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Hartman-Stampacchia theorem, Gale-Nikaido-Debreu lemma, and Brouwer and Kakutani fixed-point theorems[⋆]

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Abstract

This paper uses the Hartman-Stampacchia theorems as primary tool to prove the Gale-Nikaido-Debreu lemma. It also establishes a full equivalence circle among the Hartman Stampacchia theorems, the Gale-Nikaido-Debreu lemmas, and Kakutani and Brouwer fixed point theorems.
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1. Introduction

The existence of an economic equilibrium was proven by Arrow and Debreu (1954) and McKenzie (1954). While the existence result could be established in many different ways, all classic proofs of the existence theorem rely on Kakutani fixed point argument (see Debreu (1982)'s survey). In the so called *excess demand* approach, the existence result is yielded from the existence of prices satisfying the *Walras's law*. The core of the proof of *excess demand* approach is a celebrated result known as the Gale-Nikaido-Debreu lemma (henceforth, GND lemma) (see Gale (1955), Debreu (1956) and Nikaidô (1956) or Debreu (1959), Gale and Mas-Colell (1975, 1979)) whose proofs[1] make use of the Kakutani fixed point theorem. The present paper's main aim is, by the means of the Hartman-Stampacchia theorems (Hartman and Stampacchia (1966)) to provide a proof for the GND lemma and its generalized version given by Geistdoerfer-Florenzano (1982).

In recent years, there has been a renewed interest in the fundamental existence results of equilibrium in economics and games. In particular, the GND lemma has been extended

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[1]We refer the interested reader to Debreu (1982) and Florenzano (2003) for excellent treatments of the existence of equilibrium.

to the class of discontinuous demand functions/correspondences in various settings, such as finite-dimensional settings (e.g.,Maskin and Roberts (2008), Tian (2016) and Cornet (2020)), infinite-dimensional settings (e.g., He and Yannelis (2017)). Additionally, there have been many efforts to provide a proof of the existence result without Kakutani's fixed point argument, including studies by Greenberg (1977), Barbolla and Corchon (1989), John (1999), Quah (2008), Frayssé (2009), Maćkowiak (2010). These studies often fall into two categories, both of which impose conditions on *excess demand function* or *correspondence*. The first category is relied on *gross substitutes* assumptions (e.g., (Greenberg, 1977) Greenberg (1977), Barbolla and Corchon (1989) and Frayssé (2009)); the second one is based on *weak axiom of revealed preference* assumption (e.g., Quah (2008) and Maćkowiak (2010)). This paper aims to contribute to such line of research.

As mentioned in Duppe and Weintraub (2014) and Khan (2021), Debreu wanted to explore the possibility of proving the GND lemma and its generalization without the Kakutani's fixed point argument. Le et al. (2022) have been among the first to investigate the question. The authors have used Sperner's lemmas as the primary argument to prove the GND lemma. However, they do not give a proof of the generalized version of GND lemma, given by Geistdoerfer-Florenzano (1982) (or Florenzano (2003)) by means of Sperner lemma. The first goal of the paper is to provide an alternative argument for the proof of not only GND lemma but also its generalization. In contrast to the proof in Geistdoerfer-Florenzano (1982), the present paper provides the direct proof. The second goal is to make an equivalence circle among the Hartman Stampacchia theorems, GND lemmas and some related fixed point theorems (See below).

As for the second goal, we introduce a new version of Hartman-Stampacchia theorem (Theorem 2) associating with upper semi-continuous correspondence. The original Hartman Stampacchia theorem for continuous mapping (Theorem 1, HS1) implies the one for upper-semi continuous correspondence (Theorem 2, HS2), which yields, in turn, the GND lemma (Theorem 4). As showed in Geistdoerfer-Florenzano (1982), the GND lemma implies the Kakutani theorem, which straightforward entails the Brouwer theorem. Finally, the original Hartman Stampacchia for continuous mapping can be obtained from the Brouwer theorem, leading to a full equivalence circle. Moreover, the Kakutani and Brouwer theorems are shown to be themselves a consequences of the original Hartman Stampacchia theorem (Theorem 1, HS1). All these results are illustrates in Figure 1.

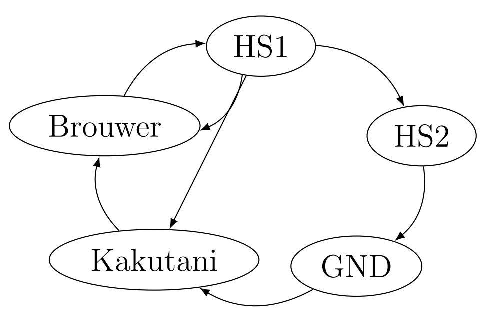

Figure 1: The full equivalence circle: Hartman Stampacchia theorem (Theorem 1, HS1), generalized version of Hartman Stampacchia for correspondence(Theorem 2, HS2), GND lemma, Kakutani and Brouwer fixed point theorems.

In order to achieve our purposes, the generalized version of Hartman-Stampacchia theorem (Theorem 2) associated with upper semi-continuous and convex-valued correspondence is provided and proven. This theorem, together with the original Hartman-Stampacchia theorem (Theorem 1), allows us to prove not only the GND lemma in Debreu (1959) but also the generalized version of GND lemma in Geistdoerfer-Florenzano (1982) (or Florenzano (2003)). To make easy the reading of our paper, we first consider the case of a continuous mapping. In the second stage, we deal with the case of an upper semi-continuous correspondence with convex, compact, non-empty values. In the case of a correspondence, some additional ingredients such as finite covering of a compact set, partition of unit subordinated to a covering and Carathéodory convexity theorem (Carathéodory (1907)) have been added.

We end this introduction by discussing more precisely link this paper and others on the same subject. First, we do not assume the so called *weak axiom of revealed preference* or its generalization as in Quah (2008) and Maćkowiak (2010) respectively. The results of Quah (2008) and Maćkowiak (2010) are special cases of the GND lemmas (Theorems 3 or 4) below. Quah (2008) established the existence result under the standard assumptions and the *weak axiom of revealed preference* assumption on the excess demand correspondence. The later condition means that the excess demand correspondence ζ, whose domain is the set of prices $P \subset \mathbb{R}^L$, obeys the *weak axiom of revealed preference* if the following statement is true:

$$\textit{for all } p \textit{ and } p' \textit{ in } P, \textit{ whenever there is } z' \in \zeta(p') \textit{ such that } \langle p, z' \rangle \leq 0, \textit{ then } \langle p', \zeta(p) \rangle \geq 0.$$

Second, the models of Greenberg (1977), Barbolla and Corchon (1989) and Frayssé (2009) Frayssé (2009) are more specific than one considered here. Indeed, they assume the stronger assumption that the excess demand correspondence is a continuous *single-valued mapping* with the *(strong) gross substitute* property.

The paper proceeds as follows. In the next section, we begin with some notations and definitions, the Hartman-Stampacchia theorem (Theorem 1) and its generalization (Theorem 2), which is of fundamental importance for the proof of the GND lemma and its extended version. Section 3 contains the main result of the paper, namely the proofs of the GND lemma and its extended version without the Kakutani fixed point argument. Section 4 is dedicated to show how the Hartman-Stampacchia theorem can be used to prove Brouwer and Kakutani theorems. Additionally, we demonstrate in Section 5 that Hartman-Stampacchia theorem is a consequence of Brouwer theorem. Finally, some extended proofs are given the Appendix A.

2. Preliminaries

We start by introducing some notations using through this paper.

2.1. General Notations and Definitions

We shall denote

- by $\langle x, y \rangle$ the inner product between x and y, and $||x||$ the norm of x for any $x, y \in \mathbb{R}^N$,
- by $0_N, 0_m$ the zero vector in $\mathbb{R}^N$, $\mathbb{R}^m$ respectively,

- by B and $\bar{B}$ the open and closed unit-ball in $\mathbb{R}^N$ centered at 0_N respectively, S the unit-sphere associated with $\bar{B}$,
- by $B(x, r)$ and $\bar{B}(x, r)$ open and closed balls with center at x and radius r respectively for any $x \in \mathbb{R}^N$ and $r > 0$,
- by $P^\circ = \{z \in \mathbb{R}^N : \langle p, z\rangle \leq 0, \ \forall p \in P\}$ the polar cone of P,
- by Δ the unit-simplex of $\mathbb{R}^N$,
- by $N_C(x)$ the normal cone to the set C at the point x for any $C \subset \mathbb{R}^N$ and $x \in C$.
- by P^{c} the complement of P for any $P \subset \mathbb{R}^N$,
- by $\mathrm{int}(P)$, $\mathrm{ri}(P)$ and $\mathrm{Bd}^{\mathrm{r}}(P)$ the interior, relative interior and relative boundary sets of a given set P respectively.

Let us recall the definition and some properties of *upper-semi correspondence*. Let X, Y be non-empty topological spaces. A correspondence $\Gamma : X \to Y$ is *upper semi-continuous* (u.s.c) at point x if for every open set V of Y for which $\Gamma(x) \subset V$, there exists a neighborhood U of x such that $\Gamma(z) \subset V \ \forall z \in U$. The correspondence Γ is said to be *upper semi-continuous* on X if it is upper continuous at every point of X.

Notice that if X is compact then Γ is upper semi-continuous if and only if Γ is closed, namely, the graph of Γ is closed. It is also clear that if Γ is upper semi-continuous and $K \subset X$ is compact, then $\Gamma(K)$ is compact. Recall that if Γ is single-valued, the notions of *continuity*, *upper semi-continuity*, and the *lower semi-continuity* turn out to be equivalent.

2.2. Hartman-Stampachia Theorem and its Generalization

Let us first recall the theorem from Hartman and Stampacchia (1966) in the finite dimensional setting.

Theorem 1 (Hartman-Stampacchia theorem)**.** [2] *Let K be a convex and compact set of $\mathbb{R}^N$, f a continuous mapping from K into $\mathbb{R}^N$. Then there exists $\bar{u} \in K$ such that*

$$\langle v, f(\bar{u})\rangle \leq \langle \bar{u}, f(\bar{u})\rangle \quad \forall v \in K. \tag{2.1}$$

Remark 1. *One of the possible proof of Theorem 1 is to make use of the index theory or the topological degree theory.*

Remark 2. *The point $f(u_0) \in N_K(u_0)$, namely, it belongs to the normal cone to the set K at u_0.*

Before going to a detailed generalization of the Hartman-Stampacchia theorem, we introduce a useful lemma[3] describing the value of a continuous mapping in terms of a finite linear

[2] See Lemma 3.1 in Hartman and Stampacchia (1966) on page 276.

[3] A very similar idea in Lemma 1 can be found in Cellina (1969). However, the formulation of the mapping f in the lemma differs from that of Theorem 1 in Cellina (1969) since both mappings are based on 2 different structures of finite covering. Besides, this version was written before reading his paper.

combination of vectors. We will later see that Lemma 1 below is the critical factor in the proof of a generalized Hartman-Stampachia theorem and enables us to use the compactness argument.

Lemma 1. *Let C be a non-empty and compact set of $\mathbb{R}^N$, ζ a non-empty valued correspondence from C into $\mathbb{R}^N$. Let $r > 0$. There exists a continuous mapping f from C into $\mathbb{R}^N$ satisfying the following condition:*

Condition R. *For each $x \in C$, there are at most $N+1$ vectors $z^1, \ldots, z^{N+1}$ in $\zeta\big(B(x,r)\big)$ and positive numbers $\beta_1, \ldots, \beta_{N+1}$ such that* [4]

$$f(x) = \sum_{i=1}^{N+1} \beta_i z^i \tag{2.2}$$

with $\sum_{i=1}^{N+1} \beta_i = 1$.

See the Proof of Lemma 1 in Appendix A on page 14.
Note that the $N+1$ vectors $z^1, \ldots, z^{N+1}$ and numbers $\beta_1, \ldots, \beta_{N+1}$ are allowed to depend on the parameters of x, r and M functions $(\alpha_i)_{i=1}^M$ (see these functions in the proof). However for simplicity, these elements are omitted.

Remark 3. *We could explain this lemma as follows: the value $f(x)$ of a continuous mapping is expressed as a convex combination of, at most, length $N+1$ of elements belonging to $\zeta\big(B(x,r)\big)$. The feature, in which the length of the combination is fixed, enable us to deploy the "compactness argument".*

Remark 4. *The conclusion of Lemma 1 still holds under two extra assumptions. More precisely, in addition to the hypotheses of Lemma 1, we assume that C is convex in $\mathbb{R}^N$ and the correspondence ζ is from C into C. The conclusion of the existence of a continuous mapping f satisfying Condition R is the same. Besides, the mapping f is admitted from C into C.*

We now introduce an extension of Hartman-Stampacchia theorem (Theorem 1) to correspondence. Such extension concerns some characteristics of the correspondence. The extension of Hartman-Stampacchia to *upper semi-continuous correspondence* seems to be the key in getting a proof of the generalized GND lemma. In this case, the generalization of the theorem is precisely stated in Theorem 2. The corresponding proof of the theorem will be relied on the original theorem and the "compactness argument", involving the concept of *unity subordinated to a covering* and *Caratheodory convexity theorem*. We also provide another generalized Hartman-Stampacchia theorem with *lower semi-continuity*, expressed in Corollary 1, where the proof makes use of a continuous selection theorem, based on the work of Michael (1956).

[4] Note the convention that superscripts are used for labelling vectors while subscripts give real numbers. For example, as in Condition R , the parameters $\beta_1, \ldots, \beta_{N+1}$ are real numbers and $z^1, \ldots z^{N+1}$ are vectors belonging to the finite dimensional space $\mathbb{R}^N$.

Theorem 2 (Hartman-Stampacchia theorem for convex-valued and upper semi-continuous correspondence)**.** *Let C be a compact and convex set in $\mathbb{R}^N$, ζ a non-empty, convex and compact valued correspondence from C into $\mathbb{R}^N$. If ζ is upper semi-continuous, then there are some $x \in C$ and $z \in \zeta(x)$ such that*

$$\langle p, z\rangle \leq \langle x, z\rangle \quad \forall p \in C.$$

See Proof of Theorem 2 in the Appendix A on page 14.

Corollary 1 (Hartman-Stampacchia theorem for lower semi-continuous correspondence)**.** *Let C be a compact and convex set in $\mathbb{R}^N$, ζ a non-empty, convex and compact valued correspondence from C into $\mathbb{R}^N$. If ζ is lower semi-continuous, then there are some $x \in C$ and $z \in \zeta(x)$ such that*

$$\langle p, z\rangle \leq \langle x, z\rangle \quad \forall p \in C.$$

See Proof of Corollary 1 in Appendix A on page 15.

3. Gale-Nikaido-Debreu Lemmas

We aim in this section to provide proofs of not only Gale-Nikaido-Debreu lemma (Theorem 3) but also its generalized version (Theorem 4) by the means of Hartman-Stampacchia theorem. These lemmas are precisely recalled in Section 3.1. The main arguments involved in the proofs are Lemma 1 or Hartman-Stampacchia theorem for convex-valued and upper semi-continuous correspondence and the concept of retract mapping. The former is introduced in Section 2.2. The latter and a supporting lemma are given in Section 3.2. Finally, by the means of Hartman-Stampacchia theorem (Theorem 1) and its generalized version (Theorem 2), we give the direct proofs in Sections 3.3 and 3.4 respectively.

3.1. Gale-Nikaido-Debreu Lemma and its Generalized Version

Arrow and Debreu (1954) prove a fundamental equilibrium existence result of theoretical economics. Later with the papers of Gale (1955) Debreu (1956) and Nikaidô (1956), a celebrated formulation so-called Gale-Nikaido-Debreu lemma provides more inside economic explanation. The core argument for the classic proof[5] of the lemma is based on the Kakutani fixed point theorem. Let us recall the GND lemma.

Theorem 3 (Gale-Nikaido-Debreu lemma)**.** *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the following condition:*

$$\forall p \in \Delta, \;\; \forall z \in \zeta(p), \;\; \langle p, z\rangle \leq 0. \tag{3.1}$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

[5]We refer to Debreu (1959) or Debreu (1982) for more details.

Remark 5. *An equivalent statement of Theorem 3 is obtained by replacing condition (3.1) by condition (3.2) below*

$$\forall p \in \Delta,\ \exists z \in \zeta(p) \text{ such that } \langle p, z\rangle \leq 0. \tag{3.2}$$

It is clear that condition (3.1) implies condition (3.2). Conversely, assume that the correspondence ζ satisfies condition (3.2). We define the correspondence $\zeta' : \Delta \to \mathbb{R}^N$ by $\zeta'(p) = \{z \in \zeta(p) : \langle p, z\rangle \leq 0\}$. It follows that ζ' is non-empty, convex, compact valued and upper semi-continuous correspondence from Δ into $\mathbb{R}^N$ such that

$$\forall p \in \Delta,\ \forall z \in \zeta'(p), \quad \langle p, z\rangle \leq 0.$$

From Theorem 3, there exits $\bar{p} \in \Delta$ such that $\zeta'(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$. Since $\zeta'(\bar{p}) \subset \zeta(\bar{p})$, it follows $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

If the correspondence ζ is a mapping, Theorem 3 is restated as follows:

Theorem 3′. *Let Δ be the unit-simplex of $\mathbb{R}^N$, ζ a continuous mapping from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the following condition:*

$$\forall p \in \Delta, \quad \langle p, \zeta(p)\rangle \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \in \mathbb{R}^N_-$.

Now we turn our attention to a generalization of Gale-Nikaido-Debreu lemma which is established by Geistdoerfer-Florenzano (1982). One of the interest of Geistdoerfer-Florenzano (1982) is to give a proof of the lemma without Kakutani fixed point argument, where she provides the proof by contradiction[6]. It makes use of the strict separation theorem for two disjoint convex sets (one of them is closed, the other one is compact) and a partition of unity subordinated to a covering, together with the Brouwer theorem.

Theorem 4 (Florenzano(1982))**.** *Let P be a closed convex cone with vertex 0_N in $\mathbb{R}^N$. Let ζ be an upper semi-continuous and non-empty, compact convex valued correspondence from $\bar{B} \cap P$ into $\mathbb{R}^N$. If ζ satisfies the condition*

$$\forall p \in S \cap P, \quad \exists z \in \zeta(p) \text{ such that } \langle p, z\rangle \leq 0, \tag{3.3}$$

then there exists $\bar{p} \in \bar{B} \cap P$ such that $\zeta(\bar{p}) \cap P^\circ \neq \emptyset$.

Remark 6. *Obviously, without loss of generality, we can replace condition (3.3) by*

$$\forall p \in S \cap P, \quad \forall z \in \zeta(p) \text{ such that } \langle p, z\rangle \leq 0. \tag{3.4}$$

Note that Florenzano and Le Van (1986) provided the following example, showing that in general the vector $\bar{p}$ of Theorem 4 might be 0_N.

[6] See Lemma 1 in Geistdoerfer-Florenzano (1982) on page 115 or Lemma 2.1.1 in Florenzano (2003) on page 45

Example 1. *Consider a cone $P = \mathbb{R}^2$ and a single-valued correspondence ζ from $\bar{B}(0,1)$ into $\mathbb{R}^2$ defined as follows: $\zeta(p) = -p$ for all $p \in \bar{B}(0,1)$. Obviously, all conditions of Theorem 4 hold.*

Indeed, it is easy to see that $\zeta(\bar{p}) \cap P^\circ \neq \emptyset$ if and only if $\bar{p} = 0_2$. In the case ζ is a single-valued correspondence, we could rewrite Theorem 4 as follows:

Theorem 4′. *Let P be a closed convex cone with vertex 0_N in $\mathbb{R}^N$. Let ζ be a continuous mapping from $\bar{B} \cap P$ into $\mathbb{R}^N$. If ζ satisfies the condition*

$$\forall p \in S \cap P, \quad \langle p, \zeta(p) \rangle \leq 0, \tag{3.5}$$

then there exists $\bar{p} \in \bar{B} \cap P$ such that $\zeta(\bar{p}) \in P^0$.

3.2. Retract Mapping and Supporting Lemma

By preparing the proof in the next sections, we introduce the concept of retract mapping. The existence of such a mapping is showed by structuring an explicit one in Lemma 2. This concept is used in the proof of Theorem 4 in the cases where the cone P is not a linear subspace of $\mathbb{R}^N$. Besides, we state and prove Lemma 3, a supporting lemma.

Definition 1 (Retract mapping)**.** *A subspace A of a topological space X is a* ***retract of*** *X if there is a continuous mapping $f : X \mapsto A$ such that $f(y) = y$ for all $y \in A$. The mapping f is called a* ***retraction of*** *X onto A.*

Lemma 2. *Let P be a closed convex cone in $\mathbb{R}^N$. If $P \subsetneq span(P)$, then there exists a retract r from $\bar{B} \cap P$ into $S \cap P$.*

See Proof of Lemma 2 in Appendix A on page 15.

Remark 7. *Lemma 2 means that if the cone P is not a vector space, then there is some retract mapping from $\bar{B} \cap P$ into $S \cap P$.*

Lemma 3. *(Supporting Lemma) Let P be a closed convex cone with vertex 0_N in $\mathbb{R}^N$. Let ζ be a correspondence from $\bar{B} \cap P$ into $\mathbb{R}^N$ satisfying condition:*

$$\forall p \in S \cap P, \quad \forall z \in \zeta(p) : \quad \langle p, z \rangle \leq 0. \tag{3.6}$$

If there are some $x \in \bar{B} \cap P$ and $z \in \zeta(x)$ such that

$$\langle p, z \rangle \leq \langle x, z \rangle \quad \forall p \in \bar{B} \cap P, \tag{3.7}$$

then $z \in P^\circ \cap \zeta(x)$.

See the Proof of Lemma 3 in Appendix A on page 17.

Remark 8. *Normally, the class of correspondence for which conditions (3.6) and (3.7) hold is upper semi-continuous. However, it is not the case in Lemma 3.*

Remark 9. *In the case ζ is single-valued, the conclusion of Lemma 3 means that there is some $x \in \bar{B} \cap P$ such that $\zeta(x) \in P^\circ$.*

3.3. Proof of Theorem 3 (Gale-Nikaido-Debreu Lemma)

In Section 3.3, we present the proof of Theorem 3. The proof splits into 2 cases associating with the correspondence ζ being either single- or multi-valued. Section 3.3.1 carries out single-valued correspondence. As for multi-valued correspondence, the proof is shown in Section 3.3.2. We directly apply Theorem 1 and the generalized Hartman-Stampacchia theorem (Theorem 2) to the cases respectively.

3.3.1. The Correspondence ζ is Single-valued Continuous

Proof. [7] In this case, we need to seek $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \in \mathbb{R}^N_-$. Indeed, applying Hartman-Stampacchia theorem to the mapping ζ on Δ, we obtain some $\bar{p} \in \Delta$ such that

$$\langle p, \zeta(\bar{p})\rangle \leq \langle \bar{p}, \zeta(\bar{p})\rangle \quad \forall p \in \Delta.$$

Since the hypothesis on ζ of Theorem 3 (or condition (3.1)) implies $\langle \bar{p}, \zeta(\bar{p})\rangle \leq 0$, we see that

$$\langle p, \zeta(\bar{p})\rangle \leq 0 \quad \forall p \in \Delta.$$

It is obvious that this implies $\zeta(\bar{p}) \in \Delta^\circ = \mathbb{R}^N_-$. □

3.3.2. The Correspondence ζ is Upper Semi-continuous with Non-empty, Compact, Convex Values

Proof. Since ζ is upper semi-continuous, we apply Theorem 2 with C replaced by Δ; consequently, we obtain $x \in \Delta$ and $z \in \zeta(x)$ such that

$$\langle p, z\rangle \leq \langle x, z\rangle \quad \forall p \in \Delta.$$

Since $z \in \zeta(x)$, by the hypotheses on ζ of Theorem 3 (or condition (3.1)), it follows $\langle x, z\rangle \leq 0$. Therefore,

$$\langle p, z\rangle \leq 0, \quad \forall p \in \Delta.$$

Equivalently, $z \in \Delta^\circ = \mathbb{R}^N_-$. We have proved that $z \in \zeta(x) \cap \mathbb{R}^N_-$. □

3.4. Proof of Theorem 4 (Florenzano(1982))

Section 3.4 is dedicated to prove Theorem 4 with condition (3.3) replaced by condition (3.4). One of the main aims of this paper is to give alternative and direct proofs for not only the original GND lemma (Theorem 3) but also its generalized version (Theorem 4). As a result of the extension, the proof of Theorem 4 is done with cost, i.e., the complicated proof with more tools deployed, compared with that of Theorem 3. To make the proof easy to read, each section below deals with separate cases of the correspondence: ζ is either single- or multi-valued. In both cases, when the cone P is not linear space of $\mathbb{R}^N$, the proofs are based on the concept of retract mapping together with Theorem 1 and Theorem 2 respectively; the existence result is a direct consequence of Lemma 3. When the cone P is a subspace of $\mathbb{R}^N$, in both cases, the existence results are reduced from Theorem 1 and Theorem 2 respectively.

[7] For the sake of providing intuition, we provide the proof for this case. Actually, the proof for this case could be viewed as to be included in the case of correspondence

3.4.1. The Correspondence ζ is Single-valued Continuous

Proof. In this case, we need to seek $\bar{p} \in \bar{B} \cap P$ such that $\zeta(\bar{p}) \in P^{\circ}$. The proof splits into 2 separate cases:

Case 1. $\boldsymbol{P \subsetneq span(P)}$

According to the hypothesis on ζ, the mapping ζ is continuous. By Lemma 2 on page 8, there is some retract $r : \bar{B} \cap P \to S \cap P$. Since ζ and r are continuous on $\bar{B} \cap P$, it follows that so is the mapping $\zeta \circ r$. We apply Theorem 1 to the mapping $\zeta \circ r$ on $\bar{B} \cap P$, and thus obtain some $\bar{x} \in \bar{B} \cap P$ such that

$$\langle p, \zeta \circ r(\bar{x}) \rangle \leq \langle \bar{x}, \zeta \circ r(\bar{x}) \rangle \quad \forall p \in \bar{B} \cap P. \tag{3.8}$$

We now deploy Lemma 3 with ζ replaced by $\zeta \circ r$, x by $\bar{x}$, z by $\zeta(r(\bar{x}))$ to prove that $\bar{p} = r(\bar{x})$ satisfies Theorem 4. It remains to verify conditions (3.6) and (3.7) of Lemma 3. On one hand, for condition (3.6), let $p \in S \cap P$. Since r is a retract mapping, it follows $r(p) = p$, and consequently $\zeta \circ r(p) = \zeta(p)$. Combining this with condition (3.4) (more precisely condition (3.5)), we obtain

$$\langle p, \zeta \circ r(p) \rangle = \langle p, \zeta(p) \rangle \leq 0,$$

implying that condition (3.6) holds. On the other hand, by inquality (3.8), condition (3.7) holds for $x = \bar{x}$ and $z = \zeta(r(\bar{x}))$. The proof is over.

Case 2. $\boldsymbol{P = span(P)}$

Appy Theorem 1 to the mapping ζ on $\bar{B} \cap P$, and obtain some $\bar{p} \in \bar{B} \cap P$ such that

$$\langle p, \zeta(\bar{p}) \rangle \leq \langle \bar{p}, \zeta(\bar{p}) \rangle \quad \text{for all } p \in \bar{B} \cap P. \tag{3.9}$$

We want to conclude that $\zeta(\bar{p}) \in P^0$; we split the argument into three subcases:

- If $\bar{p} \in int(\bar{B} \cap P)$, since $\zeta(\bar{p}) \in N_{\bar{B} \cap P}(\bar{p})$, where $N_{\bar{B} \cap P}(\bar{p})$ is the normal cone to $\bar{B} \cap P$ at $\bar{p}$, we conclude $\zeta(\bar{p}) = 0_N$. Consequently, $\zeta(\bar{p}) \in P^0$. Note that this circumstance happens only if the cone P is $\mathbb{R}^N$.

- If $\bar{p} \in \mathrm{ri}(\bar{B} \cap P)$, i.e., $\bar{p}$ belongs to the relative interior[8] of $\bar{B} \cap P$ then $\zeta(\bar{p}) \in N_{\bar{B} \cap P}(\bar{p})$. We know that since P is the subspace, it follows $N_{\bar{B} \cap P}(\bar{p}) = N_P(\bar{p}) = P^{\perp} = P^0$. Hence $\zeta(\bar{p}) \in P^0$.

- If $\bar{p} \notin \mathrm{ri}(\bar{B} \cap P)$, then $\bar{p} \in \mathrm{Bd^r}(\bar{B} \cap P)$, i.e., the relative boundary [9] of $\bar{B} \cap P$. Since P is the subspace, $\mathrm{Bd^r}(\bar{B} \cap P) = S \cap P$. Consequently, $\bar{p} \in S \cap P$. By the hypotheses on ζ, we deduce that $\langle \bar{p}, \zeta(\bar{p}) \rangle \leq 0$. Combining this with inequality (3.9), we obtain,

$$\langle p, \zeta(\bar{p}) \rangle \leq 0 \quad \text{for any } p \in \bar{B} \cap P. \tag{3.10}$$

  Since P is a cone , it follows that inequality (3.10) can be extended to any $p \in P$. As a result, $\zeta(\bar{p}) \in P^0$.

[8] For notion of relative interior and relative boundary, see, for example, Florenzano and Le Van (2001)'s Section 1.2.2 on page 11.

[9] See footnote 8

In conclusion, there is some $\bar{p} \in \bar{B} \cap P$ such that $\zeta(\bar{p}) \in P^0$. The proof for the mapping is over.

3.4.2. The Correspondence ζ is Upper Semi-continuous with Non-empty, Compact, Convex Values

Case 1. $\boldsymbol{P \subsetneq span(P)}$ By Lemma 2 on page 8, there is some retract mapping r from $\bar{B} \cap P$ into $S \cap P$. Since r is continuous and according to the hypotheses of Theorem 4, ζ is upper semi-continuous, we obtain that $\zeta \circ r$ is upper semi-continuous. Obviously, $\zeta \circ r$ is also non-empty, convex, compact valued.
We are now applying Theorem 2 with C replaced by $\bar{B} \cap P$, ζ by $\zeta \circ r$, and thus obtain $x \in \bar{B} \cap P$ and $z \in \zeta \circ r(x)$ such that

$$\langle p, z \rangle \leq \langle x, z \rangle \quad \forall p \in \bar{B} \cap P. \tag{3.11}$$

We verify that conditions (3.6) and (3.7) of Lemma 3 hold with ζ replaced by $\zeta \circ r$. On one hand, inequality (3.11) leads to condition (3.7) holding (with ζ replaced by $\zeta \circ r$). On the other hand, noting that $\zeta(x) = \zeta \circ r(x)$ for all $x \in S \cap P$, from the hypothesis on ζ of Theorem 4 (or condition (3.4)), we see that condition (3.7) of Lemma 3 holds. As a result of Lemma 3, we obtain $z \in P^\circ \cap \zeta \circ r(x)$.

Case 2. $\boldsymbol{P = span(P)}$
On one hand, according to Theorem 2 with C replaced by $\bar{B} \cap P$, we obtain that there are some $x \in \bar{B} \cap P$ and $z \in \zeta(x)$ such that

$$\langle p, z \rangle \leq \langle x, z \rangle \quad \forall p \in \bar{B} \cap P.$$

On the other hand, by the hypothesis on ζ of Theorem 4

$$\forall p \in S \cap P, \quad \forall z \in \zeta(p) \quad \langle p, z \rangle \leq 0. \tag{3.12}$$

Lemma 3 implies that $z \in P^\circ \cap \zeta(x)$. This concludes Theorem 4. $\square$

Remark 10. *In the statement of Theorem 4, it is possible that $\bar{p}$ equals 0_N. It is worth pointing out that in the direct proof of Theorem 4, $\bar{p}$ is different from 0_N when the cone P is not a linear subspace of $\mathbb{R}^N$.*

4. Brouwer and Kakutani Fixed Point Theorems

In Section 4, we first show in Proposition 1 that the Brouwer theorem is a direct consequence of the Hartman-Stampacchia theorem. Second, we demonstrate the proofs of the Kakutani theorem using the Brouwer fixed point and Hartman-Stampacchia arguments in Proposition 2.

Proposition 1 (Brouwer theorem)**.** *Let C be a non-empty, convex, compact set in $\mathbb{R}^N$. Let f be a continuous mapping from C into itself. Then, there exists a fixed point of f.*

Proof. Define $g(x) = f(x) - x$. Applying Hartman-Stampacchia theorem to the mapping g, we obtain some $\bar{x} \in C$ such that

$$\langle p, g(\bar{x})\rangle \leq \langle \bar{x}, g(\bar{x})\rangle \leq 0 \quad \forall p \in C.$$

We claim that $\bar{x}$ is a fixed point of f. Indeed, take $p = f(\bar{x}) \in C$. Then

$$\langle f(\bar{x}) - \bar{x}, f(\bar{x}) - \bar{x}\rangle \leq 0.$$

In other words, $f(\bar{x}) = \bar{x}$. □

Proposition 2 (Kakutani theorem). *Let C be a non-empty, convex, compact, subset of $\mathbb{R}^N$. Let ζ be a non-empty, convex, compact, valued correspondence from C into itself. If ζ is an upper semi-continuous, then there exists a fixed point of the correspondence ζ. That is, there exists some $x \in C$ such that $x \in \zeta(x)$.*

Proof of Proposition 2 using the Brouwer's fixed point theorem. Let (ε_k) be a non-negative sequence being decreasing and convergent to 0. According to Remark 4 on page 5, for any $k \in \mathbb{N}^*$, there is some continuous mapping f^k for which Condition R (on page 5) holds. From Proposition 1, there exists some fixed point x^k of the mapping f^k for any $k \geq 1$. Again, by Remark 4, there exist at most $N+1$ vectors $z^{1,k} \dots, z^{N+1,k}$ in $\zeta\big(B(x^k, \varepsilon_k)\big)$ and strictly positive numbers $\beta_{1,k}, \dots, \beta_{N+1,k}$ such that

$$f^k(x^k) = \sum_{i=1}^{N+1} \beta_{i,k} z^{i,k}, \tag{4.1}$$

with $\sum_{i=1}^{N+1} \beta_{i,k} = 1$. Now using the compactness argument shows the existence of a fixed point of ζ. Indeed, note that there exists $u^{i,k} \in \bar{B}$ such that $z^{i,k} \in \zeta(x^k + \varepsilon_k u^{i,k})$ for any $i = 1, \dots, N+1$ and $k \in \mathbb{N}^*$. Since the sequence $\Big(\big(x^k, (\beta_{i,k})_{i=1}^{N+1}, (u^{i,k})_{i=1}^{N+1}\big)\Big)_k$ in a compact set $C \times [0,1]^{N+1} \times \bar{B}^{N+1}$, without of loss generality, we might assume that the sequence converges to $\big(x, (\beta_i)_{i=1}^{N+1}, (u^i)_{i=1}^{N+1}\big)$. For any $i = 1, \dots, N+1$, since $\lim\limits_{k\to\infty} \big(x^k + \varepsilon_k u^{i,k}\big) = x$, by the compactness of $\zeta(x)$ and upper semi-continuity of ζ, we conclude that there is some $z^i \in C$ and a subsequence $(z^i_{n_k})$ such that $\lim\limits_{k\to\infty} z^i_{n_k} = z^i$ and $z^i \in \zeta(x)$. It is clear that $\sum_{i=1}^{N+1} \beta_i = 1$. The convexity of $\zeta(x)$ implies $\sum_{i=1}^{N+1} \beta^i z^i \in \zeta(x)$. As proved above, x^k is the fixed point of f^k and $\lim\limits_{k\to\infty} x^{n_k} = x$, implying that

$$\lim_{k\to\infty} f^{n_k}(x^{n_k}) = x.$$

On the other hand, the convergences of $\{\beta_i^{n_k}\}$ and $\{z^i_{n_k}\}$ implies

$$\lim_{k\to\infty} \sum_{i=1}^{N+1} \beta_i^{n_k} z^i_{n_k} = \sum_{i=1}^{N+1} \beta_i z^i.$$

Combining the above convergences with identity (4.1) proves $x = \sum_{i=1}^{N+1} \beta_i z^i$. Since the set $\zeta(x)$ is convex and $z^i \in \zeta(x)$ for $i = 1, \dots, N+1$, it follows that $\sum_{i=1}^{N+1} \beta_i z^i \in \zeta(x)$. As a result, $x \in \zeta(x)$. This concludes the existence of a fixed point of ζ. □

Proof of Proposition 2 using Hartman-Stampacchia theorem.
Let (ε_k) be a decreasing non-negative sequence converging to 0. By Remark 4, for any $k \in \mathbb{N}^*$, there is some continuous mapping f^k satisfying Condition R . Applying the Hartman-Stampacchia gives x^k such that

$$\langle f^k(x^k) - x^k, p - x^k \rangle \leq 0 \quad \forall p \in C.$$

Substituting p for $f^k(x^k)$ into the above inequality implies $f^k(x^k) = x^k$. Then we repeat the procedure of the proof of Proposition 2 using fixed point theorem on page 12 and conclude that there exists a fixed point of ζ. □

5. Hartman-Stampacchia and Brouwer Theorems

Theorem 5 (Generalized Hartman-Stampacchia theorem)**.** *Let C be a non-empty, compact and convex set of $\mathbb{R}^N$. Let ζ be a correspondence from X into $\mathbb{R}^N$. If ζ is lower or upper semi-continuous, then there exist $x \in C$ and $z \in \zeta(x)$ such that*

$$\langle p, z \rangle \leq \langle x, z \rangle \quad \forall p \in C. \tag{5.1}$$

Proof of Theorem 5. We consider three cases:

Case 1. ζ is single-valued. Let $g(x) = \pi_C(x + \zeta(x))$ for any $x \in C$, where π_C denotes the convex projection of $\mathbb{R}^N$ onto C. The mapping g is continuous from C into C. From Proposition 1 (Brouwer theorem), there is a fixed-point of g, i.e., $\bar{x} = g(\bar{x})$ or equivalently $\bar{x} = \pi_C(\bar{x} + \zeta(\bar{x}))$. In this case $\zeta(\bar{x}) = \bar{x} + \zeta(\bar{x}) - \bar{x}$ belongs to normal cone of C at $\bar{x}$. We get inequality (2.1) of Hartman-Stampacchia theorem.

Case 2. ζ is multi-valued upper semi-continuous See Theorem 2 on page 6.

Case 3. ζ is multi-valued lower semi-continuous See Corollary 1 on page 6. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

A. Appendix

A.1. Proof of Lemma 1

First we build a mapping f using the partition of unity subordinated to a covering. Second, we show that such a mapping satisfies Condition R .

By the compactness of C, there exists a finite covering of C, say $B(x^i, r)$, $i = 1, \ldots, M$. Let (α_i) be a partition of unity[10] over C subordinated to the covering $\{B(x^i, r)\}_{i=1}^M$. Take $y^i \in \zeta(x^i)$ for all $i = 1, \ldots, M$. Set

$$f(x) = \sum_{i=1}^{M} \alpha_i(x) y^i$$

for all $x \in C$. Obviously, the mapping f is continuous on C. Now we prove that Condition R holds for the mapping f. Indeed, fix some x in C. Let $J = \{i \in \mathbb{N} : 1 \leq i \leq M \text{ and } x \in B(x^i, r)\}$. Observe that $x \in \cap_{i \in J} B(x^i, r)$ implying that $x^i \in B(x, r)$ and $y^i \in \zeta(B(x, r))$ for all $i \in J$. Note that if $i \notin J$, it follows that $x \notin B(x^i, r)$ and thus that the partition of unity over C subordinated to the covering $\big(B(x^j, r)\big)_{j=1}^M$ implies $\alpha_i(x) = 0$. Consequently,

$$\begin{aligned} f(x) = \sum_{i=1}^{M} \alpha_i(x) y^i &= \sum_{i \in J} \alpha_i(x) y^i + \sum_{i \notin J} \alpha_i(x) y^i \\ &= \sum_{i \in J} \alpha_i(x) y^i. \end{aligned}$$

Since $\sum_{i \in J} \alpha_i(x) = 1$, we conclude that $f(x) \in co\Big(\zeta\big(B(x, r)\big)\Big)$. According to Carathéodory's convexity theorem [11], there exist at most $N+1$ vectors $z^1, \ldots, z^{N+1}$ in $\zeta\big(B(x, r)\big)$ and strictly positive numbers $\beta_1, \ldots, \beta_{N+1}$ such that

$$f(x) = \sum_{i=1}^{N+1} \beta_i z^i$$

with $\sum_{i=1}^{N+1} \beta_i = 1$. $\square$

A.2. Proof of Theorem 2

Let $(\varepsilon_k)_k$ be a non-negative sequence converging to 0. For any $k \geq 1$, apply Lemma 1 on page 5 with $r = \varepsilon_k$ and obtain some continuous mapping $f^k : C \to \mathbb{R}^N$ satisfying Condition

[10] For the notion of partition of unity, see, for instance, Aliprantis and Border (2006)'s Section 2.19. on page 66.

[11] Carathéodory (1907)'s convexity theorem states that: In an n-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination using no more than $n+1$ vectors from the set. For a simple proof, see Florenzano and Le Van (2001)'s Proposition 1.1.2 or Aliprantis and Border (2006)'s Theorem 5.32.

R. Applying Hartman-Stampacchia theorem on page 4 to the mapping f^k on C, we obtain some $x^k \in C$ such that

$$\langle p, f^k(x^k)\rangle \le \langle x^k, f^k(x^k)\rangle \quad \forall p \in C. \tag{A.1}$$

Again, according to Lemma 1, there exist[12] at most $(N+1)$ vectors $z^{1,k}, \dots, z^{N+1,k}$ in $\zeta\big(B(x^k, \varepsilon_k)\big)$ and positive numbers $\beta_{1,k} \dots, \beta_{N+1,k}$ such that

$$f^k(x^k) = \sum_{i=1}^{N+1} \beta_{i,k} z^{i,k} \tag{A.2}$$

with $\sum_{i=1}^{N+1} \beta_{i,k} = 1$. For any $i = 1, \dots, N+1$ and $k \in \mathbb{N}^*$, since $z^{i,k} \in \zeta\big(B(x^k, \varepsilon_k)\big)$, there is some $u^{i,k}$ in closed unit ball $\bar{B}$ such that $z^{i,k} \in \zeta\big(x^k + \varepsilon_k u^{i,k}\big)$. Observe that the sequence $\Big(\big(x^k, (\beta_{i,k})_{i=1}^{N+1}, (u^{i,k})_{i=1}^{N+1}\big)\Big)_k$ is in the compact set $C \times [0,1]^{N+1} \times \bar{B}^{N+1}$ implying that, without loss of generality, the sequence converges to $\big(x, (\beta_i)_{i=1}^{N+1}, (u^i)_{i=1}^{N+1}\big)$. Note that $\lim\limits_{k\to\infty} \varepsilon_k = 0$. Consequently, $\lim\limits_{k\to\infty} x^k + \varepsilon_k u^{i,k} = x$. In addition, since $\zeta(x)$ is compact and $z^{i,k} \in \zeta(x^k + \varepsilon_k u^{i,k})$, the upper semi-continuity of ζ implies that there is a subsequence $\big(z^{i,n_k}\big)_k$ being convergent to $z^i \in \zeta(x)$ for all $i = 1, \dots, N+1$. It is obvious from identity (A.2) that

$$\lim_{k\to\infty} f^{n_k}(x^{n_k}) = \sum_{i=1}^{N+1} \beta_i z^i := z. \tag{A.3}$$

Note that $z^i \in \zeta(x)$ and $\sum_{i=1}^{N+1} \beta_i = 1$. By the convexity of $\zeta(x)$, we obtain $z \in \zeta(x)$. Since $\lim\limits_{k\to\infty} x^k = x$, combining inequality (A.1) with identity (A.2) and convergence (A.3), we obtain

$$\langle p, z\rangle \le \langle x, z\rangle, \quad \forall p \in \Delta.$$

The proof of Theorem 2 is over. $\square$

A.3. Proof of Corollary 1

Indeed, since ζ is lower semi-continuous, due to Michael (1956)(Theorem 3.1‴), we obtain[13] a continuous selection mapping f of ζ. Applying Theorem 1 to mapping f with K replaced by C, we obtain $x \in C$ such that

$$\langle p, f(x)\rangle \le \langle x, f(x)\rangle \quad \forall p \in C. \tag{A.4}$$

Since f is a selection mapping of ζ, it follows that $f(x) \in \zeta(x)$. Define $z = f(x)$ to end the proof. $\square$

A.4. Proof of Lemma 2

Since $P \subsetneq span(P)$, then P is not a subspace of $\mathbb{R}^N$, consequently, there exists $a \in -P \backslash P$. We show in Claim 1 below that it is possible to choose such a satisfying $a \in P^0$.

[12] Upper indices mark vectors and lower indices real numbers.

[13] This is a particular case of Theorem 3.1‴ in Michael (1956). For detailed proof, see, e.g., Proposition 10 in Florenzano (1981) or Proposition 1.5.3 in Florenzano (2003) on page 31.

Claim 1. *There is some $a \in P^\circ \cap (-P) \cap P^{\mathsf{c}}$ and $a \neq 0_N$.*

Proof of Claim 1. Since P is not a subspace, there is some $x \in P$, but $x \notin -P$. Define y to be the orthogonal projection of x onto $-P$. Let $a = y + (-x)$. Since $x \notin -P$ and $-P$ is closed, it follows that $a \neq 0_N$. On one hand, because $-P$ is a convex cone, y and $-x$ belong to $-P$, hence a belongs to $-P$. On the other hand, by the choice of y,

$$\langle y - x, y - \bar{y} \rangle \leq 0 \text{ for all } \bar{y} \in -P.$$

Note that $\bar{y} = -nz \in -P$ with $n > 0$ and $z \in P$, and that $a = y - x$. Substituting them into the above inequality, we get

$$\langle a, \bar{y} + nz \rangle \leq 0.$$

This leads to

$$\langle a, z \rangle \leq -\frac{1}{n}\langle a, y \rangle \quad \text{for all } n > 0 \text{ and } z \in P.$$

Letting n go to infinity proves that$\langle a, z \rangle \leq 0$ for all $z \in P$. In other words, $a \in P^\circ$. Furthermore, since $a \in P^\circ$ and $\langle a, a \rangle > 0$, we can deduce that $a \notin P$ or $a \in P^{\mathsf{c}}$. The proof of Claim 1 is over.

Now we construct a retract mapping r. According to Claim 1, there exists $a \in P^\circ \cap (-P)$. Fix $x \in \bar{B} \cap P$. Consider the following equation with some real variable $\lambda_a(x)$:

$$||x + \lambda_a(x)(x - a)|| = 1.$$

This leads the quadratic equation:

$$||x - a||^2 \lambda_a^2(x) + 2\langle x, x - a \rangle \lambda_a(x) + ||x||^2 - 1 = 0. \tag{A.5}$$

Since $||x - a||^2 \left(||x||^2 - 1\right) \leq 0$, the quadratic equation has at least one non-negative solution. We are able to compute an explicit formula for this solution as follows:

$$\lambda_a(x) = \frac{-\langle x, x - a \rangle + \sqrt{\langle x, x - a \rangle^2 + (1 - ||x||^2)||x - a||^2}}{||x - a||^2}. \tag{A.6}$$

Let[14]

$$r(x) = x + \lambda_a(x)(x - a). \tag{A.7}$$

On one hand, by the construction, $||r(x)|| = 1$. On the other hand, $r(x)$ can be alternately described as $r(x) = \big(1 + \lambda_a(x)\big)x + \lambda_a(x)(-a)$. Because x and $-a$ are in the convex cone P and $\lambda_a(x) \geq 0$, it follows that $r(x)$ belongs to P. Therefore, we have constructed the well-defined mapping r from $\bar{B} \cap P$ to $S \cap P$. Since $\lambda_a(x)$ is continuous with respect to x on $\bar{B} \cap P$, then so is the mapping r. To end the proof, it remains to show that $r_{|S \cap P} = id_{S \cap P}$. Indeed, consider $x \in S \cap P$, then $||x|| = 1$. Since $a \in P^\circ$ and $x \in P$ then $\langle x, a \rangle \leq 0$ implying $\langle x, x - a \rangle \geq 0$. From (A.6) we get $\lambda_a(x) = 0$. Consequently, (A.7) leads to $r(x) = x$. $\square$

[14]By the construction, the retract r is dependent on the vector a.

A.5. Proof of Lemma 3

First we claim that

$$\langle x, z \rangle \leq 0.$$

Indeed, the proof of the claim splits into two cases:

Case 1. $\|x\| = 0$ **or** $\|x\| = 1$

- If $||x|| = 0$, then obvious $\langle x, z \rangle = 0$.
- If $||x|| = 1$, in the case $x \in S \cap P$. By condition (3.6), we consequently obtain $\langle x, z \rangle \leq 0$.

Case 2. $0 < \|x\| < 1$

Take $p = \frac{x}{||x||}$. Since P is a cone, $x \in P$, and $||p|| = 1$, it follows $p \in \bar{B} \cap P$. Inequality (3.7) implies that

$$\left(1 - \frac{1}{||x||}\right) \langle x, x \rangle \geq 0.$$

Note that $1 - \frac{1}{||x||} > 0$, hence $\langle x, z \rangle \leq 0$. We have finished proving the claim.

We now turn to show that $z \in P^\circ \cap \zeta(x)$. Since $\langle x, z \rangle \leq 0$, it follows from inequality (3.7) that

$$\langle p, z \rangle \leq 0 \quad \forall p \in \bar{B} \cap P. \tag{A.8}$$

Since P is a cone and $\{x \in P : ||x|| \leq 1\} \subset \bar{B} \cap P$, we could extend inequality (A.8) to all $p \in P$. In other words, $z \in P^\circ$. According to condition (3.7), $z \in \zeta(x)$. Therefore, we obtain $z \in P^\circ \cap \zeta(x)$, and this concludes the proof of Lemma 3. $\square$

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