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# The Market-Based Statistics of "Actual" Returns of Investors

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**ABSTRACT**

The paper presents the unified theoretical description of three levels of the market-based statistical moments of "actual" returns, which Investors gain within their market sales. The market-based statistics of "actual" returns takes into account the size of the trade sale values, purchased values and volumes of stocks and that differs it from conventional regular statistics based on frequency analysis of returns time-series. We start with description of statistical moments of returns, which Investor gains via a single sale due to his multiple purchases in the past. The second level describes statistics of returns, which Investor gains performing numerous market sales during the "trading day". The third level describes statistics of returns that different Investors gain during the "trading day". We derive dependence of statistical moments of returns on statistical moments of market sale values, purchased values and volumes of stocks. In its turn, statistical moments of trade values and volumes for finite number of market trades during the "trading day" are assessed via regular frequency-based probability.

Keywords : asset pricing, stock returns, volatility, correlations, probability, market trades

JEL: C1, E4, F3, G1, G12

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## 1. Introduction

The literature describes two kinds of stock returns. For convenience we note them as "anticipated" and as "actual" returns. The most studied "anticipated" returns $r(t,\tau)$ are determined by ratio $r(t,\tau)=p(t)/p(t-\tau)$ of stock price $p(t)$ traded "today" at time $t$ and price $p(t-\tau)$ traded at $t-\tau$ in the past. "Anticipated" stock returns $r(t,\tau)$ describe the expected, anticipated gains or losses which investors could get if they bought stocks at time $t-\tau$ and then sold them at $t$. That is the most common treatment of stock returns. Modelling and predictions of "anticipated" stock returns define the core issues of financial economics (Fisher and Lorie, 1964; Mandelbrot, Fisher and Calvet, 1997; Campbell, 1985; Brown, 1989; Fama,1990; Fama and French, 1992; Lettau and Ludvigson, 2003; Greenwood and Shleifer, 2013; van Binsbergen and Koijen, 2015; Martin and Wagner, 2019). Description of probabilistic properties of "anticipated" stock returns during any specific averaging time interval $\Delta$ "today" or as we note it "trading day" and predictions of probability of returns at horizon $T$ "next day" delivers most desired results for investors and traders. The frequency-based analysis assesses probability distributions of "anticipated" returns during "trading day" (Amaral, et al., 2000; Andersen et al., 2001; Knight and Satchell, 2001; Tsay, 2005; Andersen, and Benzoni, 2009) and in many others.

However, "anticipated" stock returns $r(t,\tau)=p(t)/p(t-\tau)$ with time shift $\tau$ describe gains or losses that may or may not match real, "actual" returns of Investors. As "actual" we consider returns, which particular Investor gains at time $t$ selling stocks he has purchased 10 minutes, a day, a week or any time in the past. It is obvious, that not all stocks that are sold by Investors at time $t$ were purchased at time $t-\tau$. Some stocks were purchased earlier or later then $t-\tau$ and at price that differs from $p(t-\tau)$. Thus, Invertors who sell stocks at time $t$ gain returns that are different from "anticipated" returns $r(t,\tau)=p(t)/p(t-\tau)$. So, "anticipated" returns $r(t,\tau)$ describe an option Investors may gain and "actual" returns describe realized benefits of Investors. During a "trading day" traders and Investors sell stocks that were initially purchased at different times before the sale in the past. Investors sale stocks during the "trading day" and gain returns on stocks they purchased 10 minutes, a week or any time in the past. To assess average returns or statistical moments of "actual" returns, which Investors gain one should take into account returns with different time shifts. That differs description of statistical properties of "anticipated" and "actual" stock returns. Investigation of "actual" returns of institutional, professional or individual Investors forms a separate problem. Different aspects of "actual" returns were studied by (Schlarbaum, Lewellen and Lease, 1978; Stanley, Lewellen and Schlarbaum, 1980; Baker and Wurgler, 2004; Ivković, Sialm and Weisbenner, 2004; Gabaix, et al 2005; Daniel and Hirshleifer, 2016; Koijen, Richmond and Yogo, 2020; Hardouvelis, Karalas and Vayanos, 2021) and others.

However, statistical properties of "anticipated" and "actual" returns mostly are studied in the same way. To assess probability $P(r)$ of "anticipated" or "actual" returns during the time interval $\Delta$ one studies time-series of returns and assesses the frequency $m_r/N$ of returns:

$$P(r) \sim \frac{m_r}{N} \tag{1.1}$$

In (1.1) $m_r$ denotes number of returns equal $r$ and $N$ is the total number of terms of time-series of returns during $\Delta$. That is the regular assessment of probability of any events analyzing its time-series during $\Delta$ that follows the firm ground of the probability theory (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). For convenience we further call such assessments (1.1) as frequency-based probabilities of stock returns. It is regular and completely correct assessment of probability of returns, if time-series of returns $r(t_i,\tau)$ during the averaging interval $\Delta$ are considered as independent, self-reliant variables. However, it is obvious that returns at time $t_i$ are determined by stock price $p(t_i)$ at $t_i$ and stock price $p(t_i-\tau)$ at time $t_i-\tau$. Moreover, stock prices $p(t_i)$ themselves are determined by market trade values $C(t_i)$ and volumes $U(t_i)$ performed at time $t_i$

$$C(t_i) = p(t_i)U(t_i) \tag{1.2}$$

For convenience in this paper we take all prices as present prices. We consider market trade values, volumes, market prices and returns as random variables during the selected time averaging interval $\Delta$. It is well known, that the set of $n$-$th$ statistical moments for all $n=1,2,\ldots$ completely describes statistical properties of a random variable, determines its characteristic function and probability density function (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). Finite number of statistical moments approximates stochastic properties, characteristic function and probability density function of a random variable.

We believe, that consideration of the market prices and stock returns as financial and economic matters should take into account impact of the size of the trade values $C(t_i)$ and volumes $U(t_i)$ (1.2) on average price, returns, volatility and statistics of returns. The well-known Volume Weighted Average Price (VWAP) (Berkowitz et al., 1988; Duffie and Dworczak, 2018) that differs from the frequency-based assessment of the mean price demonstrates impact of the size of trade volumes $U(t_i)$ on definition of the average price. Thus it is reasonable that statistical properties of the market trade values $C(t_i)$ and volumes $U(t_i)$ should determine statistics of the market prices and hence price statistics should determine statistics of stock returns. Probability of asset price and stock returns determined by stochasticity of market trade time-series differ from one determined by the frequency-based probability of price and returns time-series. That distinguishes assessments of the market-based probability of stock returns from probabilities determined by frequencies of time-series of returns. We note statistical properties of stock returns determined by random time-series of the market trade values $C(t_i)$ and volumes $U(t_i)$ during $\Delta$ as the market-based probability of stock returns. Description of statistical properties of market prices and "anticipated" stock returns determined by statistical moments of the market trade values $C(t_i)$ and volumes $U(t_i)$ is given in (Olkhov, 2021-2023) and we use main results to describe statistical moments of "actual" returns.

This pure theoretical paper describes the market-based statistical moments of the "actual" stock returns of Investors as functions of statistical moments of the market trade values $C(t_i)$ and volumes $U(t_i)$. In its turn statistical moments of the market trade values and volumes are assessed using their trade time-series via regular frequency-based (1.1) probability (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). The market-based probability of "actual" stock returns takes into account impact of the size of Investor's trade values and volumes on average return, volatility and etc.

In Sec.2 for convenience we briefly present main relations which model the market-based statistical moments of prices and "anticipated" stock returns. In Sec.3 we consider the market-based statistical moments of "actual" returns of a single trade sale. Sec.4 presents statistical moments of "actual" returns which Investor gains if he performs a lot of sales during $\Delta$. Sec.5 describes the market-based statistical moments of "actual" returns averaged by the set of $Q$ different Investors those perform trade sales during $\Delta$. Sec.6 Conclusion. We assume that readers are familiar with basics of probability theory, statistical moments and etc.

## 2. The market-based statistical moments of "anticipated" stock returns

As "anticipated" we consider stock returns $r(t_i,\tau)= p(t_i)/p(t_i\text{-}\tau)$ determined as ratio of market trade price $p(t_i)$ at time $t_i$ with respect to price $p(t_i\text{-}\tau)$ at time $t_i\text{-}\tau$. We consider the trade of identical stocks and take all prices adjusted to present. Let us consider market time-series of the trade values $C(t_i)$ and volumes $U(t_i)$ at time $t_i$ and assume that time-series determine initial time discreteness of the problem with constant shift ε

$$t_{i+1} - t_i = \varepsilon \quad ; \quad \varepsilon - const$$

Market trade time-series deliver irregular and highly variable data. To evaluate trade forecasts of stock returns one should proceed with averaged variables. To do that let us choose time averaging interval $\Delta>>\varepsilon$. The choice of the interval $\Delta$ introduces new time discreteness multiple of $\Delta$ at times $t_k$, $k=0,1,\ldots$ As $t_0=t$ we consider present time $t$ "today" and time $t_k=t-k\,\Delta$ describes $k\,\Delta$ intervals in the past:

$$\Delta_k=\left[t_k-\frac{\Delta}{2};t_k+\frac{\Delta}{2}\right] \quad ; \quad t_k=t-\Delta\cdot k \quad ; \quad k=0,1,2,\ldots \; ; \; \Delta_0=2n\cdot\varepsilon \quad (2.1)$$

Let us renumber initial trade time-series and take that

$$t_{i,k}\in\Delta_k \quad ; \quad i=1,2,..N \quad ; \quad t_{i+1,k}-t_{i,k}=\varepsilon \quad ; \quad t_{i,k+1}-t_{i,k}=\Delta \qquad (2.2)$$

Each interval $\Delta_k$, $k=0,1,..$ contains same number $N$ of terms of trade time-series. We consider trade values $C(t_{i,k})$, volumes $U(t_{i,k})$ and prices $p(t_{i,k})$ during each interval $\Delta_k$ (2.1; 2.2) as random variables. To describe probabilistic properties of a random variable one can equally take probability density function, characteristic function or the set of statistical moments of random variable (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We take the set of statistical moments to assess probabilistic properties of trade values, volumes, prices and returns. We present statistical moments of the market trade value $C(t;n)$ and volume $U(t;n)$ averaged during $\Delta$ (2.1) "today" using regular frequency-based probability (1.1) as

$$C(t;n)\equiv E[C^n(t_i)]\sim\frac{1}{N}\sum_{i=1}^{N}C^n(t_i) \qquad (2.3)$$

$$U(t;n)\equiv E[U^n(t_i)]\sim\frac{1}{N}\sum_{i=1}^{N}U^n(t_i) \quad ; \quad n=1,2,. \qquad (2.4)$$

We use $E[\ldots]$ to denote mathematical expectation (2.3; 2.4) during $\Delta$ (2.1) and symbol "~" to underline that finite number $N$ of market trades during $\Delta$ determines assessments, the estimators of the statistical moments of the market trade value $C(t;n)$ and volume $U(t;n)$ at time $t=t_0$ "today" averaged during $\Delta$. The sets of statistical moments (2.3; 2.4) for all $n=1,2,..$ describe statistical properties of the trade value $C(t_i)$ and volume $U(t_i)$ as random variables during $\Delta$ "today". However, any reasonable averaging interval $\Delta$ contains only finite number of terms of market trade time-series. Thus relations (2.3; 2.4) can assess only finite number of statistical moments of the market trade value $C(t;n)$ and volume $U(t;n)$. Statistical moments (2.3; 2.4) for finite $n$ assess only approximations of probability density functions and characteristic functions of random variables $C(t_i)$ and $U(t_i)$. Thus, finite number of statistical moments (2.3; 2.4) can describe only approximations of probability density functions and characteristic functions of the price and returns as random variables during $\Delta$. To avoid repeats and reduce complexity in this paper we skip similar derivation of the approximations of characteristic functions and probability density functions of "actual" returns and refer (Olkhov, 2021; 2022a; 2023) for details.

To describe how statistical moments (2.4; 2.5) define statistical moments $r(t,\tau;n)=E[r^n(t_i,\tau)]$ of "anticipated" returns we transform (1.2) as:

$$C(t_i) = p(t_i)U(t_i) = \frac{p(t_i)}{p(t_i-\tau)}p(t_i-\tau)U(t_i) = r(t_i,\tau)\, C_a(t_i,\tau)$$

We take "anticipated" returns $r(t_i,\tau)$ with time shift $\tau$ as ratio of prices (2.5):

$$r(t_i,\tau) = \frac{p(t_i)}{p(t_i-\tau)} \tag{2.5}$$

We obtain equations (2.6) which links up sale value $C(t_i)$, returns $r(t_i,\tau)$ and past value $C_a(t_i)$:

$$C(t_i) = r(t_i,\tau)\, C_a(t_i,\tau) \quad ; \quad C_a(t_i,\tau) = p(t_i-\tau)U(t_i) \tag{2.6}$$

We remind that consider all prices as present prices today at $t$. We introduce past value $C_a(t_i,\tau)$ (2.6) determined by trade volume $U(t_i)$ at $t_i$ and "adjusted" by price $p(t_i-\tau)$ of the trade at $t_i-\tau$ in the past. In other words, $C_a(t_i,\tau)$ (2.6) expresses the value of the volume $U(t_i)$ at purchase price $p(t_i-\tau)$ at $t_i-\tau$ in the past. Equation (2.6) is similar to trade equation (1.2) but the price $p(t_i-\tau)$ of the purchase at $t_i-\tau$ defines an "adjusted" value $C_a(t_i,\tau)$. Similar to (2.3) we introduce frequency-based assessments of statistical moments $C_a(t,\tau;n)$ of value $C_a(t_i,\tau)$:

$$C_a(t,\tau;n) \equiv E[C_a^n(t_i,\tau)] \sim \frac{1}{N}\sum_{i=1}^{N} C_a^n(t_i,\tau) \tag{2.7}$$

Equation (1.2; 2.6; 2.5) allow derive market-based *n-th* statistical moments of the market price $p(t;n)$, "adjusted" market price $p_a(t,\tau;n)$ and statistical moments of returns $r(t,\tau;n)$. To do that for $n=1,2,\ldots$ let us take *n-th* degrees (2.8-2.10) of equations (1.2; 2.6; 2.5):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \tag{2.8}$$

$$C_a^n(t_i,\tau) = p^n(t_i-\tau)U^n(t_i) \tag{2.9}$$

$$C^n(t_i) = r^n(t_i,\tau)\, C_a^n(t_i,\tau) \tag{2.10}$$

Then, using (2.3; 2.4; 2.7) we introduce (2.11; 2,12) statistical moments $p(t;n)$ of market price $p(t_i)$:

$$C(t;n) = p(t;n)U(t;n) \tag{2.11}$$

$$p(t;n) \equiv E[p^n(t_i)] = \frac{C(t;n)}{U(t;n)} \sim \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \tag{2.12}$$

Relations (2.13; 2.14) introduce statistical moments $p_a(t,\tau;n)$ of "adjusted" market price

$$C_a(t,\tau;n) = p_a(t,\tau;n)U(t;n) \tag{2.13}$$

$$p_a(t,\tau;n) \equiv E[p_a^n(t_i-\tau)] = \frac{C_a(t,\tau;n)}{U(t;n)} \sim \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i-\tau)U^n(t_i) \tag{2.14}$$

Relations (2.15; 2.16) define statistical moments $r(t,\tau;n)$ of "anticipated" returns with shift $\tau$:

$$C(t;n) = r(t,\tau;n)C_a(t,\tau;n) \tag{2.15}$$

$$r(t,\tau;n) \equiv E[r^n(t_i,\tau)] = \frac{C(t;n)}{C_a(t,\tau;n)} \sim \frac{1}{\sum_{i=1}^{N} C_a^n(t_i,\tau)} \sum_{i=1}^{N} r^n(t_i,\tau)C_a^n(t_i,\tau) \tag{2.16}$$

Relations (2.11-2.16) introduce dependence of statistical moments of price *p(t;n),* "adjusted" price $p_a(t,\tau;n)$ and "anticipated" returns *r(t,τ;n)* on statistical moments of trade values *C(t;n)* (2.3), "adjusted" values $C_a(t,\tau;n)$ (2.7) and volumes *U(t;n)* (2.4) similar to definition of VWAP (Berkowitz et al., 1988; Duffie and Dworczak, 2018). For convenience we note as volume weighted average "adjusted" price $p_a(t,\tau;1)$ (VWAPa) (2.14; 2.17)

$$p_a(t,\tau;1) = \frac{C_a(t,\tau;1)}{U(t;1)} \sim \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i, -\tau) U(t_i) \tag{2.17}$$

We call average return *r(t,τ;1)* as "adjusted" value weighted average returns ($V_a$WAR) (2.18)

$$r(t,\tau;1) = \frac{C(t;1)}{C_a(t,\tau;1)} \sim \frac{1}{\sum_{i=1}^{N} C_a(t_i,\tau)} \sum_{i=1}^{N} r(t_i,\tau) C_a(t_i,\tau) \tag{2.18}$$

Volatility $\sigma_r^2(t,\tau)$ of "anticipated" stock returns takes form:

$$\sigma_r^2(t,\tau) \equiv E[((r(t_i,\tau) - r(t,\tau;1))^2] = r(t,\tau;2) - r^2(t,\tau;1) \tag{2.19}$$

$$\sigma_r^2(t,\tau) = \frac{C(t;2)}{C_a(t,\tau;2)} - \frac{C^2(t;1)}{C_2^2(t,\tau;1)} \tag{2.20}$$

Relations (2.11-2.20) describe market-based statistical moments of price and "anticipated" stock returns in a similar way and we refer (Olkhov, 2021; 2022a; 2023) for further details. In Sections 3, 4, 5 we use (2.8-2.20) to derive market-based statistical moments of "actual" returns of Investors.

## 3. "Actual" returns of a single sale

In this Section we consider statistical moments of "actual" returns which Investor can gains within a single sale of volume $U(t_i)$ of stocks. We propose that Investor at time $t_i$ during *Δ* - *"*today*"*- sells $U(t_i)$ stocks at price $p(t_i)$. We take that Investor purchased this amount of stocks $U(t_i)$ by pieces $U(t_j(i))$ at different times $t_j(i)$, $j=1,2, ..M(i)$ in the past at price $p(t_j(i))$. As we mentioned above, we consider all prices $p(t_j(i))$ at times $t_j(i)$ in the past expressed in present price at *t*. Investor at time $t_j(i)$ purchased value $C_p(t_j(i))$ of the volume $U(t_j(i))$ of stocks at price $p(t_j(i))$:

$$C_p(t_j(i) = p\left(t_j(i)\right) U\left(t_j(i)\right) \tag{3.1}$$

For each volume $U(t_j(i))$ of stocks purchased by Investor in the past at price $p(t_j(i))$ we can take equation (3.2) similar to (2.6).

$$C\left(t_i, t_j(i)\right) = p(t_i) U\left(t_j(i)\right) = \frac{p(t_i)}{p\left(t_j(i)\right)} p\left(t_j(i)\right) U\left(t_j(i)\right) = r\left(t_j(i)\right)\ C_p\left(t_j(i)\right)$$

$$r\left(t_j(i)\right) \equiv \frac{p(t_i)}{p\left(t_j(i)\right)} \quad ; \quad C_p\left(t_j(i)\right) \equiv p\left(t_j(i)\right) U\left(t_j(i)\right)$$

$$C\left(t_i, t_j(i)\right) = r\left(t_j(i)\right)\ C_p\left(t_j(i)\right) \tag{3.2}$$

Equation (3.2) links up current $C(t_i,t_j(i))$ and past $C_p(t_j(i))$ trade values and "actual" returns $r(t_j(i))$ which Investor gains selling the piece of stocks $U(t_j(i))$ of trade volume $U(t_i)$ at price $p(t_i)$ "today". The value $C_p(t_j(i))$ (3.2) is a value of purchase at price $p(t_j(i))$ expressed in present price. Volume $U(t_i)$ (3.3) was purchased in the past by pieces $U(t_j(i))$ at prices $p(t_j(i))$. At moment of sale at $t_i$ volume $U(t_i)$ (3.6) has purchase value $C_p(t_i)$ (3.3) in present prices:

$$C_p(t_i) = \sum_{j=1}^{M(i)} C_p(t_j(i)) = \sum_{j=1}^{M(i)} p\left(t_j(i)\right) U\left(t_j(i)\right) \tag{3.3}$$

$$U(t_i) = \sum_{j=1}^{M(i)} U\left(t_j(i)\right) \tag{3.4}$$

It is obvious, that purchases of stocks in the past at different prices $p(t_j(i))$ result in different "actual" returns $r(t_j(i))$. If total number $M(i)$ of purchases is sufficiently large then equation (3.2) allows derive statistical moments of the "actual" returns $r(t_j(i))$ of a single market sale at time $t_i$. One should follow Sec 2. and similar to (2.10) take *n-th* degree of (3.2):

$$C^n(t_i, t_j(i)) = r^n\left(t_j(i)\right) C_p^n\left(t_j(i)\right) \tag{3.5}$$

We introduce statistical moments $C(t_i;n)$ (3.6) at $t_i$ of the sale value $C(t_i,t_j(i))$ and statistical moments $C_p(t_i;n)$ (3.7) at $t_i$ of the purchased value $C_p(t_j(i))$ as:

$$C(t_i;n) \equiv E\left[C^n(t_i,t_j(i))\right] \sim \frac{1}{M(i)} \sum_{j=1}^{M(i)} C^n(t_i,t_j(i)) \tag{3.6}$$

It is clear that average sale value $C(t_i;1)$ (3.6) for $n=1$ multiplied by $M(i)$ equals the value $C(t_i)$ of the sale at $t_i$.

$$C(t_i) = \sum_{j=1}^{M(i)} C^n(t_j(i)) = M(i)C(t_i;1)$$

Statistical moments $C_p(t_i;n)$ (3.7) of the purchased value $C_p(t_j(i))$ take form:

$$C_p(t_i;n) \equiv E\left[C_p^n\left(t_j(i)\right)\right] \sim \frac{1}{M(i)} \sum_{j=1}^{M(i)} C_p^n(t_j(i)) \tag{3.7}$$

Then, similar to (2.11-2.16), obtain statistical moments $r(t_i;n)$ of "actual" returns which Investor gains via a single sale of volume $U(t_i)$ at $t_i$ and numerous purchases in the past

$$C(t_i;n) = r(t_i;n)C_p(t_i;n) \tag{3.8}$$

$$r(t_i;n) \equiv E\left[r^n\left(t_j(i)\right)\right] \tag{3.9}$$

$$r(t_i;n) = \frac{C(t_i;n)}{C_p(t_i;n)} \sim \frac{1}{\sum_{j=1}^{M(i)} C_p^n(t_j(i))} \sum_{j=1}^{M(i)} r^n\left(t_j(i)\right) C_p^n\left(t_j(i)\right) \tag{3.10}$$

Average sale value $C(t_i;1)$ (3.6) and average purchase value $C_p(t_i;1)$ determine average returns $r(t_i;1)$ (3.8; 3.10) of a singe trade at $t_i$. Volatility $\sigma_r^2(t_i)$ of returns of a single trade at $t_i$ takes form

$$\sigma_r^2(t_i) \equiv E\left[r^2\left(t_j(i)\right)\right] - E^2\left[r\left(t_j(i)\right)\right] = r(t_i;2) - r^2(t_i;1) \tag{3.11}$$

$$\sigma_r^2(t_i) = \frac{C(t_i;2)}{C_p(t_i;2)} - \frac{C^2(t_i;1)}{C_p^2(t_i;1)} \tag{3.12}$$

## 4. "Actual" returns of a single Investor

Now propose that Investor *q, q=1,2,…Q* performs *N(q)* market sales during *Δ* "today" at time $t_i$, $i=1,…N(q)$. We remind that $C(t_i;1)$ (3.6) notes the average sale value at $t_i$ of the average purchase value $C_p(t_i;1)$ (3.7) and $r(t_i;1)$ (3.8-3.10) for *n=1*, denotes average "actual" returns of the single sale at $t_i$. Investor can gain different average returns $r(t_i;1)$ for different sales at times $t_i$, $i=1,…N(q)$ during *Δ* "today". If number of sales *N(q)>>1* then one can assess statistics of "actual" average returns $r(t_i;1)$(3.8-3.10) which Investor gains during *Δ*. To do that we take equation (3.8) as generating equation similar to (2.6) and derive statistical moments $r_q(t;1|n)$ during *Δ* of average returns $r(t_i;1)$ as:

$$C^n(t_i;1) = r^n(t_i;1)C_p^n(t_i;1) \ ; \ n = 1,.. \tag{4.1}$$

We introduce statistical moments $C_q(t;1|n)$ (4.2) of average value $C(t_i;1)$ (3.6) and statistical moments $C_{qp}(t;1|n)$ (4.3) of average purchased value $C_p(t_i;1)$ (3.7) of Investor *q* :

$$C_q(t;1|n) \equiv E[C^n(t_i;1)] \sim \frac{1}{N(q)}\sum_{i=1}^{N(q)} C^n(t_i;1) \tag{4.2}$$

$$C_{qp}(t;1|n) \equiv E\left[C_p^n(t_i;1)\right] \sim \frac{1}{N(q)}\sum_{i=1}^{N(q)} C_p^n(t_i;1) \tag{4.3}$$

The number *N(q)* of trade sales performed by Investor *q* during *Δ* "today" determines max *n* of statistical moments (4.2; 4.3) which can be assessed. If Investor *q* makes a single sale during *Δ* "today" then one can assess his "actual" returns of a single sale only.

However, if Investor *q* performs a lot of sales *N(q)>>1* then different "actual" average returns of sales set up statistics and we determine statistical moments $r_q(t;1|n)$ (4.4-4.6) of "actual" average returns $r(t_i;1)$(3.8-3.10) of Investor *q*:

$$C_q(t;1|n) = r_q(t;1|n)C_{qp}(t;1|n) \tag{4.4}$$

$$r_q(t;1|n) \equiv E[r^n(t_i;1)] \tag{4.5}$$

$$r_q(t;1|n) = \frac{C_q(t;1|n)}{C_{qp}(t;1|n)} \sim \frac{1}{\sum_{i=1}^{N(q)} C_p^n(t_i;1)} \sum_{i=1}^{N(q)} r^n(t_i;1)C_p^n(t_i;1) \tag{4.6}$$

Investor *q* who performs a lot *N(q)>>1* trade sales during the "trading day" *Δ* gains average "actual" returns $r(t;1|1)$ (4.7), 2-d statistical moment $r(t;1|2)$ (4.8) and volatility of returns $\sigma_r^2(t;1)$ (4.9) during *Δ*:

$$r_q(t;1|1) = \frac{C_q(t;1|1)}{C_{qp}(t;1|1)} \sim \frac{1}{\sum_{i=1}^{N(q)} C_p(t_i;1)} \sum_{i=1}^{N(q)} r(t_i;1)C_p(t_i;1) \tag{4.7}$$

$$r_q(t;1|2) = \frac{C_q(t;1|2)}{C_{qp}(t;1|2)} \sim \frac{1}{\sum_{i=1}^{N(q)} C_p^2(t_i;1)} \sum_{i=1}^{N(q)} r^2(t_i;1)C_p^2(t_i;1) \tag{4.8}$$

$$\sigma^2_{qr}(t;1) \equiv r_q(t;1|2) - r^2_q(t;1|1) = \frac{C_q(t;1|2)}{C_{qp}(t;1|2)} - \frac{C^2_q(t;1|1)}{C^2_{qp}(t;1|1)} \qquad (4.9)$$

It is important to underline, that Investors sales during the "trading day" $\Delta$ and assessments of statistical moments (4.2-4.9) cause that there are no correlations between time-series of average returns $r(t_i;1)$ (3.8-3.10) and the average purchased values $C_p(t_i;1)$ (3.7). We underline that the assessment of correlations depends on the choice of probability, and is not determined by given time-series. Indeed, market-based correlations $corr_{rCp}(t,1|t,1)$ between returns $r(t_i;1)$ (3.8-3.10) and $C_p(t_i;1)$:

$$corr_{r\,C_p}(t;1|t;1) \equiv E\left[r(t_i;1)C_p(t_i;1)\right] - E[r(t_i;1)]\,E\left[C_p(t_i;1)\right]$$

From (4.2; 4.3; 4.4 - 4.6) obtain

$$corr_{r\,C_p}(t;1|t;1) = C_q(t;1|n) - r_q(t;1|n)C_{qp}(t;1|n) = 0$$

This result is similar to no correlations between time-series of the trade volumes and prices for the VWAP (Olkhov, 2021; 2022a). However, some researcher may assess correlations between two random variables not taking into account their market-based statistical moments, but using frequency-based analysis of two given time-series only. Such assessments for sure will be different from described above market nature of stochasticity of returns determined by randomness of the market trade values and volumes. Those who studies econometric market time-series should make a choice between investigation of the market nature of stochasticity of prices and returns described above and conventional assessments of frequency-based relations which have few economic and financial sense.

We should underline that above model allows describe statistics of volatility of returns of a single trade, or statistics of *n-th* statistical moments of a single trade, which Investor has during the "trading day" $\Delta$. Taking into account (3.6-3.9; 4.4-4.9; 4.11-4.13) Investor $q$ has average volatility $\sigma_{qr}^2(t)$ (4.10; 4.14) during $\Delta$ of volatilities $\sigma_r^2(t_i)$ (3.11; 3.12) of trades at $t_i$ :

$$\sigma^2_{qr}(t) \equiv E[r(t_i;2)] - E[r^2(t_i;1)] = r_q(t;2|1) - r_q(t;1|2) \qquad (4.10)$$

$$C_q(t;2|1) \equiv E[C(t_i;2)] \sim \frac{1}{N(q)}\sum_{i=1}^{N(q)} C(t_i;2) \qquad (4.11)$$

$$C_{qp}(t;2|1) \equiv E\left[C_p(t_i;2)\right] \sim \frac{1}{N(q)}\sum_{i=1}^{N(q)} C_p(t_i;2) \qquad (4.12)$$

$$r_q(t;2|1) \equiv \frac{C_q(t;2|1)}{C_{qp}(t;2|1)} \sim \frac{1}{\sum_{i=1}^{N(q)} C_p(t_i;2)} \sum_{i=1}^{N(q)} r(t_i;2)C_p(t_i;2) \qquad (4.13)$$

$$\sigma^2_{qr}(t) = \frac{C_q(t;2|1)}{C_{qp}(t;2|1)} - \frac{C_q(t;1|2)}{C_{qp}(t;1|2)} \qquad (4.14)$$

## 5. "Actual" returns of all Investors

In this Sec. we consider assessments of statistical moments of time-series of average "actual" returns as result of trade sales of all Investors during the "trading day" $\Delta$.

Statistical moments $r(t;1|n)$ (4.4-4.7), average "actual" returns $r(t;1|1)$ and volatility $\sigma_r^2(t;1)$ (4.9) of different Investors during $\Delta$ can vary significantly. If number of Investors $Q>>1$ then one can assess statistical moments of the sale values, purchased values and "actual" returns of different Investors and assess average "actual" returns and volatility which all Investors gain within their sales during the "trading day" $\Delta$. We remind, that the "trading day" or averaging interval $\Delta$ can be equal an hour, a day, a week or whatever. Assume that Investor $q$, $q=1,..Q$ performs $N(q)$ sales and $N$ - total number of sales during $\Delta$:

$$N = \sum_{q=1}^{Q} N(q) \tag{5.1}$$

Average values of sales $C(t_i;1)$ (3.6), average purchased values $C_p(t_i;1)$ (3.7) and average "actual" returns $r(t_i;1)$ (3.8-3.10) could be presented by irregular, changeable time-series during $\Delta$. Above methods allow describe their statistical moments averaged during $\Delta$. We introduce statistical moments $C(t;1|n)$ (5.2) of average values of sales $C(t_i;1)$ (3.6) and statistical moments $C_p(t;1|n)$ (5.3) of average purchased values $C_p(t_i;1)$ (3.7) as:

$$C(t;1|n) \equiv E[C^n(t_i;1)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_i;1) \tag{5.2}$$

$$C_p(t;1|n) \equiv E\left[C_p^n(t_i;1)\right] \sim \frac{1}{N}\sum_{i=1}^{N} C_p^n(t_i;1) \tag{5.3}$$

We underline that all statistical moments $C(t_i;n)$ (3.6) of values of sales, statistical moments $C_p(t_i;n)$ (3.7) of purchased values for $n=1,2,..$ could be presented by irregular, volatile time-series during the "trading day" $\Delta$. Thus one can assess *m-th* statistical moments $C(t;n|m)$ of *n-th* statistical moments $C(t_i;n)$ (3.6) of values of sales, purchased values similar to (5.2; 5.3). Such description can provide fine details of the stochasticity of Investors sales and returns. Equations (3.8; 4.1) generate statistical moments $r(t;1|n)$ (5.4; 5.5) of average "actual" returns $r(t_i;1)$ (3.8-3.10):

$$C(t;1|n) = r(t;1|n)C_p(t;1|n) \tag{5.4}$$

$$r(t;1|n) \equiv E[r^n(t_i;1)]$$

$$r(t;1|n) = \frac{C(t;1|n)}{C_p(t;1|n)} \sim \frac{1}{\sum_{i=1}^{N} C_p^n(t_i;1)} \sum_{i=1}^{N} r^n(t_i;1)C_p^n(t_i;1) \tag{5.5}$$

Average "actual" returns $r(t;1|1)$ (5.6) of all sales, 2-d statistical moment $r(t;1|2)$ (5.7) and volatility of returns $\sigma_r^2(t;1)$ (5.8) of all sales as result of all $N$ trade sales during $\Delta$ take form:

$$r(t;1|1) = \frac{C(t;1|1)}{C_p(t;1|1)} \sim \frac{1}{\sum_{i=1}^{N} C_p(t_i;1)} \sum_{i=1}^{N} r(t_i;1)C_p(t_i;1) \tag{5.6}$$

$$r(t;1|2)=\frac{C(t;1|2)}{C_p(t;1|2)}\sim\frac{1}{\sum_{i=1}^{N}C_p^2(t_i;1)}\sum_{i=1}^{N}r^2(t_i;1)C_p^2(t_i;1) \tag{5.7}$$

$$\sigma_r^2(t;1)=r(t;1|2)-r^2(t;1|1)=\frac{C(t;1|2)}{C_p(t;1|2)}-\frac{C^2(t;1|1)}{C_p^2(t;1|1)} \tag{5.8}$$

However, statistics generated by sale values, purchase values and "actual" returns of different Investors are different from (5.2-5.8). If number $Q$ of different Investors during $\Delta$ is big enough $Q>>1$ then one can assess returns statistics of different Investors. To assess average "actual" returns $r_q(t;1|1)$ (5.9) of different Investors let us take (4.2-4.7) for $n=1$ as:

$$C_q(t;1|1)=r_q(t;1|1)C_{qp}(t;1|1) \tag{5.9}$$

Equations (4.4; 5.9) link up average value $C_q(t;1|1)$ (4.2), average purchased value $C_q(t;1|1)$ (4.3), and average "actual" returns $r_q(t;1|1)$ (4.4-4.7). Similar to (4.1) take the *m-th* degree of (5.9) and obtain:

$$C_q^m(t;1|1)=r_q^m(t;1|1)C_{qp}^m(t;1|1) \tag{5.10}$$

We define *m-th* statistical moments $C_I(t,1|1|m)$ (5.11) of average values $C_q(t,1|1)$ (4.2) and *m-th* statistical moments $C_{Ip}(t,1|1|m)$ (5.12) of average purchased values $C_{qp}(t,1|1)$ (4.3) of different Investors $q$, $q=1,\dots Q$ via frequency-based probability:

$$C_I(t;1|1|m)\equiv E\left[C_q^m(t;1|1)\right]\sim\frac{1}{Q}\sum_{q=1}^{Q}C_q^m(t;1|1) \tag{5.11}$$

$$C_{Ip}(t;1|1|m)\equiv E\left[C_{qp}^m(t;1|1)\right]\sim\frac{1}{Q}\sum_{q=1}^{Q}C_{qp}^m(t;1|1) \tag{5.12}$$

Using (5.10) we introduce *m-th* statistical moments $r_I(t;1|1|m)$ (5.13; 5.14) of "actual" average returns $r_q(t;1|1|m)$ of all Investors as:

$$C_I(t;1|1|m)=r_I(t;1|1|m)C_{Ip}(t;1|1|m) \tag{5.13}$$

$$r_I(t;1|1|m)\equiv E[r_q^m(t;1|1)]$$

$$r_I(t;1|1|m)=\frac{C_I(t;1|1|m)}{C_{Ip}(t;1|1|m)}\sim\frac{1}{\sum_{q=1}^{Q}C_{qp}^m(t;1|1)}\sum_{q=1}^{Q}r_q^m(t;1|1)C_{qp}^m(t;1|1) \tag{5.14}$$

$C_I(t;1|1|m)$ (5.10) define *m-th* statistical moment of the average sale value $C_q(t;1|1)$ (4.2) averaged over $Q$ Investors. $C_{Ip}(t;1|1|m)$ (5.11) define *m-th* statistical moment of the average purchased value $C_{qp}(t;1|1)$ (4.3) averaged by $Q$ Investors. $r_I(t;1|1|m)$ (5.12; 5.13) define *m-th* statistical moment of average "actual" returns $r_q(t;1|1)$ (4.6; 4.7) averaged over $Q$ Investors. Volatility $\sigma_{rI}^2(t;1|1)$ (5.15) of average "actual" returns of different Investors takes form:

$$\sigma_{rI}^2(t;1)\equiv r_I(t;1|1|2)-r_I^2(t;1|1|1)=\frac{C_I(t;1|1|2)}{C_{Ip}(t;1|1|2)}-\frac{C_I^2(t;1|1|1)}{C_{Ip}^2(t;1|1|1)} \tag{5.15}$$

Volatility $\sigma_{rI}^2(t;1|1)$ (5.15) describes the scale of fluctuations of average returns obtained by different Investors $q$ during $\Delta$.

## 6. Conclusion

This paper presents the unified theoretical description of statistical properties of "actual" returns, which Investors gain during the "trading day". Sec. 2 we briefly present description of the market-based statistical moments of prices and "anticipated" stock returns. Sec. 3, 4 and 5 describe three levels of statistics of "actual" returns, which Investors gain within their market sales.

Theoretical description of "actual" stock returns could be important for analysis of statistical properties of econometric time-series of market trades within the unified approach. The main issue of our model – assessments of statistical moments of asset prices, "anticipated" and "actual" stock returns uniformly takes into account impact of the size of market trade values and volumes. We describe dependence of returns statistics upon the market statistics. That is completely different from assessments of statistical properties of stock returns based conventional frequency analysis of returns time-series.

Econometric analysis of the market-based statistical moments of particular Investor using available historic time-series and assessments of volatility of returns of different Investors during the "trading day" could be important for Investors, researchers and financial authorities. Assessments of "anticipated" stock returns using the market-based statistical moments are available for large Investors and portfolio managers. They could use market-based statistical moments of "anticipated" returns as benchmarks to manage performance of their investment portfolios.

Let us underline some problems, important for further understanding of financial markets.

1. It is interesting figure out how "anticipated" average return $r(t,\tau;1)$ (2.17) and volatility $\sigma_r^2(t,\tau)$ (2.18; 2.19) depend on time-shifts $\tau$. We assume that analysis of the market trade time-series should reveal fluctuations of average return and volatility as functions of $\tau$ and that dependence should vary for different averaging intervals $\Delta$.

2. Investors could use assessments of "anticipated" returns with different time shifts $\tau$ as benchmarks for their investment decisions. Analysis of market time-series could help Investors compare the "anticipated" average returns $r(t,\tau;1)$ (2.17)

$$r(t,\tau;1) = \frac{C(t;1)}{C_a(t,\tau;1)} \sim \frac{1}{\sum_{i=1}^{N} C_a(t_i,\tau)} \sum_{i=1}^{N} r(t_i,\tau) C_a(t_i,\tau)$$

with "actual" returns $r_q(t;1|1)$ (4.7) which Investor $q$ gains during $\Delta$.

$$r_q(t;1|1) = \frac{C_q(t;1|1)}{C_{qp}(t;1|1)} \sim \frac{1}{\sum_{i=1}^{N(q)} C_p(t_i;1)} \sum_{i=1}^{N(q)} r(t_i;1) C_p(t_i;1)$$

Comparison of "anticipated" volatility $\sigma_r^2(t,\tau)$ (2.19) for various time shifts $\tau$ with volatility of $\sigma_{rI}^2(t;1|1)$ of average "actual" returns (5.15) of different Investors could uncover relations between uncertainty of "anticipated" returns and variability of "actual" returns gained by Investors.

3. Fluctuations of "anticipated" returns due to variations of time shift $\tau$ could impact the duration of stock holding by Investors. That in its turn could change the scales and fluctuations of "anticipated" returns determined by time shift $\tau$. Investigation of the hidden mutual dependence between the market-based statistics of "actual" returns of Investors and statistics of "anticipated" returns could help increase efficiency of portfolio performance and market management.

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