sigma}\right)}$

$y_{t+1} - b_L L_{t+1} = \beta_a a_{t+1} + \beta_k k_{t+1} + g(P_t, \phi_t - \beta_a a_t - \beta_k k_t) + \gamma_k k_t + \gamma_a a_t + \xi_{t+1} + \eta_{t+1}$

1974/'78 and 1982/'87 $\to$ three - step kernel estimation
OLS 1974/'78 = 1.05 $\nearrow$; 3-step = .9 $\searrow$;
OLS 1982/'87 = .95; 3-step = .95;
switch makers $\to$ OLS = 1.04 $\nearrow$; 3-step = 0.97;
non-switch makers $\to$ OLS = .99; 3-step = .96 $\searrow$

bias $\searrow$ kernel;
fourth - order polynomial;

plant-level productivity

$p_{it} = \exp\{y_{it} - b_l l_{it} - \beta_k k_{it} - \beta_a a_{it}\}$;
$p_{it} = \exp\{\omega_{it} + \eta_{it}\} \to$ error $\to$ unobserved productivity
bureau of labour statistics
Δ regulatory environment;
cost of a new process; exit $\to$ low productivity growth;

$\{ \underbrace{\eta_{it}}_{\text{measurement error}} \}$ and $\{ \underbrace{\omega_{it}}_{\text{plant level productivity}} \}$

efficiency of the output allocation among plants
share weighted average of the plant- level productivity measure
$\to$ weights $\equiv$ plant level output shares;
aggregate productivity index;

distribution of fixed factors versus variable costs

$C(Y_t, K_i, a_i, p_i, w_i) = \min_{L_t} w_i L_i$ s.t. $Y_t \leq L_t^{\beta_l} K_t^{\beta_k} e^{\beta_a a_i} e^{p_i}$
static efficiency index; intrafirm and interfirm efficiency;
$\min_{Y_1,\ldots,Y_N} \sum_{i=1}^N C(Y_i, K_i, a_i, p_i, w_i)$ s.t. $\sum_{i=1}^N Y_i = Y$
$p_t = \sum_{i=1}^N s_{it} \cdot \underbrace{p_{it}}_{\text{plant-level productivity}}$

1987 $\to$ competitive structure of the market

[4] Φ is the cdf, ϕ the pdf of x for a standard normal distribution.

sample covariance

plant-level capital and plant-level productivity $\rightarrow$ reduced form evidence;
state vector triple; dynamic general equilibrium model; capital-productivity correlation;
survival probabilities; probit analysis;
shift in productivity
telephone equipment
transmission equipment
switching equipment
copper wire; glass fiber;
serial correlation of plant's efficiency $\rightarrow$ rates of expansion of plants;
military space satellites;
$\underbrace{\text{buildings and equipment}}_{\text{capital}}\rightarrow$ inventory method;

sampling design.

Levinsohn and Petrin 2003 study unobserved firm - specific productivity, looking at intermediate inputs as a choice set; they also focus on adjustment costs as well as non convex kink points, and propose an estimation method based on lagged instrumental variables;

$y_{it} = f(x_{it}, \varepsilon_{it}; \beta)$, $\{\varepsilon_{it}\}_{t=1}^{+\infty}$
hicks neutral productivity shocks

$\varepsilon_{it} \perp X_{it} \rightarrow$ inputs and shocks
applied researchers;

$y_{it} = \beta_0 + \beta_l l_{it} + \beta_k k_{it} + \varepsilon_{it}$

$\hat{\beta}_L = \beta_l + \frac{\hat{\sigma}_{kk}\hat{\sigma}_{l\varepsilon} - \hat{\sigma}_{lk}\hat{\sigma}_{k}}{\hat{\sigma}_{ll}\hat{\sigma}_{kk} + \sigma_{lk}^2}$
$\sigma_{l\varepsilon} > 0$
unobserved firm - specific fixed effects;

$y_t = \beta_0 + \beta_l l_t + \beta_k k_t + \omega_t + \eta_t$
structural model of the optimizing firm
$i_t = i_t(\omega_t, k_t)$
$\omega_t = \omega_t(i_t, k_t)$
$y_t = \beta_l l_t + \phi_t(i_t, k_t) + \eta_t$

$\{X_{it}, \varepsilon_{it}\}_{t=1}^{+\infty}$
$\phi_t(i_t, k_t) = \beta_0 + \beta_k k_t + \omega_t(i_t, k_t)$
$\rightarrow$ non - parametric estimator in $\phi_t(\cdot)$
fourth order polynomial in i_t and k_t
to approximate $\phi(\cdot)$ with ols;

$\mathbb{E}[y_t|i_t, k_t] = \beta_l \mathbb{E}[l_t|i_{it}, k_t] + \phi_t(i_t, k_t)$
$\eta_t \perp i_t, k_t$; $\mathbb{E}[\phi_t(i_t, k_t)|i_t, k_t]$
$y_t - \mathbb{E}[y_t|i_t, k_t] = \beta(l_t - \mathbb{E}[l_t|i_t, k_t]) + \eta_t$

$\omega_t \rightarrow$ first-order markov process;
$\eta_t \perp l_t$
$\xi_t = \omega_t - \mathbb{E}[\omega_t|\omega_{t-1}]$

$y_t^* = y_t - b_l l_t = \beta_0 + \beta_k k_t + \mathbb{E}[\omega_t|\omega_{t-1}] + \eta_t^*$
$\eta_t^* \to \xi_t + \eta_t$
OLS within IV
$\omega_t \geq \bar{\omega}_t(k_t) \to$ kink point;
$i_t(\omega_t, k_t) \to$ investment is a function of productivity and capital;

inertial behaviour in investment data from US and UK plant-level surveys $\to$ non - convex adjustment costs

$\eta_t + (\omega_t - \bar{\omega}_t(k_t))$, when $\omega_t \geq \bar{\omega}_t(k_t)$; $i_t = i_t(\bar{\omega}_t, k_t)$

investment is a control or a state variable $\to$ costly to adjust; $\exists$ non - convex adjustment costs; intermediate inputs $\to$ materials or energy; additive separability assumption on the inputs

$\underbrace{y_t}_{\text{revenues}} = \beta_0 + \underbrace{\beta_l l_t}_{\text{labour}} + \underbrace{\beta_k k_t}_{\text{capital}} + \underbrace{\beta_e e_t}_{\text{energy}} + \omega_t + \eta_t \to$ three regressors
$\iota_t = \iota_t(\omega_t, k_t) \to$ inverting $\omega_t = \omega_t(\iota_t, k_t)$
$\mathbb{E}[\iota_{t-1}\eta_t^*] = 0 \to$ identification

$\text{sign}\left(\frac{\partial t}{\partial \omega}\right) = \text{sign}(f_{\iota l} f_{t\omega} - f_{ll} f_{l\omega})$
$f_{ll} = \frac{\partial^2 f}{\partial t^2}$; $f_{ll} < 0$; $f_{l\omega} \geq 0$
$\phi_t(\iota_t, k_t) = \beta_0 + \beta_k k_t + \beta_l l_t + \omega_t(l_t, k_t)$

$y_t^* = \beta_0 + \beta_k k_t + \beta_l l_t + \mathbb{E}[\omega_t|\omega_{t-1}] + \eta_t^*$
$\mathbb{E}[\iota_t, \eta_t^*] \neq 0$; $f_{l\omega} \geq 0$

cobb - douglas or CES $\to$ investment is monotonic in productivity; firm's dynamic problem:
$\iota_t = \iota_t(\omega_t, k_t) \to$ intermediate input;
demand equation $\to$ time, region, firm;
eight year panel from chile; many firm - level variables with a reasonable time - series dimension;
6,665 plants $\equiv$ firms; multi-plant firms are neglected; revenue $\equiv$ plant output;
revenue $\to$ monetary value;
labour $\to$ man - years hired for production;
$k_{jt} = (1 - \delta_j)k_{j,t-1} + i_{jt}$
$k_t = \sum_j k_{jt}$
capital adjustment process $\to \omega_{t-1}$
1979 - 1986; $\omega_t(\iota_t, k_t)$
$t = 1, 2, 3$
projected initial capital stock
food, metals, textiles, wood
capital intensive telecommunication industry

materials, electricity, fuels
$\iota_t(\omega_t, k_t)$
specification check;
$(l_t, \xi_{t+1}) = 0$

skilled and unskilled labour; capital; materials; fuels; electricity;
$y_t = \beta_0 + \beta_k k_t + \beta_s l_t^s + \beta_u l_t^u + \beta_e e_t + \beta_f f_t + \beta_m m_t + \omega_t + \eta_t$
locally weighted quadratic least squares approx.
$y = \beta_s l_t^s + \beta_u l_t^u + \beta_e e_t + \beta_f f_t + \phi_t(m_t, k_t) + \eta_t$
$\phi_t(m_t, k_t) = \beta_0 + \beta_m m_t + \beta_k k_t + \omega_t(m_t, k_t)$

OLS and a polynomial expansion
$\mathbb{E}[y_t|k_t, m_t]$; $\mathbb{E}[l_t^u|k_t, m_t]$;
$\mathbb{E}[l_t^s|k_t, m_t]$; $\mathbb{E}|e_t|k_t, m_t]$; $\mathbb{E}[f_t|k_t, m_t]$;

locally weighted least squares regression; OLS - with - a - polynomial - approximation approach;
$\neq$ sub periods of the sample; various macroeconomic cycles;
capital coefficient; a plant - level measure of productivity;

two moment conditions $\rightarrow \beta_m$ and β_k, capital does not respond immediately to the innovation in productivity ξ_t; also materials purchases in the last period; $\mathbb{E}[\omega_t|\omega_{t-1}]$
(β_m^*, β_k^*)
$\mathbb{E}[(\xi_t + \eta_t)Z_t]$
GMM
$\mathbb{E}[(\xi_t + \eta_t)m_{t-1}] = \mathbb{E}[\xi_{t-1}] = 0$
$\beta^* = (\beta_m^*, \beta_k^*)$
a set of firm - level observations
$Z_t = \{k_t, m_{t-1}, l_{t-1}^s, l_{t-1}^u, e_{t-1}, f_{t-1}, k_{t-1}, m_{t-2}\}$

$$y_{it} = \beta' X_{it} + \underbrace{\gamma_t}_{\text{time fixed effects}} + (\underbrace{\eta_i}_{\text{firm fixed effect}} + \underbrace{\nu_{it}}_{\text{AR(1)}} + \underbrace{m_{it})}_{\text{MA(0)}};$$

clothing; machinery; mechanical components;
metal and wood products $\rightarrow \nearrow$ returns
food and textiles $\rightarrow \searrow$ returns
materials and electricity;
$\omega_t = \omega_t(m_t, k_t)$
the three macroeconomic cycles in the data $\rightarrow$ 1979/'81; 1982/'83; 1984/'86; specification changes $\rightarrow$ proxy choice $\rightarrow$ satisfying monotonicity; inputs$_{t-1} \leftrightarrow$ productivity shocks
over-identifying restrictions

loosening the functional restrictions on $\phi(k_t, m_t, \omega_t)$
proxy controlling for an unobserved transmitted productivity shock;
proxy $\rightarrow$ electricity and materials;

OLS vs. LP method

skilled labour
$\hat{\beta}_{OLS} - \hat{\beta}_{LP}$
the simultaneity story

inputs lagged one period are instruments;
arellano and bond dynamic panel estimation;
fixed effects and instrumental variables;
boundary of the parameter space;
$H_0 : \beta_1 = \beta_2$
6,115 obs.
you have 48 obs.
proxies to address simultaneity biases
$\iota_t(\omega; p_l, p_k, k)$

$Y = f(K, L, \iota, \omega) : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$(k, l, \iota, \omega) \in \mathbb{R}^4$

$\iota(\omega_l; p_l, p_\iota, k) - \iota(\omega_1; p_l, p_\iota, k) = \int_{\omega_1}^{\omega_2} \frac{\partial \iota}{\partial \omega}(\omega; p_l, p_\iota, k) p(d\omega|k)$
weakly separable production function
$y_t^* = \beta_k k_t + \mathbb{E}[\omega_t|\omega_{t-1}]\eta_t^*$
$\eta_t^* = \xi_t + \eta_t$
$y_t^* = y_t - \beta_l l_t - s_{\iota t}$
$\int_{\omega_1}^{\omega_2} \frac{\partial \iota}{\partial \omega}(\omega; p_l, p_\iota, k) P(d\omega|k) > \int_{\omega_1}^{\omega_2} 0 P(d\omega|k) = 0$
$\mathbb{E}[y_t|m_t, k_t]$
$\mathbb{E}[l_t^u|m_t, k_t]$

$\phi_t(m_t, k_t)$, $t = 1, 2, 3$

rudi dornbusch

Erwin Diewert and Kevin J. Fox 2008 tried to distinguish between returns to scale and technological progress, knowing that there exists a degree of monopolistic elements in the economy $\rightarrow$ US manufacturing in $1950, \ldots, 2000$;
multiple outputs and inputs;
productivity growth; growth of output relative to the growth of inputs;
measure the performance of economies; growth and business cycle;
procyclical nature of productivity growth;
standard econometric methods and aggregate annual data; index of total factor productivity growth $\rightarrow$ index numbers techniques;
local returns to scale $\rightarrow$ multiplier;
cost function $\rightarrow$ dual of a production function;
$\sum_{m=1}^{M} w_m \cdot x_m \rightarrow$ inner product;
$R(w, y, t) = \sum_{n=1}^{N} \frac{\partial \ln C(w,y,t)}{\partial \ln y_n}$
trans-log joint cost function;
bias in technical progress term;
$R(w, y, t) = \sum_{n=1}^{N} \frac{\partial \ln C(w,y,t)}{\partial \ln y_n}$
$\beta + \gamma \ln y + \phi \ln w + \kappa t$
$P_n^t(y_n)$
technical progress
$T(w, y, t) \equiv \frac{\partial ln C(w,y,t)}{\partial t}$
productivity possibility set
wedge $\rightarrow \ln P_n^t(y_n) \equiv a_n^t - c_n^t \ln y_n \rightarrow$ markup;
index numbers techniques
$Y^t \equiv \ln Q_T(p^{t-1}, p^t, y^{t-1}, y^t)$
$X^t \equiv \ln Q_T^*(w^{t-1}, w^t, x^{t-1}, x^t)$
$(y'y)^{-1}x'y$
$\rho_y \leq \rho_x$
$X^t = -\tau + \rho Y^t$
$Y^t = \tau/\rho + 1/\rho X^t$
$\rho = \frac{\sum_n p_n y_n M_n}{\sum_m w_m x_m}$
$\rho_v \equiv [\sum_{n=1}^{N} P_n y_n M_n - \sum_{k=1}^{K} w_k x_k] / \sum_{m=k+1}^{M} w_m x_m$
$\rho_v < \rho < 1$; $1 < \rho < \rho_v$
value added or gross output
primary input share $\rightarrow$value added returns to scale parameters;
productivity shocks $\rightarrow \omega_{it}$

national income accounting;
have varying levels of technical efficiency for $i = 1, \ldots, N$ and $t = 1, \ldots, T$;

$Y^t = \rho^{-1}\tau + \rho^{-1}X^t + \rho^{-1}\varepsilon^t$, $t = 1, 2, \ldots, T$;
input growth and output growth
$\psi^t = \theta\xi^t$
$X^t = \alpha + \xi^t + \eta^t$
$Y^t = \beta + \psi^t\varepsilon^t$
errors in variables model
$\theta = \frac{1}{\rho}$
$Y^t = \mu^t + \theta X^t$; $t = 1, 2, \ldots, T$; $\mu^t = \rho^{-1}\tau^t$
$Y^t = (\beta - \alpha\theta) + \theta X^t + \varepsilon^t - \theta\eta^t$
$Y^t \equiv \ln Q_t(p^{t-1}, p^t, t^{t-1}, y^t)$
törnqvist output quantity index
shadow input vector price
cost functions $\rightarrow$ factor demand equations
deflated sales $\approx$ output
unobserved quality or $\neq$ prices
$\lambda \equiv \frac{\sigma_\varepsilon^2}{\sigma_\eta^2}$ $\rightarrow$ linear or quadratic splines
$\theta_{ML} \equiv \{y'y - \lambda x'x + [(y'y - \lambda x'x) + 4\lambda(x'y)^2]^{1/2}\}/2x'y$
$\underbrace{Y_t}_{\text{real output}} = A_t \underbrace{K_t}_{\text{capital}}^{\theta} (\eta_t \underbrace{H_t}_{\text{hours worked}})^{1-\theta}$
$M \equiv I' - Z(Z'Z)^{-1}Z'$
projection matrix
OLS $\equiv$ MLE
$\mu = \beta - \alpha\theta$
$\mu_{ML} \equiv \beta_{ML} - \alpha_{ML}\theta_{ML}$
$Z\alpha$ and $Z\beta$
$\psi = \theta\xi$; $X = Z\alpha + \xi + \eta$
$Y = Z\beta + \psi + \varepsilon$
$=_k = \xi'Z = \psi'Z$
$\theta_x \leq \theta_y$
empirical cobb - douglas estimation
$\psi = \theta\xi$
$X = Z\alpha + \xi + \eta$
$Y = Z\beta + \psi + \varepsilon$
e^{tT}, $Z\alpha$, e^{tT}, $Z\beta$
$\sigma_{11} = \theta^2\sigma_\xi 2 + \sigma_\varepsilon^2$
$\sigma_{12} = \theta\sigma_\xi^2$
$\sigma_{22} = \sigma_\xi^2 + \sigma_\eta^2$
capital, labour, energy, materials, purchased business services
engineering production function
piece-wise linear time trends
$Q_T(\cdot) = \frac{\tau}{\rho} + \frac{1}{\rho}\ln Q_T^*(\cdot)$
all the other sectors have $\nearrow$ returns to scale;
why reverse regressions? $M^t = \frac{\rho(w^t \cdot x^t)}{P^t \cdot y^t}$
aggregate manufacturing
generalized leontief cost function

- returns to scale
- technological progress
- monopolistic markups

→ multiple inputs/outputs
revenue function
trade and industry
productivity and returns to scale

2 Firms at hand

I analyze data for four manufacturing firms that are clients of the office where I work as a chartered professional accountant in Bologna, Italy; the four of them are operating in the vicinity of the town, two in the Appennines area and two on the Pianura Padana; I am going to describe the business they operate in before showing the descriptive statistics and performing the estimation procedures.

2.1 Firm 1

The first firm whose data I analyze throughout the paper, which I call FIRM 1 was founded in 1977 as a "società in nome collettivo", based in the Southern part of the Bolognese Appenninens, Italy, due to the will of the actual shareholders, who developed the company in order to make it a significant reality in the field of small metal parts. The company makes small metal parts both in steel and in brass, as well as aluminium, based on the drawings provided by the clients, with multi-spindle equipment. In such an activity, the firm has reached a quite high level of specialization, both in the workings and in the product and process controls, keeping its competitiveness in the specific reference market.

Throughout the years, thanks to the detailed requests of the clients, the firm acquired a deeper degree of consciousness upon the required level of quality; as such, it qualified as a certified company according to the ISO 9001 quality standard as well as the ISO TS 16949 in the field of the automotive industry; its main clients operate in the automobile sector indeed, especially in Germany and Sweden; the reference market for the company is therefore the one of builders of mechanical components, especially in the car industry: such a sector is extremely specialized and the request for quality of products and of specialization of the employees dedicated to the production process is fairly high, and thus the firm is able to properly respond to the expectations of the clientele with the certification of its quality system.

The firm is operating with its multi-spindle automatic lathes Gildemeister and Mori-Say for the main operations; and it is also endowed with a drying system, separation of the chip from the oil and washing of the products. The company also has an automatic weighing and batch identification system via bar-code, directly connected to the corporate information system for production batch control and better identification and traceability of lots.

As far as the checks are concerned, these are carried out using a profilometer linked to a computerized statistical program which automatically issues the control charts. The checks are also carried out through measuring instruments connected to a program for the construction of control charts which allows safety and speed in the formalization of the surveys.

2.2 Firm 2

the second firm I include in the data set of firms performance is one operating in the sector of shoe parts making, in particular, heels for female luxury shoes; born in the 1930s it claimed to have launched the female heel in the modern stylistic conception and industrialized it; the company designs and constructs innovative heels requiring specialized machinery such as high quality 3D printers; since the 1970s, firm 2 has been a partner of numerous fashion brands, mostly Italian, such as Ferragamo, Gucci and Prada, but also foreign brands, especially from Germany and Portugal, among others.

The firm dominates and manages the whole set of technological tools needed to produce heels at the highest quality standard, in order to satisfy clients' requirements, ranging from the design, modeling, construction of moulds, shape mouldings and painting; the expertise acquired by the firm allows it to hand in samples of products at any time and guarantee quality and precisions in all the phases of the productive process.

They are focused to be the ideal partner for the most important and renowned women's footwear, that are recognized by the market to produce high volumes and which make of innovation and quality their main competitive advantages. The disciplined application of CRM-Customer Relationship Management allows Tacchificio di Molinella to pursue the continuous improvement in the Total Customer Satisfaction.

The core business strategy is to always have the best R and D and styling team for creativity and problem solving. The skill of developing in a proactive approach new samples and technical proposals in real time is the main goal of their business strategy. The skilled human resources, from senior to juniors working together, allow them to manage quickly all the customer's inquiries, while developing new R and D developing projects, new materials tests and continuously field training job experience enrichment of their technicians.

Also for that, firm 2 is ,and will keep being for the years to come, a worldwide leader in the fashion footwear heels market. The highest focus on the total quality management, lead time improvement and outstanding execution are their main target and allowed them to grow the marked share in the top footwear players market.

The daily capacity of production amounts to about 16,000 pairs of heels; total quality management is applied and the organization allows it to be among the top supplier of heels in Northern Italy as well as on the global market (Europe, mostly), competing with other five/six similar companies in a radius of about two hundred kilometers.

2.3 Firm 3

The third firm in our sample is located in the plain nearby Bologna, and it designs, constructs and sells packaging machinery mostly for the food and beverage industry. Founded in 1974 by a former manager of a bigger firm working in the same business, it rapidly started to grow, to expand in foreign markets, mostly European, but also extra European.

Since the beginning of its operations, its field of operation has been the design and construction of packaging machinery exploiting the technology of wrapping with shrink film starting from a film folded in the center and the wrapping stretch. The firm soon acquired the property of having an ample productive range, which covers a variety of industries, from the multiple pack of food products (cans, jars, bottles and boxes with standardized supplies and systems), to that of the wood working products, from the packaging of expanded polystyrene products to that of the most diverse industrial products.

Nowadays, the productive range has been further expanded through the insertion in production of combined tray/carton machines with "wrap around"system and high-speed machines without "film launch"sealing bar. In 2018, the firm was acquired by a bigger player in a similar market, from Rimini[5], as part of a company restructuring which allowed it to overcome a moment of financial distress, due to poor economic results in the past years; the acquisition allowed firm 3 to integrate in a productive ensemble that covers additional markets, namely those of cardboard and film solutions at medium-low speed.

Together with other business units of the group, the firm is able to supply complete systems for the end of line. In light of this fact, the firm has always been endowed with a technical office able to promote a continuous technological updating of the products and to ensure a personalized study of the clients' need: this further allowed yhe firm to elaborate and supply the entire end of the line in a productive process.

In parallel with the electronic progress, the company also powered the software advancement which enables the softwares to be constantly up to date and properly manage the hardware of electronical components, always finalized to ease the operativity of the machines. The whole system is managed in a fast paced and flexible structure which favours technical and technological progress.

2.4 Firm 4

The history of firm 4 is rooted in the early years of the 1960s; it is a story made of commitment, research, passion and hard work. The protagonist is its founder, who, as a turner first, then as a mechanical designer, guided the growh of the company upon solid values. First of all, reliability.

The imprinting given by the founder characterizes the firm even tady. The familiarity, the sharing of challenges, the exchange of visions based on sociability and the desire to find innovative solutions, are the elements which make unique and high performing the group of the firms' employees. The presence of these great human values reflects in the action of each one in the company, creating a system generating quality, non just productive, but also relational. This is the true receipt characterizing FRB, besides the high level of technical ability and a natural tendency to the pursuit of innovative solutions, other inheritance of the founder.

In the plant located in the South West of Bologna, nearby the birth place of Guglielmo Marconi, the inventer of the first wireless telegraph and of the radio, Nobel prize winner for physics in 1909, a highly qualified personnel operates with a technologically advanced set of machineries and tools to fulfil the clients' requests to construct machine tools, in Italy and abroad. The specialty of the firm is tailstocks, the adjustable part of a lathe holding the fixed spindle, and draggers.

The company was characterized by having some patents covering its inventions, new products conceived by the founder and by his son Marco; at present there are still two patents in course of validity entitled to the company. Two years ago, the firm evaluated the possibile acquisition by an North American multinational met at an international fair in Dusseldorf in Germany; the negotiations went on for months, without eventually leading to the transaction.

3 Data construction and summary

I use data on revenues, fixed and variable costs for the firm at hand (who is a client of the chartered accountant studio where the author works) to estimate the following model in order to better understand and formally quantify the relations among the three financial variables:

$$\underbrace{Y}_{\text{revenues}} = \underbrace{X_1}_{\text{variable costs}} \beta_1 + \underbrace{X_2}_{\text{fixed costs}} \beta_2 + \underbrace{\varepsilon}_{\text{stochastic error}}$$

[5]A so called pocket multinational.

with the variables defined in levels, in euros, at annual frequency, for the years $t = 2011, \ldots, 2021$; notice that the balance sheet from which the data have been retrieved are the analytical financial statements of the end of the solar year, for all but 2021, for which I have data only up to the end of the third quarter (September 30); therefore, up to now, the length of the time series is $T = 11$ years with yearly observation and $k = 2$, the number of parameters to be estimated, due to the omission of the intercept term in all the five regressions I run.

In particular, I considered as components of **variable costs** $\longrightarrow$ the following voices of the income statements of the firm:

`purchases` $\longrightarrow$ *purchase of raw materials, namely iron, aluminium, brass, inox; purchase of finite goods from both italy and abroad*; `production costs` $\longrightarrow$ *external processing, industrial lubricants, equipment and small parts, cleaning and garbage collection, compressor maintenance, petrol/diesel trucks, consumables, treatments, car fuel, truck fuel, truck insurance*; `sales costs` $\longrightarrow$ *transport for sales, travel and transfers, packaging for sales, commercial expenses, passive commissions*; `general expenses` $\longrightarrow$ *postal and telegraph expenses, telephone expenses, revenue stamps, bank expenses, administrative services, mobile phone expenses, various rentals*;

as part of the **fixed costs**, I included the following components of the income statements: `cost of productive labour` $\longrightarrow$ *gross workers' salaries, INPS and INAIL social security contributions for workers, severance indemnity*; `production costs` $\longrightarrow$ *electricity, maintenance and repairs, heating, water consumption, insurance, car insurance, computer rental fees, truck insurance, sylos system maintenance and repairs, truck maintenance and repairs, forklift truck repairs, heating system maintenance, electrical system maintenance, washing machine maintenance and repair*; `general expenses` $\longrightarrow$ *stationery and printed matter, legal and notary consultancy*[6]*, administrative consultancy, directors' fees, computer programming assistance services, contribution of 10% for self-employed workers, compliance with law 626*[7]*, ISO 9002 compliance, board of statutory auditors compensation*; `cost of administrative labour` $\longrightarrow$ *administrative salaries, INPS and INAIL social security contributions for employees, severance pay for employees*;

Finally, as part of the **revenues**, I chose the following: `miscellaneous revenues and income` $\longrightarrow$ *sales of production in Italy, exports, sale of scrap and various scraps, recovery of expenses and other indemnities, bank interest income, interest income (coupons), contingent assets, capital gains.*

I report in numerical and graphical form the data employed in the analysis, arising from the illustrated aggregations, respectively in tables 1, 2, 3 and 4 below, and in figures 1 to 9 in the appendix.

As a comment to the summary statistics, we may notice that the firm with the highest mean revenues is firm 3, with an average over the last twelve years of about eight and a half million euros, followed by firm 2 with about six million euros of revenue on average since 2011, and lastly firms 1 and 4; in terms of costs, the variable costs are highly collinear with revenues for the whole sample, even though this can be seen via a detailed inspection of revenues and variable costs in the data set, available as a supplement to the paper and via a graphical representation of the relevant variables. Fixed costs are particularly high for firm 3, around 3 and a half million euros on average.

Capital for firm 3 is the lowest paradoxically, being a bit higher than half a million euro; labour cost, on the contrary in highest for firm 3, being of about two and a half million euros, followed by firm 2, 4 and 1, respectively. The number of employees is of about 30/40 persons per firm per year on average in the sample period. Finally, purchases of materials as intermediate inputs, are highest for firm 3 as well, being of about 4 million euros on average, followed by firm 2, 1 and 4.

[6]Due to their occasional occurrence.
[7]Safety on the job for the workers.

Tabel 1: Descriptive statistics for **firm 1** metals turner and cutter in levels over $t = 2011, \ldots, 2021$

Variable	Obs	Mean	Std. Dev.	Min	Max
$y \rightarrow$ revenues	11	2,429,347	579,151.1	1,814,393	3,863,768
$X_1 \rightarrow$ variable costs	11	1,180,501	425,585.4	705,503.7	2,265,687
$X_2 \rightarrow$ fixed costs	11	1,112,846	111,622.4	866,459	1,288,783
$K \rightarrow$ capital	11	1,288,783	304,253.2	1,041,229	2,014,860
$L \rightarrow$ labour	11	784,828.1	102747.6	580,403.1	998,017.4
$M \rightarrow$ materials	11	814,736.5	285,877.2	556,881	1,564,775

Tabel 2: Descriptive statistics for **firm 2** shoes' heels producer in levels over $t = 2011, \ldots, 2021$

Variable	Obs	Mean	Std. Dev.	Min	Max
$y \rightarrow$ revenues	11	6,114,797	1,577,543	3,514,682	8,675,828
$X_1 \rightarrow$ variable costs	11	3,348,968	962,382.6	1,881,723	5,001,197
$X_2 \rightarrow$ fixed costs	11	534,365.1	69,694.92	414,012.2	650,225.6
$K \rightarrow$ capital	11	1,483,513	203,196.4	1,081,358	1,828,179
$L \rightarrow$ labour	11	1,554,276	281,281.5	864,448.8	1,828,018
$M \rightarrow$ materials	11	2,119,399	501,035.9	1,481,902	3,237,446

Tabel 3: Descriptive statistics for **firm 3** or packaging machines producer in levels over $t = 2011, \ldots, 2021$

Variable	Obs	Mean	Std. Dev.	Min	Max
$y \rightarrow$ revenues	12	8,625,000	1,917,444	6,800,000	1.20e+07
$X_1 \rightarrow$ variable costs	12	4,787,536	1,940,533	389,905.9	7,807,371
$X_2 \rightarrow$ fixed costs	12	3,445,353	528,552.7	2,743,034	4,715,607
$K \rightarrow$ capital	12	691,247	379,873	158,742	1,200,000
$L \rightarrow$ labour	12	2,500,000	343,775.8	2,100,000	3,300,000
$M \rightarrow$ materials	12	4,141,667	1,142,134	2,700,000	6,100,000

4 Cost functions estimation

I hereby express the various specifications of the multivariate linear regression models which I estimate through STATA 13.0 SE, that depart from the basic version of the bivariate equation with the variables in contemporaneous time, allowing for some lagged independent variables to appear on the right hand side of the equations:

$$y_t = \underbrace{X_{1t}\beta_1}_{\text{no lags}} + \underbrace{X_{2t}\beta_2}_{\text{no lags}} + \varepsilon_t \tag{1}$$

$$y_t = \underbrace{X_{1t-1}\beta_1}_{\text{one lag}} + \underbrace{X_{2t-1}\beta_2}_{\text{one lag}} + \varepsilon_t \tag{2}$$

Tabel 4: descriptive statistics for **firm 4** tailstocks and drivers producer in levels over $t = 2011, \ldots, 2021$

Variable	Obs	Mean	Std. Dev.	Min	Max
$y \rightarrow$ revenues	22	2,357,330	722,897.7	1,068,269	3,954,619
$X_1 \rightarrow$ variable costs	11	1,174,144	472,311.4	197,644.9	2,009,533
$X_2 \rightarrow$ fixed costs	11	1,071,414	408,217.3	184,431	1,594,414
$K \rightarrow$ capital	14	1,824,297	530,329.2	1,013,524	2,439,805
$L \rightarrow$ labour	22	795,686.5	125,505.3	591,510	1,070,519
$M \rightarrow$ materials	18	314,472.7	85,200.19	204,741	452,746

$$y_t = \underbrace{X_{1t-1}\beta_{11}}_{\text{one lag}} + \underbrace{X_{1t}\beta_{12}}_{\text{no lags}} + \underbrace{X_{2t-1}\beta_{21}}_{\text{one lag}} + \underbrace{X_{2t}\beta_{22}}_{\text{no lags}} + \varepsilon_t \tag{3}$$

$$y_t = \underbrace{X_{1t-2}\beta_{11}}_{\text{two lags}} + \underbrace{X_{1t-1}\beta_{12}}_{\text{one lag}} + \underbrace{X_{2t-2}\beta_{21}}_{\text{two lags}} + \underbrace{X_{2t-1}\beta_{22}}_{\text{one lag}} + \varepsilon_t \tag{4}$$

$$y = \underbrace{X_{1t-2}\beta_{11}}_{\text{two lags}} + \underbrace{X_{1t-1}\beta_{12}}_{\text{one lag}} + \underbrace{X_{1t}\beta_{13}}_{\text{no lags}} + \underbrace{X_{2t-2}\beta_{21}}_{\text{two lags}} + \underbrace{X_{2t-1}\beta_{22}}_{\text{one lag}} + \underbrace{X_{2t}\beta_{23}}_{\text{no lags}} + \varepsilon_t \tag{5}$$

The idea is that, fixed costs especially may take time to deploy their positive effects on revenues, since they are connected with fixed assets, which have a relatively long economic life and therefore a longer time of transmission into products, sales and revenues; therefore, I consider equations with one to two lags, combining various layers of complication; below are synthetically reported the results of the regressions, where the number of each column corresponds to each one of the five equations.

Equation (1) is the baseline specification as well as the most relevant one, as far as it allows to best exploit the information at hand, regressing the current values of revenues on those of two categories of costs within the same year (i.e. contemporaneous effect of variable and fixed costs on revenues); while in equation (2) I basically regress the revenues each year on the fixed and variable costs of the previous year; in equation (3) I analyze the effect of the variable costs of a year on the revenues of the same year and of the subsequent year as well; in equation (4), I consider the independent variables lagged of two and of one year; and, finally, in equation (5) I set up a lag of two years, one year, and no lag of each of the two categories of costs on revenues.

Despite the lagged structure of fixed costs on output, for **firm 1** the most significant estimated coefficients appear to be those arising from the first regression, which points to the direction that marginal increases in variable costs lead to higher revenues, while, paradoxically, increases in fixed costs tend to reduce revenues, on impact; in particular, a marginal increase in variable costs of 1 euro should lead to an increase in revenues of about 1.185 euros; while an increase in fixed costs seems to be negatively related with revenues, being the associated coefficient point estimate slightly lower than 1, of about 0.763; both the estimates are highly statistically significant.

The trend seems to invert itself when we move on to consider once lagged variable and fixed costs on revenues in column 2 of table 5: as a matter of fact, marginally higher lagged fixed costs have, ceteris paribus, a negative but statistically insignificant effect on revenues, reducing them of a double amount, but this seems truly implausible; while holding fixed costs constant, and raising of one currency unit the variable costs leads to only 0.817 higer revenues, with a decent degree of statistical significance (90%).

Tabel 5: Exploratory regressions of revenues on variable and fixed costs, with and without time lags in the independent variables of **firm 1**, close to Riola di Vergato, Bologna, Italy

	{1}	{2}	{3}	{4}	{5}
VARIABLES	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$
$\texttt{variable costs}_t$	1.185***		0.932**		0.671
	(0.125)		(0.235)		(0.271)
$\texttt{fixed costs}_t$	0.763		0.822		1.370
	(0.478)		(0.512)		(0.625)
$\texttt{variable costs}_{t-1}$		0.817*	0.288	0.853	0.0195
		(0.414)	(0.165)	(1.060)	(0.342)
$\texttt{fixed costs}_{t-1}$		-2.212	-0.669	-1.836	-0.371
		(2.233)	(0.871)	(3.358)	(0.965)
$\texttt{variable costs}_{t-2}$				-0.0469	0.0221
				(0.654)	(0.195)
$\texttt{fixed costs}_{t-2}$				-1.590	-0.794
				(3.299)	(1.025)
constant	181,638	3.810e+06	795,746	5.233e+06	1.308e+06
	(435,551)	(2.142e+06)	(827,337)	(3.031e+06)	(1.042e+06)
observations	11	10	10	9	9
r-squared	0.969	0.430	0.954	0.409	0.977

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Tabel 6: Exploratory regressions of revenues on variable and fixed costs, with and without time lags in the independent variables of **firm 2**, Molinella, Bologna, Italy

VARIABLES	{1} revenues$_t$	{2} revenues$_t$	{3} revenues$_t$	{4} revenues$_t$	{5} revenues$_t$
variable costs$_t$	1.569***		1.534***		1.459**
	(0.0702)		(0.113)		(0.172)
fixed costs$_t$	1.159		1.198		-1.664
	(0.969)		(1.777)		(1.982)
variable costs$_{t-1}$		0.565	0.0305	-0.806	0.112
		(0.829)	(0.122)	(0.429)	(0.174)
fixed costs$_{t-1}$		7.811	0.741	38.94***	3.953
		(12.05)	(1.679)	(6.950)	(4.049)
variable costs$_{t-2}$				-3.107***	-0.449
				(0.462)	(0.297)
fixed costs$_{t-2}$				27.92**	2.141
				(6.431)	(3.032)
constant	241,599	-129,117	-173,219	-1.636e+07**	34,809
	(371,001)	(5.104e+06)	(1.067e+06)	(4.315e+06)	(2.305e+06)
observations	11	10	10	9	9
r-squared	0.994	0.283	0.995	0.945	0.999

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

In the third model, with the regressors both contemporaneous to the dependent variable and lagged of one period, it appears that the variable costs increase revenues less than proportionally to a marginal increase in their entity, and that the fixed costs raise revenues on impact, but reduce them if fixed costs enter the model with one lag, again, a counterintuitive effect, which should be taken with caution, due to its statistical non significance, contrarily than in the first specification; here, only the coefficient estimate associated with contemporaneous variable costs seems to be statistically significant, though, overall raising doubts on the meaningfulness of such specification.

Similar considerations hold for the fourth regression model, where, again the lagged fixed costs seems to cause negative effects on revenues, which is harly likely to be a plausible case; the fifth regression presents similar results than the third one, with the exception of contemporaneous fixed costs which have a very positive effects on revenues on impact, a result which seem very meaningful and significant.

I am led to conclude from the analysis that the bivariate contemporaneous regression (1) is the best suited to capture the relationships between fixed and variable costs on the one side, and revenues on the other side, with the side condition that the typical assumptions of the exogeneity of the independent variables (the regressors), sphericity and homoschedasticity of the errors, and full column rank of the matrix of regressors[8] hold; only if such a case happens to be holding, these econometric estimates could have a causal interpretation.

If that indeed happens to be the case, it would be worthwhile for such a firm to raise variable costs with the reasonable expectation of producing a higher share of output and thus obtaining

[8] The first three assumptions for the consistency and unbiasedness of the OLS estimator, namely the conditions of applicability of the Gauss-Markov theorem.

Tabel 7: Exploratory regressions of revenues on variable and fixed costs, with and without time lags in the independent variables of **firm 3**, Anzola Emilia, Bologna, Italy

	{1}	{2}	{3}	{4}	{5}
VARIABLES	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$
$\texttt{variable costs}_t$	0.196		0.118		0.430
	(0.240)		(0.262)		(1.053)
$\texttt{fixed costs}_t$	2.583**		3.054**		2.673
	(0.882)		(1.085)		(3.495)
$\texttt{variable costs}_{t-1}$		0.261	0.372	0.0984	0.401
		(0.433)	(0.266)	(0.392)	(0.489)
$\texttt{fixed costs}_{t-1}$		1.957	-1.575	3.512	-1.779
		(2.119)	(1.547)	(1.969)	(3.211)
$\texttt{variable costs}_{t-2}$				0.835*	-0.122
				(0.403)	(0.571)
$\texttt{fixed costs}_{t-2}$				-4.749*	-1.371
				(2.330)	(2.104)
constant	-1.213e+06	1.011e+06	1.101e+06	8.489e+06	6.239e+06
	(2.352e+06)	(6.035e+06)	(3.719e+06)	(8.085e+06)	(6.330e+06)
observations	12	11	11	10	10
r-squared	0.753	0.288	0.811	0.666	0.906

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Tabel 8: Exploratory regressions of revenues on variable and fixed costs, with and without time lags in the independent variables of **firm 4**, Pontecchio Marconi, Bologna, Italy

	{1}	{2}	{3}	{4}	{5}
VARIABLES	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$	$\texttt{revenues}_t$
$\texttt{variable costs}_t$	0.227		0.177		0.295
	(0.374)		(0.156)		(0.164)
$\texttt{fixed costs}_t$	1.310**		0.556**		0.442
	(0.433)		(0.210)		(0.203)
$\texttt{variable costs}_{t-1}$		0.817***	0.724***	0.872**	0.844**
		(0.210)	(0.153)	(0.261)	(0.164)
$\texttt{fixed costs}_{t-1}$		0.804**	0.571**	0.550	0.378
		(0.245)	(0.194)	(0.356)	(0.205)
$\texttt{variable costs}_{t-2}$				0.316	0.373
				(0.474)	(0.304)
$\texttt{fixed costs}_{t-2}$				0.185	0.157
				(0.406)	(0.229)
constant	1.135e+06*	1.133e+06**	654,759*	757,183	64,081
	(606,994)	(340,042)	(324,098)	(582,544)	(471,768)
observations	11	10	10	9	9
r-squared	0.566	0.816	0.936	0.837	0.975

standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

higher revenues, especially on the voices of costs most related with raw materials purchases.

The stochastic frontier model with the same contemporaneous variables as before happens to produce extremely similar results to the ordinary least squares estimation, with the same point estimates and differing standard errors. For more details, see the STATA replication code.

For **firm 2**, the one making heels for women's shoes, the effects of both fixed and variable costs on revenues are stronger, especially noting that, in column 1, a marginal increase in variable costs of 1 euro led to higher revenues of 1.57 euros over the sample period, while a marginal increase of one ECU in fixed costs led to 1.16 euro higher revenues over the same period. With one lag, it seems that fixed costs led to higher revenues by a factor of 7, which seems quite too high, hopefully non statistically significant - we may think of the positive effects on prototyping and production the purchase of a 3D printer may have.

Moving onwards to column 3 of table 6, we find reinforcement of the results exposed in column 1, especially for the contemporaneous effects of fixed and variable costs; in column 4, we get some strange results, with coefficients extremely high associated with fixed costs lagged once, and twice; the only statistically significant point estimate is a negative coefficient equal to -3.12 associated with twice lagged variable costs; in column 5, the results go back to normal magnitudes, with positive effects of contemporaneous variable costs and once and twice lagged fixed costs of about 3.9 and 2.12 respectively.

Firm 3, the producer of packaging machines, is characterized by a very high coefficient associated with fixed costs: a marginally higher fixed cost of a unit led to 2.58 higher revenues on average over the past twelve years; while an increase in variable costs had very little impact on revenues, perhaps due to the structure of production, whereby the firm is exploiting some complex types of machinery which are all investments in fixed capital, and hence connected with fixed costs; while variable costs might just be higher purchases of raw materials which do not significantly contribute to generate output, or at least, not as much as investments in fixed capital. Similar trends energe by reading the estimates in the subsequent columns of table 7.

Turning to **firm 4** is also strangely characterized by the negative association between variable costs on impact and revenues, and with the positive and statistically significant relation between fixed costs and revenues (table 8, column 1). Introducing a distributed lag structure in the regressors, both fixed and variable costs contribute to raise revenues to an extent that is less than proportional to their increase (column 2); similar trends emerge also from column 3 and 4; as well as from five. At least, for firm 4, which, as already stated, operates in the high precision mechanical components industry, all categories of costs, lagged or not, have a positive impact on revenues. Very similar results to those of column 1 hold for the stochastic frontier model as well, which is reported in the replication files.

As a final note on this exercise of estimation of cost functions, we decided to keep the intercept, as it allows a more precise estimation of the model parameters, the value of the estimated intercept is high in all specification for all the fours firms because the data are in levels and in millions of euros, and thus they should not be seen with suspicion.

5 Properties of production sets

PRODUCTION SETS[9] $\longrightarrow$ in an economy with L commodities, a production vector is a vector $\mathbf{y} = (y_1, \ldots, y_L) \in \mathbb{R}^L$, that describes the net output of the L commodities from a production process;

$Y \subset \mathbb{R}^L \longrightarrow$ production set; any $y \in Y$ is possible, any $y \notin Y$ isn't. a production set is a primitive datum of the theory; technological constraints $\rightarrow$ legal restrictions or prior contractual

[9]this section is largely based upon MAS-COLELL ET AL., 1995, pp. 128 - 136.

commitment $\rightarrow F(.)$: transformation function, $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$ and $F(0) = 0$ if and only if y is an element of the boundary of Y;

$\{y \in \mathbb{R}^L : F(y) = 0\} \rightarrow$ boundary $\equiv$ transformation frontier;

$MRT_{lk}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_l}}{\frac{\partial F(\bar{y})}{\partial y_k}}$, $\forall l, k$, $l \neq k$ goods, marginal rate of transformation of good l for good k at $\bar{y}$;

$\frac{\partial F(\bar{y})}{\partial y_k} \cdot dy_k + \frac{\partial F(\bar{y})}{\partial y_l} \cdot dy_l = 0$

$\mathbf{z} = (z_1, \ldots, z_{L-M}) \geq 0 \rightarrow$ firm's $L - M$ inputs;

$\mathbf{q} = (q_1, \ldots, q_M) \geq 0 \rightarrow$ outputs;

single output technology $\rightarrow f(z) \rightarrow$ max. amount of q that can be produced using input amounts $\mathbf{z} = (z_1, \ldots, z_{L-1}) \geq 0$; if the output is good $L \rightarrow Y = \{(z_1, \ldots, z_{L-1}, q) : q - f(z_1, \ldots, z_{L-1}) \leq 0$ and $(z_1, \ldots, z_{L-1}) \geq 0\}$;

$MRTS_{lk}(\bar{z}) = \frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}} \rightarrow$ marginal rate of technical substitution;

the cobb - douglas production function can be expressed as $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, and, if $\alpha + \beta = 1$ $\rightarrow f(z_1, z_2) = z_1^{\alpha} z_2^{1-\alpha}$

(i) **Y is non-empty**[10]$\rightarrow$ the firm has something to plan to do;

(ii) **Y is closed** $\longrightarrow$ the set Y includes its boundary, the limit of a sequence of feasible input - output vectors is also feasible; $y^n \rightarrow y$ and $y^n \in Y \rightarrow y \in Y$;

(iii) **no free lunch** $\longrightarrow y \in Y$ and $y \geq 0$ so that y doesn't use any inputs; this property $\longrightarrow$ this production vector cannot produce output either; $Y \cap \mathbb{R}^L_+ \subset \{0\}$

it's not possible to produce something out of nothing;

(iv) **possibility of inaction** $\longrightarrow 0 \in Y$; the point in time at which production possibilities are being analyzed is often important for the validity of this assumption; if we see a firm that could access a set of technological possibilities but hasn't yet been organized $\longrightarrow$ inaction is clearly possible; but otherwise (decisions already taken or irrevocable contracts signed), inaction isn't possible $\rightarrow$ sunk costs;

the firm is already committed to use at least $-\bar{y}_1$ units of good 1;

$\searrow$ the set is a restricted production set, reflecting the firm's remaining choices from some original production set Y like the ones in the previous graphs;

v. free disposal $\rightarrow$ holds if the absorption of any additional amount of inputs without any $\searrow$ in output is always possible, if $y \subset Y$ and $y' \leq y$ (so that y' produces at most the same amount of outputs using at least the same amount of inputs) $\rightarrow y' \in Y$; $Y - \mathbb{R}^L_+ \subset Y \leftrightarrow$ the extra amounts of inputs (or outputs) can be disposed of or eliminated at no cost;

vi. irreversibility $\rightarrow y \in Y$ and $y \neq 0$;

$-y \notin Y$; it's impossible to reverse a technologically possible production vector to transform an amount of output into the same amount of input that was used to generate it;

drawing 5 — drawing 6 — drawing 7

vii. non $\nearrow$ returns to scale $\rightarrow$ the production technology Y exhibits non $\nearrow$ returns to scale if for any $y \in Y$, we've $\alpha y \in Y$ for all scalars $\alpha \in [0, 1]$;

any feasible input - output vector can be scaled down;

viii. non $\searrow$ returns to scale $\rightarrow$ if $\forall y \in Y \rightarrow \alpha y \in Y$ for any scale $\alpha \geq 1$. any feasible input - output vector can be scaled up;

ix. constant returns to scale $\rightarrow$ the production set Y exhibits constant returns to scale if $y \in Y \rightarrow \alpha y \in Y$, for any scalar $\alpha \geq 0$. Y is a cone;

for single output technologies $\rightarrow$ properties of the production set translate into properties of the production function, $f(.)$; Y satisfies constant returns to scale if and only if $f(.)$ is homogeneous of degree 1. $f(2z_1, 2z_2) = 2^{\alpha+\beta} z_1^{\alpha} z_2^{\beta} = 2^{\alpha+\beta} f(z_1, z_2)$;

[10] the production set.

when $\alpha + \beta < 1 \rightarrow \searrow$ returns to scale;
when $\alpha + \beta = 1 \rightarrow$ constant returns to scale;
when $\alpha + \beta > 1 \rightarrow \nearrow$ returns to scale; ■

x. additivity → or free entry $\rightarrow y \in Y$ and $y' \in Y \rightarrow y + y' \in Y \leftrightarrow Y + Y \subset Y \rightarrow$ e.g. $ky \in Y$, $\forall k \in \mathbf{N}_+$; output here is available only in integer amounts. perhaps because of indivisibilities, the economic interpretation is that both y and y' are possible $\rightarrow$ one can set up two plants that don't interfere with each other and carry out production plans y and y' independently. the result is the production vector $y + y'$;

additivity $\rightarrow$ entry: if a firm produces $y \in Y \rightarrow$ net result $\rightarrow y+y' \rightarrow$ the aggregate production set $\rightarrow$ must satisfy additivity when ever unrestricted entry or free entry is possible;

xi. convexity $\rightarrow$ one of the fundamental assumptions of micro-economics $\rightarrow$ production set Y is convex $\rightarrow$ if $y, y' \in Y$ and $\alpha \in [0,1]$, $\alpha y + (1-\alpha)y' \in Y \rightarrow$ non $\nearrow$ returns, if inaction is possible, i.e. if $0 \in Y \rightarrow$ convexity $\rightarrow Y$ has non increasing returns to scale; hence if any $\alpha \in [0,1]$ $\rightarrow \alpha y = \alpha y + 0(1-\alpha)$, if $y \in Y$ and $0 \in Y \rightarrow \alpha y \in Y$, by convexity;

"unbalanced" inputs combinations aren't more productive than balanced ones;

if production plans y and y' produce exactly the same amount of output but use $\neq$ input combinations $\rightarrow$ a production vector that uses a level of each input that's the average of the levels used in these two plans can do at least as well as either y or y'.

ex. 5.B.3[11]: Y is convex if $f(z)$ is concave. suppose Y is convex; $z, z' \in \mathbf{R}_+^{L-1}$ and $\alpha \in [0,1]$ $\rightarrow (-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. by convexity

$\{-[\alpha z + (1-\alpha)z], \alpha f(z) + (1-\alpha)f(z)\} \in Y$

by convexity $\alpha f(z) + (1-\alpha)f(z) \leq f[\alpha z + (1-\alpha)z] \rightarrow f(z)$ is concave

suppose $f(z)$ is concave.

$(q, -z) \in Y$, $(q', -z') \in Y$, $\alpha \in [0,1]$ $q \leq f(z)$ and $q' \leq f(z) \rightarrow \alpha q + (1-\alpha)q' \leq \underbrace{\alpha f(z) + (1-\alpha)f(z')}$

by concavity

$\rightarrow \underbrace{\alpha f(z) + (1-\alpha)f(z')} \leq f[\alpha z + (1-\alpha)z']$ $\alpha q + (1-\alpha)q' \leq f[\alpha z + (1-\alpha z')]$

$\rightarrow \{-[\alpha z + (1-\alpha)z'], \alpha q + (1-\alpha)q\} = \alpha(-z, q) + (1-\alpha)(-z', q') \in Y$

$\rightarrow Y$ is convex. ■

xii. Y is a convex cone $\rightarrow$ convexity $\cap$ CRS. if for any production vector $y, y' \in Y$ and constants $\alpha \geq 0$ and $\beta \geq 0 \rightarrow \alpha y + \beta y' \in Y$.

proposition 5.B.1 the production set Y is additive and satisfies the non $\nearrow$ returns condition iff it's a convex cone.

proof $\alpha y + \beta y' \in Y$; $k > \max\{\alpha, \beta\}$,
$ky \in Y$, $ky' \in Y$; $\frac{\alpha}{k} < 1$ and $\alpha y = \frac{\alpha}{k} ky \rightarrow \alpha y \in Y$, similarly for β.

feasible input - output combination can be scaled down, and simultaneous operation of several technologies w/out mutual interference is possibile $\rightarrow$ convexity! production set $\rightarrow$ technology. $\searrow$ returns reflect the scarcity of an underlying, unlisted input of production.

proposition 5:B.2: for any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant return convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y\}$

additional input $\rightarrow$ entrepreneurial factor - whose return's precisely the firm's profit. $Y' = \{y' \in \mathbb{R}^{L+1} : y' = \alpha(y, -1)$ for some $y' \in Y$ and $\alpha \geq 0\}$.

[11] of Mas-Colell, Whinston, Green, 1995.

6 Estimation of the exponents of a Cobb-Douglas

I set up an empirical estimation of a Cobb-Douglas production function with the data of the firm which I have at hand; the function is of the form:

$$Y_{it} = AK_{it}^{\alpha}L_{it}^{\beta}e^{\varepsilon_{it}}$$

where the variables Y_{it}, K_{it} and L_{it}, for $i = 1, 2, 3, 4$ and $t = 2011, \ldots, 2021$, represent, respectively, revenues, capital and labour cost; ε is a mean zero iid random shock to productivity, which is unobserved by both the econometrician and the firms' managers; Y_{it}, a flow variable, is as of the ones resulting from the income statements; K_{it}, is here defined as the balance sheet value of net fixed assets; while L_{it} is intended as the sum of both productive and administrative labour cost, inclusive of the social security contributions; the data are composed of observations spanning the period t = 2011,. . . ,2021, as in the previous exercise.

A is the so called technological augmenting factor, namely a measure of the technological intensity of the productive process of the firm at hand.

My main point is connected with the exercise 3.B.1 of the Mas-Colell et al., 1995 handbook: I attempt to find out whether the sum of the estimated exponents α and β is $\gtreqless 1$, in order to conclude whether the returns to scale are respectively increasing, constant or decreasing.

In the appendix, I display, for the sake of understanding, some additional three dimensional graphs made with MATLAB 2022a, which may give an indication of the geometric properties of the production set of the firm which we are analyzing.

I first wrote a stata code that was based on the estimation of the Cobb-Douglas in level, thus with the command `nl`, non linear least squares; but then, I show in the paper only the estimates of the production function transformed in logarithms, which seems more meaningful, with two version, one with just capital and labour as inputs, another one also including materials purchases as an intermediate input as in Olley and Pakes 1996 as well as in Levinsohn and Petrin 2003, among others.

I do the estimation separately for each firm, of equations (6) and (7), and I also try an instrumental variable method, where the instruments are lagged values of labour and materials, given the endogeneity of input choices in a simple production function model.

$$\ln Y_{it} = \ln A + \alpha \ln K_{it} + \beta \ln L_{it} + \varepsilon_{it} \tag{6}$$

However, the four firms that we are considering here are heavily relying on the supply of raw materials, metals for firms 1, 3 and 4 and plastic components for firm 2, which uses it to create moulds. This led me to consider a version of the Cobb-Douglas production function which also includes an additional factor of production: purchase cost of raw materials, variable also extrapolated from the analytical income statements of the firms at hand, of which we dispose, as part of our profession[12]. Such new equation is:

$$Y_{it} = AK_{it}^{\alpha}L_{it}^{\beta}M_{it}^{\gamma}e^{\nu_{it}}$$

As usual, I log transform the equation, to obtain the second important estimating equation of the Cobb-Douglas production function:

$$\ln Y_{it} = \ln A + \alpha \ln K_{it} + \beta \ln L_{it} + \gamma \ln M_{it} + \nu_{it} \tag{7}$$

The determinant of the analysis is whether

$$\hat{\alpha} + \hat{\beta} + \hat{\gamma} \longrightarrow \begin{cases} > 1 & \longrightarrow \nearrow \text{ returns to scale} \\ = 1 & \longrightarrow \text{constant retursn to scale} \\ < 1 & \longrightarrow \searrow \text{ returns to scale} \end{cases}$$

[12] Borowski and Borwein, 1989.

Tabel 9: Estimates for the returns to scale of FIRM 1 with OLS with and without intermediate inputs (materials) and with the INSTRUMENTAL VARIABLES method where the instruments are the lagged labour and materials

variables	{OLS_1} REVENUES	{OLS_2} REVENUES	{IV_3} REVENUES
CAPITAL	-0.0471	-0.0962	-0.0889
	(0.219)	(0.132)	(0.373)
LABOUR	1.303***	0.261	-0.775
	(0.344)	(0.336)	(2.272)
MATERIALS		0.564***	1.370
		(0.144)	(1.408)
constant	-2.332	4.841	7.889
	(6.433)	(4.260)	(21.71)
observations	11	11	10
R-squared	0.682	0.901	0.020

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

6.1 Firm 1

I hereby report the results of the estimation of equation 6 and 7 as well as of an instrumental variable two stages least squares regression where the instruments are lagged labour and materials, following Levinsohn and Petrin 2003 and Ackerberg et al. 2015.

In the first case, eq. (1), table 9, column 1, I find that $\hat{\alpha} + \hat{\beta} = 1.2559 \rightarrow\ > 1 \rightarrow$ increasing returns to scale; in the second case, eq. (2), table 9, column 2, I find that $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 0.7288 \rightarrow$ decreasing returns to scale.

The two estimated equations are thus:

$$\ln Y = \underset{(6.433)}{-2.332} - \underset{(0.219)}{0.0471} \ln K + \underset{(0.344)}{1.303} \ln L$$

$$\ln Y = \underset{(4.260)}{4.841} - \underset{(0.132)}{0.0962} \ln K + \underset{(0.336)}{0.261} \ln L + \underset{(0.144)}{0.564} \ln M$$

$\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are equal to the shares of revenues due to capital, labour, and materials respectively; the marginal product of each factor is proportional to its mean product.

Here, for firm 1, strangely, returns to scale switch from increasing to decreasing after the inclusion of raw materials as further input of production.

In the case of the IV estimation, I find a very high, though not statistically significant, coefficient for materials, of about 1.37, but a negative coefficient for both capital and labour.

Overall the IV estimation would lead me to accept the hypothesis of decreasing returns, though, consistently with the estimation of the Cobb-Douglas in logarithms with capital, labour and raw materials.

6.2 Firm 2

In the first case, eq. (1), table 10, column 1, I find that $\hat{\alpha} + \hat{\beta} = 1.237 \rightarrow\ > 1 \rightarrow$ increasing returns to scale; in the second case, eq. (2), column 2, I find that $\hat{\alpha}+\hat{\beta}+\hat{\gamma} = 1.3277 \rightarrow$ increasing returns to scale as well.

Tabel 10: Estimates for the returns to scale of FIRM 2 with OLS with and without intermediate inputs (materials) and with the INSTRUMENTAL VARIABLES method where the instruments are the lagged labour and materials

	{OLS_1}	{OLS_2}	{IV_3}
variables	REVENUES	REVENUES	REVENUES
CAPITAL	0.106	0.0177	0.0462
	(0.338)	(0.206)	(0.282)
LABOUR	1.131***	0.664***	0.739
	(0.224)	(0.182)	(0.588)
MATERIALS		0.646***	0.580
		(0.168)	(0.587)
constant	-2.015	-3.504	-4.023
	(6.499)	(3.958)	(5.827)
observations	11	11	10
R-squared	0.772	0.927	0.925

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Here, for firm 2, I find that the returns to scale are increasing with both the Cobb-Douglas specification, which leads me to conclude that firm 2 has indeed increasing returns to scale; perhaps this may be due to the fact that it operates in the luxury industry, and is able to sell its products to great brands of women's fashion which extract a significant surplus to the final customers.

The two former estimated equations thus look like

$$\ln Y = \underset{(6.499)}{-2.015} + \underset{(0.338)}{0.106} \ln K + \underset{(0.224)}{1.131} \ln L$$

$$\ln Y = \underset{(3.957)}{-3.504} + \underset{(0.206)}{0.0177} \ln K + \underset{(0.182)}{0.664} \ln L + \underset{(0.168)}{0.646} \ln M$$

The third estimated equation does nothing but confirming the conjecture arising from the former two.

6.3 Firm 3

In the first vccase, eq. {1}, table 11, column 1, I find that $\hat{\alpha} + \hat{\beta} = 1.072 \rightarrow > 1 \rightarrow$ slightly more than increasing returns to scale; in the second case, eq. {2}, column 2, I find that $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 0.6776 \rightarrow$ decreasing returns to scale.

Here, as in the case of firm 1, I find first increasing returns to scale and then decreasing ones, after the inclusion of raw materials as an additional input of production; the estimated simultaneous equations are therefore:

$$\ln Y = \underset{(5.4288)}{0.151} - \underset{(0.0611)}{0.0120} \ln K + \underset{(0.341)}{1.084^{**}} \ln L$$

$$\ln Y = \underset{(4.492)}{-5.639} - \underset{(0.0458)}{0.0192} \ln K + \underset{(0.435)}{0.0808} \ln L + \underset{(0.217)}{0.616^{**}} \ln M$$

This result means that a marginal increase in one of the two productive inputs (K or L) would lead, in these conditions (those holding, on average, in the past twelve years) to an increase in

Tabel 11: Estimates for the returns to scale of FIRM 3 with OLS with and without intermediate inputs (materials) and with the INSTRUMENTAL VARIABLES method where the instruments are the lagged labour and materials

	{OLS$_1$}	{OLS$_2$}	{IV$_3$}
variables	REVENUES	REVENUES	REVENUES
CAPITAL	-0.0120	-0.0192	0.0149
	(0.0611)	(0.0458)	(0.162)
LABOUR	1.084**	0.0807	-2.102
	(0.341)	(0.435)	(8.346)
MATERIALS		0.616**	2.333
		(0.217)	(5.698)
constant	0.151	5.640	11.25
	(5.428)	(4.492)	(39.07)
observations	12	12	11
R-squared	0.593	0.798	

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

output (here measured as revenues, but one could also opt for other monetary measures such as the value added[13]) slightly more than proportional to the increase in the input (or factor) of production; adding materials as an intermediate input to the estimation, lead us to conclude an opposite fact, namely that an increase in one of the three inputs (K, L or M) leads to an increase in revenues (our monetary proxy for output) less than proportional than the increase in inputs.

6.4 Firm 4

In the first case, eq. {1}, table 12, column 1, I find that $\hat{\alpha} + \hat{\beta} = 1.894 \rightarrow > 1 \rightarrow$ significantly increasing returns to scale; in the second case, eq. {2}, column 2, I find that $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 0.374$ $\rightarrow$ significantly decreasing returns to scale.

While in the first specification I find very high returns to scale, in the second one, including raw materials, I find very low returns to scale - I do not know which could be the source of such a polarization, perhaps the fact that materials are costly to treat and to transform into a finite product.

Therefore, based on the previous results, the two Cobb-Douglas in logarithms for firm 4 are[14]:

$$\ln Y = -\underset{(2.818)}{11.25}^{***} + \underset{(0.135)}{0.235} \ln K + \underset{(0.248)}{1.659}^{**} \ln L$$

$$\ln Y = \underset{(6.086)}{9.020} - \underset{(0.146)}{0.158} \ln K + \underset{(0.209)}{1.248}^{***} \ln L - \underset{(0.203)}{0.716}^{***} \ln M$$

The IV estimates more or less confirm the predictions generated by the estimation of equations (6) and (7).

[13]Francesco Giovanardi and Marco Luca Pinchetti pointed out that it would be more appropriate to consider as a measure of both inputs and outputs of production a physical measure of the units employed, such as hours worked and amount of pieces of product which have been generated; unfortunately, we do not this dispose of this piece of information, therefore, we have to proxy these variable with their corresponding monetary counterpart, as pointed out in ACKERBERG ET AL. 2015.

[14]Standard errors in parentheses.

Tabel 12: Estimates for the returns to scale of FIRM 4 with OLS with and without intermediate inputs (materials) and with the INSTRUMENTAL VARIABLE method where the instruments are the lagged labour and materials

	{OLS_1}	{OLS_2}	{IV_3}
variables	REVENUES	REVENUES	REVENUES
CAPITAL	0.235	-0.158	-0.462
	(0.135)	(0.146)	(0.259)
LABOUR	1.659***	1.248***	1.039***
	(0.248)	(0.209)	(0.311)
MATERIALS		-0.716***	-1.224**
		(0.203)	(0.390)
constant	-11.25***	9.020	22.63*
	(2.818)	(6.086)	(10.99)
observations	14	14	14
R-squared	0.888	0.950	0.918

standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

6.5 Panel data estimates

As a final exercise, I pool the data of the four firms at hand in a panel with about 48 observation to compare the different results which I obtain by applying the various methods of simple OLS, Levinsohn and Petrin 2003, panel data method with fixed effects, with random effects, an instrumental variable method, and a generalized method of moments estimation method, where the instruments for the endogenous inputs are their one period lagged values.

Tabel 13: comparison among OLS, Levinsohn and Petrin 2003, fixed effects, random effects and instrumental variables estimation methods of estimating a production function with inputs capital, labour and materials

	OLS	*LP*	*FE*	*RE*	*IV*	*GMM*
	REVENUES	REVENUES	REVENUES	REVENUES	REVENUES	REVENUES
CAPITAL	0.130***	0.0174	0.0567	0.130***	0.123**	.150**
	(0.0472)	(0.160)	(0.0531)	(0.0472)	(0.0486)	(0.0795)
LABOUR	1.197***	0.738***	1.420***	1.197***	1.230***	1.236***
	(0.0901)	(0.248)	(0.134)	(0.0901)	(0.108)	(0.0775)
MATERIALS	0.0364	2.12e-41	-0.0317	0.0364	0.0142	0.0187
	(0.0458)	(0.267)	(0.0902)	(0.0458)	(0.0545)	(0.0516)
CONST.	-3.887***		-5.044**	-3.887***	-3.954***	-4.480***
	(1.244)		(2.264)	(1.244)	(1.338)	(1.564)
observations	48	48	48	48	45	44
R-squared	0.942	χ^2=0.61	0.741	0.7216	0.7147	
number of id			4	4	4	4

comparison of six $\neq$ estimation methods
*** p<0.01, ** p<0.05, * p<0.1

LEVINSOHN AND PETRIN 2003 method leads me to more conservative estimates of the returns to scale of the four firms together, thus, pointing towards lower returns to scale, if compared with the simple OLS (baseline) method. As a matter of fact, with OLS, I obtain evidence in favour of increasing returns to scale for the sample at hand, considering capital, labour and raw materials as inputs. Given the structure of the pooled data, I experiment fixed and random effects, which both yield results in line with the OLS method, especially with regards to the random effects model. IV and GMM both lead to higher point estimates than the OLS. Additional results based on dynamic panel data models (the so called Arellano and Bond 1991 estimation method) are further illustrated as part of the replication codes.

7 Conclusions

I carried out a double empirical exercise with some balance sheet data of four firms operating in various manufacturing sectors in the neighbourhood of Bologna, Northern Italy. I first reviewed the theory and empirics of Cobb-Douglas production functions estimation with firm/sector level data, as well as that of cost functions estimation; then specified and estimated a set of five bivariate linear regressions of revenues on fixed and variable costs, both contemporaneous and lagged for the four firms at hand separately; I found evidence for positive impact of variable costs on revenues on impact on average for the whole four firms.

Afterwards, I reviewed a part of the theory of production following Mas-Colell et al., 1995, ch.5, stressing the relevance of the geometrical properties of production sets, with specific reference to the returns to scale with the Cobb-Douglas production function; in particular, assuming the Cobb-Douglas is a good fit in approximating the firm's production process, I tried to find out whether the returns to scale have been, for the past eleven/twelve years of operations, increasing, decreasing or constant, depending on the magnitude of the summed exponents estimates of the production function.

The results are mixed, depending on the industry in which the firm operate, and the inclusion in the log linear production function of materials as an intermediate input besided capital and labour; I also pool the data of the four firms together, trying different types of estimation: simple OLS as the baseline, Levinsohn and Petrin 2003 method[15], panel data fixed effects models, random effects, instrumental variable method and generalized method of moments. The most significant trend is the downward pressure on the estimates due to the LP 2003 method.

More research is needed however on this topic, perhaps on estimating the returns to scale of entire sectors of activity with coutry or region level data, as well as trying to estimate some other forms of production functions such as the constant elasticity of substitution (CES as in Ramsey and Zarembka 1971), of the form

$$y = [\delta_1 K^\rho + \delta_2 L^\rho]^{\nu/\rho} \tag{8}$$

or a generalized production function,

$$ye^{\gamma y} = AK^\alpha L^\beta \tag{9}$$

or a quadratic production function,

$$y = A + \alpha K + \beta L + \gamma K^2 + \delta L^2 + \lambda KL \tag{10}$$

[15] The Olley and Pakes 1996 method is inapplicable to our case, because we shall need a data set with many firms, and some entering and some exiting the market, which we do not have.

Supplementary information

The paper has an appendix with the STATA code and data set used to produce the estimation reported in the main text as well as the MATLAB code written to produce the figures contained in the paper.

Acknowledgments

I thank a collaborator in my studio who helped me making available the financial statement of the firm from which the data are taken.

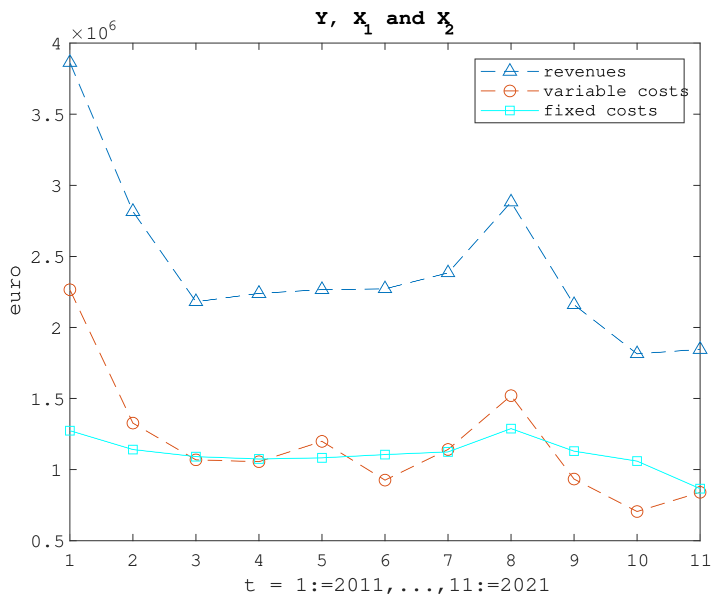
Declarations

- no conflict of interests
- consent for publication of the data
- STATA data set and codes available for replications
- MATLAB code available for replications

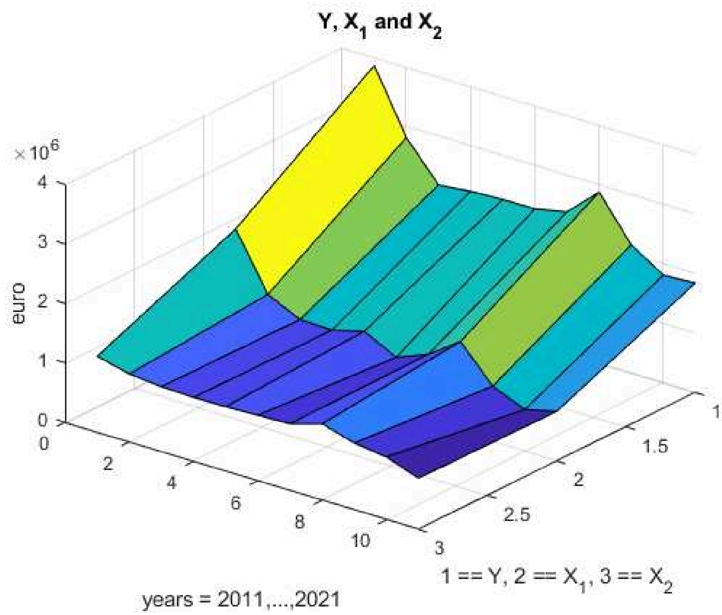
Appendix

An appendix contains supplementary information that is not an essential part of the text itself but which may be helpful in providing a more comprehensive understanding of the research problem or it is information that is too cumbersome to be included in the body of the paper.

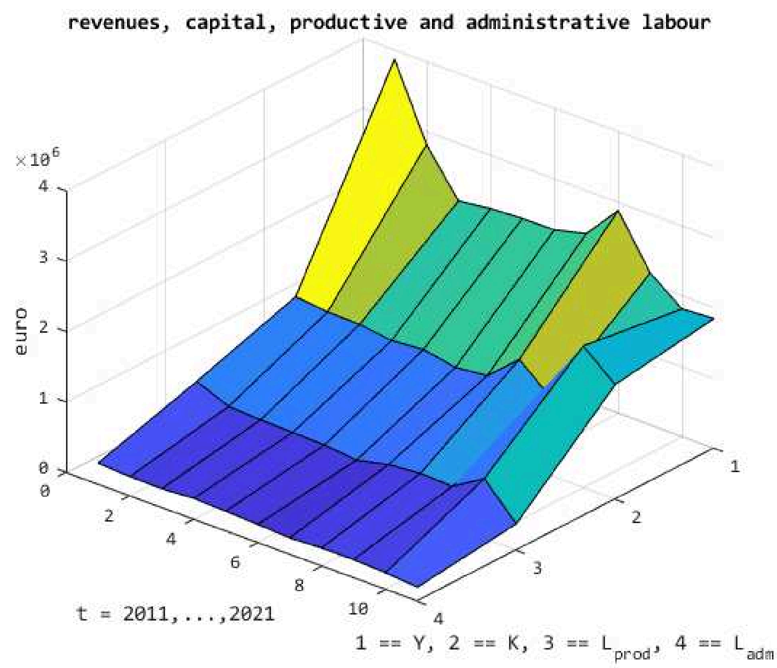
Figuur 1: revenues, variable and fixed costs for firm 1



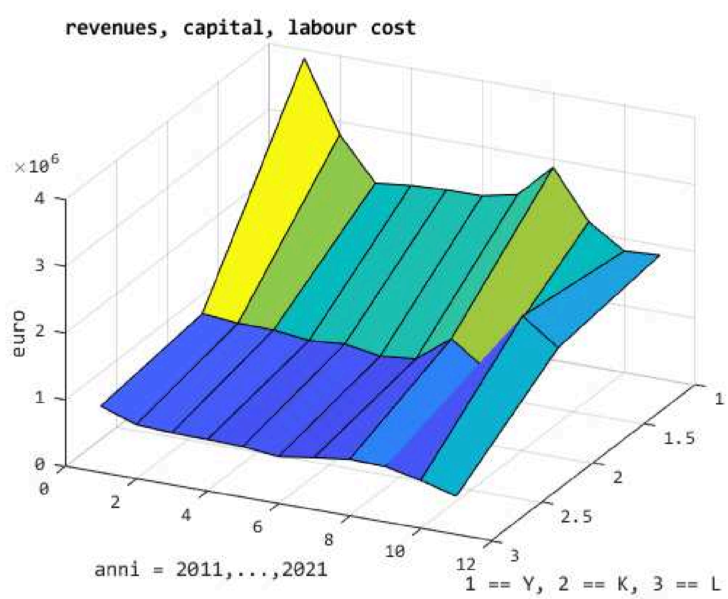
Figuur 2: revenues, variable and fixed costs for firm 1



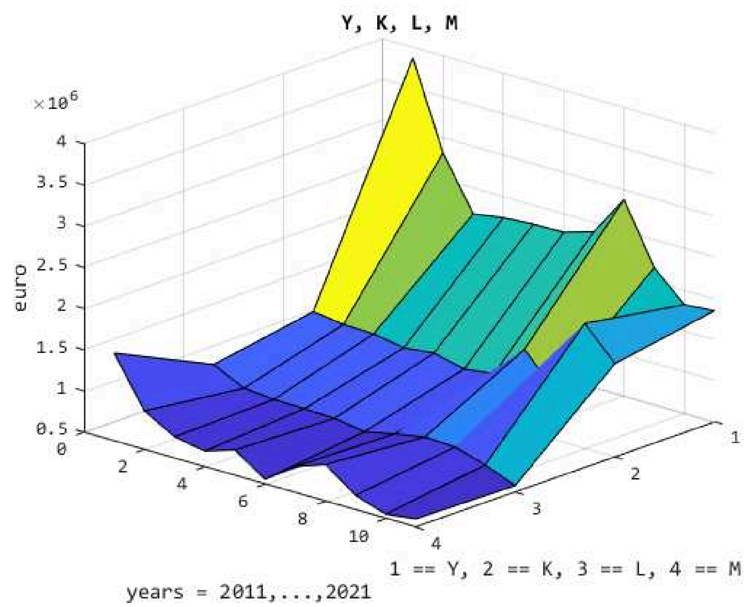
Figuur 3: revenues, capital, disaggregated labour cost for firm 1



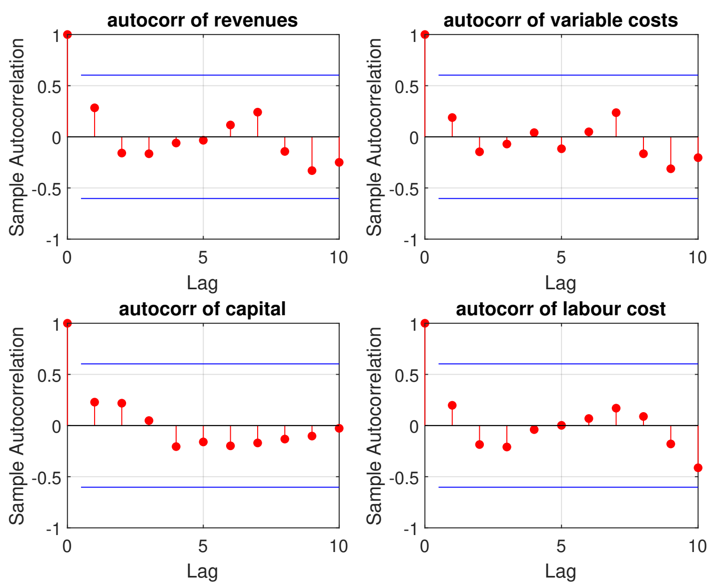
Figuur 4: revenues, capital, labour for firm 1



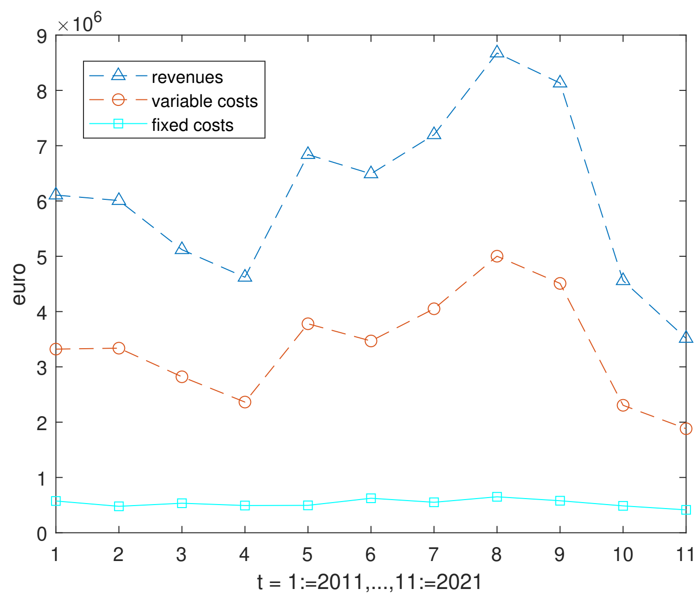
Figuur 5: revenues, capital, labour costs and purchases of raw materials for firm 1



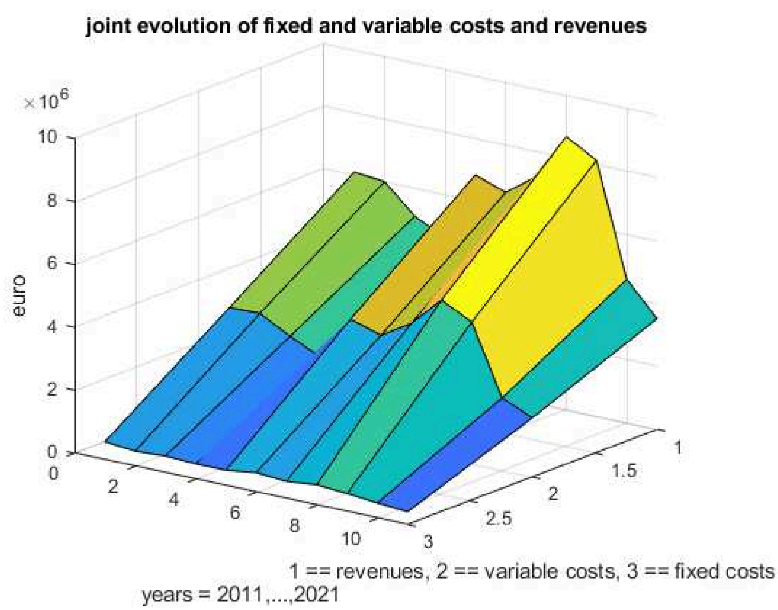
Figuur 6: sample autocorrelation functions for firm 1



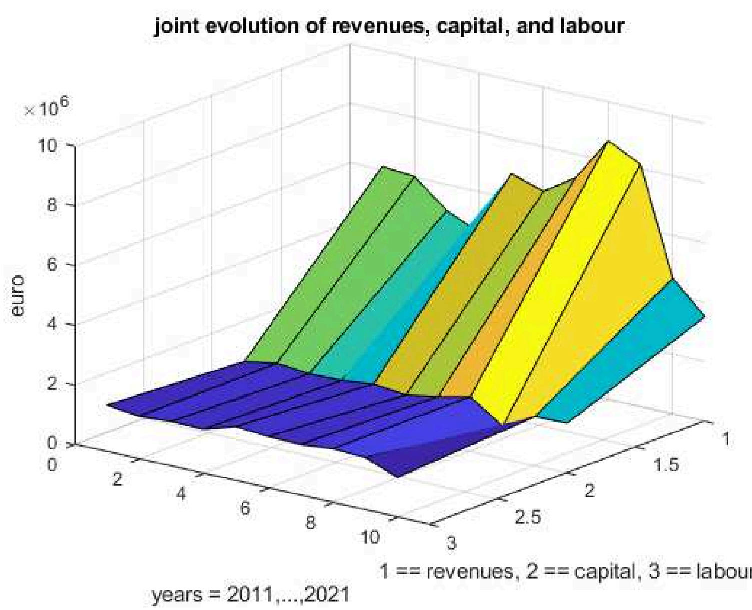
Figuur 7: revenues, fixed and variable costs for firm 2



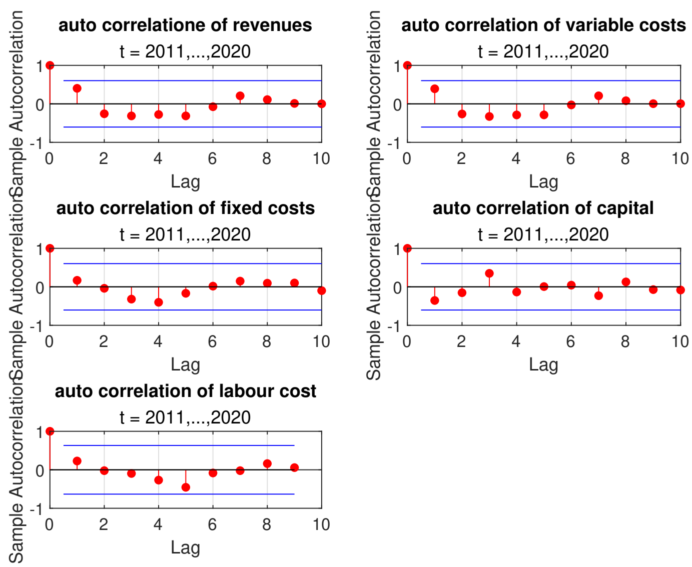
Figuur 8: revenues, fixed and variable costs for firm 2



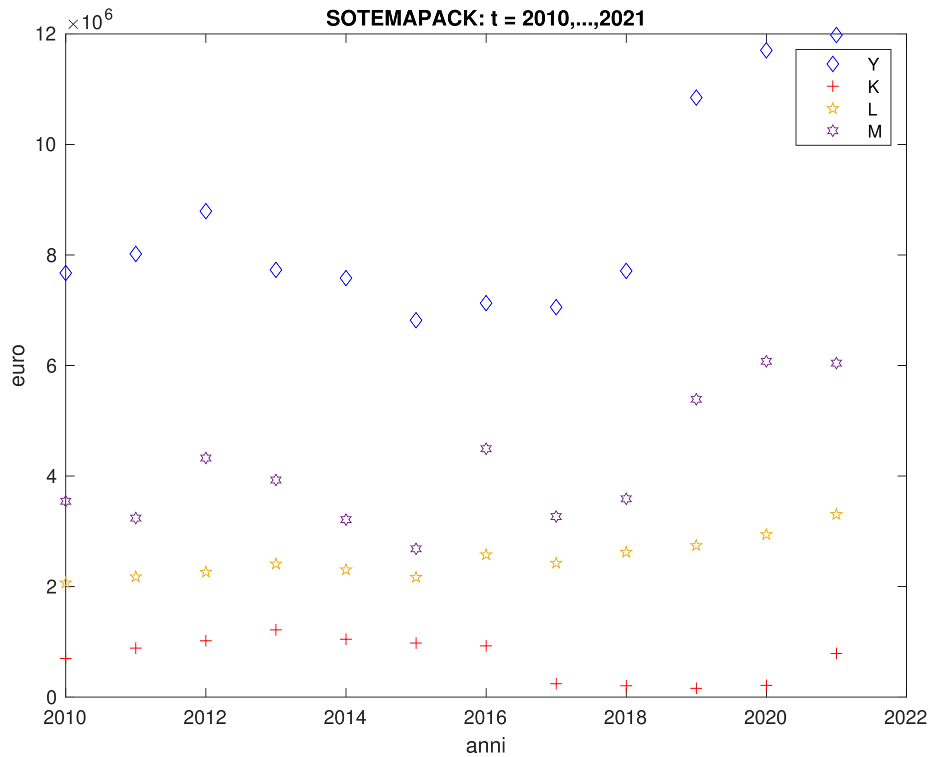
Figuur 9: revenues, capital and labour for firm 2



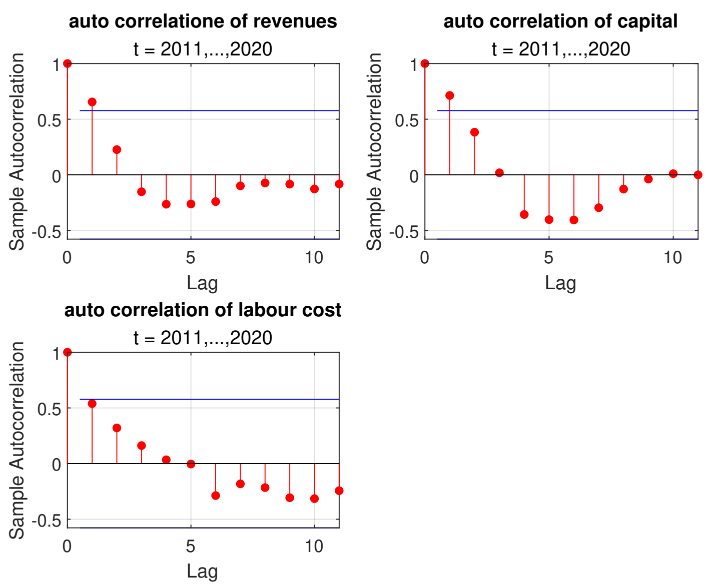
Figuur 10: sample auto - correlation functions for firm 2



Figuur 11: revenues, capital, labour and materials for firm 3



Figuur 12: auto correlation for revenues, capital, labour and materials for firm 3



A.1 Properties of the Cobb-Douglas

c. w. cobb - p. h. douglas, a theory of production, american economic review, march 1928[16]

$X = f(L, K, t)$

$X = aL^{\alpha}K^{\beta}e^{\gamma t}$

$\ln X = \ln a + \alpha \ln L + \beta \ln K + \gamma t$

α and β are equal to the shares of output received by labour and capital, respectively. the marginal product of each factor is proportional to its average product, since

$\frac{\partial X}{\partial L} = a\alpha L^{\alpha-1}L^{\beta}e^{\gamma t} = \alpha\frac{X}{L} \to \alpha = \frac{\partial X}{\partial L} \cdot \frac{L}{X}$

$\frac{\partial X}{\partial K} = aL^{\alpha}\beta K^{\beta-1}e^{\gamma t} = \beta\frac{X}{K} \to \beta = \frac{\partial X}{\partial K} \cdot \frac{K}{X}$

$\frac{\partial X}{\partial L} = \frac{w}{p}$ and $\frac{\partial X}{\partial K} = \frac{r}{p}$, where

$w \to$ wage rate; $r \to$ unit price of capital,

$p \to$ product price level, for each purely competitive, profit maximizing firm.

$\to \quad \underbrace{\alpha = \frac{wL}{pX}}_{\text{labour share of output}}$ and $\underbrace{\beta = \frac{rK}{pX}}_{\text{capital share of output}}$ in current prices.

α and β must sum to unity under this last assmpt., since $wL + rK = pX$.

$\to$ unduly restrictive functional form-
capital/labour ration $\leftrightarrow$ strictly proportional
to the factor price ratio;

$\frac{\alpha}{\beta} = \frac{\frac{\partial X}{\partial L}\cdot\frac{L}{X}}{\frac{\partial X}{\partial K}\cdot\frac{K}{X}} = \frac{\partial K}{\partial L} \cdot \frac{L}{K} \to \frac{\alpha}{\beta}\frac{\partial L}{\partial K} = \frac{L}{K} \leftrightarrow$

$\frac{\beta}{\alpha} \cdot \frac{\partial K}{\partial L} = \frac{K}{L}$; under cost minimization

$\frac{\partial X}{\partial L} \cdot \frac{1}{w} = \frac{\partial X}{\partial K}\frac{1}{r} \leftrightarrow \frac{\partial K}{\partial L} = \frac{w}{r} \to$

$\to \frac{K}{L} = \frac{\beta}{\alpha} \cdot \frac{w}{r} \to \ln\left(\frac{K}{L}\right) = \ln\left(\frac{\beta}{\alpha}\right) - \ln\left(\frac{\partial L}{\partial K}\right)$

$\frac{d\left(\ln\frac{K}{L}\right)}{d\left(\ln\frac{\partial L}{\partial K}\right)} = -1 \to$ elasticity of substitution $\to \sigma$. the cobb - douglas is said to have a unitary elasticity of substitution.

CES $\to$ constant elasticity of substitution $\to \sigma$ can take any constant value. k. j. arrow, h. b. cherney, b. s. minhas, and r. m. solow, capital - labour substitution and economic efficiency; review of economics and statistics, vol. 43, no. 3, august 1961, pp. 225 - 250.

$\ln\left(\frac{X}{L}\right) = \ln a + \sigma \ln w$;

$\sigma :=$ elasticity of substitution;

pure competition and π max $\to$ constant returns to scale $\to$

$X = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{1}{\rho}}$

$\gamma \to$ efficiency parameter $\to$ scale factor;

$\delta \to$ distribution parameter $\to a^{-1/\sigma} \cdot \gamma^{\frac{\sigma}{1-\sigma}}$

$\rho \to$ substitution parameter $\to \frac{1}{\sigma} - 1$

***

derivation of the CES production function $\to$ l. r. klein and r. s. preston, the measurement of capacity utilization, american economic review- march 1967.

$\sigma = 1 \to$ cobb - douglas function

$\sigma = \infty \to$ linear production function

$\sigma = 0 \to$ fixed - proportion or leontief function (right - angle isoquants).

[16] This section is based on Michael K. Evans, 1969, chapter 10, Aggregate Supply Components and Factor Shares.

$\ln\left(\frac{X}{L}\right) = \ln a + \sigma \ln w$

$X = wL + rK,$

$w = \frac{X}{L} - r\frac{K}{L} = \frac{X}{L} - \frac{dX}{dK}\cdot\frac{K}{L}$

$\to \ln\left(\frac{X}{L}\right) = \ln a + \sigma\ln\left(\frac{X}{L} - \frac{dX}{dK}\cdot\frac{K}{L}\right)$

$\frac{X}{L} = a\left(\frac{X}{L} - \frac{dX}{dK}\cdot\frac{K}{L}\right)^{\sigma}$

$\left(\frac{X}{L}\right)^{1/\sigma} = a^{1/\sigma}\cdot\frac{X}{L} - a^{1/\sigma}\cdot\frac{K}{L}\cdot\frac{d(X/L)}{d(K/L)}$

$\to \frac{d(X/L)}{d(K/L)} = -\frac{a^{-\frac{1}{\sigma}}(X/L)^{1/\sigma}}{K/L} + \frac{X/L}{K/L}$

$\searrow \frac{d(X/L)}{d(K/L)} = \frac{X/L - a^{-1/\sigma}(X/L)^{1/\sigma}}{K/L}$

$$\searrow \underbrace{\frac{d(X/L)}{X/L - a^{-1/\sigma}\cdot\left(X/L\right)^{1/\sigma}}}_{\downarrow} = \frac{d(K/L)}{K/L}\,()$$

$\frac{d(X/L)}{X/L} + \frac{[a^{-1/\sigma}(X/L)^{\frac{1}{\sigma}-2}]d(X/L)}{1-a^{-1/\sigma}\cdot(X/L)^{1/\sigma-1}}$

to integrate the second term on the right hand side

$1 - a^{-\frac{1}{\sigma}}\cdot\left(\frac{X}{L}\right)^{\frac{1}{\sigma}-1} = z$

$\searrow -a^{-\frac{1}{\sigma}}\cdot\left(\frac{1}{\sigma} - 1\right)\left(\frac{X}{L}\right)^{\frac{1}{\sigma}-2}\cdot d\left(\frac{X}{L}\right) = dz$

$\int \frac{1}{X/L}d\left(\frac{X}{L}\right) - \int\frac{\sigma}{1-\sigma}\frac{dz}{z} = -\int\frac{1}{K/L}d\left(\frac{K}{L}\right)$

$\ln\left(\frac{X}{L}\right) - \frac{\sigma}{1-\sigma}\ln\left[1-a^{-\frac{1}{\sigma}}\left(\frac{X}{L}\right)^{\frac{1}{\sigma}-1}\right] = \ln\left(\frac{K}{L}\right) - \frac{\sigma}{1-\sigma}\ln\beta$

$\searrow \frac{K}{L} = \frac{X}{L}\left[1-a^{-\frac{1}{\sigma}}\left(\frac{X}{L}\right)^{\frac{1-\sigma}{\sigma}}\right]^{-\frac{\sigma}{1-\sigma}}\cdot\beta^{\frac{\sigma}{1-\sigma}}$

$\leftrightarrow \frac{K}{L} = \frac{X}{L}\left[1-\alpha\left(\frac{X}{L}\right)^{\rho}\right]^{-\frac{1}{\rho}}\cdot\beta^{\frac{1}{\rho}}$

$\rho = \frac{1-\sigma}{\sigma}$; $\alpha = a^{-1/\sigma}$; $\beta \to$ a constant of integration.

$\left(\frac{K}{L}\right)^{\rho} = \left(\frac{X}{L}\right)^{\rho}\cdot\left[1-\alpha\left(\frac{X}{L}\right)^{\rho}\right]^{-1}\cdot\beta$

$\beta\left(\frac{K}{L}\right)^{-\rho} = \left(\frac{X}{L}\right)^{-\rho}\cdot\left[1-\alpha\left(\frac{X}{L}\right)^{\rho}\right]$

$\beta\left(\frac{K}{L}\right)^{-\rho} = \left(\frac{X}{L}\right)^{-\rho} - \alpha$

$\left(\frac{X}{L}\right)^{\rho} = \beta\left(\frac{K}{L}\right)^{-\rho} + \alpha$

$\left(\frac{X}{L}\right) = \left[\beta\left(\frac{K}{L}\right)^{-\rho} + \alpha\right]^{-\frac{1}{\rho}}$

$X = \left(L^{-\rho}\right)^{-\frac{1}{\rho}}\cdot\left[\beta\left(\frac{K}{L}\right)^{-\rho} + \alpha\right]^{-\frac{1}{\rho}}$

$X = \left[\beta K^{-\rho} + \alpha L^{-\rho}\right]^{-\frac{1}{\rho}}$

if we choose an efficiency parameter γ, and let $\delta = \alpha\gamma^{\rho-1}$, the function can be rewritten as

$$X = \gamma\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\frac{1}{\rho}}$$

$$= \gamma\left[\alpha\gamma^{\rho-1}K^{-\rho} + (1+\alpha\gamma^{\rho-1})L^{-\rho}\right]^{-\frac{1}{\rho}}$$

when $\sigma = 1$, integrate the function

$\ln\left(\frac{X}{L}\right) = \ln a + \ln w$

$\frac{X}{L} = a\left(\frac{X}{L} - \frac{dX}{dK} \cdot \frac{K}{L}\right)$

$\frac{X}{L} \cdot a^{-1} = \frac{X}{L} - \frac{dX}{dK} \cdot \frac{K}{L}$

$\frac{dX}{dK} \cdot \frac{K}{L} = \frac{X}{L} - \frac{X}{L} \cdot a^{-1}$

$\frac{dX}{dK} = \frac{X/L}{K/L} - \frac{X/L}{K/L} \cdot a^{-1}$

$\frac{d(X/L)}{d(K/L)} = \frac{X/L}{K/L} - \frac{X/L}{K/L} \cdot a^{-1}$

$\frac{d(X/L)}{d(K/L)} = \frac{X/L}{K/L} \cdot \left[1 - a^{-1}\right]$

$\frac{d(X/L)}{X/L \cdot [1-a^{-1}]} = \frac{d(K/L)}{K/L}$

$\frac{a}{a-1} \cdot \ln\left(\frac{X}{L}\right) = \ln\left(\frac{K}{L}\right) + \frac{a}{a+1} \ln \beta$

$\beta \in \mathbb{R}$, constant of integration.
$\searrow \left(\frac{X}{L}\right)^{\frac{a}{a-1}} = \frac{K}{L} \cdot \beta^{\frac{a}{a-1}}$

$\frac{X}{L} = \beta \cdot \left(\frac{K}{L}\right)^{\frac{a-1}{a}}$

$X = \beta \cdot L^{\frac{1}{a}} \cdot K^{\frac{a-1}{a}}$
$\rightarrow$ the cobb - douglas pro=
duction function w/cons=
tant returns to scale

the fixed proportional leontief is even simpler; $\sigma = 0 \rightarrow$
$\ln\left(\frac{X}{L}\right) = \ln a$ or $\frac{X}{L} = a \rightarrow X = aL$,
for $K > K_0$; also we can write
$\ln\left(\frac{X}{K}\right) = \ln b$ or $\frac{X}{K} = b \rightarrow X = bK$
for $L > L_0 \rightarrow$ when $\exists$ excess capital,
output is uniquely determined by labour
and viceversa $\rightarrow$ right angled isoquant for
which profit maximizing firms produce
only at the corner. a change in factor
prices for a given output will then have no ef=
fect on the ratio of factors demanded. $\nexists$
any substitutability of one factor for ano=
ther relative prices change.
as $\sigma \rightarrow \infty \rightarrow$ the same method of integration
cannot be used; $\rho = \frac{1}{\sigma} - 1$, $\sigma \rightarrow \infty \rightarrow \rho = -1$
$\searrow X = (\beta K^{-\rho} + \alpha L^{-\rho})^{-\frac{1}{\rho}} \rightarrow$ becomes
$X = \beta K + \alpha L$

– • –

the CES function is more general than the cobb - douglas because it can take on any real value. also, factor shares need not be constant as factor intensities vary.

$X^{-\rho} = \gamma^{-\rho}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]$

differentiate

$-\rho X^{-\rho-1} \cdot \frac{\partial X}{\partial L} = \gamma^{-\rho}[-\rho(1-\delta)L^{-\rho-1}]$

$-\rho X^{-\rho-1} \cdot \frac{\partial X}{\partial K} = \gamma^{-\rho}[-\rho\delta K^{-\rho-1}]$

$\frac{\partial X}{\partial L} = (1-\delta)\left(\frac{X}{L}\right)^{\rho+1} \cdot \gamma^{-\rho}$

$\frac{\partial X}{\partial K} = \delta\left(\frac{X}{K}\right)^{\rho+1} \cdot \gamma^{-\rho}$

$\frac{\frac{\partial X}{\partial L}}{\frac{\partial X}{\partial K}} = \frac{\partial K}{\partial L} = \frac{1-\delta}{\delta} \cdot \left(\frac{K}{L}\right)^{\rho+1}$

$(\rho+1)\ln\left(\frac{K}{L}\right) = \ln\frac{1-\delta}{\delta} - \ln\left(\frac{\partial L}{\partial K}\right)$

$\frac{d(\ln K/L)}{d(\ln \partial L/\partial K)} = -\frac{1}{1+\rho} = -\sigma$

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