Sensitivity Analysis between Commodity and Budget: Utility Maximization Case

Mohajan, Devajit and Mohajan, Haradhan

Department of Civil Engineering, Chittagong University of Engineering Technology, Chittagong, Bangladesh, Department of Mathematics, Premier University, Chittagong, Bangladesh

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Devajit Mohajan
Department of Civil Engineering, Chittagong University of Engineering & Technology, Chittagong, Bangladesh
Email: devajit1402@gmail.com
Mobile: +8801866207021

Haradhan Kumar Mohajan
Department of Mathematics, Premier University, Chittagong, Bangladesh
Email: haradhan1971@gmail.com
Mobile: +8801716397232

Abstract

In this article, sensitivity analysis between commodity and total budget are discussed. The property of a commodity that enables it to satisfy human wants is called utility. In economics, utility maximization method is essential for the welfare of the organizations and society. This study deals with four commodities and two constraints, such as budget constraint, and coupon constraint. In this article, 6×6 Hessian and 6×10 Jacobian are operated for the sensitivity analysis. Throughout the paper scientific method of optimization are applied.

Keywords: Budget, Lagrange multipliers, sensitivity analysis, utility maximization

1. Introduction

Mathematical modeling in economics is the application of mathematics in economics to explain economic behavior of optimization (Samuelson, 1947). It plays an important role in modern economics for the development of global financial structure (Ferdous & Mohajan, 2022). In social sciences and mathematical economics, the property of a commodity that enables to satisfy human wants is called utility (Fishburn, 1970). In the society a rational individual wants to maximize his/her utility (Gauthier, 1975). The concept of utility was developed in the late 18th century by the English moral philosopher, jurist, and social reformer Jeremy Bentham (1748-1832) and English philosopher, political economist, Member of Parliament (MP) and civil servant John Stuart Mill (1806-1873) (Bentham, 1780; Chisholm, 1911). According to Bentham, utility is the tendency of an object or action to increase or decrease overall happiness (Bentham, 1780). Producers always want to increase the utility among the consumers (Mohajan, 2021a). In modern economics, utility is a measure of a consumer's preferences on an alternative set of commodities or services (Coleman & Fararo, 1992). Utility maximization policy is the best way for the sustainability of the organizations (Kirsh, 2017).

Lagrange multipliers method is a very useful and powerful practice in multivariable calculus. It has been used to facilitate the determination of necessary conditions. This method is considered as a device for transforming a constrained problem to a higher dimensional unconstrained problem (Baxley & Moorhouse, 1984; Islam et al., 2010). In a running firm, the achievement of maximum profit is depended on efficient use of inputs, factor shares in total output, degree of returns to scale, and moreover, on utility maximization (Khatun & Afroze, 2016). On the other hand, sensitivity analysis plays an important role to predict on future production of the commodities (Islam et al., 2010).

In this study we have tried to discuss the utility maximization policy of an organization. We have stressed on the sensitivity analysis between commodity and budget (Mohajan & Mohajan, 2022c,g). We have used both 6×6 bordered Hessian and 6×10 Jacobian to show the mathematical calculations of optimization clearly (Mohajan & Mohajan, 2022a,b).

2. Literature Review

The literature review is an introductory section of a scholarly research, which tries to indicate the contributions of other scholars in the same research area (Polit & Hungler, 2013). American economists John V. Baxley and John C. Moorhouse have discussed the utility maximization subject to a budget constraint. They have also provided a mathematical formulation for nontrivial constrained optimization problem with special reference to the application in economics (Baxley & Moorhouse, 1984). Distinguished mathematician Jamal Nazrul Islam and his coauthors have discussed utility maximization by considering reasonable interpretation of the two Lagrange multipliers (Islam et al., 2010, 2011). Jannatul Ferdous and Haradhan Kumar Mohajan have developed a profit maximization problem. In their article they have considered three inputs, such as capital, labor, and raw materials and other inputs (Ferdous & Mohajan, 2022). Lia Roy and her coauthors have analyzed cost minimization problem of an industry. In their study they have observed that for the sustainability of an industry, it should use the inputs efficiently, and run the industry through the green and sustainable environment (Roy et al., 2021).

F. Thomas Juster has provided a brief history of the development of utility theory. He has tried to simplify the conceptual structure at the cost of complicating the measurement problem (Juster, 1990). American mathematician Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976) have derived the functional distribution of income between capital and labor in 1928 (Cobb & Douglas, 1928). Pahlaj Moolio and his coworkers have introduced the Cobb-Douglas production function to determine the maximization of an output subject to a budget constraint (Moolio et al., 2009).

Haradhan Kumar Mohajan has considered the maximization of utility problem of consumers of Bangladesh subject to two constraints: budget constraint and coupon constraint (Mohajan, 2022). In two studies he has explored interpretation of Lagrange multiplier to predict the cost minimization policy using Cobb-Douglas production function. He tried to show the production of garments in minimum cost by using statistical analysis (Mohajan, 2021b,c). Devajit Mohajan and Haradhan Kumar Mohajan have discussed profit maximization problem, where they have used four variable inputs, such as capital, labor, principal raw materials, and other inputs to develop the mathematical structure (Mohajan & Mohajan, 2022a-i). Later, they have analyzed the sensitivity analysis among commodities, coupons, and prices (Mohajan & Mohajan, 2023a-e).

3. Methodology of the Study

Research is a hard-working search, scholarly inquiry, and investigation aimed at the discovery of new facts and findings (Adams et al., 2007). Methodology is an organized and meaningful procedural works (Ojo, 2003). Therefore, research methodology is the systematic procedure adopted by researchers to solve a research problem (Kothari, 2008). Research can be classified into three main categories as: i) quantitative research, ii) qualitative research, and iii) mixed method research. Our study falls in the category of qualitative research (Creswell, 2011; Mohajan, 2018b, 2020).

During the sensitivity analysis first, we have used 6×6 bordered Hessian, and later 6×10 Jacobian (Mohajan & Mohajan, 2022e,f). Reliability and validity are two most important and fundamental features in a good research (Mohajan, 2017b,c, 2018c, 2022b). In the study we have depended on the secondary data that are collected from various research papers, books, internet, etc. (Mohajan, 2012a,b, 2013a,b, 2014, 2015a,b, 2018a; Rahman & Mohajan, 2019)

4. Objective of the Study

The major objective of this article is to discuss the sensitivity analysis between commodity and total budget during the utility maximization analysis. The other minor objectives are as follows:

- to show the nature of the bordered Hessian and Jacobian in economic models,
- to interpret the results precisely and concisely, and
- to demonstrate mathematical calculations in some details.

5. Economic Model of Utility

Let us consider an economic world where there are only four commodities that are A_1, A_2, A_3, and A_4 (Moolio et al., 2009; Mohajan & Mohajan, 2022b). Let a consumer wants to buy only x_1, x_2, x_3, and x_4 amounts from these four commodities A_1, A_2, A_3, and A_4, respectively. The utility function on these four commodities can be written as (Roy et al., 2021; Mohajan & Mohajan, 2022b),

$$u(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 . \quad (1)$$

The budget constraint of the consumers can be represented as,

$$B(x_1, x_2, x_3, x_4) = p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 \quad (2)$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities x_1, x_2, x_3, and x_4, respectively. Now the coupon constraint will be,

$$C(x_1, x_2, x_3, x_4) = \kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3 + \kappa_4 x_4 \quad (3)$$

where κ_1, κ_2, κ_3, and κ_4 are the coupons necessary to purchase a unit of commodity of x_1, x_2, x_3, and x_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $L(x_1, x_2, x_3, x_4, \eta_1, \eta_2)$ as (Baxley & Moorhouse, 1984; Ferdous & Mohajan, 2022; Mohajan & Mohajan, 2022b),

$$L(x_1, x_2, x_3, x_4, \eta_1, \eta_2) = x_1x_2x_3x_4 + \eta_1(B - p_1x_1 - p_2x_2 - p_3x_3 - p_4x_4)$$
$$+ \eta_2(C - \kappa_1x_1 - \kappa_2x_2 - \kappa_3x_3 - \kappa_4x_4). \tag{4}$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where η_1 and η_2 are two Lagrange multipliers that are used as devices of mathematical procedures. Now we consider the bordered Hessian (Mohajan, 2021a; Mohajan & Mohajan, 2022c),

$$|H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -C_1 & -C_2 & -C_3 & -C_4 \\ -B_1 & -C_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}. \tag{5}$$

Now taking first and second order and cross-partial derivatives in (4) we obtain (Islam et al. 2009a,b; Mohajan & Mohajan, 2022d);

$$B_1 = p_1, \quad B_2 = p_2, \quad B_3 = p_3, \quad B_4 = p_4.$$
$$C_1 = \kappa_1, \quad C_2 = \kappa_2, \quad C_3 = \kappa_3, \quad C_4 = \kappa_4. \tag{6}$$
$$L_{11} = 0, \quad L_{12} = L_{21} = x_3x_4, \quad L_{13} = L_{31} = x_2x_4,$$
$$L_{14} = L_{41} = x_2x_3, \quad L_{22} = 0, \quad L_{23} = L_{32} = x_1x_4,$$
$$L_{24} = L_{42} = x_1x_3, \quad L_{33} = 0, \quad L_{34} = L_{43} = x_1x_2, \quad L_{44} = 0. \tag{7}$$

We use $p_3 = p_1$ and $p_4 = p_2$, i.e., a pair of prices are same, and $\kappa_3 = \kappa_1$ and $\kappa_4 = \kappa_2$, i.e., a pair of coupon numbers are same. Now we consider that in the expansion of (5) every term contains $p_1p_2\kappa_1\kappa_2$, then (5) becomes (Mohajan & Mohajan, 2022e);

$$|H| = -2p_1p_2\kappa_1\kappa_2 < 0. \tag{8}$$

For x_1, x_2, x_3, x_4, η_1, and η_2 in terms of p_1, p_2, p_3, p_4, κ_1, κ_2, κ_3, κ_4, B, and C we can calculate the sixty partial derivatives, such as $\frac{\partial \eta_1}{\partial p_1}$, $\frac{\partial \eta_2}{\partial p_1}$, …, $\frac{\partial \eta_1}{\partial \kappa_1}$, $\frac{\partial \eta_2}{\partial \kappa_1}$, …, $\frac{\partial x_1}{\partial p_1}$, …, $\frac{\partial x_1}{\partial \kappa_1}$, …, $\frac{\partial \eta_1}{\partial B}$, …, $\frac{\partial \eta_1}{\partial C}$, etc. (Islam et al., 2010; Mohajan, 2021c). Now we consider 6×6 Hessian and Jacobian matrix (Mohajan & Mohajan, 2022a; Mohajan, 2021b);

$$J = H = \begin{bmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -C_1 & -C_2 & -C_3 & -C_4 \\ -B_1 & -C_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \qquad (9)$$

which is non-singular at the optimum point $\left(x_1^*, x_2^*, x_3^*, x_4^*, \eta_1^*, \eta_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (9) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem. We have total 16 variables in our study, such as η_1, η_2, x_1, x_2, x_3, x_4 p_1, p_2, p_3, p_4, $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, B, and C. By the implicit function theorem, we can write (Moolio et al., 2009; Islam et al., 2011; Mohajan, 2021c),

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4, B, C\right). \qquad (10)$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by (Mohajan et al., 2013; Mohajan, 2021a),

$$J_G = \begin{bmatrix} \frac{\partial \eta_1}{\partial p_1} & \frac{\partial \eta_1}{\partial p_2} & \frac{\partial \eta_1}{\partial p_3} & \frac{\partial \eta_1}{\partial p_4} & \frac{\partial \eta_1}{\partial \kappa_1} & \frac{\partial \eta_1}{\partial \kappa_2} & \frac{\partial \eta_1}{\partial \kappa_3} & \frac{\partial \eta_1}{\partial \kappa_4} & \frac{\partial \eta_1}{\partial B} & \frac{\partial \eta_1}{\partial C} \\ \frac{\partial \eta_2}{\partial p_1} & \frac{\partial \eta_2}{\partial p_2} & \frac{\partial \eta_2}{\partial p_3} & \frac{\partial \eta_2}{\partial p_4} & \frac{\partial \eta_2}{\partial \kappa_1} & \frac{\partial \eta_2}{\partial \kappa_2} & \frac{\partial \eta_2}{\partial \kappa_3} & \frac{\partial \eta_2}{\partial \kappa_4} & \frac{\partial \eta_2}{\partial B} & \frac{\partial \eta_2}{\partial C} \\ \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \frac{\partial x_1}{\partial p_3} & \frac{\partial x_1}{\partial p_4} & \frac{\partial x_1}{\partial \kappa_1} & \frac{\partial x_1}{\partial \kappa_2} & \frac{\partial x_1}{\partial \kappa_3} & \frac{\partial x_1}{\partial \kappa_4} & \frac{\partial x_1}{\partial B} & \frac{\partial x_1}{\partial C} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \frac{\partial x_2}{\partial p_3} & \frac{\partial x_2}{\partial p_4} & \frac{\partial x_2}{\partial \kappa_1} & \frac{\partial x_2}{\partial \kappa_2} & \frac{\partial x_2}{\partial \kappa_3} & \frac{\partial x_2}{\partial \kappa_4} & \frac{\partial x_2}{\partial B} & \frac{\partial x_2}{\partial C} \\ \frac{\partial x_3}{\partial p_1} & \frac{\partial x_3}{\partial p_2} & \frac{\partial x_3}{\partial p_3} & \frac{\partial x_3}{\partial p_4} & \frac{\partial x_3}{\partial \kappa_1} & \frac{\partial x_3}{\partial \kappa_2} & \frac{\partial x_3}{\partial \kappa_3} & \frac{\partial x_3}{\partial \kappa_4} & \frac{\partial x_3}{\partial B} & \frac{\partial x_3}{\partial C} \\ \frac{\partial x_4}{\partial p_1} & \frac{\partial x_4}{\partial p_2} & \frac{\partial x_4}{\partial p_3} & \frac{\partial x_4}{\partial p_4} & \frac{\partial x_4}{\partial \kappa_1} & \frac{\partial x_4}{\partial \kappa_2} & \frac{\partial x_4}{\partial \kappa_3} & \frac{\partial x_4}{\partial \kappa_4} & \frac{\partial x_4}{\partial B} & \frac{\partial x_4}{\partial C} \end{bmatrix}. \qquad (11)$$

$$
= -J^{-1}\begin{bmatrix} -x_1 & -x_2 & -x_3 & -x_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & -x_4 & 0 & 1 \\ -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 \end{bmatrix}. \tag{12}
$$

The inverse of Jacobian is, $J^{-1} = \frac{1}{|J|}C^T$, where $C = (C_{ij})$, the matrix of cofactors of J, and T indicates transpose, then (12) becomes (Mohajan, 2017a; Islam et al., 2009b, 2011),

$$
J_G = -\frac{1}{|J|}C^T\begin{bmatrix} -x_1 & -x_2 & -x_3 & -x_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & -x_4 & 0 & 1 \\ -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_1 & 0 & 0 & 0 & -\eta_2 & 0 & 0 \end{bmatrix}. \tag{13}
$$

Now 6×6 transpose matrix C^T can be represented by,

$$
C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \tag{14}
$$

Using (14) we can write (11) as a 6×10 Jacobian matrix (Mohajan & Mohajan, 2022b);

$$
J_G = -\frac{1}{|J|}\left[\begin{matrix} -x_1C_{11}-\eta_1C_{31} & -x_2C_{11}-\eta_1C_{41} & -x_3C_{11}-\eta_1C_{51} & -x_4C_{11}-\eta_1C_{61} & -x_1C_{21}-\eta_2C_{31} \\ -x_1C_{12}-\eta_1C_{32} & -x_2C_{12}-\eta_1C_{42} & -x_3C_{12}-\eta_1C_{52} & -x_4C_{12}-\eta_1C_{62} & -x_1C_{22}-\eta_2C_{32} \\ -x_1C_{13}-\eta_1C_{33} & -x_2C_{13}-\eta_1C_{43} & -x_3C_{13}-\eta_1C_{53} & -x_4C_{13}-\eta_1C_{63} & -x_1C_{23}-\eta_2C_{33} \\ -x_1C_{14}-\eta_1C_{34} & -x_2C_{14}-\eta_1C_{44} & -x_3C_{14}-\eta_1C_{54} & -x_4C_{14}-\eta_1C_{64} & -x_1C_{24}-\eta_2C_{34} \\ -x_1C_{15}-\eta_1C_{35} & -x_2C_{15}-\eta_1C_{45} & -x_3C_{15}-\eta_1C_{55} & -x_4C_{15}-\eta_1C_{65} & -x_1C_{25}-\eta_2C_{35} \\ -x_1C_{16}-\eta_1C_{36} & -x_2C_{16}-\eta_1C_{46} & -x_3C_{16}-\eta_1C_{56} & -x_4C_{16}-\eta_1C_{66} & -x_1C_{26}-\eta_2C_{36} \end{matrix}\right.
$$

$$
\left.\begin{matrix} -x_2C_{21}-\eta_2C_{41} & -x_3C_{21}-\eta_2C_{51} & -x_4C_{21}-\eta_2C_{61} & C_{11} & C_{21} \\ -x_2C_{22}-\eta_2C_{42} & -x_3C_{22}-\eta_2C_{52} & -x_4C_{22}-\eta_2C_{62} & C_{12} & C_{22} \\ -x_2C_{23}-\eta_2C_{43} & -x_3C_{23}-\eta_2C_{53} & -x_4C_{23}-\eta_2C_{63} & C_{13} & C_{23} \\ -x_2C_{24}-\eta_2C_{44} & -x_3C_{24}-\eta_2C_{54} & -x_4C_{24}-\eta_2C_{64} & C_{14} & C_{24} \\ -x_2C_{25}-\eta_2C_{45} & -x_3C_{25}-\eta_2C_{55} & -x_4C_{25}-\eta_2C_{65} & C_{15} & C_{25} \\ -x_2C_{26}-\eta_2C_{46} & -x_3C_{26}-\eta_2C_{56} & -x_4C_{26}-\eta_2C_{66} & C_{16} & C_{26} \end{matrix}\right]. \tag{15}
$$

Now we analyze the nature of consumption of commodity x_1 when total budget B increases. Taking T_{39}, (i.e., term of 3rd row and 9th column) from both sides of (15) we get (Islam et al., 2011; Mohajan & Mohajan, 2022e),

$$\frac{\partial x_1}{\partial B} = -\frac{1}{|J|}\left[C_{13}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{13}$$

$$= -\frac{1}{|J|}\begin{vmatrix} 0 & 0 & -C_2 & -C_3 & -C_4 \\ -B_1 & -C_1 & L_{12} & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{22} & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{32} & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{42} & L_{43} & L_{44} \end{vmatrix}$$

$$= -\frac{1}{|J|}\left\{ -C_2\begin{vmatrix} -B_1 & -C_1 & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{43} & L_{44} \end{vmatrix} + C_3\begin{vmatrix} -B_1 & -C_1 & L_{12} & L_{14} \\ -B_2 & -C_2 & L_{22} & L_{24} \\ -B_3 & -C_3 & L_{32} & L_{34} \\ -B_4 & -C_4 & L_{42} & L_{44} \end{vmatrix} - C_4\begin{vmatrix} -B_1 & -C_1 & L_{12} & L_{13} \\ -B_2 & -C_2 & L_{22} & L_{23} \\ -B_3 & -C_3 & L_{32} & L_{33} \\ -B_4 & -C_4 & L_{42} & L_{43} \end{vmatrix} \right\}$$

$$= -\frac{1}{|J|}\left[-C_2\left\{ -B_1\begin{vmatrix} -C_2 & L_{23} & L_{24} \\ -C_3 & L_{33} & L_{34} \\ -C_4 & L_{43} & L_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & L_{23} & L_{24} \\ -B_3 & L_{33} & L_{34} \\ -B_4 & L_{43} & L_{44} \end{vmatrix} + L_{13}\begin{vmatrix} -B_2 & -C_2 & L_{24} \\ -B_3 & -C_3 & L_{34} \\ -B_4 & -C_4 & L_{44} \end{vmatrix} - L_{14}\begin{vmatrix} -B_2 & -C_2 & L_{23} \\ -B_3 & -C_3 & L_{33} \\ -B_4 & -C_4 & L_{43} \end{vmatrix} \right\}\right.$$

$$+ C_3\left\{ -B_1\begin{vmatrix} -C_2 & L_{22} & L_{24} \\ -C_3 & L_{32} & L_{34} \\ -C_4 & L_{42} & L_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & L_{22} & L_{24} \\ -B_3 & L_{32} & L_{34} \\ -B_4 & L_{42} & L_{44} \end{vmatrix} + L_{12}\begin{vmatrix} -B_2 & -C_2 & L_{24} \\ -B_3 & -C_3 & L_{34} \\ -B_4 & -C_4 & L_{44} \end{vmatrix} - L_{14}\begin{vmatrix} -B_2 & -C_2 & L_{22} \\ -B_3 & -C_3 & L_{32} \\ -B_4 & -C_4 & L_{42} \end{vmatrix} \right\}$$

$$\left. - C_4\left\{ -B_1\begin{vmatrix} -C_2 & L_{22} & L_{23} \\ -C_3 & L_{32} & L_{33} \\ -C_4 & L_{42} & L_{43} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & L_{22} & L_{23} \\ -B_3 & L_{32} & L_{33} \\ -B_4 & L_{42} & L_{43} \end{vmatrix} + L_{12}\begin{vmatrix} -B_2 & -C_2 & L_{23} \\ -B_3 & -C_3 & L_{33} \\ -B_4 & -C_4 & L_{43} \end{vmatrix} - L_{13}\begin{vmatrix} -B_2 & -C_2 & L_{22} \\ -B_3 & -C_3 & L_{32} \\ -B_4 & -C_4 & L_{42} \end{vmatrix} \right\}\right]$$

$$= -\frac{1}{|J|}\left[B_1C_2^2L_{34}^2 - B_1C_2C_4L_{23}L_{34} - B_1C_2C_3L_{24}L_{34} - B_2C_1C_2L_{34}^2 + B_4C_1C_2L_{23}L_{34} + B_3C_1C_2L_{24}L_{34}\right.$$
$$+ B_2C_2C_4L_{13}L_{34} - B_4C_2^2u_{13}u_{34} - B_3C_2C_4L_{13}L_{24} + B_4C_2C_3L_{13}L_{24} + B_2C_2C_3L_{14}L_{34} - B_3C_2^2L_{14}L_{34}$$
$$+ B_3C_2C_4L_{14}L_{23} - B_4C_2C_3L_{14}L_{23} - B_1C_2C_3L_{24}L_{34} + B_1C_3^2L_{24}^2 - B_1C_3C_4L_{23}L_{24} + B_2C_1C_3L_{24}L_{34}$$
$$+ B_3C_1C_3L_{24}^2 - B_4C_1C_3L_{23}L_{24} - B_2C_3C_4L_{13}L_{34} + B_4C_2C_3L_{13}L_{34} + B_3C_3C_4L_{13}L_{24} - B_4C_3^2L_{13}L_{24}$$
$$- B_2C_3^2L_{14}L_{24} + B_2C_3C_4L_{14}L_{23} + B_2C_2C_3L_{14}L_{24} - B_4C_2C_3L_{14}L_{23} - B_1C_2C_4L_{23}L_{34} - B_1C_3C_4L_{23}L_{24}$$
$$+ B_1C_4^2L_{23}^2 + B_2C_1C_4L_{23}L_{34} + B_3C_1C_4L_{23}L_{24} - B_4C_1C_4L_{23}^2 - B_2C_3C_4L_{12}L_{34} + B_3C_2C_4L_{12}L_{34} - B_3C_4^2L_{12}L_{23}$$
$$\left. + B_4C_3C_4L_{12}L_{23} + B_2C_3C_4L_{13}L_{24} - B_2C_4^2L_{13}L_{23} - B_3C_2C_4L_{13}L_{24} + B_4C_2C_4L_{13}L_{23}\right]$$

$$= -\frac{1}{|J|}\Big[p_1\kappa_1^2x_1^2x_2^2 \quad - p_1\kappa_2\kappa_4x_1^2x_2x_4 \quad - p_1\kappa_2\kappa_3x_1^2x_2x_3 \quad - p_1\kappa_2^2x_1^2x_2^2 \quad - p_4\kappa_1\kappa_2x_1^2x_2x_4 \quad + p_3\kappa_1\kappa_2x_1^2x_2x_3$$
$$+ p_2\kappa_2\kappa_4x_1x_2^2x_4 \quad - p_4\kappa_2^2x_1x_2^2x_4 \quad - p_3\kappa_2\kappa_4x_1x_2x_3x_4 \quad + p_2\kappa_2\kappa_3x_1x_2x_3x_4 \quad + p_2\kappa_2\kappa_3x_1x_2^2x_3 \quad - p_3\kappa_2^2x_1x_2^2x_3$$
$$+ p_3\kappa_2\kappa_4x_1x_2x_3x_4 \quad - p_4\kappa_2\kappa_3x_1x_2x_3x_4 \quad - p_1\kappa_2\kappa_3x_1^2x_2x_3 \quad + p_1\kappa_3^2x_1^2x_3^2 \quad - p_1\kappa_3\kappa_4x_1^2x_3x_4 \quad + p_2\kappa_1\kappa_3x_1^2x_2x_3$$
$$+ p_2\kappa_1\kappa_2x_1^2x_3^2 \quad - p_4\kappa_1\kappa_3x_1^2x_3x_4 \quad - p_2\kappa_3\kappa_4x_1x_2^2x_4 \quad + p_2\kappa_2\kappa_3x_1x_2^2x_4 \quad + p_3\kappa_3\kappa_4x_1x_2x_3x_4 \quad - p_4\kappa_3^2x_1x_2x_3x_4$$
$$- p_2\kappa_3^2x_1x_2x_3^2 \quad + p_2\kappa_3\kappa_4x_1x_2x_3x_4 \quad + p_2\kappa_2\kappa_3x_1x_2x_3^2 \quad - p_2\kappa_2\kappa_3x_1x_2x_3x_4 \quad - p_1\kappa_2\kappa_4x_1^2x_2x_4 \quad - p_1\kappa_3\kappa_4x_1^2x_3x_4$$
$$+ p_1\kappa_4^2x_1^2x_4^2 \quad + p_2\kappa_1\kappa_4x_1^2x_2x_4 \quad + p_3\kappa_1\kappa_4x_1^2x_3x_4 \quad - p_1\kappa_1\kappa_4x_1^2x_4^2 \quad - p_2\kappa_3\kappa_4x_1x_2x_3x_4 \quad + p_3\kappa_2\kappa_4x_1x_2x_3x_4$$
$$- p_3\kappa_4^2x_1x_3x_4^2 + p_4\kappa_3\kappa_4x_1x_3x_4^2 + p_2\kappa_3\kappa_4x_1x_2x_3x_4 - p_2\kappa_4^2x_1x_2x_4^2 - p_2\kappa_2\kappa_4x_1x_2x_3x_4 + p_4\kappa_2\kappa_4x_1x_2x_4^2\Big]$$

$$\frac{\partial x_1}{\partial B} = -\frac{1}{|J|}\Big[\left(p_1\kappa_1^2 - p_1\kappa_2^2\right)x_1^2x_2^2 + \left(p_1\kappa_3^2 + p_2\kappa_1\kappa_2\right)x_1^2x_3^2 + \left(p_1\kappa_4^2 - p_1\kappa_1\kappa_4\right)x_1^2x_4^2 + \left(p_4\kappa_3\kappa_4 - p_3\kappa_4^2\right)x_1x_3x_4^2$$
$$+ \left(p_3\kappa_1\kappa_2 + p_2\kappa_1\kappa_3 - 2p_1\kappa_2\kappa_3\right)x_1^2x_2x_3 + \left(2p_2\kappa_2\kappa_3 - p_3\kappa_2^2\right)x_1x_2^2x_3 + \left(p_3\kappa_1\kappa_4 - 2p_1\kappa_3\kappa_4 - p_4\kappa_1\kappa_3\right)x_1^2x_3x_4$$
$$+ \left(p_2\kappa_1\kappa_4 - p_4\kappa_1\kappa_2 - 2p_1\kappa_2\kappa_4\right)x_1^2x_2x_4 \quad + \left(p_2\kappa_2\kappa_3 - p_2\kappa_3^2\right)x_1x_2x_3^2 \quad + \left(p_2\kappa_2\kappa_4 - p_4\kappa_2^2 - p_2\kappa_3\kappa_4\right)x_1x_2^2x_4$$
$$+ \left(p_3\kappa_3\kappa_4 + p_3\kappa_2\kappa_4 + p_2\kappa_3\kappa_4 - p_4\kappa_2\kappa_3 - p_4\kappa_3^2 - p_2\kappa_2\kappa_4\right)x_1x_2x_3x_4 + \left(p_4\kappa_2\kappa_4 - p_2\kappa_4^2\right)x_1x_2x_4^2\Big]. \qquad (16)$$

Using $x_1 = x_2 = x_3 = x_4 = 1$ in (16) we get,

$$\frac{\partial x_1}{\partial B} = -\frac{1}{|J|}\Big[\big(p_1\kappa_1^2 + p_1\kappa_3^2 + p_2\kappa_1\kappa_2 + p_1\kappa_4^2 + p_4\kappa_3\kappa_4 + p_3\kappa_1\kappa_2 + p_2\kappa_1\kappa_3 + 3p_2\kappa_2\kappa_3 \quad + p_3\kappa_1\kappa_4 + p_2\kappa_1\kappa_4$$
$$+ p_3\kappa_3\kappa_4 + p_3\kappa_2\kappa_4 + p_4\kappa_2\kappa_4\big) \quad - \big(p_1\kappa_2^2 + p_1\kappa_1\kappa_4 + p_3\kappa_4^2 + 2p_1\kappa_2\kappa_3 + p_3\kappa_2^2 + 2p_1\kappa_3\kappa_4 + p_4\kappa_1\kappa_3 + p_4\kappa_1\kappa_2$$
$$+ 2p_1\kappa_2\kappa_4 + p_2\kappa_3^2 + p_4\kappa_2^2 + p_4\kappa_2\kappa_3 + p_4\kappa_3^2 + p_2\kappa_4^2\big)\Big]. \qquad (17)$$

We consider, $p_1 = p_2 = p_3 = p_4 = p$, and $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa$, then $|J| = -2p^2\kappa^2$; and (17) gives,

$$\frac{\partial x_1}{\partial B} = \frac{1}{p} > 0. \qquad (18)$$

Inequality (18) indicates that if the total budget of individual/community increases, the level of consumption of commodity x_1 will also increase. We believe that commodity x_1 is not an inferior good; it may be a superior good, and it has no supplementary goods (Islam et al., 2010; Mohajan, 2021b; Mohajan & Mohajan, 2022c).

We consider $p_3 = p_1$ and $p_4 = p_2$; and $\kappa_3 = \kappa_1$, and $\kappa_4 = \kappa_2$, then $|J| = |H| = -2p_1p_2\kappa_1\kappa_2$, and from (17) we get,

$$\frac{\partial x_1}{\partial B} = \frac{2p_2 - p_1}{p_1p_2} + \frac{(p_1 - p_2)\kappa_1}{p_1p_2\kappa_2} - \frac{(p_1 + p_2)\kappa_2}{p_1p_2\kappa_1}. \qquad (19)$$

Now let $\kappa_1 = \kappa_2 = \kappa$, then from (19) we get,

$$\frac{\partial x_1}{\partial B} = -\frac{1}{p_2} < 0. \qquad (20)$$

Inequality (20) indicates that even if the total budget of individual/community increases, but the level of consumption of commodity x_1 can decrease. In this situation it seems that commodity x_1 is an inferior good.

Now we analyze the nature of consumption of commodity x_2 when the total budget B increases. Taking T_{49}, (i.e., term of 4th row and 9th column) from both sides of (14) we get (Islam et al., 2010; Mohajan & Mohajan, 2022c,e),

$$\frac{\partial x_2}{\partial B} = -\frac{1}{|J|}\left[C_{14}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{14}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & 0 & -C_1 & -C_3 & -C_4 \\ -B_1 & -C_1 & L_{11} & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{21} & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{31} & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{41} & L_{43} & L_{44} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{ -C_1\begin{vmatrix} -B_1 & -C_1 & L_{13} & L_{14} \\ -B_2 & -C_2 & L_{23} & L_{24} \\ -B_3 & -C_3 & L_{33} & L_{34} \\ -B_4 & -C_4 & L_{43} & L_{44} \end{vmatrix} + C_3\begin{vmatrix} -B_1 & -C_1 & L_{11} & L_{14} \\ -B_2 & -C_2 & L_{21} & L_{24} \\ -B_3 & -C_3 & L_{31} & L_{34} \\ -B_4 & -C_4 & L_{41} & L_{44} \end{vmatrix} - C_4\begin{vmatrix} -B_1 & -C_1 & L_{11} & L_{13} \\ -B_2 & -C_2 & L_{21} & L_{23} \\ -B_3 & -C_3 & L_{31} & L_{33} \\ -B_4 & -C_4 & L_{41} & L_{43} \end{vmatrix} \right\}$$

$$= \frac{1}{|J|}\left[-C_1\left\{ -B_1\begin{vmatrix} -C_2 & L_{23} & L_{24} \\ -C_3 & L_{33} & L_{34} \\ -C_4 & L_{43} & L_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & L_{23} & L_{24} \\ -B_3 & L_{33} & L_{34} \\ -B_4 & L_{43} & L_{44} \end{vmatrix} + L_{13}\begin{vmatrix} -B_2 & -C_2 & L_{24} \\ -B_3 & -C_3 & L_{34} \\ -B_4 & -C_4 & L_{44} \end{vmatrix} - L_{14}\begin{vmatrix} -B_2 & -C_2 & L_{23} \\ -B_3 & -C_3 & L_{33} \\ -B_4 & -C_4 & L_{43} \end{vmatrix} \right\} \right.$$

$$+ C_3\left\{ -B_1\begin{vmatrix} -C_2 & L_{21} & L_{24} \\ -C_3 & L_{31} & L_{34} \\ -C_4 & L_{41} & L_{44} \end{vmatrix} + C_1\begin{vmatrix} -B_2 & L_{21} & L_{24} \\ -B_3 & L_{31} & L_{34} \\ -B_4 & L_{41} & L_{44} \end{vmatrix} - L_{14}\begin{vmatrix} -B_2 & -C_2 & L_{21} \\ -B_3 & -C_3 & L_{31} \\ -B_4 & -C_4 & L_{41} \end{vmatrix} \right\} - C_4\left\{ -B_1\begin{vmatrix} -C_2 & L_{21} & L_{23} \\ -C_3 & L_{31} & L_{33} \\ -C_4 & L_{41} & L_{43} \end{vmatrix} \right.$$

$$\left.\left. + C_1\begin{vmatrix} -B_2 & L_{21} & L_{23} \\ -B_3 & L_{31} & L_{33} \\ -B_4 & L_{41} & L_{43} \end{vmatrix} - L_{13}\begin{vmatrix} -B_2 & -C_2 & L_{21} \\ -B_3 & -C_3 & L_{31} \\ -B_4 & -C_4 & L_{41} \end{vmatrix} \right\} \right]$$

$$= \frac{1}{|J|}\Big[B_1C_1C_2L_{34}^2 - B_1C_1C_4L_{23}L_{34} - B_1C_1C_3L_{24}L_{34} - B_2C_1^2L_{34}^2 + B_4C_1^2L_{23}L_{34} + B_3C_1^2L_{23}L_{34}$$

$$+ B_2C_1C_4L_{13}L_{34} - B_4C_1C_2L_{13}L_{34} - B_3C_1C_4L_{13}L_{24} + B_4C_1C_3L_{13}L_{24} + B_2C_1C_3L_{14}L_{34} - B_3C_1C_2L_{14}L_{34}$$

$$+ B_3C_1C_4L_{14}L_{23} - B_4C_1C_3L_{14}L_{23} + B_1C_2C_3L_{14}L_{34} + B_1C_3C_4L_{12}L_{34} + B_1C_3^2L_{14}L_{24} - B_1C_3C_4L_{13}L_{24}$$

$$+ B_2C_1C_3L_{14}L_{34} - B_4C_1C_3L_{12}L_{34} - B_3C_1C_3L_{14}L_{24} + B_4C_1C_3L_{13}L_{24} - B_2C_3^2L_{14}^2 + B_2C_3C_4L_{13}L_{14} - B_3C_2C_3L_{14}^2$$

$$+B_4C_2C_3L_{13}L_{14} \quad -B_3C_3C_4L_{12}L_{14} \quad +B_4C_3^2L_{12}L_{14} \quad -B_1C_2C_4L_{13}L_{34} \quad +B_1C_3C_4L_{12}L_{34} \quad -B_1C_3C_4L_{14}L_{23}$$
$$+B_1C_4^2L_{13}L_{23} \quad +B_2C_1C_4L_{13}L_{34} \quad -B_3C_1C_4L_{12}L_{34} \quad +B_3C_1C_4L_{14}L_{23} \quad -B_4C_1C_4L_{13}L_{23} \quad +B_2C_3C_4L_{13}L_{14}$$
$$-B_2C_4^2L_{13}^2 - B_3C_2C_4L_{13}L_{14} + B_4C_2C_4L_{13}^2 + B_3C_4^2L_{12}L_{13} - B_4C_3C_4L_{12}L_{13}\Big]$$

$$=\frac{1}{|J|}\Big[p_1\kappa_1\kappa_2x_1^2x_2^2 \quad -p_1\kappa_1\kappa_4x_1^2x_2x_4 \quad -p_1\kappa_1\kappa_3x_1^2x_2x_3 \quad -p_1\kappa_1^2x_1^2x_2^2 \quad +p_4\kappa_1^2x_1^2x_2x_4 \quad +p_3\kappa_1^2x_1^2x_2x_4$$
$$+p_2\kappa_1\kappa_4x_1x_2^2x_4 \quad -p_4\kappa_1\kappa_2x_1x_2^2x_4 \quad -p_3\kappa_1\kappa_4x_1x_2x_3x_4 \quad +p_4\kappa_1\kappa_3x_1x_2x_3x_4 \quad +p_2\kappa_1\kappa_3x_1x_2^2x_3 \quad -p_3\kappa_1\kappa_2x_1x_2^2x_3$$
$$+p_3\kappa_1\kappa_4x_1x_2x_3x_4 \quad -p_4\kappa_1\kappa_3x_1x_2x_3x_4 \quad +p_1\kappa_2\kappa_3x_1x_2^2x_3 \quad +p_1\kappa_3\kappa_4x_1x_2x_3x_4 \quad +p_1\kappa_3^2x_1x_2x_3^2 \quad -p_1\kappa_3\kappa_4x_1x_2x_3x_4$$
$$+p_2\kappa_1\kappa_3x_1x_2^2x_3 \quad -p_4\kappa_1\kappa_3x_1x_2x_3x_4 \quad -p_3\kappa_1\kappa_3x_1x_2x_3^2 \quad -p_4\kappa_1\kappa_3x_1x_2x_3x_4 \quad -p_2\kappa_3^2x_2^2x_3^2 \quad +p_2\kappa_3\kappa_4x_2^2x_3x_4$$
$$-p_3\kappa_2\kappa_3x_2^2x_3x_4 \quad +p_3\kappa_2\kappa_3x_2^2x_3^2 \quad -p_3\kappa_3\kappa_4x_2x_3^2x_4 \quad +p_4\kappa_3^2x_2x_3^2x_4 \quad -p_1\kappa_2\kappa_4x_1x_2^2x_4 \quad +p_1\kappa_3\kappa_4x_1x_2x_3x_4$$
$$-p_1\kappa_3\kappa_4x_1x_2x_3x_4 \quad +p_1\kappa_4^2x_1x_2x_4^2 \quad +p_2\kappa_1\kappa_4x_1x_2^2x_4 \quad -p_4\kappa_1\kappa_4x_1x_2x_3x_4 \quad +p_3\kappa_1\kappa_4x_1x_2x_3x_4 \quad -p_4\kappa_1\kappa_4x_1x_2x_4^2$$
$$+p_2\kappa_3\kappa_4x_2^2x_3x_4 - p_2\kappa_4^2x_2^2x_4^2 - p_3\kappa_2\kappa_4x_2^2x_3x_4 + p_4\kappa_2\kappa_4x_2^2x_4^2\Big]$$

$$=\frac{1}{|J|}\Big[\left(p_1\kappa_1\kappa_2 - p_1\kappa_1^2\right)x_1^2x_2^2 \quad +\left(p_3\kappa_2\kappa_3 - p_2\kappa_3^2\right)x_2^2x_3^2 \quad +\left(p_4\kappa_2\kappa_4 - p_2\kappa_4^2\right)x_2^2x_4^2 \quad +\left(p_4\kappa_1^2 - p_1\kappa_1\kappa_3\right)x_1^2x_2x_3$$
$$+\left(2p_2\kappa_1\kappa_4 + p_3\kappa_1^2 - p_1\kappa_1\kappa_4 - p_1\kappa_2\kappa_4 - p_4\kappa_1\kappa_2\right)x_1x_2^2x_4 + \left(2p_2\kappa_3\kappa_4 - p_3\kappa_2\kappa_3 - p_3\kappa_2\kappa_4\right)x_2^2x_3x_4$$
$$+\left(2p_2\kappa_1\kappa_3 + p_1\kappa_2\kappa_3 - p_3\kappa_1\kappa_2\right)x_1x_2^2x_3 \quad +\left(p_4\kappa_3^2 - p_3\kappa_3\kappa_4\right)x_2x_3^2x_4 \quad +\left(p_3\kappa_1\kappa_4 - 2p_4\kappa_1\kappa_3 - p_4\kappa_1\kappa_4\right)x_1x_2x_3x_4$$
$$+\left(p_1\kappa_3^2 - p_3\kappa_1\kappa_3\right)x_1x_2x_3^2 + \left(p_1\kappa_4^2 - p_4\kappa_1\kappa_4\right)x_1x_2x_4^2\Big]. \tag{21}$$

Now we use $p_3 = p_1$ and $p_4 = p_2$ where pair of prices are same, and $\kappa_3 = \kappa_1$ and $\kappa_4 = \kappa_2$, i.e., two types of coupon numbers are same. We put $x_1 = x_2 = x_3 = x_4 = 1$ then (21) becomes (Mohajan & Mohajan, 2022b,e),

$$\frac{\partial x_2}{\partial B} = \frac{1}{|J|}\left(p_2\kappa_1\kappa_2 + p_2\kappa_1^2 - p_1\kappa_1^2 - p_1\kappa_2^2\right). \tag{22}$$

Now we use, $\kappa_1 = \kappa_2 = \kappa$, and $|J| = -2p_1p_2\kappa^2$ in (22), and then we get,

$$\frac{\partial x_2}{\partial B} = \frac{p_1 - p_2}{p_1p_2}. \tag{23}$$

Now if $p_1 > p_2$ in (23) we get,

$$\frac{\partial x_2}{\partial B} > 0. \tag{24}$$

Inequality (24) indicates that if the total budget of individual/community increases, the level of consumption of commodity x_2 will also increase. Therefore, commodity x_2 is not an inferior good; it may be a superior good, and it has no supplementary goods (Islam et al., 2010; Mohajan, 2021b; Mohajan & Mohajan, 2022d).

Now if $p_2 > p_1$ in (23) we get,

$$\frac{\partial x_2}{\partial B} < 0. \tag{25}$$

Inequality (25) indicates that even if the total budget of individual/community increases, but the level of consumption of commodity x_2 can decrease. Consequently, commodity x_2 is an inferior good.

From this study we have realized that $\frac{\partial x_2}{\partial B} \neq 0$, so that, from (23) we see that $p_1 \neq p_2$, i.e., the price of two commodities x_1 and x_2 never be equal.

6. Conclusions

In this study we have taken attempts to discuss utility maximization with detail mathematical calculations. We have used two constraints: budget constraint and coupon constraint to perform the research efficiently. We have discussed the sensitivity analysis and also have tried to find relationships between commodity and total budget. We have used four commodity variables to operate the mathematical formulation efficiently. Throughout the study, we have applied the technique of Lagrange multipliers to investigate the optimization problems. In this study we have tried to show mathematical calculations in some details.

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