The per capita Shapley support levels value

Besner, Manfred

HfT Stuttgart

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Manfred Besner*

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Abstract

The per capita Shapley support levels value extends the Shapley value to cooperative games with a level structure. This value prevents symmetrical groups of players of different sizes from being treated equally. We use efficiency, additivity, the null player property, and two new properties to give an axiomatic characterization. The first property, called joint productivity, is a fairness property within components and makes the difference to the Shapley levels value. If all players of two components are only jointly productive, they should receive the same payoff. Our second axiom, called neutral collusions, is a fairness axiom for players outside a component. Regardless of how players of a component organize their power, as long as the power of the coalitions that include all players of the component remains the same, the payoff to players outside the component does not change.

Keywords Cooperative game · Level structure · Per capita Shapley support levels value · Joint productivity · Neutral collusions

1 Introduction

Probably, the most important and commonly used solution concept for games with transferable utility is the Shapley value (Shapley, 1953b). There now exist numerous axiomatic characterizations of the Shapley value that recommend its use for countless real-world applications (see, e.g., Lipovetsky (2020)). Shapley (1953b) axiomatized his value by efficiency (the final output of the grand coalition should be fully transferred to the players), the null player property (a player contributing nothing to the game also receives nothing), additivy (a player's payoff from the sum of two games is equal to the sum of the player's payoffs for the two games), and symmetry (players who contribute the same to the game should have the same payoff).

With larger player sets, groupings often occur. Ideally, these form a partition of the player set. Games in this form are called games with a coalition structure (Aumann and Drèze, 1974). The Owen value (Owen, 1977) extends the Shapley value to games with a coalition structure. Often, a coalition structure does not sufficiently reflect the actual structure of

*M. Besner
Department of Geomatics, Computer Science and Mathematics,
HFT Stuttgart, University of Applied Sciences, Schellingstr. 24
D-70174 Stuttgart, Germany
Tel.: +49 (0)711 8926 2791
E-mail: manfred.besner@hft-stuttgart.de

a player set, especially if the set is organized hierarchically, such as a large corporation or a governmental or political entity. As suggested by Owen (1977), Winter (1989) defined for such games a model called cooperative games with a level structure (LS-games) and extended the Owen value and, therefore, also the Shapley value to the Shapley levels value.

A level structure comprises a sequence of coalition structures (the different levels). At each level, the player set is divided into components with each lower level being finer than the next higher.

Both Owen (1977) and Winter (1989) extended the axiomatization of Shapley (1953b), in particular, they used a symmetry between components axiom. It stays that if two components which are subsets of the same component one level higher are symmetric in a game where these components are the players, the total payoff to all players of the first component is equal to the total payoff to all players of the second component.

In many situations, this axiom seems to be questionable. For the case where there are reasons not to treat symmetric players equally, Shapley (1953a) introduced the class of weighted Shapley values. Kalai and Samet (1987) examine games in which players represent groups of individuals and state,

> *"Such is the case for example when the players are parties, cities, or management boards. The use of the symmetric Shapley value seems to be unjustified in certain cases of this type because the players represent constituencies of different sizes. A natural candidate for a solution is the weighted Shapley value where the players are weighted by the size of the constituencies they stand for."*

The representation as a TU-game, where a player represents a group of individuals, has the disadvantage that the TU-game does not reflect the impact of the individuals personally, but only as a group. For this purpose, games with a coalition structure are required. McLean (1991) extended the class of weighted Shapley values to games with a coalition structure, in Dragan (1992) called McLean weighted coalition structure values.

Harsanyi (1977) observes that in simple bargaining processes, when two or more players join to form an acting bargaining unit, their bargaining position worsens relative to the remaining players. Moreover, Harsanyi notes that this holds for all solution concepts that satisfy efficiency and the symmetry axiom, hence also for the Shapley value.

This effect, in Vidal-Puga (2012) called the Harsanyi paradox, is, of course, also noticeable for the Owen value. For example, if all players of two components are symmetric in a game in which all coalitions achieve a cooperative win, the players of the larger component will receive a smaller payoff than those of the other component.

In light of this, Vidal-Puga (2012) introduced a value for games with a coalition structure with weights for the components determined by the size of the coalitions, which is extended by Gómez-Rúa and Vidal-Puga (2011) to games with a level structure which can be seen, similar to the Shapley levels value, as a special case of the class of weighted Shapley hierarchy levels values in Besner (2019).

Gómez-Rúa and Vidal-Puga (2010) have the merit of axiomatically comparing three extensions of the Shapley value to games with a coalition structure, the Owen value, the per capita value in Vidal-Puga (2012) and a value of the class of the McLean weighted coalition structure values, in this case also with weights for the components determined by the size of the coalitions.

The latter value is the starting point of the value for games with a level structure examined in this study. Since this value is a special case of the class of weighted Shapley support levels values in Besner (2022a), we will call it the per capita Shapley support levels value.

In today's era of increasing use of artificial intelligence and machine learning, it is becoming more and more important to adopt methods from cooperative game theory into practice. In application, the computational complexity to compute the Shapley value turns out to be a hurdle because for a player set with n players, the coalitional worths of all 2^n many possible coalitions (except those of the empty set) are needed for computation. This is one motivation for dedicating a separate study to the per capita Shapley support levels value. In Besner (2022b), it is shown that all three of the previously mentioned values for games with a level structure have a polynomial running time for their computation and are thus preferable to the Shapley value in general in this respect.

While the Shapley levels value has been extensively axiomatically investigated, and there is already a separate investigation for the value in Vidal-Puga (2012) and Gómez-Rúa and Vidal-Puga (2011), for the per capita Shapley support levels value there exists so far only the summary work by Gómez-Rúa and Vidal-Puga (2010) mentioned above (and only for the special case for games with a coalition structure). Apart from that, this value, which is very important from our point of view, especially for the applications, is unknown so far.

As argued above, both per capita values for games with a level structure are preferable to the Shapley levels value in many applications with respect to the symmetry axiom and the Harsanyi paradox. However, with the value in Gómez-Rúa and Vidal-Puga (2011), many users may be bothered by the fact that it does not satisfy the null player property. This shortcoming does not occur with the per capita Shapley support levels value. Nevertheless, here are the null players "not so null" (see the relevant comments in Vidal-Puga (2012), Gómez-Rúa and Vidal-Puga (2011), and in the Conclusion in Besner (2022a)).

To avoid a "two-step" approach with two different behaviors, one for the game between the components and one within components, we deal here from the beginning with games with a level structure and have the same behavior for each step or level. In general, this also makes the axiomatization more compact. A game with a coalition structure is considered here as a special case, just as a conventional TU-game is another special case of a game with a level structure.

For axiomatization, we use the standard axioms efficiency, the null player property, and additivity and two new axioms. Unlike in Gómez-Rúa and Vidal-Puga (2010), we do not use unanimity games. As a special feature, in contrast to all other axiomatizations of the Owen or Shapley levels value known to us, such as, e.g., in Owen (1977), Winter (1989), Calvo et al. (1996), Khmelnitskaya and Yanovskaya (2007), Alvarez-Mozos and Tejada (2011), or Casajus and Takeng (2023), our axiomatization does not need quotient games, also referred to as intermediate games, i.e., games with components as players.

Our first new axiom, called joined productivity, is a weakening and extension of symmetry. It implies that if for two components of the same level, which are subsets of the same component one level higher, all players make cooperative gains only if all players join forces, then each player should receive the same payoff.

While collusion studies in the literature (see Harsanyi (1977), Haller (1994), or Segal (2003)) are mainly concerned with how collusive arrangements affect the colluding actors, in our second new axiom, called neutral collusions, we focus on the effects on the other, non-colluding actors. This axiom then states that it does not matter how players in a component use their powers in different coalitions, as long as the total power remains the same for the coalitions involving all players in the component, nothing changes for players outside the component.

In short, it is recommended to consider the per capita Shapley support levels value as a fair payoff method when players are able to join forces to form larger actionable units, for whatever reason. These include, to name just a few application examples, the distribution of costs in large companies, the distribution of profits in company shareholdings or to participants in supply chains, payments for the generation or storage of green electricity to individual participants, who can join together regionally and locally, the weighting of votes of members of individual parties and countries in parliaments, or the scheduling of processes in computer cores.

The paper is organized as follows. Section 2 contains some preliminaries. As the main section, we give in Section 3 the definition of the per capita Shapley support levels value, introduce the new axioms and give an axiomatic characterization. Section 4 concludes our results. The Appendix (Section 5) shows the logical independence of the axioms in our axiomatization.

2 Preliminaries

An n-person cooperative game with transferable utility (**TU-game**) (N, v) on a non-empty and finite player set N is given by a coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Since throughout the paper we are only dealing with a fixed player set N, N is usually omitted as an argument. The subsets $T \subseteq N$ are called **coalitions**, $v(T)$ is the **worth** of the coalition T, and the set of all nonempty subsets of N is denoted by Ω^N. We denote the cardinality of any coalition T by $|T|$ and the set of all TU-games on N is denoted by $\mathbb{V}$.

The **dividends** $\Delta_v(T)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(T) := \begin{cases} v(T) - \sum_{S \subsetneq T} \Delta_v(S), & \text{if } T \in \Omega^N, \text{ and} \\ 0, & \text{if } T = \emptyset. \end{cases}$$

A TU-game $u_T \in \mathbb{V}$, $T \in \Omega^N$, with $u_T(S) := 1$ if $T \subseteq S$ and $u_T(S) := 0$ otherwise for all $S \subseteq N$ is called a **unanimity game**. Any coalition function v on N has a unique representation, given by

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{1}$$

A coalition $T \subseteq N$ is called **essential** in v, if $\Delta_v(T) \neq 0$. We call a player $i \in N$ a **null player** in v if $v(T \cup \{i\}) = v(T)$ for all $T \subseteq N \backslash \{i\}$ and we call two players $i, j \in N$, $i \neq j$, **mutually dependent** (Nowak and Radzik, 1995) in v if $v(T \cup \{i\}) = v(T) = v(T \cup \{j\})$ for all $T \subseteq N \backslash \{i, j\}$ or, equivalently, $\Delta_v(T \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $T \subseteq N \backslash \{i, j\}$. This means, mutually dependent players are only jointly productive.

A set $\mathcal{B} := \{B_1, ..., B_m\}$ of coalitions of players is called a **coalition structure** on N if $\mathcal{B}$ is a partition of the player set N, i.e., a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of N. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component containing the player $i \in N$.

A finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N is called a **level structure** (Winter, 1989) on N if

- $\mathcal{B}^0 = \big\{\{i\} \colon i \in N\big\}$,
- $\mathcal{B}^{h+1} = \{N\}$, and
- for each r, $0 \leq r \leq h$, $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$, i. e., $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$.

$\mathcal{B}^r$ is called the rth **level** of $\underline{\mathcal{B}}$ and $\mathbb{L}$ denotes the set of all level structures on N. A TU-game $v \in \mathbb{V}$ together with a level structure $\underline{\mathcal{B}} \in \mathbb{L}$ is an **LS-game** which we denote by $(v, \underline{\mathcal{B}})$. The set of all LS-games on N is denoted by $\mathbb{VL}$.

A **TU-value** ϕ is an operator that assigns to any $v \in \mathbb{V}$ a payoff vector $\phi(v) \in \mathbb{R}^N$. As probably the most important representative of TU-values, the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(v) := \sum_{T \subseteq N, T \ni i} \frac{\Delta_v(T)}{|T|} \quad \text{for all } i \in N,$$

distributes the dividend of each coalition equally to its members.

An **LS-value** φ is an operator that assigns to any LS-game $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$ a payoff vector $\varphi(v, \underline{\mathcal{B}}) \in \mathbb{R}^N$. As probably the most important representative of LS-values, the **Shapley levels value** Sh^L (Winter, 1989) is given by (see Calvo et al. 1996, Eq. (1))

$$Sh_i^L(v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_{\underline{\mathcal{B}},T}(i) \Delta_v(T) \text{ for all } i \in N, \tag{2}$$

where, for all $T \in \Omega^N$, $T \ni i$, we have

$$K_{\underline{\mathcal{B}},T}(i) := \prod_{r=0}^{h} \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\ B \cap T \neq \emptyset\}|}.$$

This means that from the dividend of a coalition T, all components of the hth level containing members of T initially receive an equal share. Then, the share of each such component is distributed equally among the subsets of that component that are components of the next lower level and also contain members of T, and so on, until finally only members of T itself, as members of a component of the first level, divide the share of that component equally among themselves.

It is easy to see that Sh^L coincides with Sh if we have $\underline{\mathcal{B}} = \{\mathcal{B}_0, \mathcal{B}_1\}$ and it is well-known that Sh^L coincides with the Owen value (Owen, 1977) if $\underline{\mathcal{B}} = \{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2\}$.

The following axioms for LS-values φ are simple adaptations of standard axioms for TU-values.

Efficiency, E. For all $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$, we have $\sum_{i \in N} \varphi_i(v, \underline{\mathcal{B}}) = v(N)$.

Efficiency means that the complete total payoff matches exactly the output of the grand coalition.

Null player, N. For all $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(v, \underline{\mathcal{B}}) = 0$.

According to the null player property, a player who does not contribute to the game at all should not receive a payoff.

Additivity, A. For all $(v, \underline{\mathcal{B}}), (v', \underline{\mathcal{B}}) \in \mathbb{VL}$, we have
$\varphi(v, \underline{\mathcal{B}}) + \varphi(v', \underline{\mathcal{B}}) = \varphi(v + v', \underline{\mathcal{B}})$.

Additivity requires that an LS-value be an additive function of LS-games, which means that a player's payoff from the sum of two games is the sum of the player's payoff for the two games.

3 The per capita Shapley support levels value

We can see the Shapley levels value as a special case of the class of the weighted Shapley support levels values in Besner (2022a). The following LS-value is also a special case of this class. Therefore, the algorithm for the distribution of dividends is quite similar, but instead of the shares of the components involved in each level being equal, each component always receives a share corresponding to the number of members of the component.

Definition 3.1. For all $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and for all $T \in \Omega^N$, $T \ni i$, define

$$K^{PC}_{\underline{\mathcal{B}},T}(i) := \prod_{r=0}^{h} \frac{|\mathcal{B}^r(i)|}{\sum_{\substack{B\in\mathcal{B}^r:\, B\subseteq\mathcal{B}^{r+1}(i),\\ B\cap T\neq\emptyset}} |B|}. \tag{3}$$

Then, the **per capita Shapley support levels value** Sh^{PCSL} is given by

$$Sh^{PCSL}_i(v, \underline{\mathcal{B}}) = \sum_{T\subseteq N,\, T\ni i} K^{PC}_{\underline{\mathcal{B}},T}(i)\Delta_v(T) \text{ for all } i \in N. \tag{4}$$

Also here, it is easy to see that Sh^{PCSL} coincides with Sh if we have $\underline{\mathcal{B}} = \{\mathcal{B}_0, \mathcal{B}_1\}$ and, using the presentation of the class of the McLean weighted coalition structure values (McLean, 1991), given in Dragan (1992, Sec. 2(e)), Sh^{PCSL} can be seen as a special case of this class if $\underline{\mathcal{B}} = \{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2\}$.

In what follows, we show axiomatically that the per capita Shapley support levels value, rather than just the Shapley levels value, can be viewed as a useful extension of the Shapley value.

Joint productivity, JP. For all $(v, \underline{\mathcal{B}}) \in \mathbb{VL}, \underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and two components $B_1, B_2 \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_1, B_2 \subseteq B, B \in \mathcal{B}^{r+1}$, and all players i, j, $i \in B_1$ and $j \in B_2$, are mutually dependent in v, we have $\varphi_i(v) = \varphi_j(v)$.

This axiom is a fairness property within components. It means that if all players of two components, which are subsets of the same component one level higher and are only jointly productive, they should receive the same payoff.

For our last property, we introduce a game related to an origin game, where the players of a component can make collusions of the power of all coalitions containing some players of the component. The only condition is that the coalitions that contain all players of the component have the same power as these coalitions in the origin game.

Definition 3.2. Let $(v, \underline{\mathcal{B}}) \in \mathbb{VL}, \underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and $B \in \mathcal{B}^r$, $0 \leq r \leq h$, be a component. Then an LS-game $(v_B, \underline{\mathcal{B}}) \in \mathbb{VL}$ is called a **component collusion game** to $(v, \underline{\mathcal{B}})$ if we have that

$$v_B(S) = v(S) \text{ if } S \subseteq N\backslash B \text{ or } B \subseteq S.$$

Neutral collusions, NC. For all $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, a component $B \in \mathcal{B}^r$, $0 \leq r \leq h$, and a component collusion game $(v_B, \underline{\mathcal{B}})$ to $(v, \underline{\mathcal{B}})$, we have

$$\varphi_i(v, \underline{\mathcal{B}}) = \varphi_i(v_B, \underline{\mathcal{B}}) \text{ for all } i \in N\backslash B.$$

We can also consider this axiom as a fairness property but for players outside a component. Regardless of how players within a component organize their power, as long as the power of the coalitions that include all players of the component remains the same, the payoff to players outside the component does not change.

The next theorem characterizes the per capita Shapley support levels value.

Theorem 3.3. *An LS-value φ satisfies* ***E****,* ***N****,* ***A****,* ***JP****, and* ***NC*** *if and only if φ equals Sh^{PCSL}.*

Proof. Let $(v, \underline{\mathcal{B}}) \in \mathbb{VL}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$.

I. Existence: Since Sh^{PCSL} is obviously a special case of the weighted Shapley support levels values, **E**, **N**, and **A** are satisfied by Besner (2022a).

• **JP**: Let $B_1, B_2 \in \mathcal{B}^k$, $0 \leq k \leq h$, be such that $B_1, B_2 \subseteq B, B \in \mathcal{B}^{k+1}$, and all players i, j, $i \in B_1$ and $j \in B_2$, are mutually dependent in v. Since all players from B_1, B_2 are mutually dependent, in the sum in (4), for a player $i \in B_1$ or $j \in B_2$, we have only to regard coalitions T such that $B_1 \cup B_2 \subseteq T$. All other coalitions containing a player i or j have a dividend of zero.

For each r, $0 \leq r < k$, in (3), the denominator of the fraction equals $|\mathcal{B}^{r+1}(i)|$ or $|\mathcal{B}^{r+1}(j)|$, respectively, for $r = k$, the denominators of the fractions are equal for i and j, and for $r > k$, the fractions are equal for i and j. Therefore, we have $K^{PC}_{\underline{\mathcal{B}},T}(i) = K^{PC}_{\underline{\mathcal{B}},T}(j)$ for all i, j, $i \in B_1$ and $j \in B_2$ and the claim follows by (4).

• **NC**: Let $B \in \mathcal{B}^r$, $0 \leq r \leq h$, $(v_B, \underline{\mathcal{B}})$ be a component collusion game to $(v, \underline{\mathcal{B}})$, $S \subseteq N \backslash B$ be a coalition of players, and $i \in S$ be a player outside of B. Then, in the sum in (4), there is no difference between summands containing dividends for coalitions T with $T \cap B = \emptyset$ between the games $(v, \underline{\mathcal{B}})$ and $(v_B, \underline{\mathcal{B}})$ for the player i.

Now, we regard coalitions T such that $T = S \cup R$, $R \subseteq B$, $R \neq \emptyset$. For all such T, in (4) the $K^{PC}_{\underline{\mathcal{B}},T}(i)$ are equal and depend not on the coalition function. Since we have $v_B(T) = v(T)$ if $B \subseteq T$, we have $v_B(S \cup B) = v(S \cup B)$ and $(v, \underline{\mathcal{B}}) = (v_B, \underline{\mathcal{B}})$. Therefore, if a dividend of such a coalition T changes in v_B compared to v, a dividend of one or more such other coalitions $T' = S \cup R$, $R \subseteq B$, $R \neq \emptyset$, must also change.[1] In any case, the total amount of the dividends from all such coalitions remains the same in both games. Thus, since from these amounts of dividends the player i gets always the same share in both games and this holds for all S defined above, Sh^{PCSL} satisfies **NC**.

II. Uniqueness: Let φ be an LS-value which satisfies all axioms from Theorem 3.3. By **A** and (1), it is sufficient to show that φ is unique for all games $\Delta_v(T)u_T$, $T \in \Omega^N$. For all such games such that $\Delta_v(T) = 0$, φ is unique by **N**. Let now T such that $\Delta_v(T) \neq 0$.

The following part of the proof is constructive. For each such T, we always start with the game $\Delta_v(T)u_N$, for which φ is unique by **JP**, and, using the satisfied axioms, we modify step by step the game to the game $\Delta_v(T)u_T$, preserving uniqueness.

If $T = N$, φ is already unique. Let now $T \subsetneq N$. Then there is a highest level r, $0 \leq r \leq h$, such that all players of some components $B^r \in \mathcal{B}^r$ which are subsets of the same component $B^{r+1} \in \mathcal{B}^{r+1}$ one level higher are null players in $\Delta_v(T)u_T$. Note that there exist always some components of the rth level which are also subsets of B^{r+1} where not all members are null players in $\Delta_v(T)u_T$. We delete all the players from the components B^r which contain only null players in $\Delta_v(T)u_T$ from the coalition N and obtain a new coalition T_1^r. Then, the game $\Delta_v(T)u_{T_1^r}$ is a component collusion game to $\Delta_v(T)u_N$. Therefore, by **NC**, φ is unique on the game $\Delta_v(T)u_{T_1^r}$ for all players $i \in N \backslash B^{r+1}$ again and, by **JP**, φ is also unique on $\Delta_v(T)u_{T_1^r}$ for all players $i \in B^{r+1}$.

[1] T' must always contain S because if we use instead S a proper subset $S' \subsetneq S$, the dividends must be already be "balanced" within $S' \cup B$ due to $v_B(S' \cup B) = v(S' \cup B)$.

If there are further components of the rth level where all players are null players in $\Delta_v(T)u_T$, we repeat the same procedure and obtain that φ is unique on the game $\Delta_v(T)u_{T_2^r}$ and so on. At the end, we have that φ is unique on a game $\Delta_v(T)u_{T_k^r}$.

Then there is, eventually, again another highest level ℓ, $0 \leq \ell < r$, such that all players of some components $B^\ell \in \mathcal{B}^\ell$ which are subsets of the same component $B^{\ell+1} \in \mathcal{B}^{\ell+1}$ one level higher are null players in $\Delta_v(T)u_T$. Again, we apply the same procedure and when we are done with this level we descend to the next level and so on. If there are no more components within our unanimity coalition that contain only null players in $\Delta_v(T)u_T$, the unanimity coalition is exactly our coalition T and it is shown that φ is unique on the game $\Delta_v(T)u_T$ and the proof is complete. □

4 Conclusion

To keep the theoretical foundations of the study as simple as possible, no obvious axiomatization of the per capita Shapley support levels value in terms of the weighted support levels values in Besner (2022a, Proposition 4.5) was undertaken. There is only a need to replace the weights by the size of the components for the w-weighted dependence between components property.

Purely for reasons of proof, our two new axioms could also be weakened. It would be sufficient if the joint productivity property and the neutral collusions property were formulated only for unanimity games. However, we believe that these axioms are more meaningful in the form chosen.

We can see the neutral collusions property as an important argument for all parties involved in the payoff calculation to agree on a value. In particular, smaller or weaker participants, who rarely have control over all operations at the larger partners, can be assured that activities in which they are not involved will not rip them off.

From a technical point of view, the neutral collusion property is responsible for the fact that the LS-values discussed here are preferable to the Shapley value in terms of runtime complexity. Simply, not all coalitions are needed to compute the payoff (see Besner (2022b)).

Of course, if the components on each level each have the same size, the per capita Shapley support level value also satisfies the symmetry between components axiom[2], which is always satisfied by the Shapley levels value. Moreover, in this case, the per capita Shapley support levels value, the LS-value in Gómez-Rúa and Vidal-Puga (2011), and the Shapley levels value coincide.

Table 1 shows the main characteristics of the three LS-values discussed here. For a numerical example comparing the three LS-values, we refer to Besner (2022a, Section 5, Example).

[2]Two players $i, j \in N, i \neq j$, are symmetric in v if $v(T \cup \{i\}) = v(T \cup \{j\})$ for all $T \subseteq N \backslash \{i, j\}$. The symmetry between components axiom (Winter, 1989) states that if two components of the same level that are subsets of the same component one level higher are symmetric in a game with the components as players, the total payoff to all players of the first component is equal to the total payoff to the players of the second component.

[3]The Shapley value Sh is here interpreted as an LS-value, the LS-value from Gómez-Rúa and Vidal-Puga (2011) is denoted here, as a special case of the weighted Shapley hierarchy levels values in Besner (2019), by Sh^{PCHL}.

[4]The balanced per capita contributions property (Gómez-Rúa and Vidal-Puga, 2011) states that for two components that are subsets of the same component one level higher, in a game with the components as players, the change per capita in the payoffs of the players in the first component when the second

Table 1: Properties of some LS-values[3]

LS-value	Sh	Sh^L	Sh^{PCHL}	Sh^{PCSL}
Efficiency	+	+	+	+
Null player	+	+	–	+
Additivity	+	+	+	+
Symmetry between components	–	+	–	–
Balanced per capita contributions[4]	–	–	+	–
Joint Productivity	+	–	+	+
Neutral collusions	–	+	+	+

5 Appendix

5.1 Logical independence

Remark 5.1. *The axioms in Theorem 3.3 are logically independent.*

Proof.

- **E**: The LS-value Ψ, given by

$$\Psi_i(v, \underline{\mathcal{B}}) = 2 \cdot \sum_{T \subseteq N, T \ni i} K^{PC}_{\underline{\mathcal{B}},T}(i) \Delta_v(T) \text{ for all } i \in N.$$

  where the $K^{PC}_{\underline{\mathcal{B}},T}(i)$ are the coefficients defined in (3), satisfies **N**, **A**, **JP**, and **NC** but not **E**.

- **N**: The equal division value **ED**, interpreted as an LS-value, given by

$$ED_i(v, \underline{\mathcal{B}}) = \frac{v(N)}{|N|} \text{ for all } i \in N,$$

  satisfies **E**, **A**, **JP**, and **NC** but not **N**.

- **A**: The LS-value ψ, given by

$$\psi_i(v, \underline{\mathcal{B}}) := \begin{cases} 0, \text{ if } i \text{ is a null player in } v, \\ \dfrac{v(N)}{|\{j \in N : j \text{ is no null player in } v|}, \text{ otherwise,} \end{cases}$$

  for all $i \in N$, satisfies **E**, **N**, **JP**, and **NC** but not **A**.

- **JP**: The Shapley levels value Sh^L satisfies **E**, **N**, **A**, and **NC** but not **JP**.

- **NC**: The Shapley value **Sh**, interpreted as an LS-value, given by

$$Sh_i(v \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} \frac{\Delta_v(T)}{|T|} \text{ for all } i \in N,$$

  satisfies **E**, **N**, **A**, and **JP** but not **NC**.

□

component leaves the game should be equal to the change per capita in the payoffs of the players in the second component when the first component leaves the game.

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