Unpaid family labor and self-employment: Two multi-sector models of capitalist reproduction and endogenous cycles

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Abstract[2]

This paper presents two sectoral models of endogenous cycles that illustrate the relevance of unpaid family labor and self-employment for capitalist extended reproduction. In Model A, Sector 1 produces capital goods, Sector 2 produces consumption goods, and Sector 3 includes unpaid family labor producing consumption goods to improve working-class subsistence. In addition, Model B includes Sector 4 of self-employed who produce consumption goods and accumulate capital. The paper analytically proves that unpaid labor contributes to the stability of capitalist cycles, and it illustrates how self-employment may improve working-class living standards during the upper stage of these cycles.

Keywords: Unpaid family labor, self-employment, sectoral models of extended reproduction, endogenous cycles

JEL codes: B51, C63, O41

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1. Introduction

Most of the Marxian literature on analytical sectoral models of extended reproduction of capital has focused on the complex dynamics and interactions between an economic sector producing means of production ("capital goods") and other sector producing means of consumption ("consumption goods") (Marx, 2010 [1885]; Harris, 1972; Roemer, 1978; Dutt, 1988; Cajas Guijarro, 2022). Although these and similar sectors are relevant to understand the economic complexity of capitalism, other sectors also have a structural relevance although they tend to be forgotten by most analytical models. That is the case for two crucial economic sectors focused on the subsistence of the working class and their families: unpaid family labor and self-employed workers.[3] As Naidu (2022) claims, particularly in the Global South there are situations of truncated proletarianization where wages may be supplementary to other forms of economic production and income provided by "subsistence sectors" during the reproduction of the working classes. Therefore, capitalist production overlaps with multiple non-capitalist modes of production –like unpaid family labor and self-employment– when articulating "circuits of social reproduction." Similarly, Jaramillo (2018) indicates that in concrete capitalist societies multiple goods and services are produced through non-mercantile labor relations that tend to be excluded from economic calculations. In his perspective, even wage and self-employed workers dedicate a substantial proportion of their time to improving their subsistence through non-mercantile production.

In concordance with these and similar intuitions,[4] we think that Marxian analytical models should include sectors from both capitalist and non-capitalist modes of production to get better representations of the complexity associated with the process of extended reproduction of capital and the subsistence of the members of the working class and their families. With this idea in mind, the following paper illustrates the structural relevance of unpaid family labor and self-employment for the extended reproduction of capital by presenting two sectoral models of endogenous cycles (Models A and B) based on the Marxian sectoral models presented by Dutt (1988) and Cajas Guijarro (2022), and the model of economic cycles presented by Goodwin (1967). Model A considers three sectors: Sector 1 includes firms producing capital goods, Sector 2 includes firms producing consumption goods, and Sector 3 groups unpaid family labor producing consumption goods to improve the subsistence level of working-class families. In contrast, Model B is an extended version of Model A that includes Sector 4 of self-employed workers who produce consumption goods, contribute to the subsistence of working-class families, and receive financial support from wage workers to accumulate capital. Intuitively, Sector 4 represents an alternative mode of production that may improve the subsistence of the working class in particular when there are not enough jobs available in capitalist sectors. From these models, the paper mathematically proves that the existence of unpaid family labor contributes to the cyclical stability of capitalist extended reproduction (see Appendixes A.1 and A.2) and analytically identifies other relevant patterns. In

[3] Feminist economists have made relevant contributions to the analytical study of these sectors. For a review of recent postkeynesian and neoclassical contributions, see Blecker and Braunstein (2022). From a Marxian perspective, we can mention the theoretical model of unpaid housework and super-exploitation of labor presented by Duque García (2021), who also reviews relevant literature on the issue.

[4] For an intuitive discussion about how "peasants and housewives" (associated with self-employment and unpaid family labor, respectively), may be considered as subsistence producers within the process of expanded reproduction of capital, see Bennholdt-Thomsen (1982).

addition, numerical simulations illustrate how the existence of self-employment may contribute to improving the living standards of working-class families by increasing the subsistence level as well as the employment rate during the upper stage of capitalist cycles.

The rest of the paper is organized as follows. Section 2 studies Model A in analytical terms first by assuming an exogenous real wage (as in Dutt, 1988), and later by assuming that the growth rate of the real wage depends on the employment rate (as in Goodwin, 1967). In this section, we prove the structural relevance of unpaid family labor for the endogenous cycles of capitalist reproduction. Section 3 presents a preliminary analysis of Model B for both exogenous and endogenous distribution. This analysis is based on numerical simulations because of the complexity of the model and illustrates how self-employment may improve the living standards of the working class. Finally, Section 4 summarizes the main results obtained from both models, offers insights for future discussions, and concludes.

2. Model A: Unpaid family labor

2.1. Initial scheme and dynamics with an exogenous real wage

We begin our analytical discussion with model A, where we consider an economy divided into three sectors. Sector 1 ($i = 1$) includes capitalist firms producing means of production, also named capital goods. Sector 2 ($i = 2$) includes capitalist firms producing consumption goods. Sector 3 ($i = 3$) employs unpaid family labor, that is, members of working-class families who produce consumption goods for the subsistence of their own families without receiving economic remuneration. Given a fixed-coefficients technology, we may write labor productivity of sector i (q_i) as the ratio between production (Q_i) and effective labor (L_i):

$$q_i = \frac{Q_i}{L_i}, \qquad i = 1,2,3 \quad (1)$$

For the sake of simplicity, assume q_i is constant.[5] All sectors require capital goods for production,[6] therefore we may define capital-output ratios (σ_i) as follows:

$$\sigma_i = \frac{K_i}{Q_i}, \qquad i = 1,2,3 \quad (2)$$

where K_i is the fixed capital stock in sector i. Assume K_i depreciates at a rate δ_i. Following the Marxian sectoral model presented by Dutt (1988), assume labor and fixed capital are the only inputs into production for all sectors and they are used at full capacity. Therefore, σ_i is constant for all i.

To describe how fixed capital distributes between capitalist sectors, define the sectoral distribution of capital (for Sector 2) as the following ratio:

[5] Labor productivity depends on technology as well as on multiple social relations. For instance, productivity may depend on how workers and capitalists dispute the level of labor intensity during production (Cajas Guijarro and Vera, 2022).

[6] The term "capital goods" does not seem appropriate to represent the means of production employed by Sector 3. However, we consider both terms as synonyms for simplifying the exposition.

$$k = \frac{K_2}{K_1} \quad (3)$$

Since Sectors 1 and 2 are capitalist, the distribution of their income is given by:

$$p_i Q_i = wL_i + p_1 K_i(\delta_i + r_i), \qquad i = 1,2 \quad (4)$$

where p_i is the price of the good produced by sector i, workers in both sectors receive the same wage ratio (w) (we assume labor mobility), capitalists obtain a rate of profit (r_i) computed as the ratio between profits and money representing the value of fixed capital ($p_1 K_i$),[7] and depreciation costs ($p_1 K_i \delta_i$) are also considered.

It is useful to represent prices in relative terms. In this sense, define the relative price of capital goods to consumption goods (p) and the real wage (ω) as:

$$p = \frac{p_1}{p_2} \quad (5)$$

$$\omega = \frac{w}{p_2} \quad (6)$$

For the moment, assume the real wage is an exogenous constant. If we assume Sector 3 does not accumulate capital (it uses capital only for the subsistence of working-class families), then the demand for capital goods (D_1) includes the depreciation of capital from all sectors and the desires of capitalists from Sectors 1 and 2 to increase their capital stock. This demand is represented by:

$$D_1 = (\delta_1 + g_1)K_1 + (\delta_2 + g_2)K_2 + \delta_3 K_3 \quad (7)$$

where g_i is the desired growth rate of capital in sector i. Following Dutt (1988) and Cajas Guijarro (2022), the desired growth rates of capital in Sectors 1 and 2 are given by:

$$g_2 = g_1 + \mu(r_2 - r_1), \qquad \mu > 0 \quad (8)$$

This formulation assumes that, if the rate of profit in Sector 2 is higher than the rate of profit in Sector 1 ($r_2 > r_1$), then capitalists will invest more in Sector 2, and vice versa. Also, we assume capitalists save a constant proportion (s_i) of their income and they invest all their savings, as indicated by:

$$s_1 r_1 K_1 + s_2 r_2 K_2 = g_1 K_1 + g_2 K_2 \quad (9)$$

Concerning the demand for consumption goods, consider the following assumption. Workers use a constant fraction ($1 - x$) of their wages to buy consumption goods from Sector 2, and they use the rest of their income (x) to buy new capital goods that replace depreciated capital in Sector 3 (for instance, working-class families buy new stoves to replace the depreciated ones).[8] These and previous assumptions imply:

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[7] We assume firms pay wages at the end of the production period. Therefore, the wage fund is not considered as part of the initial capital.

[8] In a more complex setting, x may be the result of a maximization problem where workers split up their consumption expenditure between Sectors 2 and 3 given their budget constraints and their material needs for subsistence.

$$D_2 = (1-x)\omega(L_1 + L_2) + p[(1-s_1)K_1 r_1 + (1-s_2)K_2 r_2] \quad (10)$$

$$xw(L_1 + L_2) = \delta_3 p_1 K_3 \quad (11)$$

We complete this initial scheme by assuming that the real income used by workers to buy consumption goods from Sector 2 ($(1-x)\omega(L_1 + L_2)$) and the entire production of Sector 3 (Q_3) guarantee an average level of subsistence (θ) for all the labor power contributed by the members of working-class families. This assumption implies:

$$(1-x)\omega(L_1 + L_2) + Q_3 = \theta(L_1 + L_2 + L_3) \quad (12)$$

where L_3 is unpaid family labor.

Following Dutt (1988), we assume capital and production are fixed in the short run. Therefore, markets clear through price variations until demand equals output in each sector, implying the equivalence between desired and actual growth rates of capital in Sectors 1 and 2. For instance, assume the relative price adjusts in the short run according to the following dynamic equation:

$$p' = h\left(\frac{D_1 - Q_1}{Q_1}\right) \quad (13)$$

where $p' = \frac{dp}{dt}$ is the time derivative of p and h represents the sensibility of the relative price to the excess demand for capital goods. By combining (1) to (13) and assuming a simplified situation where sectors have equal productivities ($q_1 = q_2 = q$) and equal capital-output ratios ($\sigma_1 = \sigma_2 = \sigma_3 = 1$), capitalists from Sectors 1 and 2 have the same saving rate ($s_1 = s_1 = s$), fixed capital does not depreciate in capitalist sectors ($\delta_1 = \delta_2 = 0$), and $\mu = 1$, we obtain:[9]

$$p' = \frac{q\alpha[sk - p(1-s)] - \alpha\omega(s-x)(1+k)}{qp} \quad (14)$$

In short-run equilibrium, $p' = 0$, implying the demand-supply balance for Sector 1:

$$D_1 = Q_1 \quad (15)$$

Combining equations (4)-(7), (9)-(11), and (15), we get the balance for Sector 2:

$$D_2 = Q_2 \quad (16)$$

Therefore, the balance for Sector 1 implies a general equilibrium in the market of goods (Walras' law). In this context, the equilibrium value of the relative price is:

$$p = \frac{qsk - \omega(s-x)(1+k)}{q(1-s)} \quad (17)$$

Since p' is a decreasing function of p, the equilibrium price given by (17) is stable (Strogatz, 2015). At this price level, the sectoral rates of profit and the average subsistence of working-class families are equal to:

[9] All deductions were obtained using a *Wolfram Mathematica* notebook. Details are available upon request to the author.

$$r_1 = \frac{qsk - \omega[1 - x + k(s - x)]}{qsk - \omega(s - x)(1 + k)} \quad (18)$$

$$r_2 = \frac{(1 - s)(q - \omega)}{qsk - \omega(s - x)(1 + k)} \quad (19)$$

$$\theta = \frac{\omega\{q[s\delta_3 k(1 - x) + x(1 - s)] - \omega\delta_3(s - x)(1 - x)(1 + k)\}}{qs\delta_3 k - \omega[\delta_3 s + \delta_3 k(s - x) - x(1 - s + \delta_3)]} \quad (20)$$

Positive values for p, r_i, and θ are guaranteed if k and ω satisfy:

$$k > \frac{\omega(1 - x)}{qs - \omega(s - x)} \quad (21)$$

$$\omega < \frac{qs}{s - x} \quad (22)$$

where it is assumed that $s > x$ and $0 < \delta_3 < 1$. Concerning the desired growth rates of capital g_i at the short-run equilibrium, they are given by:

$$g_1 = \frac{qk[s(2 + k) - 1] + s\omega(1 + k)[s(1 - x) + k(s - x)]}{(1 + k)[qsk - \omega(s - x)(1 + k)]} \quad (23)$$

$$g_2 = \frac{(1 - s)[q - \omega x(1 + k)]}{(1 + k)[qsk - \omega(s - x)(1 + k)]} \quad (24)$$

Following Dutt (1988), in the long run, fixed capital in each sector changes over time because of capital accumulation. As a result, the sectoral distribution of capital (k) moves according to:

$$\frac{k'}{k} = g_k = g_2 - g_1 \quad (25)$$

where $\frac{k'}{k} = g_k$ is the growth rate of k. Substituting (23) and (24) into (25) results in the following differential equation:

$$\frac{k'}{k} = g_k = \frac{q[1 - s(1 + k)] + \omega(s - x)(1 + k)}{qsk - \omega(s - x)(1 + k)} \quad (26)$$

Setting $g_k = 0$ in (26) gives a long-run equilibrium value for k:[10]

$$k^{E1} = \frac{q(1 - s) + \omega(s - x)}{qs - \omega(s - x)} \quad (27)$$

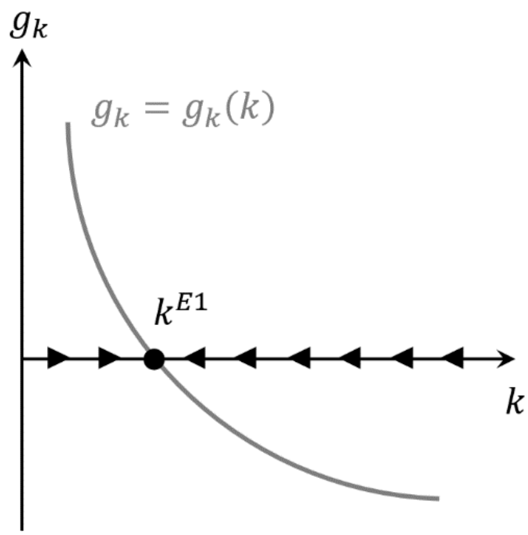
And taking the partial derivative of the right-hand side of (26) with respect to k gives:

$$\frac{\partial g_k}{\partial k} = -\frac{q(1 - s)[qs - \omega(s - x)]}{[qsk - \omega(s - x)(1 + k)]^2} \quad (28)$$

[10] We implicitly assume that q is constant. This assumption is removed in the next section.

Condition (22) guarantees that $\frac{\partial g_k}{\partial k} < 0$. Therefore, the equilibrium value k^{E1} is stable, as Figure 1 illustrates.

Figure 1. Long-run stable equilibrium with exogenous wage (model A)



In this context, we identify a relevant influence of unpaid family labor on the specialization of the economy between Sectors 1 and 2 in the long run. As equation (27) suggests, a higher proportion of wages used to buy capital goods for Sector 3 ($\uparrow x$) tends to reduce the equilibrium value of the sectoral distribution of capital ($\downarrow k^{E1}$). In other words, since Sector 3 requires capital goods to produce consumption goods and to guarantee the subsistence of working-class families, when this sector becomes bigger it increases the demand for capital goods produced by Sector 1 ($\uparrow D_1$). As a result, the scale of production of Sector 1 tends to grow faster than the scale of production of Sector 2. This tendency is represented by a lower long-run equilibrium value of the sectoral distribution of capital ($\downarrow k^{E1}$).

Concerning the average level of subsistence of working-class families (θ), at the long-run equilibrium with exogenous distribution this subsistence level takes the following value:

$$\theta^{E1} = \frac{q\omega[x(1-\delta_3)+\delta_3]}{\delta_3 q + x\omega} \quad (29)$$

The partial derivative of θ^{E1} with respect to x is:

$$\frac{\partial \theta^{E1}}{\partial x} = \frac{\delta_3 q\omega[q(1-\delta_3)-\omega]}{(\delta_3 q + x\omega)^2} \quad (30)$$

Therefore, if the depreciation rate δ_3 is sufficiently low to satisfy:

$$\delta_3 < \frac{q-\omega}{q} \quad (31)$$

Then, in the long run, a higher proportion of wages used to finance Sector 3 ($\uparrow x$) tends to increase the average subsistence level of working-class families ($\uparrow \theta^{E1}$) for a given real wage (ω). In fact, when $x = 0$, by (29) we get $\theta^{E1} = \omega$, and since (31) guarantees a positive partial derivative in (30), then when $x > 0$ we have $\theta^{E1} > \omega$. These results imply that the existence of unpaid family labor is crucial to expand the subsistence level of working-class families beyond the limits of the real wage, even in the long run. However, this long-run perspective becomes more complex when the real wage is endogenous.

2.2. Long-run dynamics with an endogenous wage

The previous section assumed an exogenous real wage as in Dutt (1988). However, Marxian interpretations like Goodwin (1967) and others suggest an endogenous real wage that changes over time depending on the negotiation power of the working class represented by the dynamics of the employment rate.[11] In addition, within a sectoral framework, the employment rate adjusts to changes in the sectoral distribution of capital. Therefore, from a Marxian perspective, it is plausible to assume that the real wage ω changes when k converges toward its long-run equilibrium value.

To illustrate this intuition, define the following employment rate for wage labor:

$$v = \frac{L_1 + L_2}{N} \quad (32)$$

where N is the labor supply. Combining (32) with previous equations gives:

$$v = \frac{K_1(1 + k)}{qN} \quad (33)$$

Applying logarithms and time differentiation to (33) results in a differential equation that describes the dynamics of the employment rate:

$$\frac{v'}{v} = \frac{k'}{1 + k} + g_1 - (n + \alpha), \qquad 0 < \alpha < 1, \qquad 0 < n < 1 \quad (34)$$

where $\alpha = \frac{q'}{q}$ and $n = \frac{N'}{N}$ are the growth rates of labor productivity and labor supply, respectively, and $g_1 = \frac{K_1'}{K_1}$ is the effective growth rate of capital in Sector 1 given by (23).

To represent the distributive effect of the employment rate, as in Goodwin (1967), we assume the following linear version of a real wage Phillips curve:

$$\frac{\omega'}{\omega} = -\gamma_0 + \gamma_1 v \quad (35)$$

where γ_0 and γ_1 may be interpreted as simple representations of the distribution of power between capitalists and workers when they negotiate the growth rate of the real wage (Cajas Guijarro and Vera, 2022). In addition, we define the wage share (with respect to capitalist production) as:

[11] Marx (2010 [1867]) recognized relevant interactions between capital accumulation and employment that may have consequences in terms of wages, labor extension, labor intensity, mechanization, and other dimensions of capitalist production. For some textual references see Eagly (1972), Cámara Izquierdo (2022), Cajas Guijarro and Vera (2022).

$$u = \frac{w(L_1 + L_2)}{p_1 Q_1 + p_2 Q_2} \quad (36)$$

Combining (36) with previous results gives:

$$u = \frac{(1-s)(1+k)\omega}{qk - \omega(s-x)(1+k)} \quad (37)$$

Applying logarithms and time differentiation to (37) we obtain a differential equation for the growth rate of the wage share:

$$\frac{u'}{u} = \frac{q\{k\omega'(1+k) - \omega[\alpha k(1+k) + k']\}}{\omega(1+k)[qk - \omega(s-x)(1+k)]} \quad (38)$$

Combining (23), (26), (34), (35), (37), and (38) gives a dynamical system in terms of the endogenous variables u, v, and k equal to:

$$\frac{u'}{u} = \frac{[1-s+u(s-x)]\{k[s-u(s-x)][1-(\alpha+\gamma_0-\gamma_1 v)] - k^2(\alpha+\gamma_0-\gamma_1 v)[s-u(s-x)] - (1-s) - u(s-x)\}}{k(1-s)(1+k)[s-u(s-x)]} \quad (39)$$

$$\frac{v'}{v} = \frac{s[1-(n+\alpha)(1+k)] - u\{s(1-n-\alpha) + (n+\alpha)[x-k(s-x)]\}}{(1+k)[s-u(s-x)]} \quad (40)$$

$$k' = \frac{1}{s-u(s-x)} - (1+k) \quad (41)$$

Setting $u' = v' = k' = 0$ gives non-trivial long-run equilibrium values for the three endogenous variables:

$$u^* = \frac{s-n-\alpha}{s} \quad (42)$$

$$v^* = \frac{\alpha + \gamma_0}{\gamma_1} \quad (43)$$

$$k^* = \frac{s(1-n-\alpha) - x(s-n-\alpha)}{sx + (n+\alpha)(s-x)} \quad (44)$$

To guarantee positive equilibrium values, we assume:

$$s > n + \alpha, \qquad x < \frac{s(1-n-\alpha)}{s-n-\alpha} \quad (45)$$

If the proportion x satisfies:

$$\frac{(n+\alpha)(1-s)}{s-n-\alpha} < x < \frac{s(1-n-\alpha)}{s-n-\alpha} \quad (46)$$

then it can be proved that the equilibrium point (u^*, v^*, k^*) of the dynamic system (39)-(41) is locally stable as time goes on (see Appendix A.1 for the mathematical proof). Also, it is possible to prove that when x is close to the critical value:

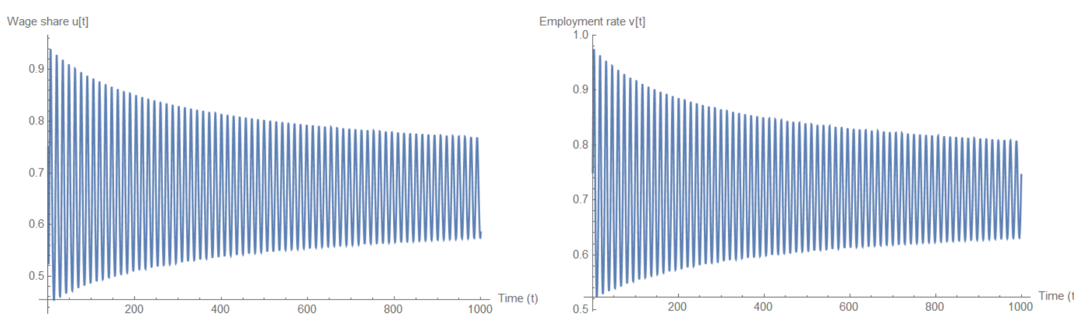
$$x^{HB} = \frac{(n+\alpha)(1-s)}{s-n-\alpha} \quad (47)$$

The dynamic system (39)-(41) has a persistent cyclical behavior (see Appendix A.2 for the mathematical proof). Numerical simulations of this system suggest that the periodic solutions identified in the neighborhood of x^{HB} are stable (supercritical Hopf bifurcation), that is, for initial values close to the equilibrium point (u^*, v^*, k^*) solutions are trapped to *stable limit cycles* when $x \to x^{HB}$, as Figures 2 and 3 suggest.[12] Numerical simulations also illustrate how economic cycles are highly sensitive to changes in the proportion x and, therefore, in the size of Sector 3. On the one hand, if x is considerably higher than the critical value x^{HB} ($x \gg x^{HB}$) (but below the upper bound imposed by (45) to obtain a positive equilibrium), the cyclical pattern of the system (39)-(41) takes the form of stable spirals that converge toward the equilibrium point (u^*, v^*, k^*) as time goes on (see Figures B.1 and B.2 in Appendix B). On the other hand, if $x \ll x^{HB}$ (but still positive), the cyclical pattern takes the form of unstable spirals that diverge from the equilibrium point until the model crashes (see Figures B.3 and B.4 in Appendix B).

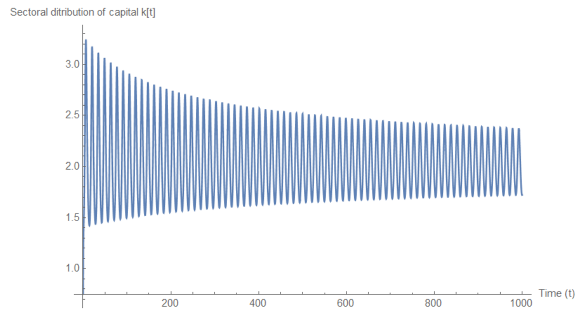
From these simulations, clockwise stable cycles can be identified in the wage share – employment rate plane $(u - v)$, like Goodwin's (1967) cycles. In contrast, counterclockwise stable cycles are observed when the wage share and the employment rate are compared with the sectoral distribution of capital (k). It is also remarkable that, on average, cycles in the $u - k$ plane exhibit a "positive slope," that is, as the wage share increases (decreases), the growth rate of capital in Sector 2 is higher (lower) than the growth rate in Sector 1, causing k to increase (decrease). In other words, when the cycle pushes income distribution in favor of the working class, there is a tendency to over-accumulate capital in Sector 2. And when the cycle pushes income distribution in favor of capitalists, there is a potential over-accumulation in Sector 1. However, a deeper theoretical discussion is needed to verify the generality of these patterns.

Figure 2. Simulation of time series and 2-D stable limit cycles (Model A)

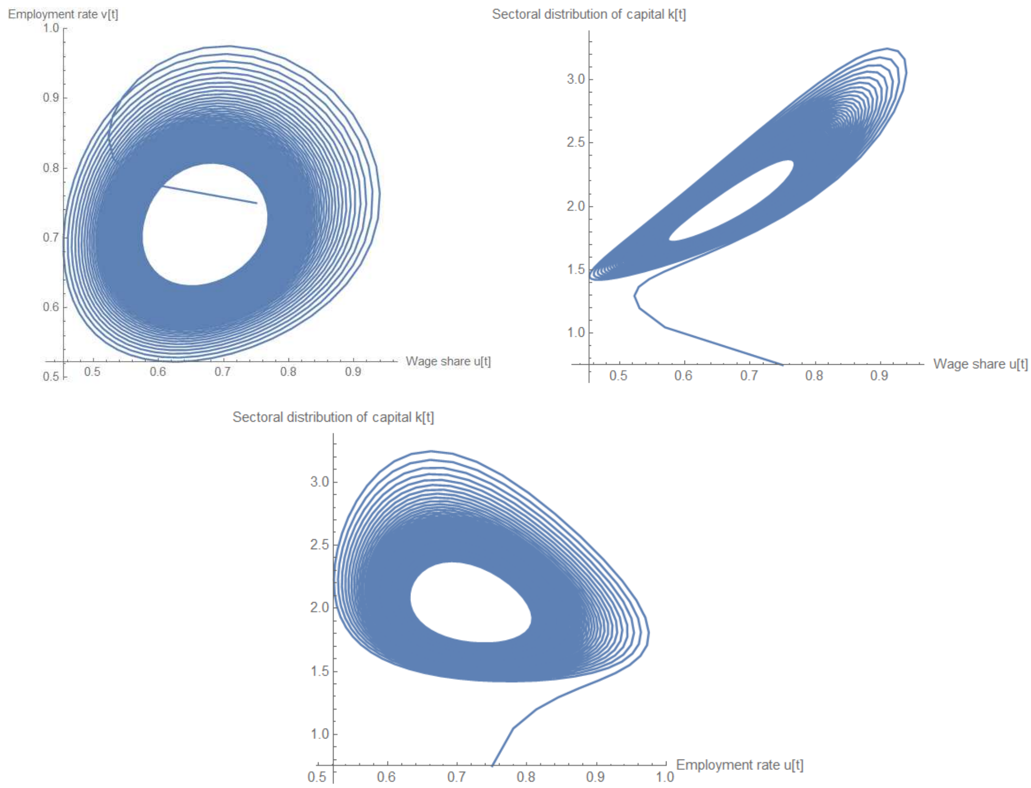
Time series



[12] All parameters and initial conditions used in this paper have been chosen only for illustrative purposes. Calibration or statistical adjustment to real economies is left for future discussion.

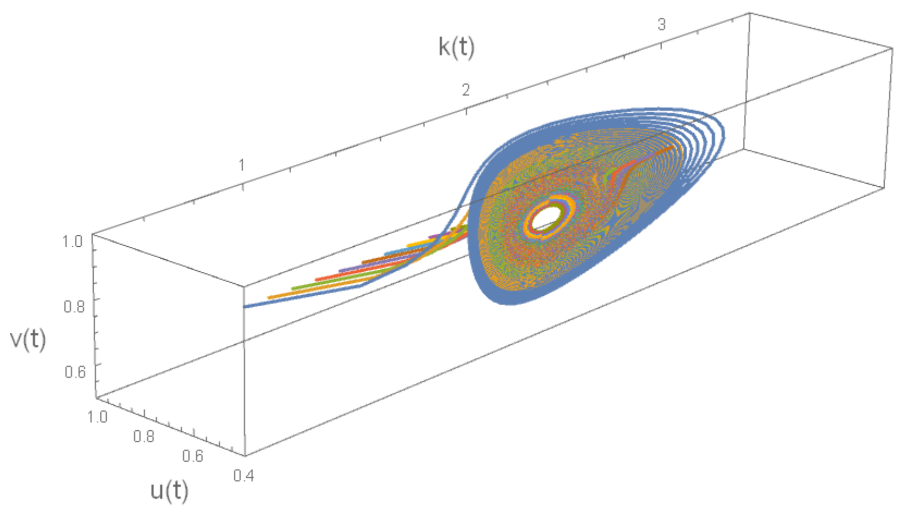


2-D Parametric plots



Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = k_0 = 0.75$

Figure 3. Simulations of 3-D stable limit cycles (Model A)



Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = 0.75, k_0 \in [0.75,\ 2.75]$

Concerning the average subsistence level of the working class, to illustrate its dynamics first define the subsistence-wage ratio (ϕ) as:

$$\phi = \frac{\theta}{\omega} \quad (48)$$

Combining (48) with (20) and (37) gives:

$$\phi = \frac{(1+k)\{x[(1-s)+u(s-x)] + \delta_3 k(1-x)[ux + s(1-u)]\}}{k\{xu + \delta_3[s(1+k) - u(s-x)(1+k)]\}} \quad (49)$$

If we give initial values to the endogenous variables u, v, and k then from (49) we know the initial value of the subsistence-wage ratio ϕ. Also, if we numerically simulate the dynamics of u, v, and k, then from (49) we can simulate the dynamics of ϕ. For simplicity, we represent this process in general terms by:

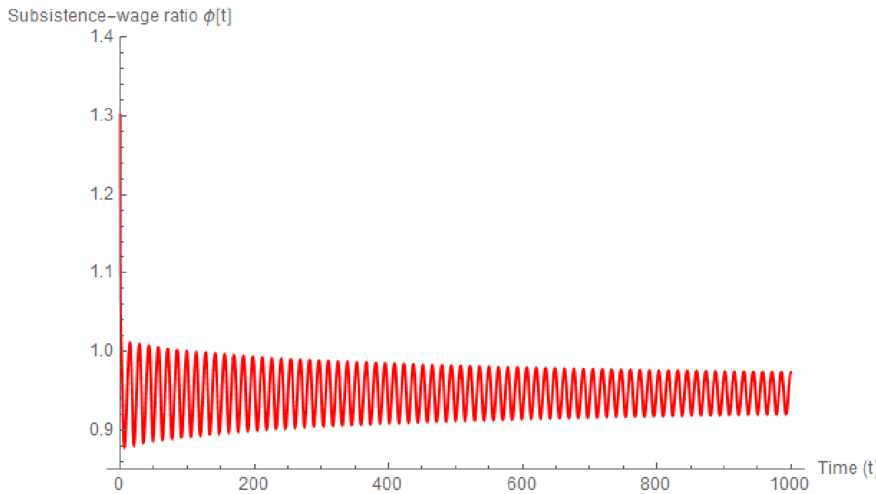
$$\phi_0 = \phi_0(u_0, v_0, k_0), \qquad \phi' = \phi'(u, v, k, u', v', k') \quad (50)$$

Numerical simulations presented above with $x = x^{HB}$ can be used to simulate (50) since ϕ does not influence the system (39)-(41). The result is illustrated in Figure 4 where the simulated time series of ϕ presents stable (limit) cycles as time goes on. These cycles suggest the existence of stages where the average subsistence level of working-class families θ grows faster (slower) than the real wage ω, so ϕ increases (decreases). It is also possible to identify stable spirals for ϕ when $x \gg x^{HB}$ and unstable spirals when $x \ll x^{HB}$.[13] Figure 4 also suggests an inverse relationship between the subsistence-real wage ratio and the sectoral distribution of capital.

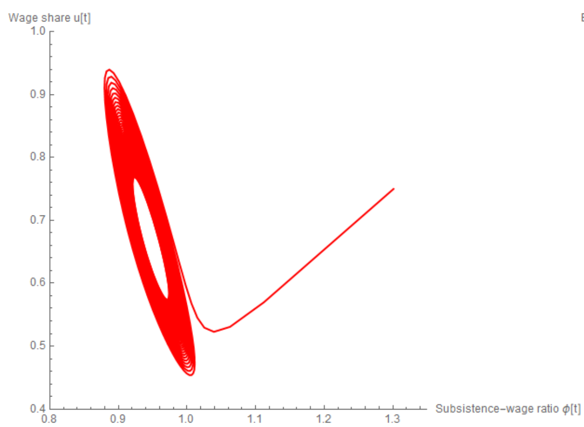
Figure 4. Simulations of stable limit cycles for the subsistence-wage ratio (Model A)

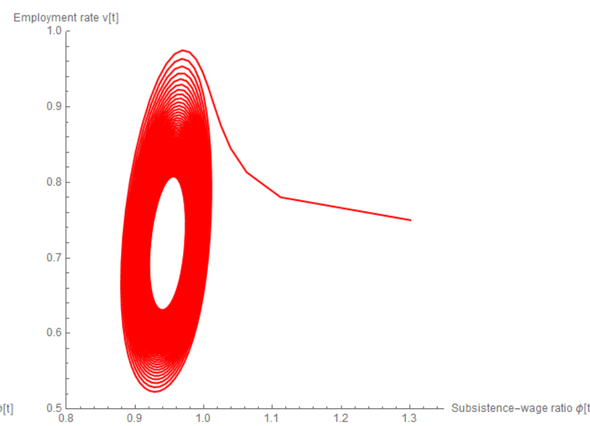
[13] These results are available upon request to the author.

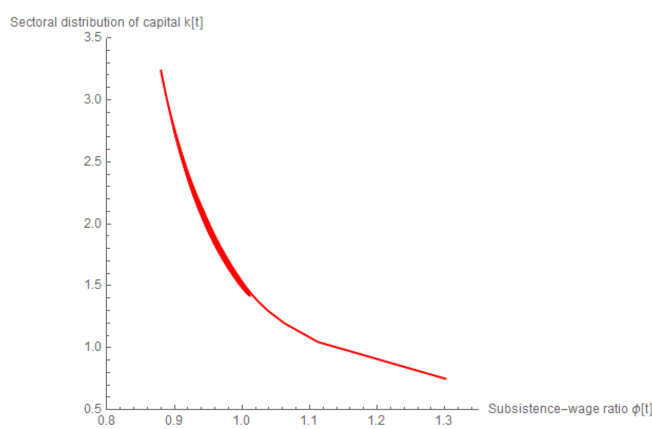
Time series



2-D Parametric plots







Note: Simulations using parameters $n = 0.1, s = 0.6, \delta_3 = 0.5, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = k_0 = 0.75$ ($\phi_0 = 1.301$)

The strong influence of x on the stability of the dynamical system (and the stability of the subsistence-real wage ratio) implies that the existence of unpaid family labor grouped in Sector 3 is not trivial. In particular, if labor supply grows as well as productivity ($n > 0, \alpha > 0$), then x should take positive values within the interval defined by (46) to guarantee stable cycles. In contrast, if $n > 0, \alpha > 0$, and $x = 0$ (unpaid family labor does not exist), then the system becomes unstable since it generates explosive spirals. Cajas Guijarro and Vera (2022) interpret this kind of instability as a potential *structural crisis* (different from the periodic crises that emerge during the business cycles) that can only be overcome through an exogenous change in the parameters of the model. Thus, when working-class families use a (positive) proportion x of their income to buy capital goods and combine them with unpaid family labor to produce part of the subsistence goods they consume, they contribute in some way to the stability of capitalist reproduction, at least within the conditions represented by Model A.

3. Model B: Unpaid family labor and self-employment

3.1. A new extended scheme with exogenous distribution

The previous section presented a model of extended reproduction with two capitalist sectors and a third sector that grouped unpaid family labor. Now, we present an extended version of the model (named model B) by including a new sector of self-employed workers (Sector 4) who produce consumption goods, compete with Sector 2, obtain an income from the sale of their products in the market, contribute to the subsistence of working-class families, and receive financial support from wage workers to accumulate capital.[14] Intuitively, Sector 4 represents an alternative mode of production that the working class may employ to improve their subsistence if there are not enough jobs available in capitalist sectors.[15] In this sense, define labor productivity and capital-output ratios as before but now consider four sectors:

$$q_i = \frac{Q_i}{L_i}, \qquad i = 1,2,3,4 \quad (51)$$

$$\sigma_i = \frac{K_i}{Q_i}, \qquad i = 1,2,3,4 \quad (52)$$

Concerning the sectoral distribution of capital between Sectors 1, 2, and 4 (those that accumulate capital), it is described by the following ratio:

$$k_i = \frac{K_i}{K_1}, \qquad i = 2,4 \quad (53)$$

For capitalist sectors ($i = 1,2$), income distribution keeps its original structure:

$$p_i Q_i = wL_i + p_1 K_i(\delta_i + r_i), \qquad i = 1,2 \quad (54)$$

[14] As in the case of Sector 3, the term “capital goods” does not seem appropriate for representing the means of production employed and accumulated by Sector 4. The term is used only to simplify the exposition.
[15] The mode of production of Sector 4 may be close to the notion of “simple commodity production” (Marx, 2010 [1867]).

In contrast, income distribution in Sector 4 includes depreciation costs and economic returns to self-employed labor, as represented by:

$$p_2 Q_4 = p_1 K_4 \delta_4 + \rho w L_4 \quad (55)$$

where we assume consumption goods produced by Sectors 2 and 4 have the same price (both sectors compete in the same market) and, for the sake of simplicity, economic returns to self-employed labor are expressed as a proportion ρ of the wage rate paid in Sectors 1 and 2. Since consumption goods from Sectors 2 and 4 have the same price, the relative price and the real wage can be represented as before:

$$p = \frac{p_1}{p_2} \quad (56)$$

$$\omega = \frac{w}{p_2} \quad (57)$$

In this context, demand for capital goods includes depreciation from all sectors and desired accumulation of capital from Sectors 1, 2, and 4:

$$D_1 = (\delta_1 + g_1)K_1 + (\delta_2 + g_2)K_2 + \delta_3 K_3 + (\delta_4 + g_4)K_4 \quad (58)$$

As in Model A, we assume the desired growth rates of capitalist sectors satisfy:

$$g_2 = g_1 + \mu(r_2 - r_1) \quad (59)$$

$$s_1 r_1 K_1 + s_2 r_2 K_2 = g_1 K_1 + g_2 K_2 \quad (60)$$

Concerning the accumulation of capital in Sector 4 ($g_4 K_4$), we assume it is financed by a proportion z of the total income of the working class (wage and self-employed workers), as suggested by:

$$g_4 p_1 K_4 = zw(L_1 + L_2 + \rho L_4) \quad (61)$$

Also, we assume wage and self-employed workers use a constant fraction x of their income to buy new capital goods that replace depreciated capital in Sector 3:

$$\delta_3 p_1 K_3 = xw(L_1 + L_2 + \rho L_4) \quad (62)$$

Given these assumptions, demand for consumption goods can be written as:

$$D_2 = (1 - x - z)\omega(L_1 + L_2 + \rho L_4) + p[(1 - s_1)K_1 r_1 + (1 - s_2)K_2 r_2] \quad (63)$$

In addition, assume the real income gained by wage and self-employed workers and the entire production of Sector 3 sustain a new average level of subsistence for all the labor power contributed by the members of working-class families:

$$(1 - x - z)\omega(L_1 + L_2 + \rho L_4) + Q_3 = \theta(L_1 + L_2 + L_3 + L_4) \quad (64)$$

For the sake of simplicity, assume the economy is already in a short-run equilibrium where the relative price adjusts to guarantee the demand-supply balance for Sector 1:

$$Q_1 = D_1 \quad (65)$$

As before, it can be proved that the demand-supply balance for capital goods guarantees the balance for consumption goods ($Q_2 + Q_4 = D_2$). Given this short-run equilibrium, we assume the desired growth rates of capital in Sectors 1, 2, and 4 are equal to the actual rates. Therefore, the sectoral distributions of capital change according to the following dynamic equation:

$$\frac{k_i'}{k_i} = g_i - g_1, \qquad i = 2{,}4 \quad (66)$$

By combining (51) to (66) and assuming a simplified version of the model ($q_i = q, \sigma_i = 1; s_i = s; \delta_1 = \delta_2 = \delta_4 = 0; \mu = 1$), we get a complex dynamical system in terms of k_2 and k_4 that, for the sake of simplicity, we represent in general terms by:

$$k_2' = k_2'(k_2, k_4), \qquad k_4' = k_4'(k_2, k_4) \quad (67)$$

Setting $k_2' = k_4' = 0$ in (67) gives three long-run equilibrium points with exogenous distribution, but only one of them gives positive values for all endogenous variables. This positive point is given by:

$$k_2^{E2} = \frac{q[s(q-\omega) - zq]}{s(q-\omega)[q(s-z) - \omega(s-x-z)]} - 1 \quad (68)$$

$$k_4^{E2} = \frac{zq\omega}{s(q-\omega)[q(s-z) - \omega(s-x-z)]} \quad (69)$$

where we assume $s > z$. A formal proof of the local stability of (k_2^{E2}, k_4^{E2}) is left for future discussion. Instead, numerical simulations of the model suggest this equilibrium is stable when productivity is sufficiently high and the real wage is sufficiently low, as Figure 5 illustrates. In other words, it is possible to identify stable situations where the four sectors presented in this model coexist. Also, Figure 5 suggests that a higher proportion of the total income of the working class used to buy consumption goods produced by Sector 4 ($\uparrow z$) tends to change the long-run sectoral distribution of capital with a relative lower accumulation in Sector 2 ($\downarrow k_2^{E2}$) and a higher accumulation in Sector 4 ($\uparrow k_4^{E2}$).

For the case of the average level of subsistence of working-class families, by combining (51) to (66) with the long-run equilibrium values with exogenous distribution given by (68) and (69), we get:

$$\theta^{E2} = \frac{sq\omega\{x + \delta_3[1-(x+z)]\}}{sx\omega + q\delta_3(s-z)} \quad (70)$$

The partial derivatives of θ^{E2} with respect to x is:

$$\frac{\partial\theta^{E2}}{\partial x} = \frac{sq\omega\delta_3[q(s-z)(1-\delta_3) - s\omega(1-z)]}{[sx\omega + q\delta_3(s-z)]^2} \quad (71)$$

If productivity is high enough to satisfy:

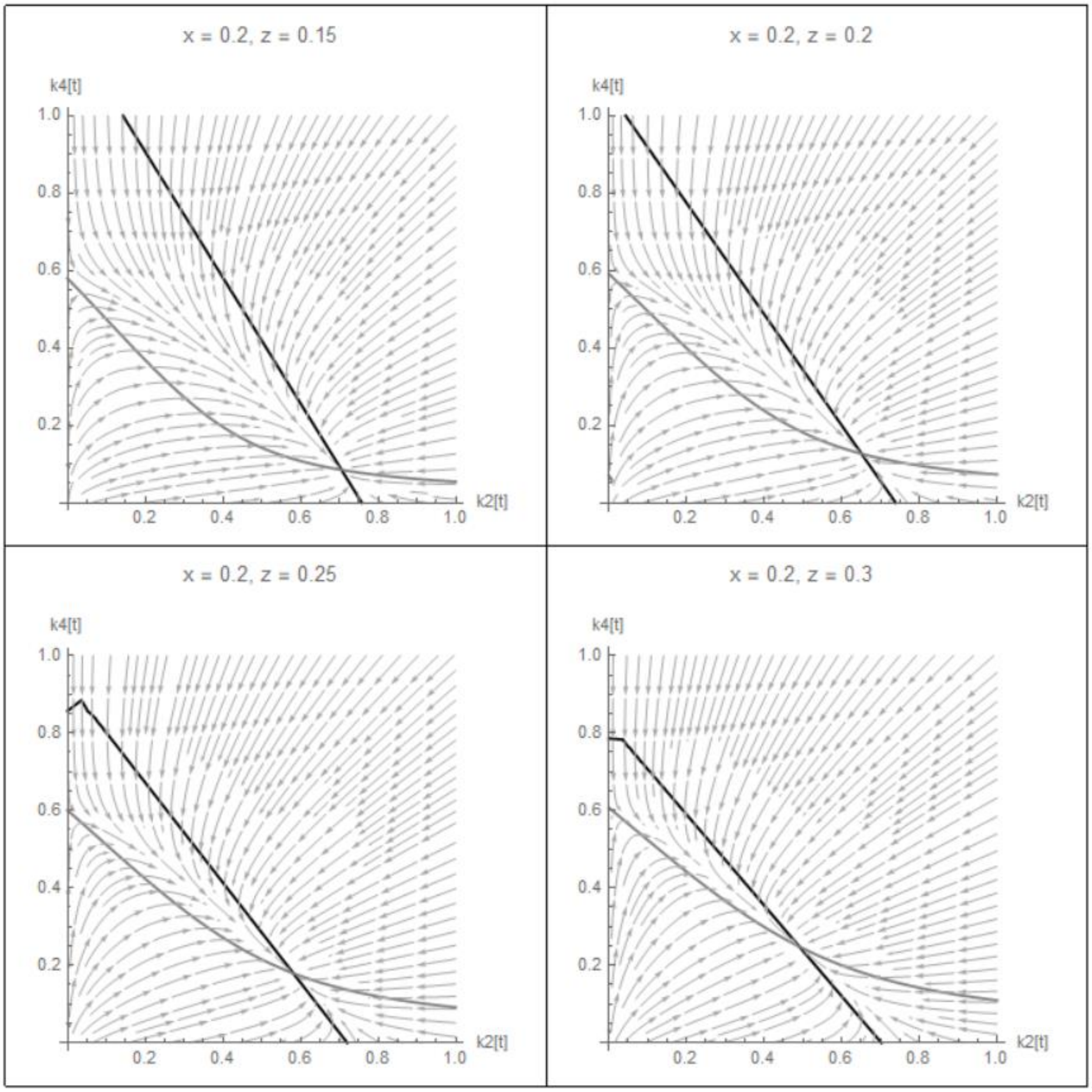
$$q > \frac{s\omega(1-z)}{(s-z)(1-\delta_3)} \quad (72)$$

Then, equation (71) represents a positive partial derivative. Therefore, as in Model A, the long-run level of subsistence θ^{E2} increases when x is higher and distribution is exogenous. On the other hand, the partial derivative of θ^{E2} with respect to z is:

$$\frac{\partial\theta^{E2}}{\partial z} = \frac{sq\omega\delta_3[x(q - s\omega) + q\delta_3(1 - s - x)]}{[sx\omega + q\delta_3(s - z)]^2} \quad (73)$$

If we assume $s + x < 1$ and $q > \omega$, then a higher proportion z of the working-class income used to finance Sector 4 also increases the subsistence level θ^{E2}. These results imply that the expansion of Sector 4 is a relevant alternative to improve the average subsistence of working-class families, with the advantage that self-employed workers in this sector receive an economic remuneration for their labor and, at the same time, the working class accumulates means of production and reinforces its competition with Sector 2.

Figure 5. Long-run stable equilibrium with exogenous wage (Model B, numerical simulations)



Note: Simulation using parameters $q = 4, s = 0.6, \omega = 0.5, x = 0.2, z \in \{0.05,0.2,0.35,0.5\}$. In all cases, the dynamical system given by (67) has a Jacobian matrix with a negative trace and a positive determinant when evaluated at (k_2^{E2}, k_4^{E2}).

3.2. Numerical simulations and sensitivity analyses with endogenous distribution

This section extends Model B, which includes unpaid family labor (Sector 3) and self-employed labor (Sector 4), to the case of endogenous distribution. In this sense, we define a new aggregated employment rate including both wage and self-employed workers, given by:

$$v = \frac{L_1 + L_2 + L_4}{N} \quad (74)$$

Combining previous equations with (74) gives:

$$v=\frac{K_1(1+k_2+k_4)}{qN} \quad (75)$$

Applying logarithms and time differentiation to (75) results in:

$$\frac{v'}{v}=\frac{k_2'+k_4'}{1+k_2+k_4}+g_1-(n+\alpha) \quad (76)$$

where, as before, α and n are the growth rates of labor productivity and labor supply, respectively. By combining equations (51) to (66), it can be proved that g_1 is equal to:

$$g_1=\frac{q\{k_4(x+z)(s+k_2)-k_2[1-s(2+k_2)]\}-\omega(1+k_2)[s(1-x-z)+k_2(s-x-z)]}{(1+k_2)\{q[sk_2+k_4(x+z)]-\omega(s-x-z)(1+k_2)\}} \quad (77)$$

To include endogenous distribution in the model, we assume an extended version of the Phillips curve used by Goodwin (1967) where both wage and self-employed workers contribute to the bargaining power of the working class when negotiating the growth rate of the real wage. The intuition behind this assumption is that a higher employment rate for both types of workers implies a lower unemployment rate, although wage and self-employed labor may have different weights during the negotiation process.[16] In this sense, assume the following real wage Phillips curve:

$$\frac{\omega'}{\omega}=-\gamma_0+\gamma_1\left(\frac{L_1+L_2}{N}\right)+\gamma_2\left(\frac{L_4}{N}\right), \qquad \gamma_1>\gamma_2 \quad (78)$$

Combining (78) with previous equations gives:

$$\frac{\omega'}{\omega}=-\gamma_0+v\frac{(\gamma_1+\gamma_1k_2+\gamma_2k_4)}{1+k_2+k_4} \quad (79)$$

As in Model A, we visualize the distributive effect of the bargaining power of the working class through the wage share in terms of capitalist production:

$$u=\frac{w(L_1+L_2)}{p_1Q_1+p_2Q_2} \quad (80)$$

Combining (80) with previous results gives:

$$u=\frac{\omega(1-s)(1+k_2)}{q[k_2+k_4(x+z)]-\omega(s-x-z)(1+k_2)} \quad (81)$$

Applying logarithms and time differentiation to (81) gives a complex differential equation that describes the dynamics of u. In general terms, this equation is represented by:

$$u'=u'(u,\omega,\omega',k_2,k_2',k_4,k_4') \quad (82)$$

[16] For instance, in many countries self-employed workers are less organized and more informal than wage workers. Therefore, the "rate of self-employment" may have a lower impact on reducing the threat of dismissal than the rate of employment for wage labor.

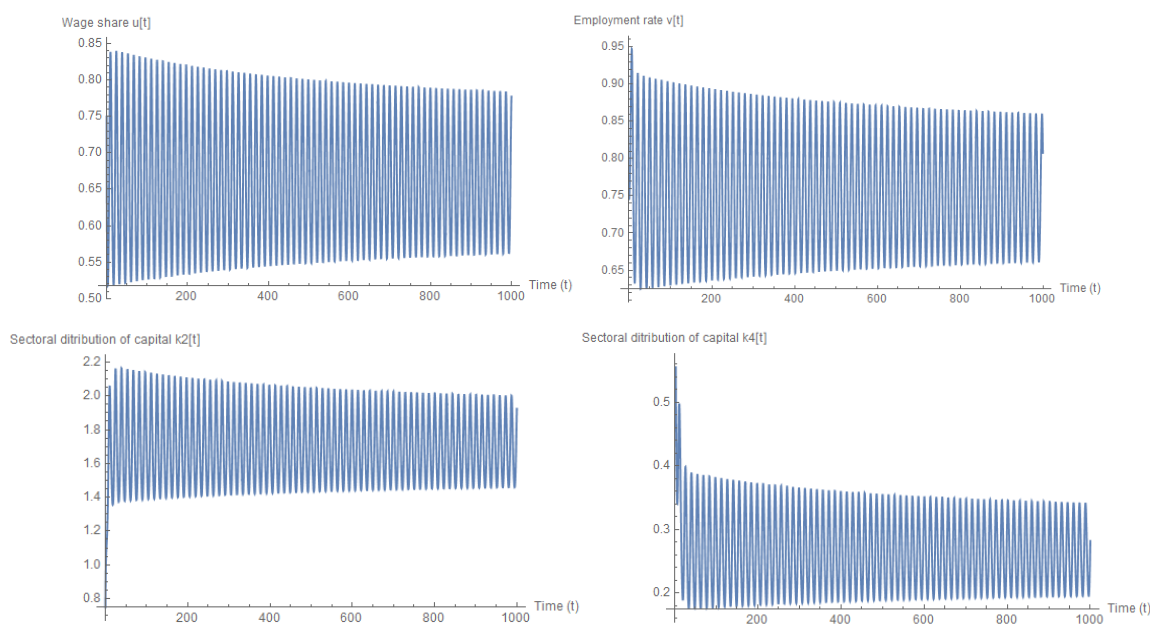
Combining (67), (76), (77), (79), (81), and (82) gives a complex dynamical system of four differential equations for the endogenous variables u, v, k_2, and k_4. This system is represented in general terms by:

$$u' = u'(u, v, k_2, k_4), \qquad v' = v'(u, v, k_2, k_4), \qquad k_2' = k_2'(u, v, k_2, k_4), \qquad k_4' = k_4'(u, v, k_2, k_4) \quad (83)$$

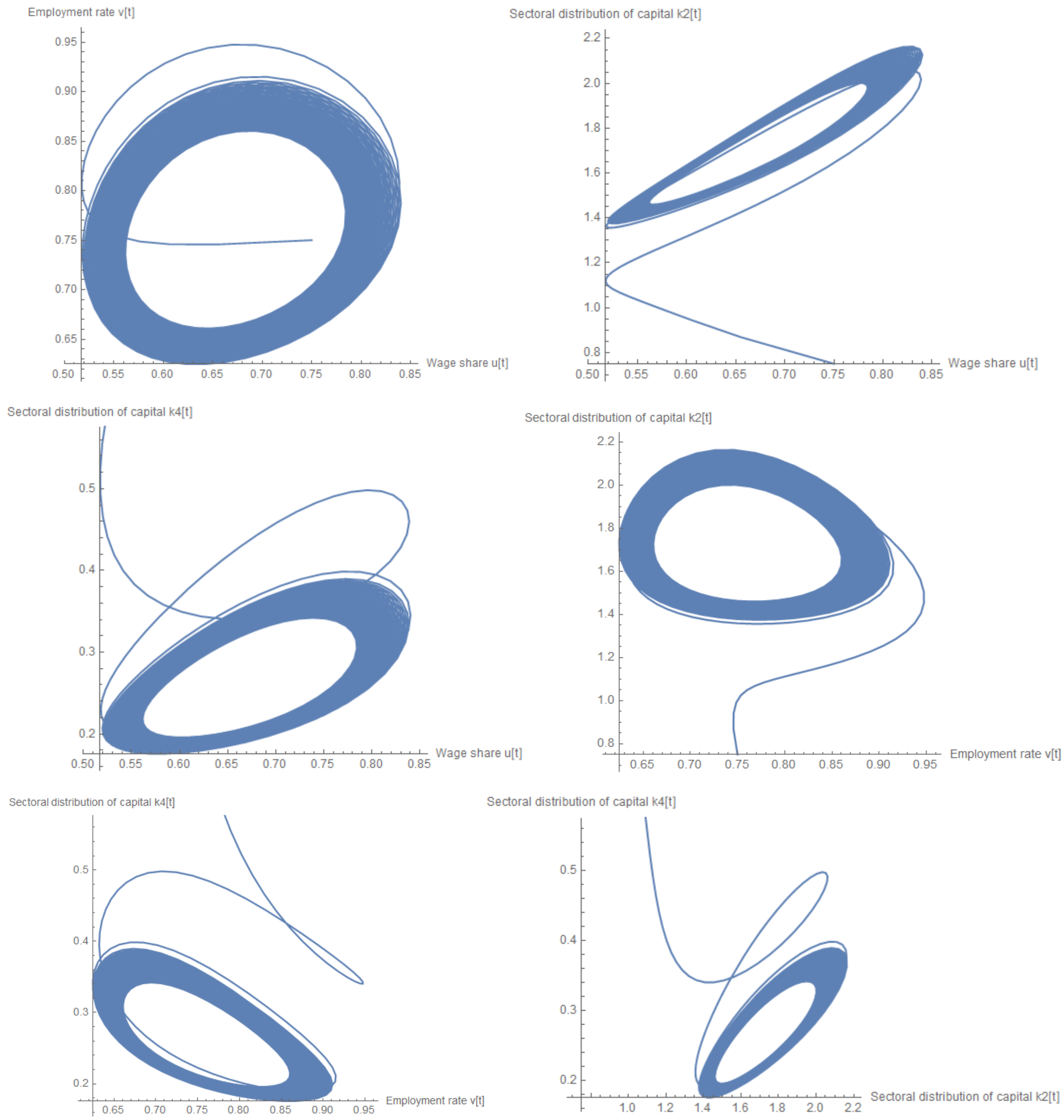
The rigorous analysis of the dynamics of (83) goes beyond the limits of this paper. Instead, we use numerical simulations with parameters and initial values close to those used when simulating Model A to illustrate the dynamics of capitalist reproduction with the coexistence of unpaid family labor and self-employment. From these simulations, we identify potentially stable limit cycles where all endogenous variables fluctuate around economically relevant long-run equilibrium points, as Figures 6 and 7 suggest. It is also possible to simulate stable spirals converging toward a long-run equilibrium point (see Figures B.5 and B.6 in Appendix B), as well as unstable spirals that may cause structural crises (see Figures B.7 and B.8 in Appendix B). These simulations suggest clockwise cycles between the wage share (u) and the employment rate (v) and counterclockwise cycles between the other pairs of variables. Also, the simulations indicate a "positive slope" for the cycles in the space defined by the wage share (u) and the ratios of sectoral distribution of capital for Sectors 2 and 4 (k_2, k_4), a result close to the cycles between v and k obtained from model A (see Figure 2). In addition, there is a "negative slope" between the employment rate (v) and the sectoral distribution of capital for Sector 4, but not for Sector 2. Finally, cycles between the two ratios of sectoral distribution of capital (k_2, k_4) have a "positive slope", suggesting these sectors tend to accelerate their capitalist accumulation during the same stages of the cycle.

Figure 6. Simulation of time series and 2-D stable limit cycles (Model B)

Time series

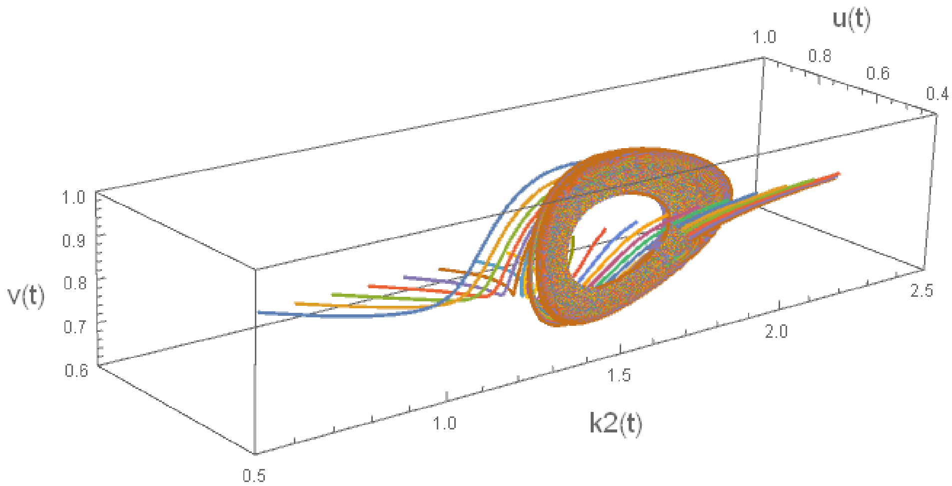


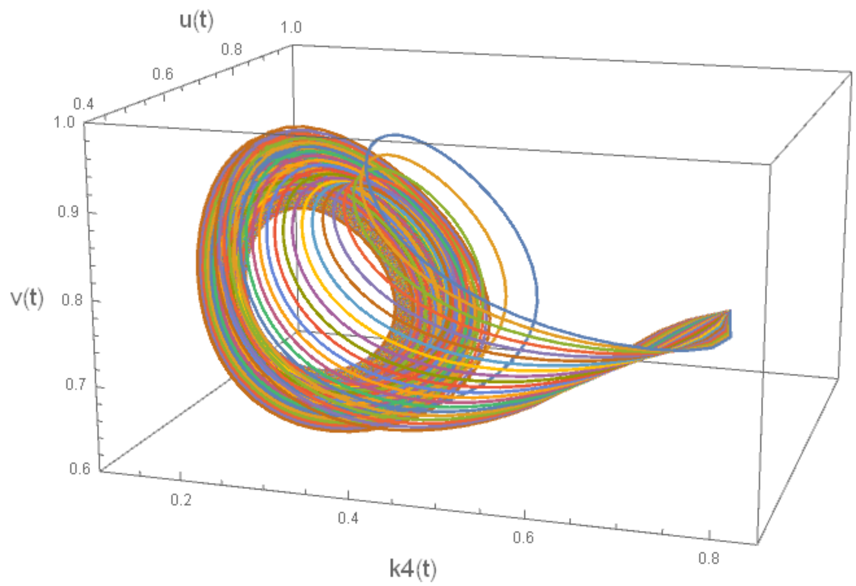
2-D Parametric plots

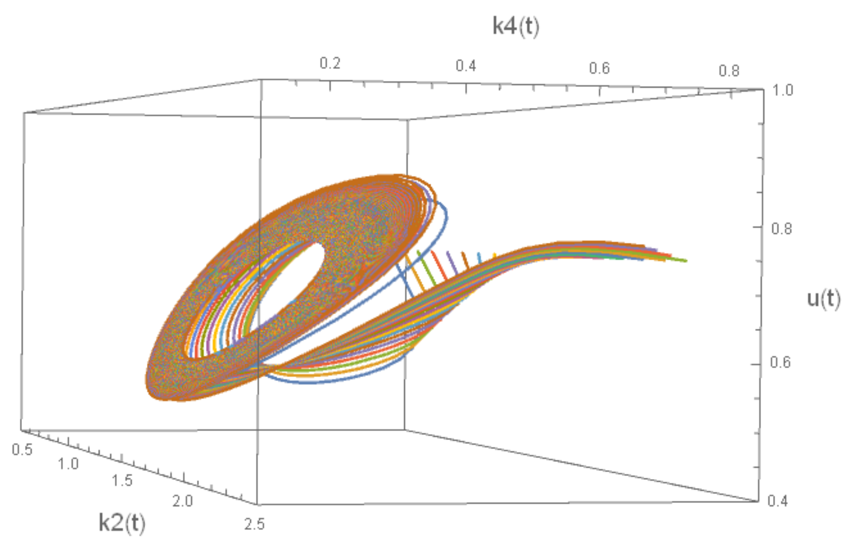


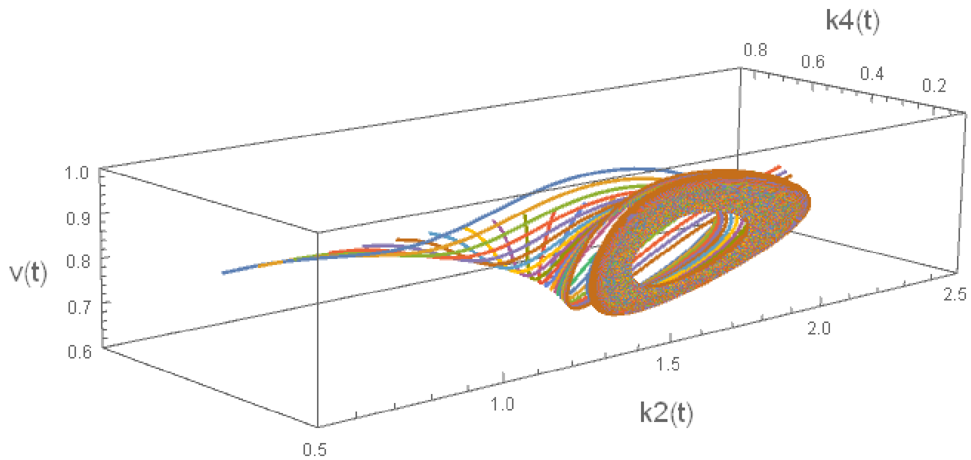
Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.2, z = 0.025$ and initial conditions $u_0 = v_0 = k_{20} = k_{40} = 0.75$

Figure 7. Simulations of 3-D stable limit cycles (Model B)


u(t)
1.0
0.8
0.6
0.4
1.0
0.9
0.8
0.7
0.6
v(t)
0.5
1.0
1.5
2.0
2.5
k2(t)


u(t)
1.0
0.8
0.6
0.4
1.0
0.9
0.8
0.7
0.6
v(t)
0.2
0.4
0.6
0.8
k4(t)


k4(t)
0.2
0.4
0.6
0.8
1.0
0.8
0.6
0.4
u(t)
0.5
1.0
1.5
2.0
2.5
k2(t)

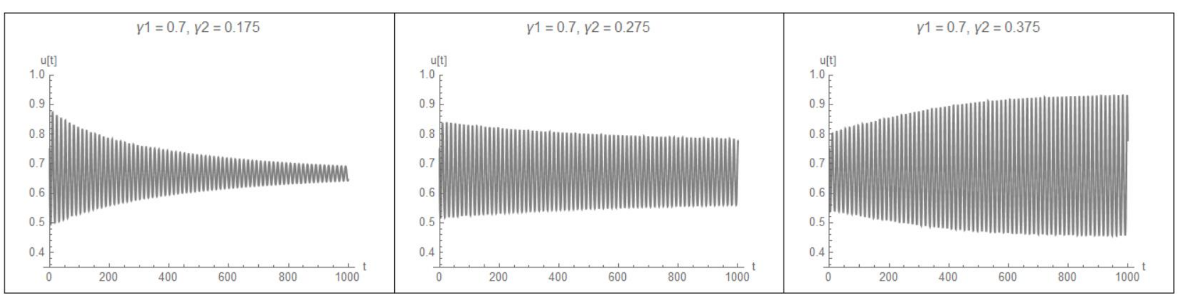


Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.2, z = 0.025$ and initial conditions $u_0 = v_0 = k_{40} = 0.75, k_{20} \in [0.75,\ 2.75]$

Numerical simulations of the dynamical system also illustrate how capitalist stability may constrain the concrete form of the bargaining power of the working class. In particular, we find that the influence of self-employed workers on the growth rate of the real wage in the extended Phillips curve (represented by γ_2) should be lower than the influence of wage labor (represented by γ_1) to get stable cycles. In fact, when γ_2 decreases, *ceteris paribus*, the simulations generate trajectories with reduced volatility, as indicated in Figures 8 and 9. This result gives insights into how it may be useful for capitalist reproduction to keep deteriorated labor conditions and weak trade unions in the case of self-employed workers. However, a deeper theoretical and analytical discussion is needed to support this intuition in more general terms.[17]

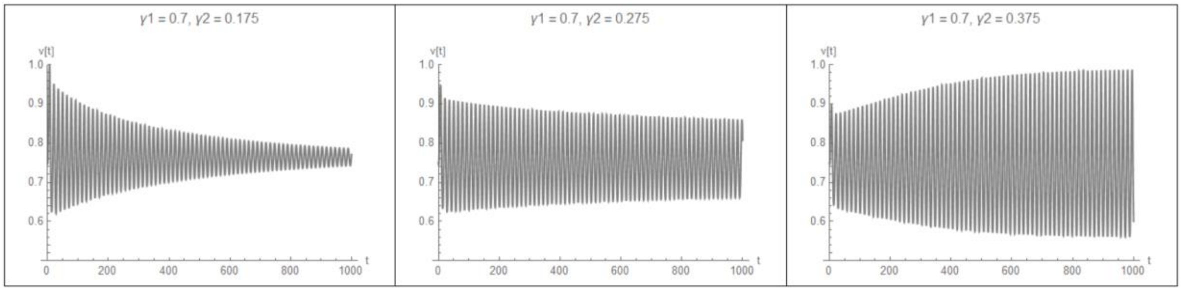
Figure 8. Simulated effect of γ_2 on the volatility of cycles (time series) (Model B)

Wage share (u)

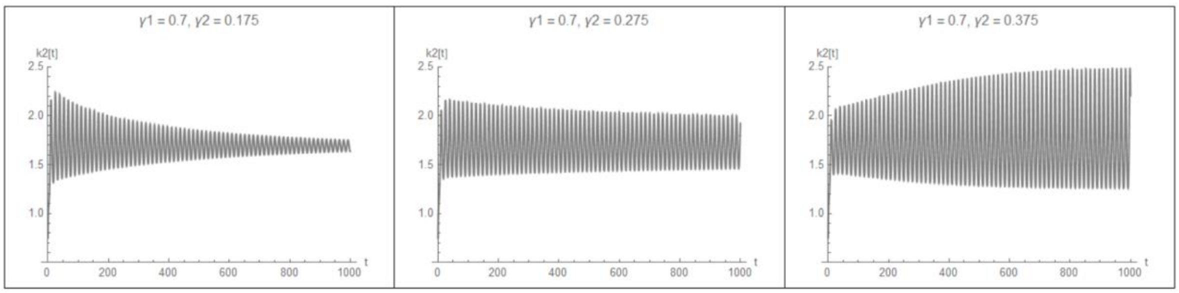


Employment rate (v)

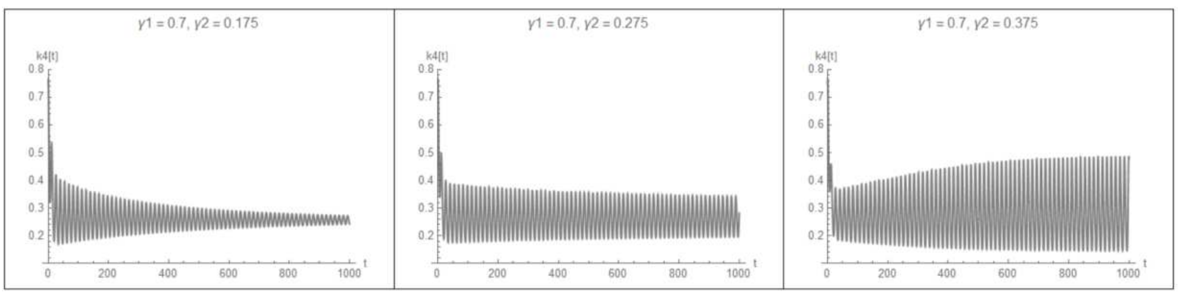
[17] Another pattern that is left for future discussion is the tendency of model B to become unstable when $z \to x$, at least for the simulations considered in this paper. These results are available upon request to the author.



Sectoral distribution of capital for sector 2 (k_2)

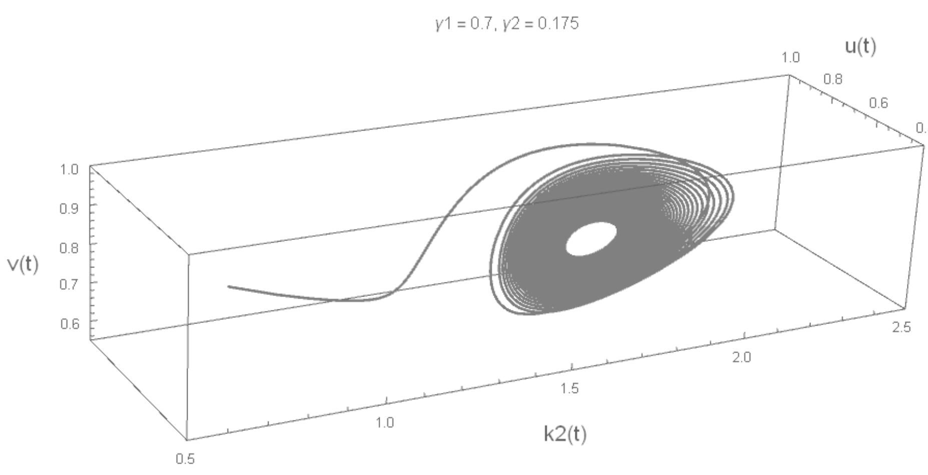


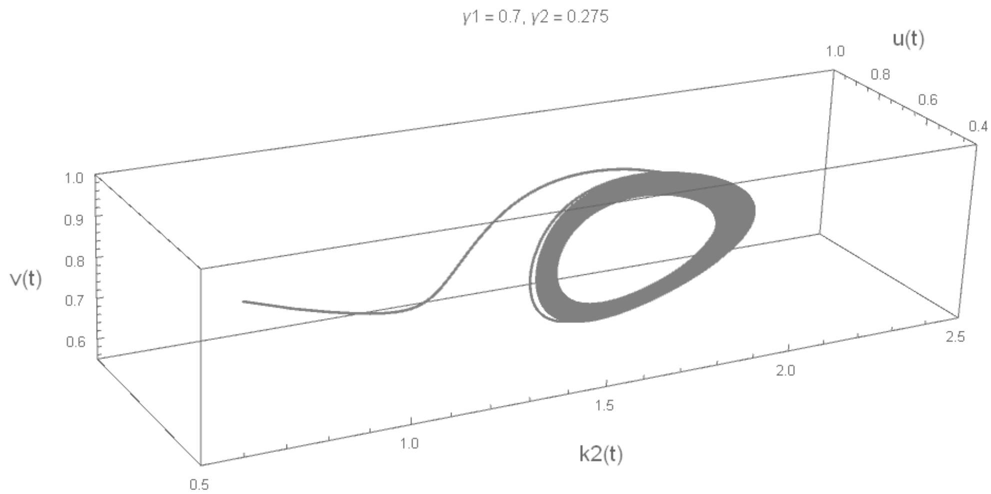
Sectoral distribution of capital for sector 4 (k_4)

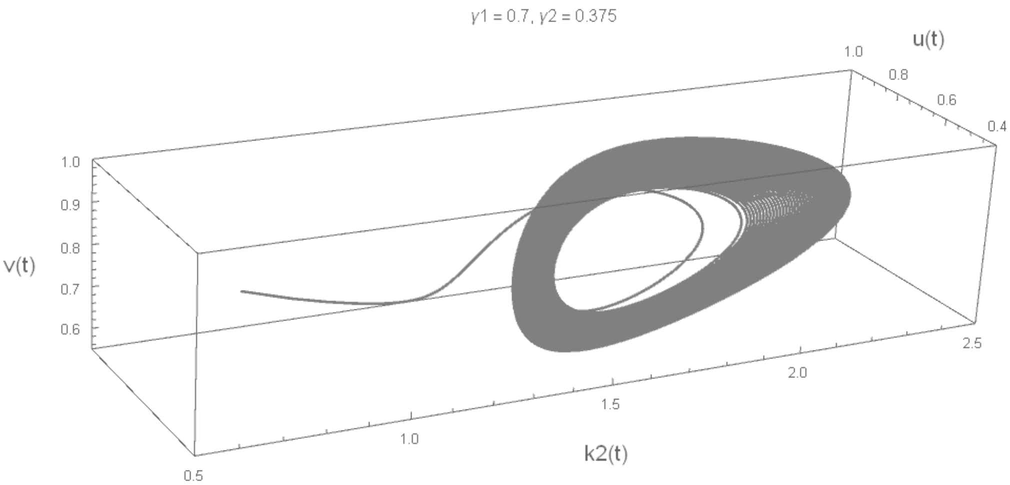


Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.2, z = 0.025$ and initial conditions $u_0 = v_0 = k_{40} = 0.75$

Figure 9. Simulated effect of γ_2 on the volatility of cycles (selected 3D parameter plots) (Model B)



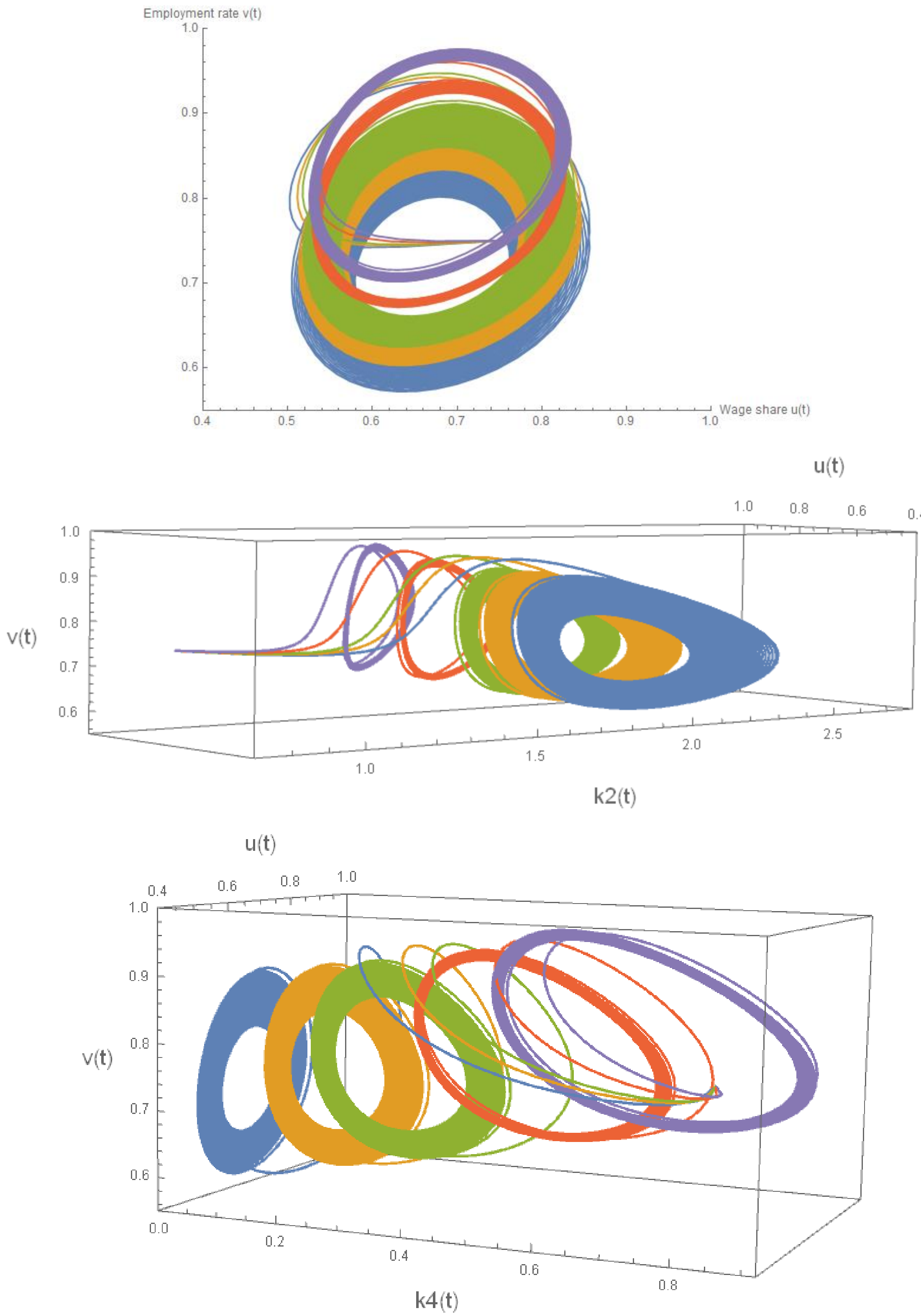




Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.2, z = 0.025$ and initial conditions $u_0 = v_0 = k_{40} = 0.75$

Also, numerical simulations are useful to illustrate the effect of changes in the proportion z on the capitalist cycles. Thus, when $z \to 0$, the sectoral distribution of capital for Sector 4 tends to zero ($k_4 \to 0$) while the sectoral distribution for Sector 2 (k_2) tends to maximum values, as indicated by the blue cycles in Figure 10. In contrast, when z increases it is possible to identify stable limit cycles where, on average, the sectoral distribution of capital for Sector 4 increases ($\uparrow k_4$), the sectoral distribution for Sector 2 decreases ($\downarrow k_2$), and the employment rate is higher ($\uparrow v$) for given values of the wage share (u) (cycles move up in the space $u - v$). In other words, at least within the context of Model B and the parameters and initial values considered in this paper, the existence of self-employed labor is a relevant alternative to improve the employment rate without decreasing the wage share. However, the model seems to become unstable when z is too large.

Figure 10. Simulated effect of z on the capitalist cycles (selected parametric plots) (Model B)



Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.2, z \in \{0, 0.015, 0.025, 0.045, 0.06\}$ and initial conditions $u_0 = v_0 = k_{20} = k_{40} = 0.75$

Concerning the average subsistence level of working-class families, as in Model A, its dynamics can be illustrated through the subsistence-wage ratio:

$$\phi = \frac{\theta}{\omega} \quad (84)$$

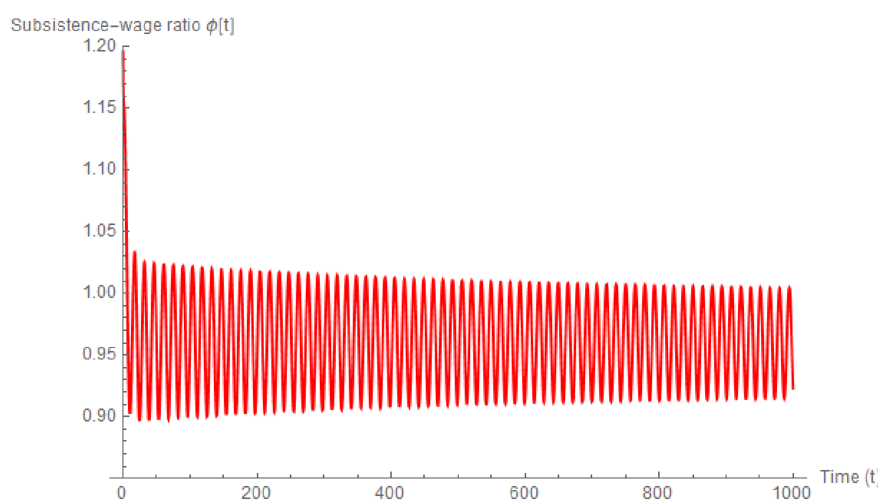
By combining (84) with (81) and with results from (51) to (66), it can be proved that both the initial value and the time derivative of ϕ depend on the endogenous variables u, v, k_2, k_4, and their dynamics, as illustrated in general terms by:

$$\phi_0 = \phi_0(u_0, v_0, k_{20}, k_{40}), \qquad \phi' = \phi'(u, v, k_2, k_4, u', v', k_2', k_4') \quad (85)$$

Therefore, numerical simulations of the system (83) can be used to simulate the dynamics of ϕ, as indicated in Figure 11 for the case of stable limit cycles.[18] When comparing these results with the stable case of Model A, it can be seen that the existence of Sector 4 creates the possibility to obtain a higher subsistence-wage ratio (ϕ) in the upper stage of the capitalist cycle almost without lowering the ratio during the lower stage of the cycle, as Figure 12 illustrates. Figure 12 also suggests that, on average, the existence of Sector 4 in Model B consolidates limit cycles with a higher employment rate (v) in comparison with cycles generated by Model A while the wage share (u) keeps similar values in the long run. However, as mentioned before, a deeper theoretical discussion is needed before accepting the generality of the existence of Sector 4 as a relevant alternative to improve the subsistence level and the employment rate of the working class and their families during the capitalist cycles. In any case, Models A and B presented in this paper are useful as preliminary approximations to the analytical study of unpaid family labor and self-employment within the context of extended capitalist reproduction and endogenous cycles.

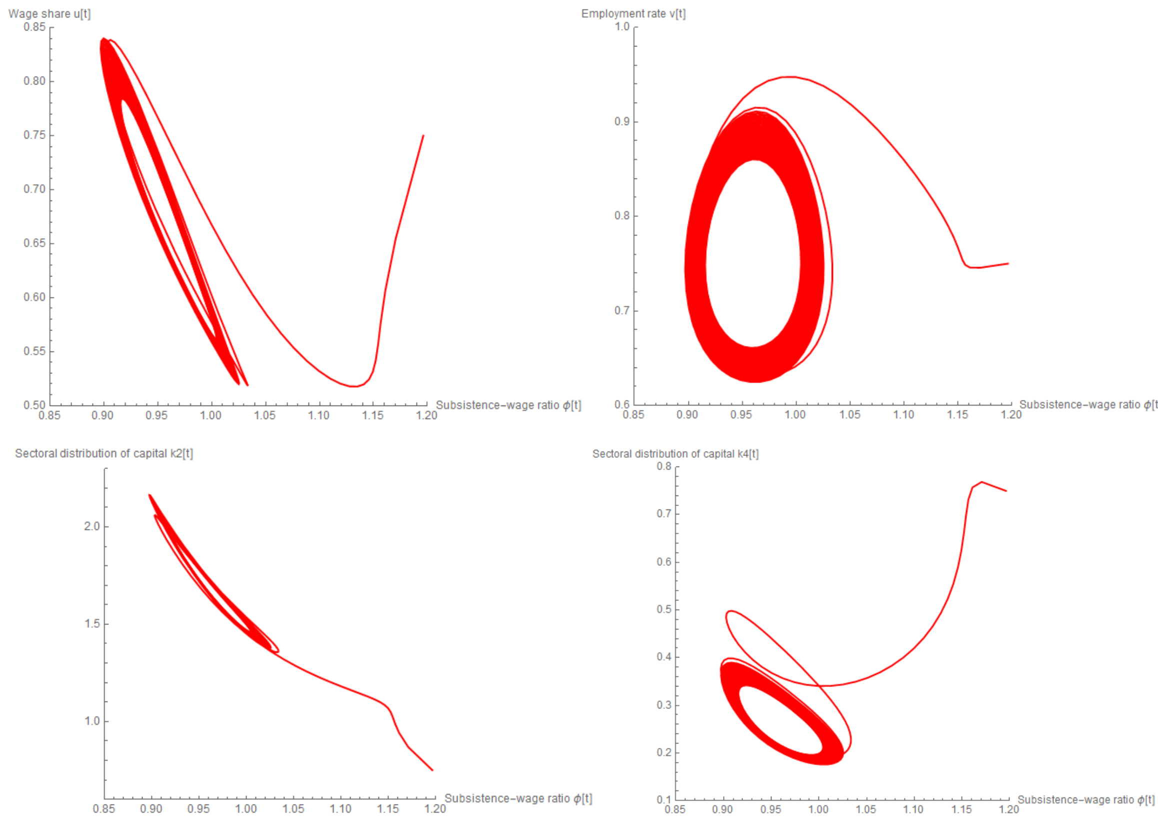
Figure 11. Simulations of stable limit cycles for the subsistence-wage ratio (Model B)

Time series



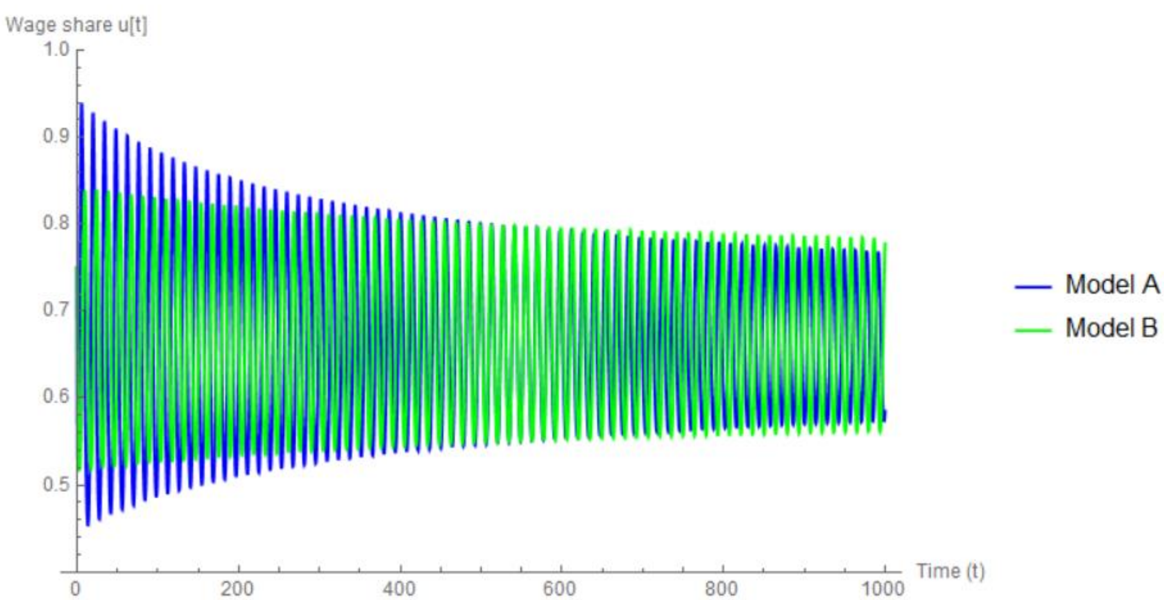
2-D Parametric plots

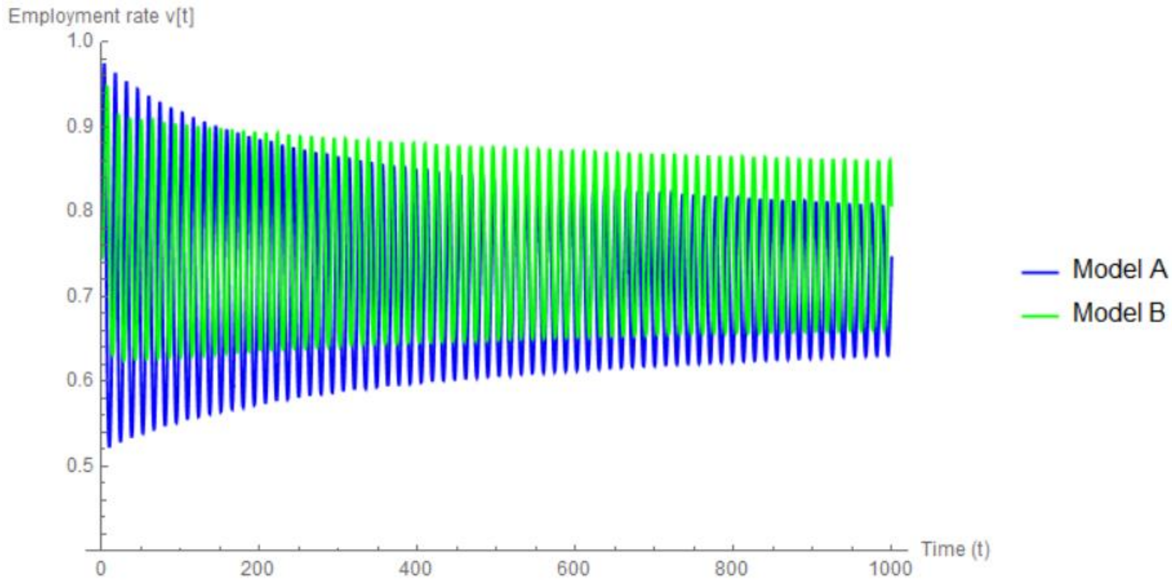
[18] Stable and unstable spirals can also be identified.

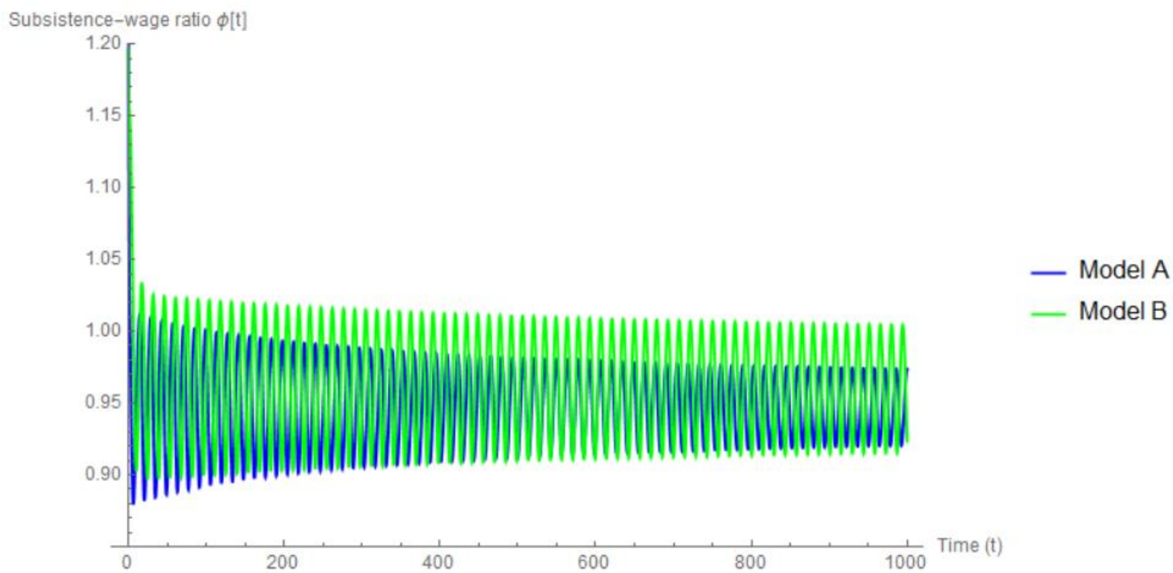


Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.2, z = 0.025$ and initial conditions $u_0 = v_0 = k_{20} = k_{40} = 0.75$ ($\phi_0 = 1.196$)

Figure 12. Simulations of stable limit cycles of Model A vs Model B (selected time series)







Note: Models A and B simulated according to parameters and initial values described in the notes of figures 4 and 11, respectively.

4. Summary and concluding remarks

This paper illustrates the structural relevance of unpaid family labor and self-employment for the process of extended reproduction of capital by presenting two sectoral models of endogenous cycles (Models A and B). Model A considers three sectors: Sector 1 includes firms producing capital goods, Sector 2 includes firms producing consumption goods, and Sector 3 groups unpaid family labor producing consumption goods to improve the subsistence level of working-class families. For Model A, we analytically prove that a higher scale of unpaid family labor (represented by a higher proportion x of wages used to buy capital goods for Sector 3) tends to accelerate capital accumulation in Sector 1 compared to Sector 2 in the long run. This result is a consequence of the higher requirements of capital goods from the sector of unpaid family labor when x increases. Thus, when Sector 3 has a larger scale, it also expands the demand for capital goods in favor of Sector 1. Also, when the capital depreciation rate in Sector 3 is sufficiently low, a larger scale of this sector increases the average subsistence level of working-class families beyond the limits of the real wage.

Using an extended version of Model A that includes a real wage Phillips curve as in Goodwin (1967), we analytically prove the existence of stable limit cycles (supercritical Hopf bifurcation) where Sectors 1, 2, and 3 coexist (see Appendix A.1 and A.2). This mathematical demonstration suggests that when labor supply and productivity increase, the stability of cycles requires a positive proportion of wages used to buy capital goods for Sector 3 ($x > 0$). In other words, the existence of unpaid family labor is not trivial since it contributes in some way to the cyclical stability of capitalist extended reproduction. In fact, if labor supply and productivity increase but unpaid family labor does not exist ($x = 0$), Model A generates unstable and explosive dynamics that may be interpreted as a form of structural crisis. We also find a cyclical behavior in the subsistence-real wage ratio, suggesting the existence of cyclical stages where the average subsistence level of working-class families grows faster than the real wage.

The paper also presents Model B as an extended version of Model A with the inclusion of Sector 4. This new sector groups self-employed workers who produce consumption goods, compete with Sector 2, obtain an income from the sale of their products in the market, contribute to the subsistence of working-class families, and receive financial support from wage workers to accumulate capital. Intuitively, Sector 4 represents an alternative mode of production that may improve the subsistence of the working class in particular when there are not enough jobs available in capitalist sectors. When an exogenous real wage is assumed, numerical simulations of Model B suggest the existence of a stable equilibrium point in the long run where the four sectors presented in this paper coexist. At this point, if the proportion ($z > 0$) of the working-class income used to finance Sector 4 increases, then this sector accumulates a higher proportion of capital than Sector 2 in the long run. Also, Model B suggests that higher proportions of working-class incomes used to finance Sectors 3 and 4 (x and z) increase the average subsistence level of their families. In this sense, the expansion of Sector 4 is relevant for the working class since it has the advantage that self-employed workers receive an economic remuneration for their labor and, at the same time, the working class accumulates means of production and reinforces its competition with Sector 2.

As with Model A, Model B is also extended to the case of endogenous distribution by using an extended Phillips curve where both wage and self-employed workers have different influences on the growth rate of the real wage. By using numerical simulations of this complete version of the model, we identify potential stable limit cycles where Sectors 1 to 4 coexist. In addition, we find relevant patterns through some sensitivity analyses. For instance, when the influence of self-employed workers on the real wage falls, capitalist cycles tend to be more stable and less volatile. This result gives insights into how it may be useful for capitalist reproduction to keep deteriorated labor conditions and weak trade unions in the case of self-employed workers. Also, if the proportion of working-class incomes used to finance Sector 4 increases, then economic cycles tend to fluctuate around a higher rate of employment without relevant changes in the wage share (although, cycles may become unstable if the proportion z is sufficiently high). Finally, when comparing Models A and B, we note that the existence of Sector 4 may increase the subsistence-real wage ratio as well as the employment rate during the upper stage of the capitalist cycle. All these results reinforce the relevance of Sector 4 as a mechanism to improve the living standards of working-class families, although a deeper theoretical discussion is needed to prove the generality of this claim, including an analytical proof of the stability and cyclical nature of Model B (or a simpler version).

In addition to illustrating the structural relevance of Sectors 3 (unpaid family labor) and 4 (self-employment), the models presented in this paper offer insights for future discussions. For instance, sectors may have different levels of productivity because of multiple factors, including different labor intensities between capitalist and non-capitalist sectors. If we assume that the growth rate of labor intensity and the employment rate have an inverse relationship (Cajas Guijarro and Vera, 2022), then it would be possible to represent complex dynamics for Models A and B by including sectoral intensities and productivities as new endogenous variables. Also, it is possible to extend these models in Kaleckian terms by assuming that sectors do not produce at full capacity while prices are rigid (at least in the short run). In this sense, it may be useful to combine ideas from the Kaleckian model of Dutt (1988), the model of Onaran et al. (2022) associated with unpaid family labor, and the model of Vasudevan and Raghavendra (2022) associated with self-employment. Concerning the proportions of working-class incomes used to finance Sectors 3 (x) and 4 (z), they could be interpreted as the result of an optimization problem from the perspective of working-class families, as in Gronau (1973) but with a focus on material subsistence rather than subjective preferences.

Another alternative for future discussion is to compare the average subsistence level of working-class families (θ) and the subsistence-real wage ratio (ϕ) with the value of the working-class labor power (Marx, 2010 [1867]). The objective would be to identify a potential relationship between the level of unpaid family labor and the super-exploitation of labor, as Duque García (2021) suggests, particularly during the stages of the capitalist cycle when the subsistence-real wage ratio falls. We also recommend exploring the structural role of the four sectors presented in this paper through a North-South perspective, with an emphasis on the assumption that the South may have a larger proportion of labor supply employed in Sector 4 compared with the North. The North-South models presented by Dutt (1989, 1990) and the intuitions proposed by Naidu (2022) may be helpful starting points. Other extensions of Models A and B may consider the inclusion of more complex real wage Phillips curves (Flaschel, 2010), the discussion of the monetary flows between the four sectors from a network perspective (Cajas Guijarro, 2022), the statistical identification of relevant numerical parameters as Grasselli and Maheshwari (2018) present for the Goodwin's (1967) model, as well as other elements available from the literature of Marxian sectoral models of extended capitalist reproduction.

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Appendix A

A.1. Proof of local stability (Model A)

Similar to Dávila-Fernández and Sordi (2019), Mariolis et al. (2021), Cajas Guijarro and Vera (2022), and others who extend the model of Goodwin (1967) for more than two dimensions, to prove the local stability of the system (39)-(41) first we linearise it around the equilibrium point given by (42)-(44):

$$\begin{bmatrix} u' \\ v' \\ k' \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{22} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} u - u^* \\ v - v^* \\ k - k^* \end{bmatrix}$$

where the elements J_{ij} of the Jacobian matrix evaluated at (u^*, v^*, k^*) are:

$$J_{11} = \left.\frac{\partial u'}{\partial u}\right|_{(u^*,v^*,k^*)} = -\frac{(s-n-\alpha)(s-x)}{s(1-s)}$$

$$J_{12} = \left.\frac{\partial u'}{\partial v}\right|_{(u^*,v^*,k^*)} = \frac{\gamma_1(s-n-\alpha)[s(1-n-\alpha)-x(s-n-\alpha)]}{s^2(1-s)}$$

$$J_{13} = \left.\frac{\partial u'}{\partial k}\right|_{(u^*,v^*,k^*)} = \frac{(s-n-\alpha)[sx+(n+\alpha)(s-x)]^2}{s^3(1-s)}$$

$$J_{21} = \left.\frac{\partial v'}{\partial u}\right|_{(u^*,v^*,k^*)} = -\frac{s^2x(\alpha+\gamma_0)}{\gamma_1[sx+(n+\alpha)(s-x)]}$$

$$J_{22} = \left.\frac{\partial v'}{\partial v}\right|_{(u^*,v^*,k^*)} = 0$$

$$J_{23} = \left.\frac{\partial v'}{\partial k}\right|_{(u^*,v^*,k^*)} = -\frac{(n+\alpha)(\alpha+\gamma_0)[sx+(n+\alpha)(s-x)]}{s\gamma_1}$$

$$J_{31} = \left.\frac{\partial k'}{\partial u}\right|_{(u^*,v^*,k^*)} = \frac{s^2(s-x)}{[sx+(n+\alpha)(s-x)]^2}$$

$$J_{32} = \left.\frac{\partial k'}{\partial v}\right|_{(u^*,v^*,k^*)} = 0$$

$$J_{33} = \left.\frac{\partial k'}{\partial k}\right|_{(u^*,v^*,k^*)} = -1$$

Therefore, if eigenvalues are noted by λ, the trace of the Jacobian is T, and its determinant is Δ, then the characteristic polynomial is equal to:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where:

$$b_1 = -\mathrm{T} = \frac{s(1-n-\alpha) - x(s-n-\alpha)}{s(1-s)}$$

$$b_2 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = \frac{x(\alpha+\gamma_0)(s-n-\alpha)[s(1-n-\alpha) - x(s-n-\alpha)]}{(1-s)[sx + (n+\alpha)(s-x)]}$$

$$b_3 = -\Delta = \frac{(\alpha+\gamma_0)(s-n-\alpha)[s(1-n-\alpha) - x(s-n-\alpha)]}{s(1-s)}$$

These results imply:

$$b_1 b_2 - b_3 = \frac{\gamma_0(s-n-\alpha)(s-x)[s(1-n-\alpha) - x(s-n-\alpha)][x(s-n-\alpha) - (n+\alpha)(1-s)]}{s(1-s)^2[sx + (n+\alpha)(s-x)]}$$

The necessary and sufficient condition for the local stability of (u^*, v^*, k^*) is that all the roots x of the characteristic equation have negative real parts. From the Routh-Hurwitz criterion, we know this is equivalent to $b_1 > 0, b_2 > 0, b_3 > 0$, and $b_1 b_2 - b_3 > 0$. These inequalities are simultaneously satisfied when:

$$\frac{(n+\alpha)(1-s)}{s-n-\alpha} < x < \frac{s(1-n-\alpha)}{s-n-\alpha}$$

Thus, it is confirmed the local stability of the equilibrium point (u^*, v^*, k^*).

A.2. Proof of periodic solutions (Model A)

Following Dávila-Fernández and Sordi (2019), based on Gandolfo (2009), we can prove the existence of periodic solutions (with the form of limit cycles) in the system (39)-(41) by using the Hopf bifurcation theorem for 3-D dynamical systems and taking x as a bifurcation parameter with critical value x^{HB}. This proof requires two conditions: (HB1) the characteristic polynomial has two purely imaginary roots and one root with a non-zero real part at the critical value x^{HB}; (HB2) the derivative of the real part of the complex root with respect to x is not null when evaluated at the critical value x^{HB}. For condition (HB1), from Appendix A.1 we can see that $b_1 > 0, b_2 > 0$, and $b_3 > 0$ when:

$$x < \frac{s(1-n-\alpha)}{s-n-\alpha}$$

Thus, a Hopf bifurcation requires $b_1 b_2 - b_3 = 0$. This condition is guaranteed when:

$$x = x^{HB} = \frac{(n+\alpha)(1-s)}{s-n-\alpha}$$

For condition (HB2), we require the derivatives of b_1, b_2, b_3 with respect to x:

$$\frac{\partial b_1}{\partial x} = -\frac{s-n-\alpha}{s(1-s)}$$

$$\frac{\partial b_2}{\partial x} = -\frac{(\alpha+\gamma_0)(s-n-\alpha)[(s-n-\alpha)^2x^2+2s(n+\alpha)(s-n-\alpha)x-s^2(n+\alpha)(1-n)]}{(1-s)[sx+(n+\alpha)(s-x)]^2}$$

$$\frac{\partial b_3}{\partial x} = -\frac{(\alpha+\gamma_0)(s-n-\alpha)^2}{s(1-s)}$$

When $x = x^{HB}$, from (HB1) we know the characteristic polynomial has one real negative root $\lambda_1 < 0$ and two purely imaginary roots $\lambda_{2,3} = A \pm Bi$ with $A = 0$. Following the procedure presented by Dávila-Fernández and Sordi (2019, Appendix A.3), we obtain the following system of equations:

$$-\frac{\partial \lambda_1}{\partial x} - 2\frac{\partial A}{\partial x} = -\frac{s-n-\alpha}{s(1-s)}$$

$$2\lambda_1\frac{\partial A}{\partial x} + 2B\frac{\partial B}{\partial x} = -\frac{(\alpha+\gamma_0)(s-n-\alpha)[(s-n-\alpha)^2x^2+2s(n+\alpha)(s-n-\alpha)x-s^2(n+\alpha)(1-n)]}{(1-s)[sx+(n+\alpha)(s-x)]^2}$$

$$-B^2\frac{\partial \lambda_1}{\partial x} - 2\lambda_1 B\frac{\partial B}{\partial x} = -\frac{(\alpha+\gamma_0)(s-n-\alpha)^2}{s(1-s)}$$

when solving the system for $\frac{\partial \lambda_1}{\partial x}, \frac{\partial A}{\partial x}$, and $\frac{\partial B}{\partial x}$, and evaluating the solution at $x = x^{HB}$, we get:

$$\left.\frac{\partial A}{\partial x}\right|_{x=x^{HB}} = \frac{(s-n-\alpha)\{B^2(n+\alpha)+(\alpha+\gamma_0)[(n+\alpha)(s-n-\alpha)-s\lambda_1(s^2-n-\alpha)]\}}{2ns(1-s)(b^2+\lambda_1^2)}$$

This derivative is null only when:

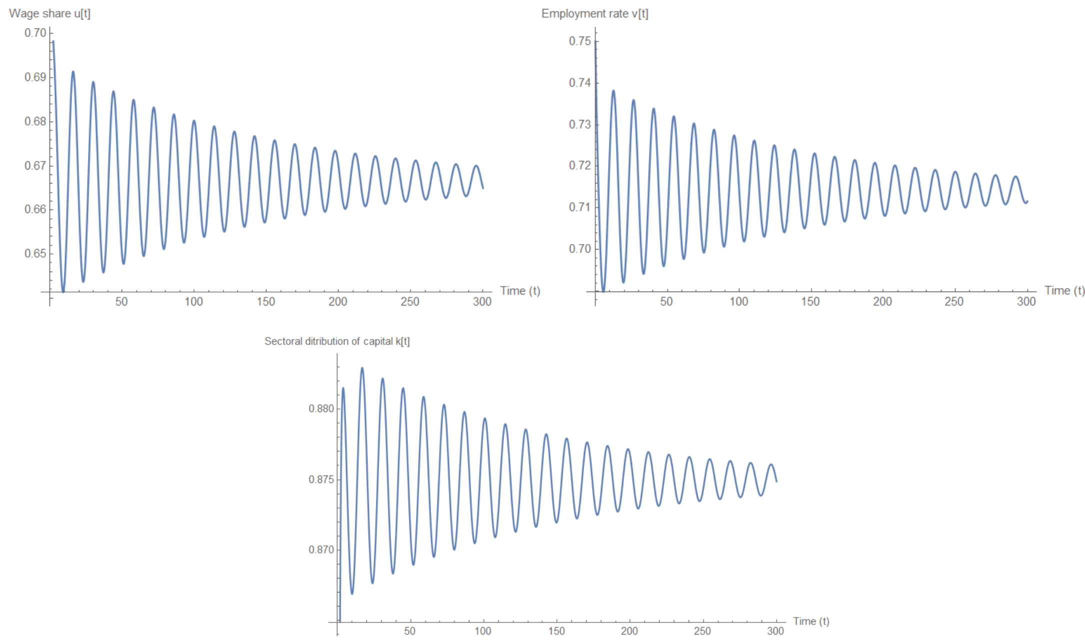
$$\gamma_0 = \gamma_0^{NULL} = \frac{B^2(n+\alpha)}{(n+\alpha)(s-n-\alpha)-\lambda_1 s(s^2-n-\alpha)} - \alpha$$

In other words, for any $\gamma_0 \neq \gamma_0^{NULL}$ the derivative $\left.\frac{\partial A}{\partial x}\right|_{x=x^{HB}}$ is different from zero. Therefore, it is confirmed the existence of periodic solutions (Hopf bifurcation) in the neighborhood of x^{HB}.

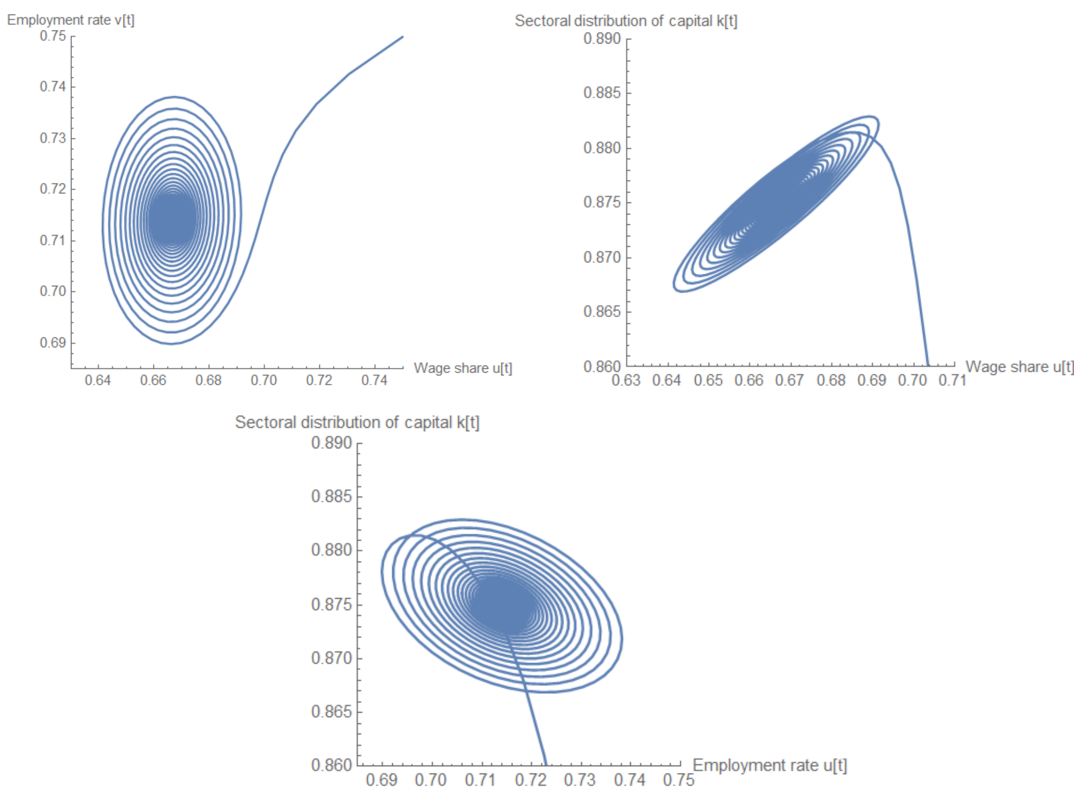
Appendix B

Figure B.1. Simulation of time series and 2-D stable spirals (Model A)

Time series

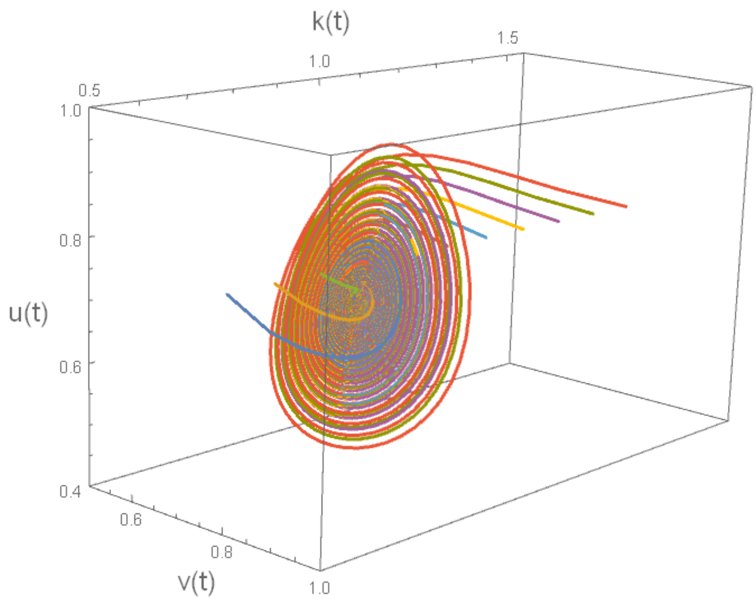


2-D Parametric plots



Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.5 \gg x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = k_0 = 0.75$

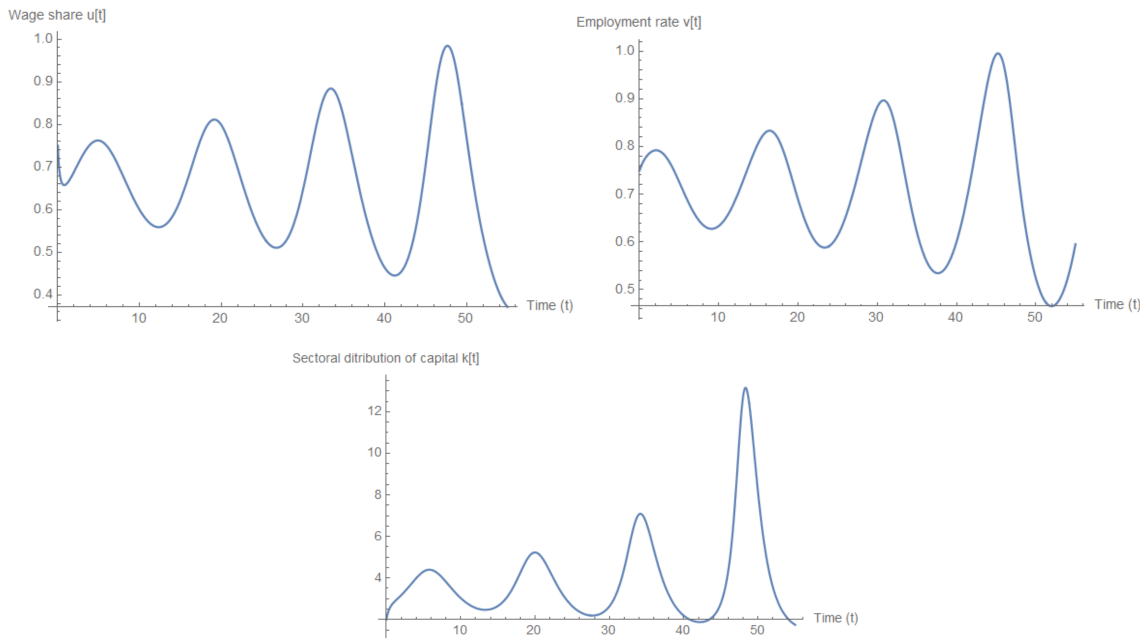
Figure B.2. Simulations of 3-D stable spirals (Model A)



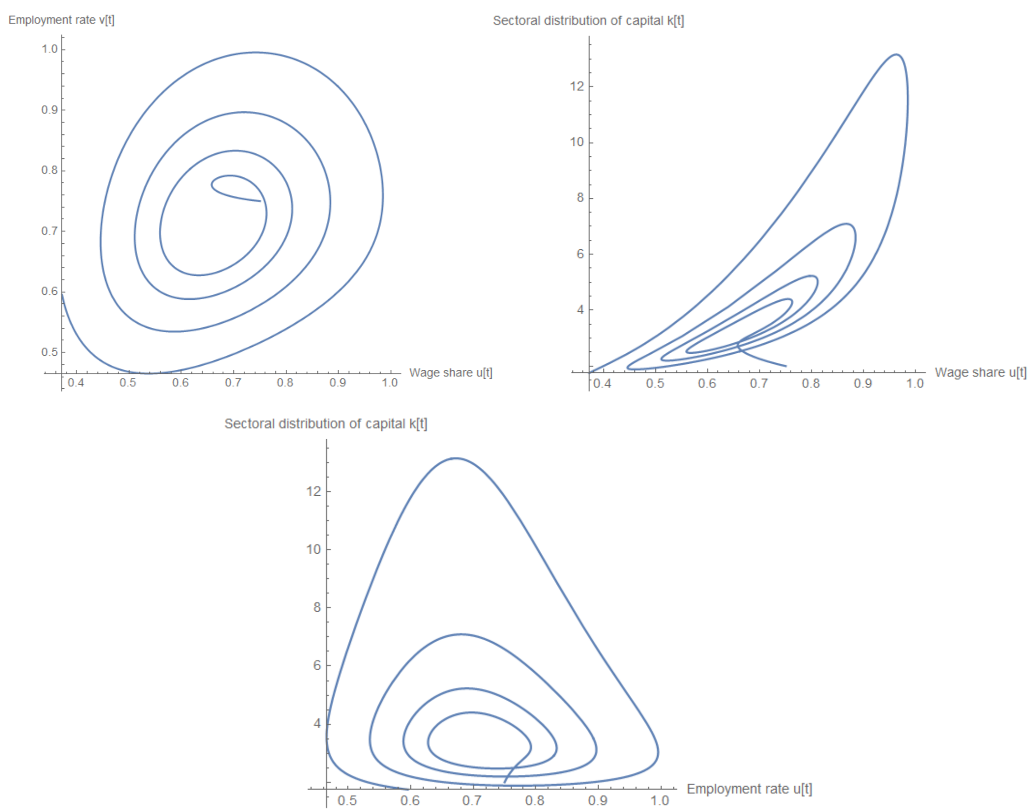
Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.5 \gg x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = 0.75, k_0 \in [0.55,\ 1.55]$

Figure B.3. Simulation of time series and 2-D unstable spirals (Model A)

Time series

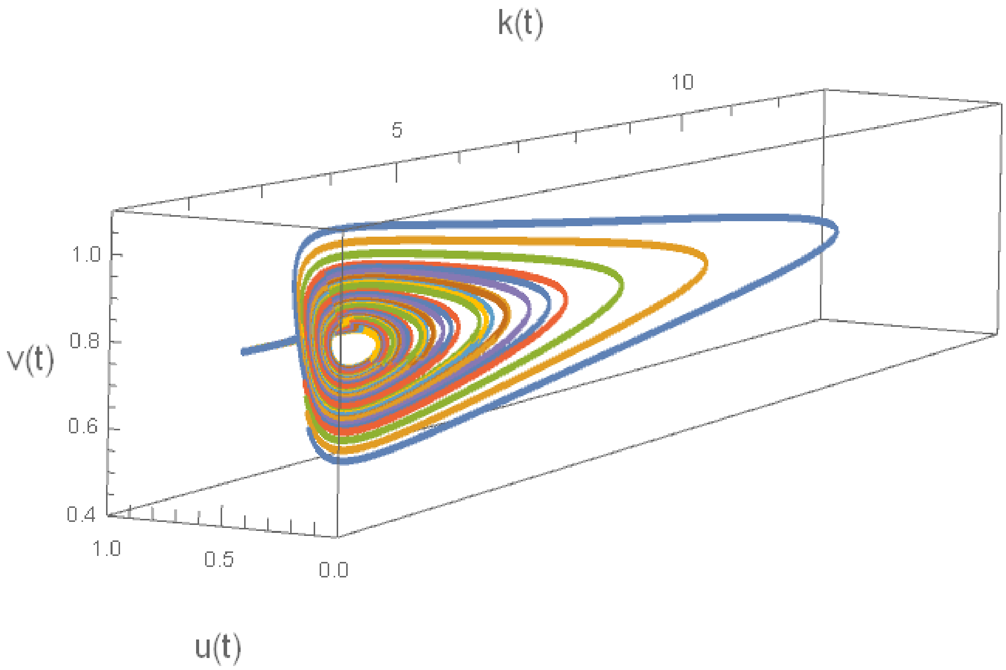


2-D Parametric plots



Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.1 \ll x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = 0.75, k_0 = 2$

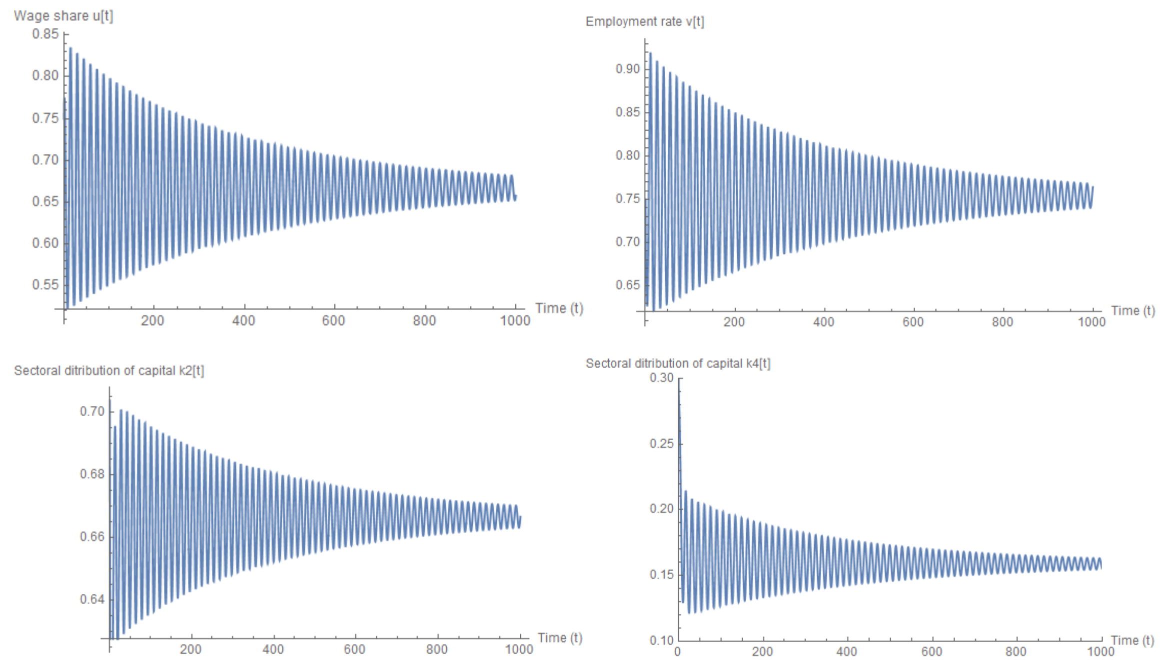
Figure B.4. Simulations of 3-D unstable spirals (Model A)



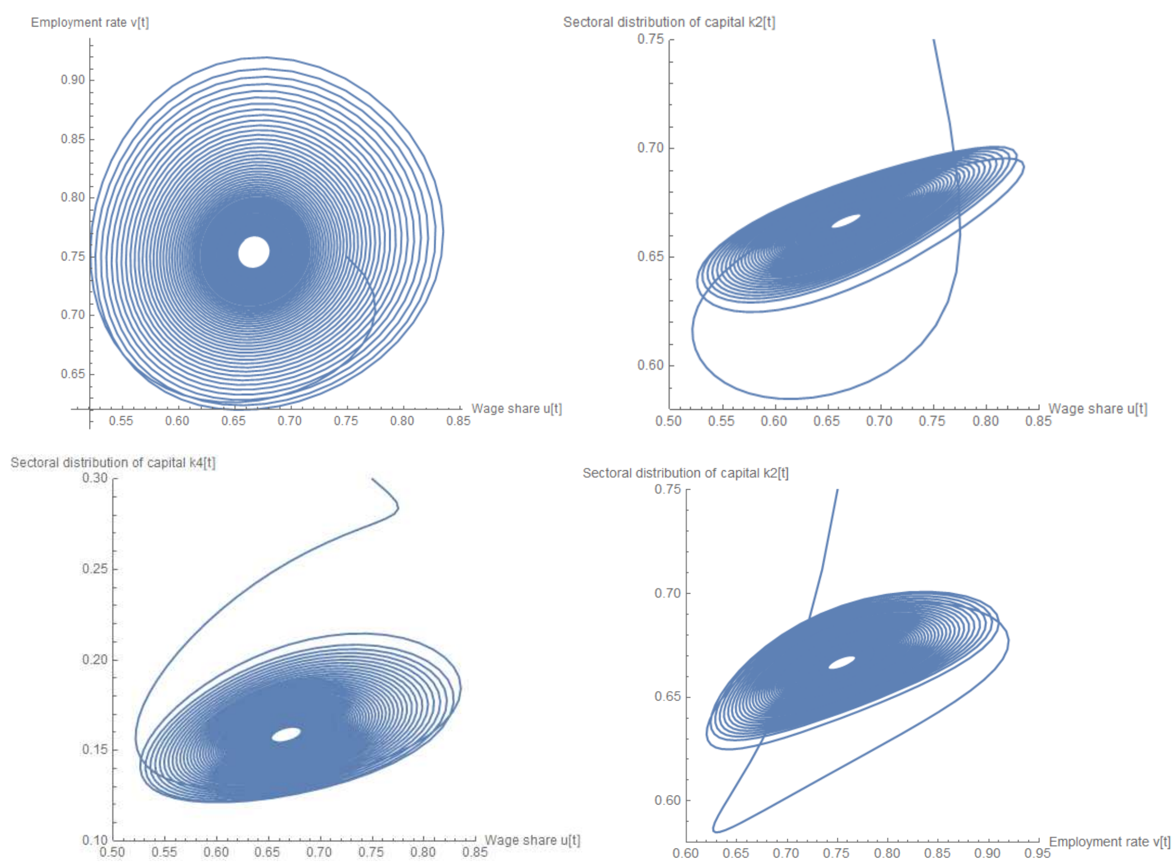
Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, x = 0.05 \ll x^{HB} = 0.2$ and initial conditions $u_0 = v_0 = 0.75, k_0 \in [2,\ 2.7]$

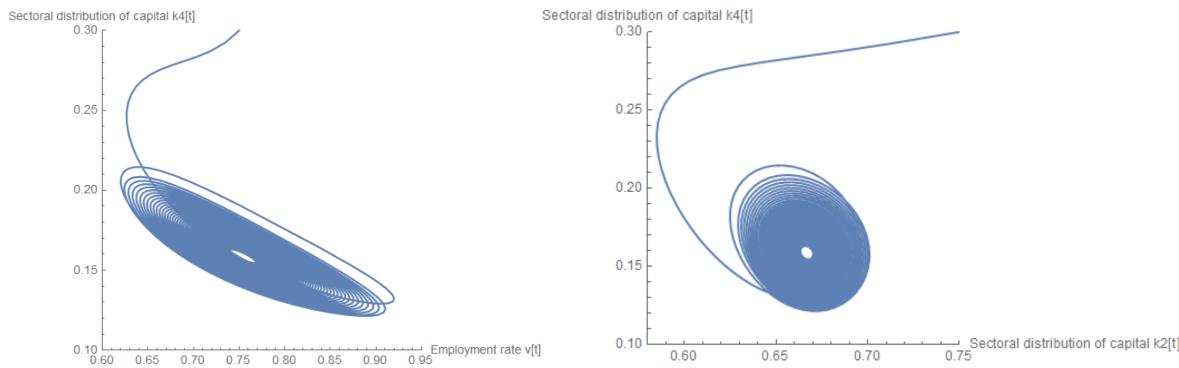
Figure B.5. Simulation of time series and 2-D stable spirals (Model B)

Time series



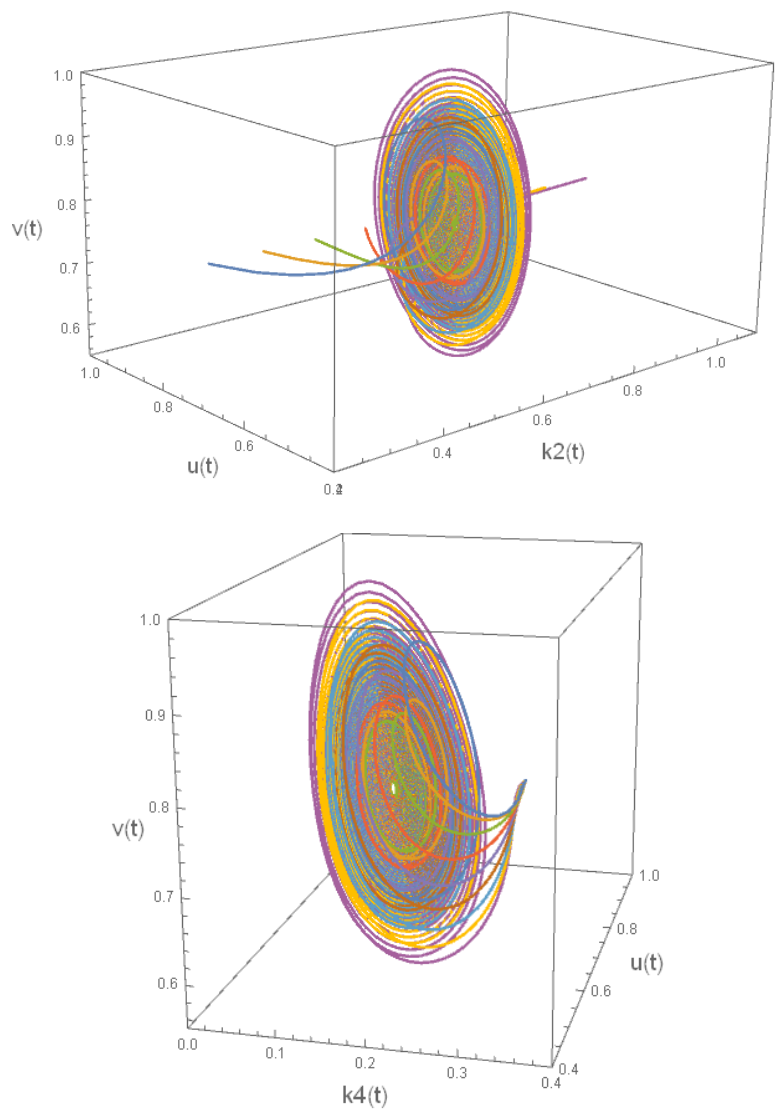
2-D Parametric plots

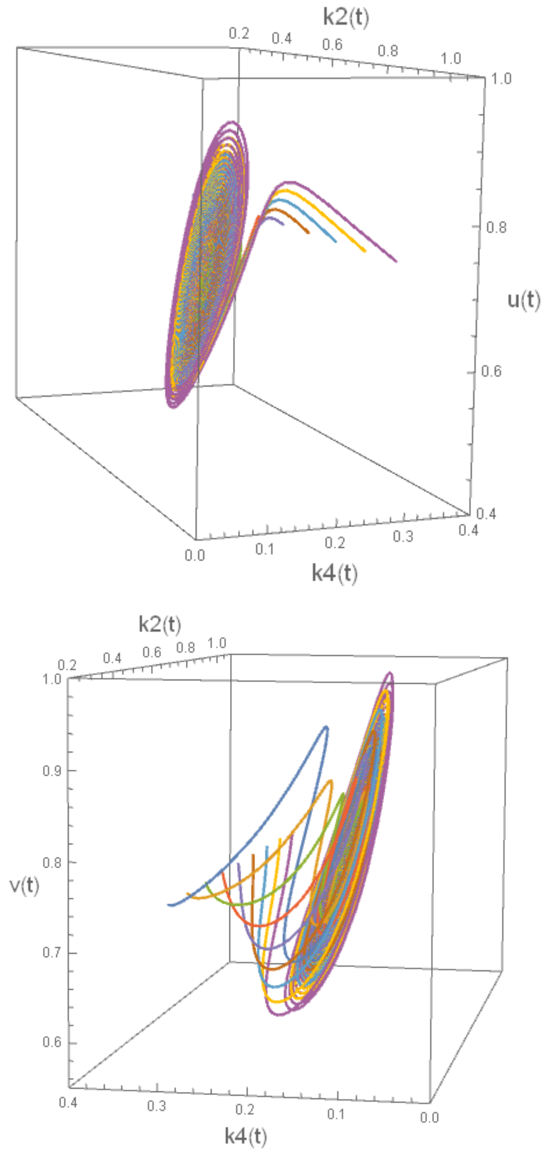




Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.5, z = 0.025$ and initial conditions $u_0 = v_0 = k_{20} = 0.75, k_{40} = 0.3$

Figure B.6. Simulations of 3-D stable spirals (Model B)

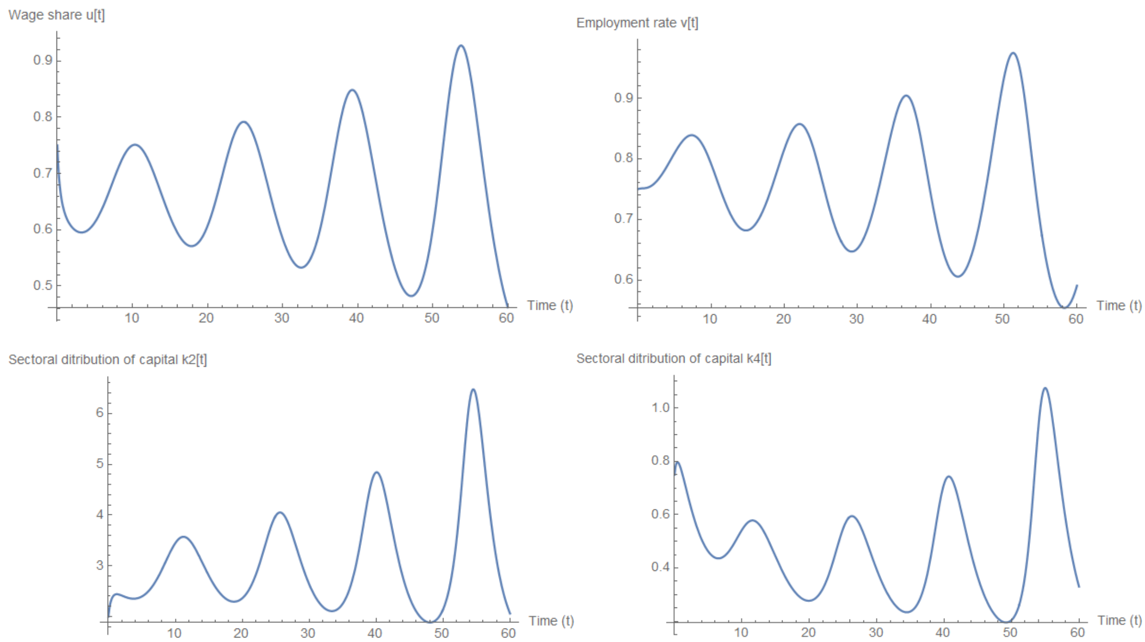




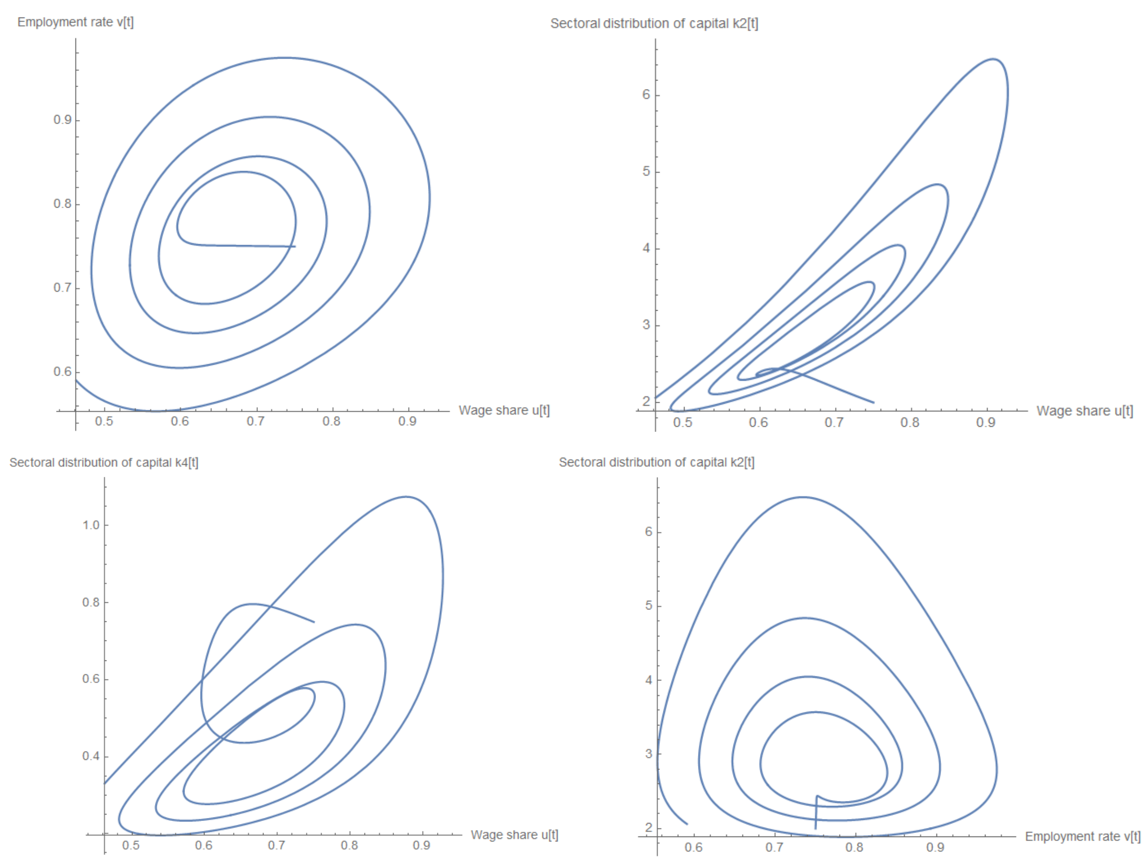
Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.5, z = 0.025$ and initial conditions $u_0 = v_0 = 0.75, k_{40} = 0.3, k_{20} \in [0.25, 1.05]$

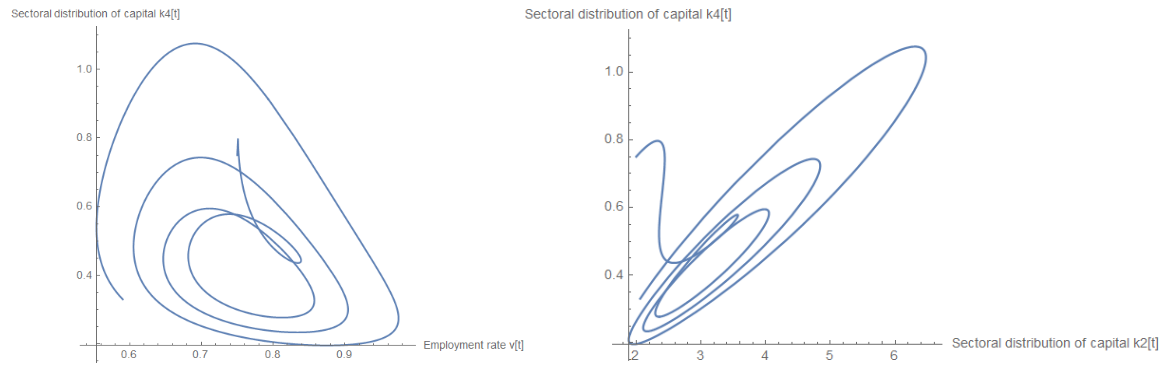
Figure B.7. Simulation of time series and 2-D unstable spirals (Model B)

Time series



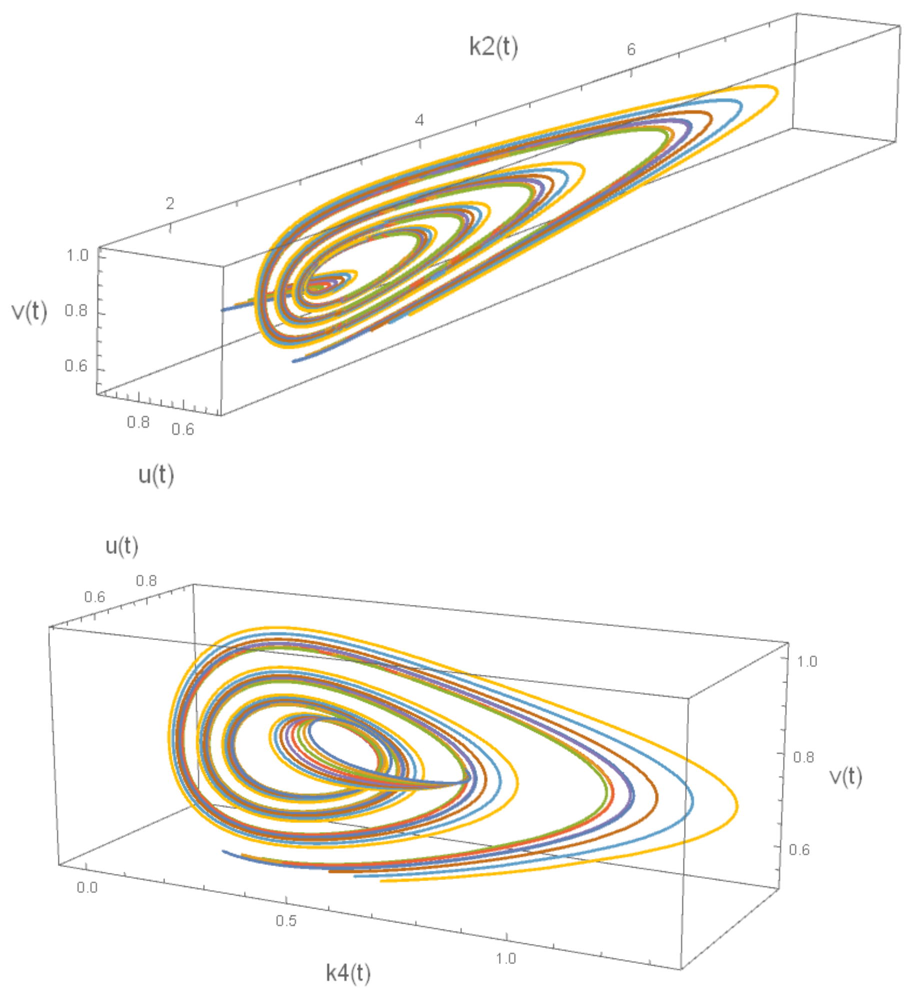
2-D Parametric plots

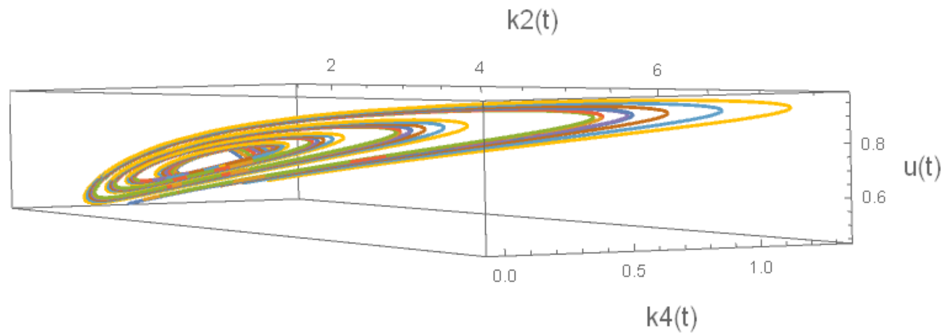


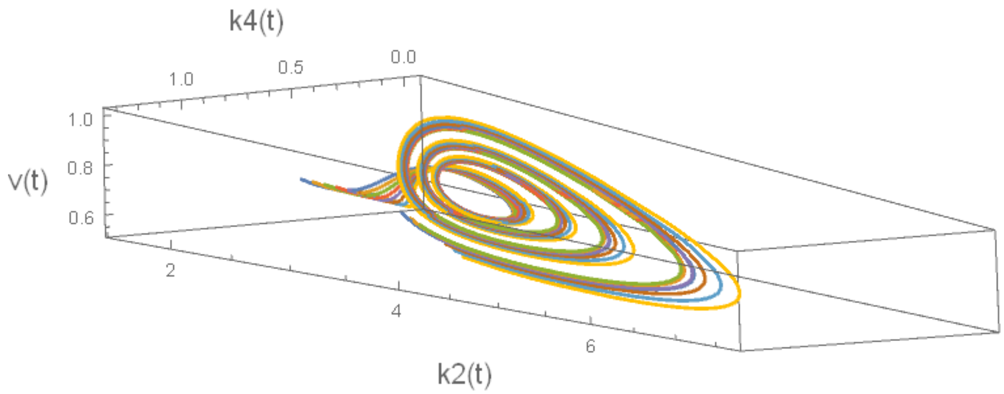


Note: Simulation using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.05, z = 0.025$ and initial conditions $u_0 = v_0 = k_{20} = k_{40} = 0.75$

Figure B.8. Simulations of 3-D unstable spirals (Model B)







Note: Simulations using parameters $n = 0.1, s = 0.6, \alpha = 0.1, \gamma_0 = 0.4, \gamma_1 = 0.7, \gamma_2 = 0.275, x = 0.05, z = 0.025$ and initial conditions $u_0 = v_0 = k_{40} = 0.75, k_0 \in [2,\ 2.7]$