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1. Introduction

First of all, the model in this paper is exactly the same as the binomial tree in my earlier paper, Brogi (2014). What differs now is that, while in my previous paper the tree was implemented by Monte Carlo simulation, i.e. simulating price trajectories along the tree, in this paper the whole (recombining) underlying price tree is calculated without resorting to Monte Carlo, just like for example the classic Cox, Ross and Rubinstein (1979) binomial tree (CRR tree). This means that the option price is obtained virtually instantly using for example Matlab on a standard PC. On the other hand, Monte Carlo simulation was rather lengthy and the resulting option price had a standard error. The main features that make the tree appealing are unchanged: excess kurtosis and negative skewness of price distribution of underlying security. For more details please see simulation in Brogi (2014).

The paper proceeds as follows. In Section 2 the binomial tree is described. Section 3 derives a formula for the risk-neutral measure. Section 4 shows how volatility is modeled. Section 5 explains how to implement the tree. Finally Section 6 concludes the paper with some remarks.

2. Binomial Tree

Time-points t_i, $i = 0,1,\ldots,n$, are equidistant, and time-periods $\Delta t = t_i - t_{i-1}$, $i = 1,\ldots,n$, and time horizon $T = n\Delta t$, which is fixed length of time of expiration of option in years. t_0 is current time-point. We also have an extra historical time-point, t_{-1}, which precedes t_0, and such that $t_0 - t_{-1} = \Delta t$.

The underlying security price can either rise or fall from one-point to the next, $i = 1,\ldots,n$:

$$S_{t_i} = S_{t_{i-1}} u_{t_i} \text{ with probability } q_i \text{ or}$$
$$S_{t_i} = S_{t_{i-1}} d_{t_i} \text{ with probability } 1 - q_i, \tag{1}$$

where u_{t_i} stands for *up*, d_{t_i} stands for *down*, and u_{t_i}, d_{t_i} are variable. q_i is the risk-neutral probability of underlying security price at t_{i-1}, $S_{t_{i-1}}$, rising to $S_{t_{i-1}} u_{t_i}$ at t_i. Further down we derive a formula for q_i.

The definition of continuously compounded return of underlying security from t_{i-1} to t_i:

$$R_{t_i} = \log S_{t_i} - \log S_{t_{i-1}}, \qquad i = 0,1,\ldots,n. \tag{2}$$

We call *current return* $R_{t_0} = \log S_{t_0} - \log S_{t_{-1}}$, where $S_{t_{-1}}$ is a known historical price, so *current return* is known too. Rearranging (1), and taking logarithms, and using (2) we define, $i = 1,\ldots,n$,

$$\log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} = \log u_{t_i} = R_{t_i}^+ \text{ with probability } q_i \text{ or}$$
$$\log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} = \log d_{t_i} = R_{t_i}^- \text{ with probability } 1 - q_i, \tag{3}$$

where

$$R_{t_i}^+ = \mu \Delta t + \sigma_{t_i} \sqrt{\Delta t}, \tag{4}$$
$$R_{t_i}^- = \mu \Delta t - \sigma_{t_i} \sqrt{\Delta t}, \tag{5}$$

with $\mu \Delta t < \sigma_{t_i} \sqrt{\Delta t}$ for large n, or equivalently small Δt. $\sigma_{t_i} \sqrt{\Delta t}$ is part of a volatility process, $\left\{ \sigma_{t_i} \sqrt{\Delta t} \right\}_{i=0}^n$, which we need to model, where σ_{t_0} is known current annual volatility.

3. Martingale Condition

Under no arbitrage, the discounted price process of the underlying security, $\left\{ \widetilde{S}_{t_i} \right\}_{i=0}^n$, must be a martingale. We now derive a formula for risk-neutral probability q_i in (1), so that $\left\{ \widetilde{S}_{t_i} \right\}_{i=0}^n$ is a martingale.

Let us introduce a sample of independent Bernoulli random variables, which are independent of $\left\{ \widetilde{S}_{t_i} \right\}_{i=0}^n$:

$$Z_i = +1 \text{ with probability } q_i \text{ or}$$
$$Z_i = -1 \text{ with probability } 1 - q_i,$$

where q_i is the risk-neutral probability in (1). Then (4) and (5) can be written as one equation:

$$R_{t_i} = \mu \Delta t + \sigma_{t_i} \sqrt{\Delta t} Z_i, \qquad i = 1,\ldots,n.$$

The martingale condition is

$$\mathrm{E}\!\left(\widetilde{S}_{t_i} \mid \widetilde{S}_{t_{i-1}}, \widetilde{S}_{t_{i-2}}, \ldots\right) = \widetilde{S}_{t_{i-1}}, \qquad\qquad i = 1, \ldots, n,$$

$$\mathrm{E}\!\left(e^{-ir\Delta t} S_{t_{i-1}} e^{R_{t_i}} \mid \widetilde{S}_{t_{i-1}}, \widetilde{S}_{t_{i-2}}, \ldots\right) = e^{-(i-1)r\Delta t} S_{t_{i-1}}$$

$$\mathrm{E}\!\left(e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}\, Z_i} \mid \widetilde{S}_{t_{i-1}}, \widetilde{S}_{t_{i-2}}, \ldots\right) = e^{r\Delta t},$$

where r is the risk-free rate of interest, which is constant during time horizon T, and

$$e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}\, Z_i} \mid \sigma_{t_i} = e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}} \text{ with probability } q_i \text{ or}$$

$$e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}\, Z_i} \mid \sigma_{t_i} = e^{\mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}} \text{ with probability } 1 - q_i,$$

so that

$$q_i e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}} + (1 - q_i) e^{\mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}} = e^{r\Delta t}.$$

Hence,

$$q_i = \frac{e^{r\Delta t} - e^{\mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}}}{e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}} - e^{\mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}}}.$$

In risk-neutral pricing we set $\mu = r$, so that

$$q_i = \frac{1 - e^{-\sigma_{t_i}\sqrt{\Delta t}}}{e^{\sigma_{t_i}\sqrt{\Delta t}} - e^{-\sigma_{t_i}\sqrt{\Delta t}}}, \qquad i = 1, \ldots, n. \tag{6}$$

For large n, or equivalently small Δt, substituting the exponentials in (6) by their series expansions ignoring terms of order $(\Delta t)^{3/2}$ or higher, we get

$$q_i = \frac{1}{2} - \frac{1}{4}\sigma_{t_i}\sqrt{\Delta t}, \qquad i = 1, \ldots, n. \tag{7}$$

So, the risk-neutral probability of $S_{t_{i-1}}$ rising is less than for $S_{t_{i-1}}$ falling. This is true for any n, or equivalently any Δt.

$$\mathrm{E}\!\left(R_{t_i} \mid \sigma_{t_i}\right) = \mu\Delta t + (2q_i - 1)\sigma_{t_i}\sqrt{\Delta t} \tag{8}$$

$$\mathrm{var}\!\left(R_{t_i} \mid \sigma_{t_i}\right) = 4q_i(1 - q_i)\sigma_{t_i}^2 \Delta t. \tag{9}$$

Notice that if $q_i = 1/2$ (which it is not), then

$$\mathrm{E}\!\left(R_{t_i} \mid \sigma_{t_i}\right) = \mu\Delta t$$

$$\mathrm{var}\!\left(R_{t_i} \mid \sigma_{t_i}\right) = \sigma_{t_i}^2 \Delta t.$$

Setting q_i as in (6) is an artificial device which forces $\{\widetilde{S}_{t_i}\}_{i=0}^n$ to be a martingale.

4. Modeling Volatility

As regards the modeling of $\sigma_{t_i}\sqrt{\Delta t}$, Black (1976) already noticed a negative correlation between returns and volatility, i.e. when returns are high, volatility is low, and when returns are low, volatility is high. Such negative correlation can be captured by the following equation:

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(R_{t_{i-1}} - \mu\Delta t\right), \qquad i = 1,\ldots,n, \qquad (10)$$

where $0 < \alpha < 1$. When $R_{t_{i-1}} = \mu\Delta t$ volatility is constant. This can be seen as an equilibrium. Whilst the more $R_{t_{i-1}}$ differs from $\mu\Delta t$, the more we move away from equilibrium, and the more pronounced changes in volatility are. It is clear that according to (10) volatility of returns, $\sigma_{t_i}\sqrt{\Delta t}$, can never be negative, because, recalling (4) and (5), if $R_{t_{i-1}} = R_{t_{i-1}}^+$, then

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(\sigma_{t_{i-1}}\sqrt{\Delta t}\right), \qquad i = 2,\ldots,n, \qquad (11)$$

$0 < \alpha < 1$. Alternatively, if $R_{t_{i-1}} = R_{t_{i-1}}^-$, then

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} + \alpha\left(\sigma_{t_{i-1}}\sqrt{\Delta t}\right), \qquad i = 2,\ldots,n, \qquad (12)$$

$0 < \alpha < 1$.

$$\begin{aligned}
\mathrm{E}\left(\sigma_{t_i}\sqrt{\Delta t}\mid\sigma_{t_{i-1}}\right) &= \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(\mathrm{E}\left(R_{t_{i-1}}\mid\sigma_{t_{i-1}}\right) - \mu\Delta t\right) \\
&= \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha(2q_{i-1}-1)\sigma_{t_{i-1}}\sqrt{\Delta t}
\end{aligned},$$

where, as in (6),

$$q_0 = \frac{1 - e^{-\sigma_{t_0}\sqrt{\Delta t}}}{e^{\sigma_{t_0}\sqrt{\Delta t}} - e^{-\sigma_{t_0}\sqrt{\Delta t}}},$$

which is known.
From (10) we see that

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_0}\sqrt{\Delta t} - \alpha\sum_{j=0}^{i-1}\left(R_{t_j} - \mu\Delta t\right), \qquad i = 1,\ldots,n.$$

Hence,

$$\mathrm{E}\left(\sigma_{t_i} \sqrt{\Delta t} \mid \sigma_{t_{i-1}}, \sigma_{t_{i-2}}, \ldots\right) = \sigma_{t_0} \sqrt{\Delta t} - \alpha \sum_{j=0}^{i-1} (2q_j - 1)\sigma_{t_j} \sqrt{\Delta t} \,.$$

At t_n, dropping $\sqrt{\Delta t}$,

$$\mathrm{E}\left(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \ldots\right) = \sigma_{t_0} - \alpha \sum_{j=0}^{n-1} (2q_j - 1)\sigma_{t_j} \,,$$

and, using (7) for q_j,

$$\lim_{n \to \infty} \mathrm{E}\left(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \ldots\right) = \sigma_{t_0} - \alpha \lim_{n \to \infty} \sum_{j=0}^{n-1} \left(-\frac{1}{2}\sigma_{t_j} \sqrt{\Delta t}\right)\sigma_{t_j} = \infty \,, \tag{13}$$

because each term in the sum is of order $\sqrt{\Delta t}$ (order $1/\sqrt{n}$), but extra terms are added to the sum at rate n. So, the expected value of σ_{t_n} has no finite limit.

$$\mathrm{var}\left(\sigma_{t_i} \sqrt{\Delta t} \mid \sigma_{t_{i-1}}\right) = \alpha^2 \, \mathrm{var}\left(R_{t_{i-1}} \mid \sigma_{t_{i-1}}\right) = 4\alpha^2 q_{i-1}(1 - q_{i-1})\sigma_{t_{i-1}}^2 \Delta t \,. \tag{14}$$

Looking at (14), we note that the greater the α, the greater the variance of volatility of returns, and the greater the variance of volatility of returns, the greater the kurtosis of the distribution of returns.

5. Implementing the Tree

Before providing commented Matlab code implementing the tree, let me first make it clear that the tree of underlying security prices is in fact recombining, because

$$d_{t_{i-1}} u_{t_i} = e^{2\mu\Delta t + \alpha\sigma_{t_{i-1}}\sqrt{\Delta t}} = u_{t_{i-1}} d_{t_i} \qquad i = 2, \ldots, n$$

Furthermore, volatilities are also recombining, because from equations (11) and (12) we have that

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t}(1 - \alpha) \ \text{ or } \ \sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t}(1 + \alpha) \qquad i = 2, \ldots, n$$

Given the recombining property of prices and volatilities, the full price tree can be obtained. Then the option value at expiry is calculated, and from the expiry nodes work backwards down the tree to obtain option values at earlier nodes in standard fashion, as in the CRR tree, with the possibility of pricing European or American options. The only difference in working backwards down the tree compared to the CRR tree is that, while in the CRR tree the probability of an 'up' move is fixed, here we have a different

probability of an 'up' move at each node calculated according to equation (6) or (7). It is these different probabilities that need to be used when working backwards down the tree. The option price equals the value obtained at the starting node. See for example Haug (2007) to see how to work backwards in the CRR tree.

Here follows the Matlab code (comments in green).

```matlab
function
optionprice=athostree(St0,Shist,k,sigt0,r,T,n,alpha,otype,earlyexercise)
%function to calculate the price of a vanilla European or American
%Put or Call option using the binomial tree presented in paper
%"A Binomial Tree to Price European and American Options" by Athos
Brogi.
%Function inputs:
% St0 - current price of underlying
% Shist - historical price of underlying
% k - strike price
% r - risk free interest rate (=mu)
% T - time horizon in years
% n - number of time steps
% otype - must be 'PUT' or 'CALL'
% earlyexercise - true for American, false for European


currentreturn=log(St0)-log(Shist);

%calculate the recombining volatility tree (vtree)
vtree=nan(n,n);
vtree(1,1)=sigt0*sqrt(T/n)-alpha*(currentreturn-r*(T/n));
%(note: check vtree(1,1)>0)
for i=2:n
    vtree(1:i-1,i)=vtree(1:i-1,i-1)*(1-alpha);
    vtree(i,i)=vtree(i-1,i-1)*(1+alpha);
end

%calculate the recombining price of underlying tree (ptree)
ptree=nan(n+1,n+1);
ptree(1,1)=St0;
for i=2:n+1
    ptree(1:i-1,i)=ptree(1:i-1,i-1).*exp(r*(T/n)+vtree(1:i-1,i-1));
    ptree(i,i)=ptree(i-1,i-1).*exp(r*(T/n)-vtree(i-1,i-1));
end

%calculate the probability tree (probtree), where each node shows
probability
%of an 'up' move. The probability of a 'down' move equals 1 minus
%probability of an 'up' move
probtree=nan(n,n);
probtree=1/2-1/4*vtree;

%calculate value at expiry
valuetree=nan(size(ptree));
switch otype
    case 'PUT'
        valuetree(:,end)=max(k-ptree(:,end),0);
    case 'CALL'
```

```
            valuetree(:,end)=max(ptree(:,end)-k,0);
end

%work backwards down the valuetree to get values at earlier nodes
for i=n:-1:1
    valuetree(1:i,i)=...
        exp(-r*T/n)*(probtree(1:i,i).*valuetree(1:i,i+1)...
        +(1-probtree(1:i,i)).*valuetree(2:i+1,i+1));
    if earlyexercise
        switch otype
            case 'PUT'
                valuetree(1:i,i)=...
                    max(k-ptree(1:i,i),valuetree(1:i,i));
            case 'CALL'
                valuetree(1:i,i)=...
                    max(ptree(1:i,i)-k,valuetree(1:i,i));
        end
    end
end

%option price
optionprice=valuetree(1);
end
```

Here are 4 examples of calculated prices.

>> optionprice=athostree(100,98,100,0.3,0.03,1,100,0.05,'PUT',false)

optionprice = 10.1273

>> optionprice=athostree(100,98,100,0.3,0.03,1,100,0.05,'CALL',false)

optionprice = 13.0822

>> optionprice=athostree(100,98,100,0.3,0.03,1,100,0.05,'PUT',true)

optionprice = 10.3303

>> optionprice=athostree(100,98,100,0.3,0.03,1,100,0.05,'CALL',true)

optionprice = 13.0822

6. Performance

Pricing performance has been measured by comparing model mean squared error (MSE) with MSE of Black-Scholes (BS) model.

Sample: Trades of S&P500 calls of Tuesday 15/1/2019 (a randomly chosen day, because it is provided by iVolatility.com free of charge as a data example) for a total of 5498 trades after having applied 2 filters, i.e. $0.9 <= St0/K <= 1.1$ and expiration < 6 months (18/1, 15/2, 15/3, 18/4, 21/6).

Limitations: Dividend yield not included and r = 0.01 for all expiration dates for greater ease.

Calibration: I calibrated my model, which has (only!) 2 parameters to be calibrated, i.e. sigmat0 and alpha. I also calibrated BS model, which has unique parameter sigma to be calibrated.

The calibration for each model was done by searching the values of above parameters that minimize the MSE between the market price and the model price by means of the fminsearch Matlab function, which uses the Nelder-Mead method.

The number of steps of my model (tree) was set to 100.

Results:
BS: sigma = 0.1513, MSE = 13.85, running time = few seconds.
My tree: sigmat0 = 0.1558, alpha = 0.0423, MSE = 4.15, running time = about 9 minutes.

The sample mean market price is 34.37.

Matlab R2009b was used on my (few years old) PC with Intel Core i5 processor.

I also calibrated BS and my model to S&P500 calls trades of 2 other days:

16/1/2019
BS: sigma = 0.1471, MSE = 9.39.
My tree: sigmat0 = 0.1514, alpha = 0.0408, MSE = 1.9107.
The sample mean market price is 30.19.

23/1/2019 (expiration < 9 months)
BS: sigma = 0.1626, MSE = 22.28.
My tree: sigmat0 = 0.1694, alpha = 0.0446, MSE = 3.3646.
The sample mean market price is 47.40.

7. Final Remarks

The greatest challenge is not implementing the tree, but to choose appropriate values of α and n for pricing. As we have seen above, equation (13), as n tends to infinity so does volatility, and therefore the tree explodes. We have also seen, equation (14), that the greater α, the greater the kurtosis of the distribution of returns. So, the tree explodes also if α is too large. Therefore, appropriate values of α and n need to be input to avoid the tree calculating meaningless prices.
However, on the positive side, the tree has a relatively small number of parameters, and to me it can also be viewed as a sound application of martingale pricing. It also calculates American and European option prices virtually instantly on a standard PC.

References

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