Sensitivity Analysis between Lagrange Multipliers and Consumer Budget: Utility Maximization Case

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17 February 2023

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Abstract

In this paper sensitivity analysis between Lagrange multipliers and total budget is discussed. The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus. In mathematical economics, utility is the vital concept that increases or decreases overall happiness of the consumers. This study tries to discuss utility maximization policy of an organization by considering two constraints: budget constraint and coupon constraint. In this article, an attempt has been taken to achieve the best result through the application of scientific method of optimization.

Keywords: Budget, Lagrange multipliers, sensitivity analysis, utility maximization

1. Introduction

In the 21^{st} century mathematical modeling in economics becomes an essential part to investigate optimization policy. Mathematical modeling in economics is considered as the application of mathematics in economics [Samuelson, 1947; Carter, 2001]. For the sustainable of an industry, utility maximization strategy is essential. The property of a commodity that enables to satisfy human necessities is called utility [Bentham, 1780]. It directly influences the demand and supply of the organizations [Fishburn, 1970]. The concept of utility was developed in the late 18^{th} century by the English moral philosopher Jeremy Bentham (1748-1832) and English philosopher John Stuart Mill (1806-1873) [Bentham, 1780; Gauthier, 1975].

Two American scholars: mathematician John V. Baxley and economist John C. Moorhouse have analyzed an example of utility maximization subject to a budget constraint from a somewhat wider perspective. They have provided a mathematical formulation for nontrivial constrained optimization problem with special reference for the application in economics [Baxley & Moorhouse, 1984].

In this article, we have tried to form a mathematical formulation of economic model for maximizing utility function subject to two constrains, such as budget constraint and coupon constraint. The method of Lagrange multipliers is considered as a device for transforming a constrained problem to a higher dimensional unconstrained problem [Islam et al., 2010]. In this study, we have worked with the determinant of 6×6 Hessian matrix and 6×10 Jacobian matrix. We have operated the article with 16 variables, such as four price vectors, four types of coupon numbers, four commodity variables, two Lagrange multipliers, one total budget variable, and one total coupon variable. In the study mathematical calculations are displayed in some details.

2. Literature Review

In any research, literature review is an introductory section, which highlights previous researches in the same field [Polit & Hungler, 2013]. In 1928, two American scholars; mathematician Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976), have worked on production functions [Cobb & Douglas, 1928]. Later in 1984, another two American professors; mathematician John V. Baxley and economist John C. Moorhouse, have given the utility maximization structure with sufficient mathematical techniques [Baxley & Moorhouse, 1984].

Pahlaj Moolio and his coauthors have given reasonable interpretation of the Lagrange multipliers and examined the behavior of the firm by analyzing comparative static results [Moolio et al., 2009]. Notable mathematician Jamal Nazrul Islam and his coauthors have analyzed utility maximization and other optimization problems by considering reasonable interpretation of the Lagrange multipliers [Islam et al., 2009a,b, 2011]. Qi Zhao and his coauthors have proposed for multi-product utility maximization as a general approach to the recommendation driven by economic principles

[Zhao et al., 2017]. Young researcher Lia Roy and her coworkers have discussed cost minimization policy of an industry, where they have provided detail mathematical formulation [Roy et al., 2021]. Haradhan Kumar Mohajan has explored utility maximization model for Bangladeshi consumers [Mohajan, 2021a]. In a published book, he and his coauthors have discussed a series of optimization problems for the social welfare [Mohajan et al., 2012, 2013].

Devajit Mohajan and Haradhan Kumar Mohajan have discussed a profit maximization problem in an industry, where they have used four variable inputs, such as capital, labor, principal raw materials, and other inputs [Mohajan & Mohajan, 2022a-j, 2023a-q]. On the other hand, Jannatul Ferdous and Haradhan Kumar Mohajan have briefly solved a profit maximization problem [Ferdous & Mohajan, 2022]. In another paper, they have calculated utility maximization policy of an organization [Mohajan & Mohajan, 2022b; Mohajan, 2020a-e, 2021a-d, 2022a-d].

3. Methodology of the Study

Research is an essential and influential device to the professors to lead the academic world [Pandey & Pandey, 2015]. Methodology is a guideline that tries to describe the types of research and the types of data [Somekh & Lewin, 2005]. Research methodology is the science and philosophy behind all researches and it provides the principles for organizing, planning, designing and conducting a good research [Remenyi et al., 1998; Legesse, 2014]. It tries to create new knowledge basis on the existing knowledge [Goddard & Melville, 2001].

In this study we have used 6×6 bordered Hessian matrix and 6×10 Jacobian matrix, and we have also used four commodities b_1, b_2, b_3, and b_4, and two Lagrange multipliers λ_1 and λ_2. We have tried to provide mathematical calculations and results very clearly [Mohajan, 2016a,b, 2017a-e, 2018a-e]. In this study we have depended on the utility maximization related mathematical secondary data sources. The data are collected from the secondary data sources, such as from published research papers, books and handbooks of famous authors, internet, websites, etc. [Islam et al., 2012a,b,c, Mohajan, 2011a,b, 2012a-f, 2013a-g, 2014a-g, 2015a-e, Rahman & Mohajan, 2019].

4. Objective of the Study

The leading objective of this paper is to discuss sensitivity analysis between Lagrange multipliers and total budget of consumers during the utility maximization and economic analysis. The other supplementary objectives are as follows:

- to develop the bordered Hessian and Jacobian,
- to provide the sensitivity results properly, and
- to display the mathematical calculations in some details.

5. An Economic Model

To study sensitivity analysis we consider four commodities: A_1, A_2, A_3, and A_4. Let the consumers in the society wants to purchase b_1, b_2, b_3, and b_4 amounts from these four commodities A_1, A_2, A_3, and A_4, respectively. The utility function for these four commodities can be written as [Islam et al., 2010; Mohajan & Mohajan, 2022b],

$$u(b_1,b_2,b_3,b_4)=b_1b_2b_3b_4. \qquad (1)$$

The budget constraint of the consumers is,

$$B(b_1,b_2,b_3,b_4)=p_1b_1+p_2b_2+p_3b_3+p_4b_4 \qquad (2)$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities b_1, b_2, b_3, and b_4, respectively. Now the coupon constraint is,

$$K(b_1,b_2,b_3,b_4)=k_1b_1+k_2b_2+k_3b_3+k_4b_4, \qquad (3)$$

where k_1, k_2, k_3, and k_4 are the coupons necessary to purchase a unit of commodity of b_1, b_2, b_3, and b_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $U(b_1,b_2,b_3,b_4,\lambda_1,\lambda_2)$ as [Baxley & Moorhouse, 1984; Ferdous & Mohajan, 2022],

$$\begin{aligned}U(b_1,b_2,b_3,b_4,\lambda_1,\lambda_2)=&\,b_1b_2b_3b_4+\lambda_1(B-p_1b_1-p_2b_2-p_3b_3-p_4b_4)\\&+\lambda_2(K-k_1b_1-k_2b_2-k_3b_3-k_4b_4).\end{aligned} \qquad (4)$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where λ_1 and λ_2 are two Lagrange multipliers.

Now taking first and second order and cross-partial derivatives in (4) we obtain [Islam et al. 2009a,b; Mohajan & Mohajan, 2022d];

$$\begin{aligned}&B_1=p_1, \quad B_2=p_2, \quad B_3=p_3, \quad B_4=p_4.\\&K_1=k_1, \quad K_2=k_2, \quad K_3=k_3, \quad K_4=k_4.\end{aligned} \qquad (5)$$

$$\begin{aligned}&U_{11}=0, \quad U_{12}=U_{21}=b_3b_4, \quad U_{13}=U_{31}=b_2b_4,\\&U_{14}=U_{41}=b_2b_3, \quad U_{22}=0, \quad U_{23}=U_{32}=b_1b_4,\\&U_{24}=U_{42}=b_1b_3, \quad U_{33}=0, \quad U_{34}=U_{43}=b_1b_2, \quad U_{44}=0.\end{aligned} \qquad (6)$$

Now we consider the bordered Hessian [Mohajan, 2021a; Mohajan & Mohajan, 2022c],

$$|H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & -K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & -K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & -K_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix}. \tag{7}$$

We use $p_3 = p_1$ and $p_4 = p_2$, i.e., amount of a pair of prices are same, and $k_3 = k_1$ and $k_4 = k_2$, i.e., a pair of coupon numbers are same. Now we consider that in the expansion of (7) every term contains $p_1 p_2 k_1 k_2$, then from (7) we can derive [Mohajan & Mohajan, 2022e];

$$|H| = -2 p_1 p_2 k_1 k_2 < 0. \tag{8}$$

For b_1, b_2, b_3, b_4, λ_1, and λ_2 in terms of p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and K we can calculate sixty partial derivatives, such as $\frac{\partial \lambda_1}{\partial p_1}$, $\frac{\partial \lambda_2}{\partial p_1}$, …, $\frac{\partial \lambda_1}{\partial k_1}$, $\frac{\partial \lambda_2}{\partial k_1}$, …, $\frac{\partial b_1}{\partial p_1}$, …, $\frac{\partial b_1}{\partial k_1}$, …, $\frac{\partial \lambda_1}{\partial B}$, …, $\frac{\partial \lambda_1}{\partial K}$, etc., [Islam et al., 2011; Mohajan, 2021c]. Now we consider 6×6 Hessian and Jacobian matrix as [Mohajan, 2021b; Mohajan & Mohajan, 2022a];

$$J = H = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & -K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & -K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & -K_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix} \tag{9}$$

which is non-singular at the optimum point $\left(b_1^*, b_2^*, b_3^*, b_4^*, \lambda_1^*, \lambda_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (9) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem. We have total 16 variables in our study, such as λ_1, λ_2, b_1, b_2, b_3, b_4 p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and K. By the implicit function theorem, we can write [Moolio et al., 2009; Islam et al., 2010];

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ b_1\\ b_2\\ b_3\\ b_4\end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, M\right). \tag{10}$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by [Mohajan, 2021a; Mohajan & Mohajan, 2022a],

$$J_G = \begin{bmatrix}
\frac{\partial \lambda_1}{\partial p_1} & \frac{\partial \lambda_1}{\partial p_2} & \frac{\partial \lambda_1}{\partial p_3} & \frac{\partial \lambda_1}{\partial p_4} & \frac{\partial \lambda_1}{\partial k_1} & \frac{\partial \lambda_1}{\partial k_2} & \frac{\partial \lambda_1}{\partial k_3} & \frac{\partial \lambda_1}{\partial k_4} & \frac{\partial \lambda_1}{\partial B} & \frac{\partial \lambda_1}{\partial K}\\
\frac{\partial \lambda_2}{\partial p_1} & \frac{\partial \lambda_2}{\partial p_2} & \frac{\partial \lambda_2}{\partial p_3} & \frac{\partial \lambda_2}{\partial p_4} & \frac{\partial \lambda_2}{\partial k_1} & \frac{\partial \lambda_2}{\partial k_2} & \frac{\partial \lambda_2}{\partial k_3} & \frac{\partial \lambda_2}{\partial k_4} & \frac{\partial \lambda_2}{\partial B} & \frac{\partial \lambda_2}{\partial K}\\
\frac{\partial b_1}{\partial p_1} & \frac{\partial b_1}{\partial p_2} & \frac{\partial b_1}{\partial p_3} & \frac{\partial b_1}{\partial p_4} & \frac{\partial b_1}{\partial k_1} & \frac{\partial b_1}{\partial k_2} & \frac{\partial b_1}{\partial k_3} & \frac{\partial b_1}{\partial k_4} & \frac{\partial b_1}{\partial B} & \frac{\partial b_1}{\partial K}\\
\frac{\partial b_2}{\partial p_1} & \frac{\partial b_2}{\partial p_2} & \frac{\partial b_2}{\partial p_3} & \frac{\partial b_2}{\partial p_4} & \frac{\partial b_2}{\partial k_1} & \frac{\partial b_2}{\partial k_2} & \frac{\partial b_2}{\partial k_3} & \frac{\partial b_2}{\partial k_4} & \frac{\partial b_2}{\partial B} & \frac{\partial b_2}{\partial K}\\
\frac{\partial b_3}{\partial p_1} & \frac{\partial b_3}{\partial p_2} & \frac{\partial b_3}{\partial p_3} & \frac{\partial b_3}{\partial p_4} & \frac{\partial b_3}{\partial k_1} & \frac{\partial b_3}{\partial k_2} & \frac{\partial b_3}{\partial k_3} & \frac{\partial b_3}{\partial k_4} & \frac{\partial b_3}{\partial B} & \frac{\partial b_3}{\partial K}\\
\frac{\partial b_4}{\partial p_1} & \frac{\partial b_4}{\partial p_2} & \frac{\partial b_4}{\partial p_3} & \frac{\partial b_4}{\partial p_4} & \frac{\partial b_4}{\partial k_1} & \frac{\partial b_4}{\partial k_2} & \frac{\partial b_4}{\partial k_3} & \frac{\partial b_4}{\partial k_4} & \frac{\partial b_4}{\partial B} & \frac{\partial b_4}{\partial K}
\end{bmatrix}. \tag{11}$$

$$= -J^{-1}\begin{bmatrix}
-b_1 & -b_2 & -b_3 & -b_4 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & 0 & 1\\
-\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0\\
0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0
\end{bmatrix}. \tag{12}$$

The inverse of Jacobian matrix is, $J^{-1} = \frac{1}{|J|}C^T$, where $C = \left(C_{ij}\right)$, the matrix of cofactors of J, and T indicates transpose, then (12) becomes [Mohajan, 2017a; Islam et al., 2009b, 2011],

$$J_G = -\frac{1}{|J|}C^T\begin{bmatrix}
-b_1 & -b_2 & -b_3 & -b_4 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & 0 & 1\\
-\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0\\
0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0
\end{bmatrix}. \tag{13}$$

Now 6×6 transpose matrix C^T can be represented by,

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \tag{14}$$

Using (14) we can write (11) as a 6×10 Jacobian matrix [Mohajan & Mohajan, 2022b];

$$J_G = -\frac{1}{|J|}\begin{bmatrix} -b_1C_{11}-\lambda_1C_{31} & -b_2C_{11}-\lambda_1C_{41} & -b_3C_{11}-\lambda_1C_{51} & -b_4C_{11}-\lambda_1C_{61} & -b_1C_{21}-\lambda_2C_{31} \\ -b_1C_{12}-\lambda_1C_{32} & -b_2C_{12}-\lambda_1C_{42} & -b_3C_{12}-\lambda_1C_{52} & -b_4C_{12}-\lambda_1C_{62} & -b_1C_{22}-\lambda_2C_{32} \\ -b_1C_{13}-\lambda_1C_{33} & -b_2C_{13}-\lambda_1C_{43} & -b_3C_{13}-\lambda_1C_{53} & -b_4C_{13}-\lambda_1C_{63} & -b_1C_{23}-\lambda_2C_{33} \\ -b_1C_{14}-\lambda_1C_{34} & -b_2C_{14}-\lambda_1C_{44} & -b_3C_{14}-\lambda_1C_{54} & -b_4C_{14}-\lambda_1C_{64} & -b_1C_{24}-\lambda_2C_{34} \\ -b_1C_{15}-\lambda_1C_{35} & -b_2C_{15}-\lambda_1C_{45} & -b_3C_{15}-\lambda_1C_{55} & -b_4C_{15}-\lambda_1C_{65} & -b_1C_{25}-\lambda_2C_{35} \\ -b_1C_{16}-\lambda_1C_{36} & -b_2C_{16}-\lambda_1C_{46} & -b_3C_{16}-\lambda_1C_{56} & -b_4C_{16}-\lambda_1C_{66} & -b_1C_{26}-\lambda_2C_{36} \end{bmatrix}$$

$$\begin{bmatrix} -b_2C_{21}-\lambda_2C_{41} & -b_3C_{21}-\lambda_2C_{51} & -b_4C_{21}-\lambda_2C_{61} & C_{11} & C_{21} \\ -b_2C_{22}-\lambda_2C_{42} & -b_3C_{22}-\lambda_2C_{52} & -b_4C_{22}-\lambda_2C_{62} & C_{12} & C_{22} \\ -b_2C_{23}-\lambda_2C_{43} & -b_3C_{23}-\lambda_2C_{53} & -b_4C_{23}-\lambda_2C_{63} & C_{13} & C_{23} \\ -b_2C_{24}-\lambda_2C_{44} & -b_3C_{24}-\lambda_2C_{54} & -b_4C_{24}-\lambda_2C_{64} & C_{14} & C_{24} \\ -b_2C_{25}-\lambda_2C_{45} & -b_3C_{25}-\lambda_2C_{55} & -b_4C_{25}-\lambda_2C_{65} & C_{15} & C_{25} \\ -b_2C_{26}-\lambda_2C_{46} & -b_3C_{26}-\lambda_2C_{56} & -b_4C_{26}-\lambda_2C_{66} & C_{16} & C_{26} \end{bmatrix}. \tag{15}$$

Now we analyze the nature of Lagrange multiplier λ_1 when total budget B of the consumers increases. Taking T_{19}, (i.e., term of 1st row and 9th column) from both sides of (15) we get [Islam et al., 2011; Mohajan & Mohajan, 2022e],

$$\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}\left[C_{11}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{11}$$

$$= -\frac{1}{|J|}\begin{vmatrix} 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -K_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix}$$

$$
=-\frac{1}{|J|}\left\{K_1\begin{vmatrix}-K_1 & U_{12} & U_{13} & U_{14}\\ -K_2 & U_{22} & U_{23} & U_{24}\\ -K_3 & U_{32} & U_{33} & U_{34}\\ -K_4 & U_{42} & U_{43} & U_{44}\end{vmatrix}-K_2\begin{vmatrix}-K_1 & U_{11} & U_{13} & U_{14}\\ -K_2 & U_{21} & U_{23} & U_{24}\\ -K_3 & U_{31} & U_{33} & U_{34}\\ -K_4 & U_{41} & U_{43} & U_{44}\end{vmatrix}\right.
$$

$$
\left.+K_3\begin{vmatrix}-K_1 & U_{11} & U_{12} & U_{14}\\ -K_2 & U_{21} & U_{22} & U_{24}\\ -K_3 & U_{31} & U_{32} & U_{34}\\ -K_4 & U_{41} & U_{42} & U_{44}\end{vmatrix}-K_4\begin{vmatrix}-K_1 & U_{11} & U_{12} & U_{13}\\ -K_2 & U_{21} & U_{22} & U_{23}\\ -K_3 & U_{31} & U_{32} & U_{33}\\ -K_4 & U_{41} & U_{42} & U_{43}\end{vmatrix}\right\}
$$

$$
=-\frac{1}{|J|}\left[K_1\left\{-K_1\begin{vmatrix}U_{22} & U_{23} & U_{24}\\ U_{32} & U_{33} & U_{34}\\ U_{42} & U_{43} & U_{44}\end{vmatrix}-U_{12}\begin{vmatrix}-K_2 & U_{23} & U_{24}\\ -K_3 & U_{33} & U_{34}\\ -K_4 & U_{43} & U_{44}\end{vmatrix}+U_{13}\begin{vmatrix}-K_2 & U_{22} & U_{24}\\ -K_3 & U_{32} & U_{34}\\ -K_4 & U_{42} & U_{44}\end{vmatrix}-U_{14}\begin{vmatrix}-K_2 & U_{22} & U_{23}\\ -K_3 & U_{32} & U_{33}\\ -K_4 & U_{42} & U_{43}\end{vmatrix}\right\}\right.
$$

$$
-K_2\left\{-K_1\begin{vmatrix}U_{21} & U_{23} & U_{24}\\ U_{31} & U_{33} & U_{34}\\ U_{41} & U_{43} & U_{44}\end{vmatrix}+U_{13}\begin{vmatrix}-K_2 & U_{21} & U_{24}\\ -K_3 & U_{31} & U_{34}\\ -K_4 & U_{41} & U_{44}\end{vmatrix}-U_{14}\begin{vmatrix}-K_2 & U_{21} & U_{23}\\ -K_3 & U_{31} & U_{33}\\ -K_4 & U_{41} & U_{43}\end{vmatrix}\right\}+K_3\left\{-K_1\begin{vmatrix}U_{21} & U_{22} & U_{24}\\ U_{31} & U_{32} & U_{34}\\ U_{41} & U_{42} & U_{44}\end{vmatrix}\right.
$$

$$
\left.+U_{12}\begin{vmatrix}-K_2 & U_{21} & U_{24}\\ -K_3 & U_{31} & U_{34}\\ -K_4 & U_{41} & U_{44}\end{vmatrix}-U_{14}\begin{vmatrix}-K_2 & U_{21} & U_{22}\\ -K_3 & U_{31} & U_{32}\\ -K_4 & U_{41} & U_{42}\end{vmatrix}\right\}-K_4\left\{-K_1\begin{vmatrix}U_{21} & U_{22} & U_{23}\\ U_{31} & U_{32} & U_{33}\\ U_{41} & U_{42} & U_{43}\end{vmatrix}+U_{12}\begin{vmatrix}-K_2 & U_{21} & U_{23}\\ -K_3 & U_{31} & U_{33}\\ -K_4 & U_{41} & U_{43}\end{vmatrix}\right.
$$

$$
\left.\left.-U_{13}\begin{vmatrix}-K_2 & U_{21} & U_{22}\\ -K_3 & U_{31} & U_{32}\\ -K_4 & U_{41} & U_{42}\end{vmatrix}\right\}\right]
$$

$$
=-\frac{1}{|J|}\left\{-K_1^2U_{23}U_{24}U_{34}\quad -K_1^2U_{23}U_{24}U_{34}\quad -K_1K_2U_{12}U_{34}^2\quad +K_1K_4U_{12}U_{23}U_{34}\quad +K_1K_3U_{12}U_{24}U_{34}\right.
$$

$$
+K_1K_2U_{13}U_{24}U_{34}-K_1K_3U_{13}U_{24}^2+K_1K_4U_{13}U_{23}U_{24}+K_1K_2U_{14}U_{23}U_{34}-K_1K_3U_{14}U_{23}U_{24}-K_1K_4U_{14}U_{23}^2
$$

$$
-K_1K_2U_{12}U_{14}U_{34}\quad +K_1K_2U_{14}U_{23}U_{34}\quad +K_1K_2U_{13}U_{24}U_{34}\quad -K_2^2U_{13}U_{14}U_{34}\quad +K_2K_4U_{12}U_{13}U_{34}
$$

$$
+K_2K_3U_{13}U_{14}U_{24}-K_2K_4U_{24}U_{13}^2-K_2^2U_{13}U_{14}U_{34}+K_2K_3U_{12}U_{14}U_{34}-K_2K_3U_{23}U_{14}^2+K_2K_4U_{13}U_{14}U_{23}
$$

$$
+K_1K_3U_{12}U_{24}U_{34}-K_1K_3U_{13}U_{24}^2+K_1K_3U_{14}U_{23}U_{24}+K_2K_3U_{12}U_{14}U_{34}-K_3K_4U_{34}U_{12}^2-K_3^2U_{12}U_{14}U_{24}
$$

$$
+K_3K_4U_{12}U_{13}U_{24}+K_1K_4U_{12}U_{23}U_{34}+K_1K_4U_{13}U_{23}U_{24}-K_1K_4U_{14}U_{23}^2+K_2K_4U_{12}U_{13}U_{34}-K_3K_4U_{34}U_{12}^2
$$

$$
+K_3K_4U_{12}U_{14}U_{23}-K_4^2U_{12}U_{13}U_{23}-K_2K_4U_{24}U_{13}^2+K_2K_4U_{13}U_{14}U_{23}+K_3K_4U_{12}U_{13}U_{24}
$$

$$
\left.-K_4^2U_{12}U_{13}U_{23}\right\}
$$

$$
=-\frac{1}{|J|}\left\{-2k_1^2b_1^3b_2b_3b_4\quad -k_1k_2b_1^2b_2^2b_3b_4\quad +k_1k_4b_1^2b_2b_3b_4^2\quad +k_1k_3b_1^2b_2b_3^2b_4\quad -k_1k_2b_1^2b_2^2b_3b_4\quad -k_1k_3b_1^2b_2b_3^2b_4\right.
$$

$$
+k_1k_4b_1^2b_2b_3b_4^2\quad +k_1k_2b_1^2b_2^2b_3b_4\quad -k_1k_3b_1^2b_2b_3^2b_4\quad -k_1k_4b_1^2b_2b_3b_4^2\quad -k_1k_2b_1b_2^2b_3^2b_4\quad +k_1k_2b_1^2b_2^2b_3b_4
$$

$$
+k_1k_2b_1^2b_2^2b_3b_4-k_2^2b_1b_2^3b_3b_4+k_2k_4b_1b_2^2b_3b_4^2+k_2k_3b_1b_2^2b_3^2b_4-k_2k_4b_1b_2^2b_3b_4^2-k_2^2b_1b_2^3b_3b_4+k_2k_3b_1b_2^2b_3^2b_4
$$

$$-k_2k_3b_1b_2^2b_3^2b_4 + k_2k_4b_1b_2^2b_3b_4^2 + k_1k_3b_1^2b_2b_3^2b_4 - k_1k_3b_1^2b_2b_3^2b_4 + k_1k_3b_1^2b_2b_3^2b_4 + k_2k_3b_1b_2^2b_3^2b_4$$
$$-k_3k_4b_1b_2b_3^2b_4^2 - k_3^2b_1b_2b_3^3b_4 + k_3k_4b_1b_2b_3^2b_4^2 + k_1k_4b_1^2b_2b_3b_4^2 + k_1k_4b_1^2b_2b_3b_4^2 - k_1k_4b_1^2b_2b_3b_4^2 + k_2k_4b_1b_2^2b_3b_4^2$$
$$-k_3k_4b_1b_2b_3^2b_4^2 + k_3k_4b_1b_2b_3^2b_4^2 - k_4^2b_1b_2b_3b_4^3 - k_2k_4b_1b_2^2b_3b_4^2 + k_2k_4b_1b_2^2b_3b_4^2 + k_3k_4b_1b_2b_3^2b_4^2 - k_4^2b_1b_2b_3b_4^3\Big\}$$

$$\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}\Big\{-2k_1^2b_1^3b_2b_3b_4 - 2k_2^2b_1b_2^3b_3b_4 - k_3^2b_1b_2b_3^3b_4 - 2k_4^2b_1b_2b_3b_4^3 + 2k_1k_4b_1^2b_2b_3b_4^2 + 2k_2k_3b_1b_2^2b_3^2b_4$$
$$+ 2k_2k_4b_1b_2^2b_3b_4^2 + k_3k_4b_1b_2b_3^2b_4^2\Big\}. \tag{16}$$

Using $b_1 = b_2 = b_3 = b_4 = 1$ in (16) we get,

$$\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}\Big\{-2k_1^2 - 2k_2^2 - k_3^2 - 2k_4^2 + 2k_1k_4 + 2k_2k_3 + 2k_2k_4 + 2k_3k_4\Big\}. \tag{17}$$

Using $k_3 = k_1$ and $k_4 = k_2$ in (17) we get,

$$\frac{\partial \lambda_1}{\partial B} = (k_1 - k_2)(3k_1 - 2k_2). \tag{18}$$

If $k_1 < \frac{2}{3}k_2$ or $k_1 > k_2$ in (18) we get,

$$\frac{\partial \lambda_1}{\partial B} > 0. \tag{19}$$

Inequality (19) indicates that if the total budget of the consumers' increases, the level of marginal utility will also increase. Therefore, in this situation the consumers will collect the coupons such that, $k_1 > k_2$ or $k_1 < \frac{2}{3}k_2$. Depending on the consumers' demand, the organization should take attempts to increase the production level.

If $\frac{2}{3}k_2 < k_1 < k_2$ in (18) we get,

$$\frac{\partial \lambda_1}{\partial B} < 0. \tag{20}$$

Inequality (20) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, in this situation the consumers will collect the coupons such that, $\frac{2}{3}k_2 < k_1 < k_2$. Depending on the consumers' demand, the organization should take attempts to decrease the production level.

In this study we observe that, $\frac{\partial \lambda_1}{\partial B} \neq 0$. Therefore, from (18) we see that, $k_1 \neq k_2$ and $k_1 \neq \frac{2}{3}k_2$; consequently, also $k_3 \neq k_4$ and $k_3 \neq \frac{2}{3}k_4$ in this model.

Now we analyze the nature of Lagrange multiplier λ_2 when total budget B of the consumers increases. Taking T_{29}, (i.e., term of 2nd row and 9th column) from both sides of (15) we get [Islam et al., 2010; Mohajan & Mohajan, 2022e,f],

$$\frac{\partial \lambda_2}{\partial B} = -\frac{1}{|J|}\left[C_{12}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{12}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{ K_1 \begin{vmatrix} -B_1 & U_{12} & U_{13} & U_{14} \\ -B_2 & U_{22} & U_{23} & U_{24} \\ -B_3 & U_{32} & U_{33} & U_{34} \\ -B_4 & U_{42} & U_{43} & U_{44} \end{vmatrix} - K_2 \begin{vmatrix} -B_1 & U_{11} & U_{13} & U_{14} \\ -B_2 & U_{21} & U_{23} & U_{24} \\ -B_3 & U_{31} & U_{33} & U_{34} \\ -B_4 & U_{41} & U_{43} & U_{44} \end{vmatrix} \right.$$

$$\left. + K_3 \begin{vmatrix} -B_1 & U_{11} & U_{12} & U_{14} \\ -B_2 & U_{21} & U_{22} & U_{24} \\ -B_3 & U_{31} & U_{32} & U_{34} \\ -B_4 & U_{41} & U_{42} & U_{44} \end{vmatrix} - K_4 \begin{vmatrix} -B_1 & U_{11} & U_{12} & U_{13} \\ -B_2 & U_{21} & U_{22} & U_{23} \\ -B_3 & U_{31} & U_{32} & U_{33} \\ -B_4 & U_{41} & U_{42} & U_{43} \end{vmatrix} \right\}$$

$$= \frac{1}{|J|}\left[K_1 \left\{ -B_1 \begin{vmatrix} U_{22} & U_{23} & U_{24} \\ U_{32} & U_{33} & U_{34} \\ U_{42} & U_{43} & U_{44} \end{vmatrix} - U_{12} \begin{vmatrix} -B_2 & U_{23} & U_{24} \\ -B_3 & U_{33} & U_{34} \\ -B_4 & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -B_2 & U_{22} & U_{24} \\ -B_3 & U_{32} & U_{34} \\ -B_4 & U_{42} & U_{44} \end{vmatrix} - U_{14} \begin{vmatrix} -B_2 & U_{22} & U_{23} \\ -B_3 & U_{32} & U_{33} \\ -B_4 & U_{42} & U_{43} \end{vmatrix} \right\} \right.$$

$$- K_2 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{23} & U_{24} \\ U_{31} & U_{33} & U_{34} \\ U_{41} & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -B_2 & U_{21} & U_{24} \\ -B_3 & U_{31} & U_{34} \\ -B_4 & U_{41} & U_{44} \end{vmatrix} - U_{14} \begin{vmatrix} -B_2 & U_{21} & U_{23} \\ -B_3 & U_{31} & U_{33} \\ -B_4 & U_{41} & U_{43} \end{vmatrix} \right\} + K_3 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{22} & U_{24} \\ U_{31} & U_{32} & U_{34} \\ U_{41} & U_{42} & U_{44} \end{vmatrix} \right.$$

$$\left. + U_{12} \begin{vmatrix} -B_2 & U_{21} & U_{24} \\ -B_3 & U_{31} & U_{34} \\ -B_4 & U_{41} & U_{44} \end{vmatrix} - U_{14} \begin{vmatrix} -B_2 & U_{21} & U_{22} \\ -B_3 & U_{31} & U_{32} \\ -B_4 & U_{41} & U_{42} \end{vmatrix} \right\} - K_4 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \\ U_{41} & U_{42} & U_{43} \end{vmatrix} + U_{12} \begin{vmatrix} -B_2 & U_{21} & U_{23} \\ -B_3 & U_{31} & U_{33} \\ -B_4 & U_{41} & U_{43} \end{vmatrix} \right.$$

$$\left.\left. - U_{13} \begin{vmatrix} -B_2 & U_{21} & U_{22} \\ -B_3 & U_{31} & U_{32} \\ -B_4 & U_{41} & U_{42} \end{vmatrix} \right\} \right]$$

$$= \frac{1}{|J|}\{ -2B_1K_1U_{23}U_{24}U_{34} \quad -B_2K_1U_{12}U_{34}^2 \quad +B_4K_1U_{12}U_{23}U_{34} \quad +B_3K_1U_{12}U_{24}U_{34} \quad +B_2K_1U_{13}U_{24}U_{34}$$

$$-B_3K_1U_{13}U_{24}^2 \quad +B_4K_1U_{13}U_{23}U_{24} \quad +B_2K_1U_{14}U_{23}U_{34} \quad +B_3K_1U_{14}U_{23}U_{24} \quad -B_4K_1U_{14}U_{23}^2 \quad -B_1K_2U_{12}U_{34}^2$$

$$
\begin{aligned}
&+B_1K_2U_{14}U_{23}U_{34} \quad +B_1K_2U_{13}U_{24}U_{34} \quad -B_2K_2U_{13}U_{14}U_{34} \quad -B_4K_2U_{12}U_{13}U_{34} \quad +B_3K_2U_{13}U_{14}U_{24} \\
&-B_4K_2U_{13}^2U_{24} -B_2K_2U_{13}U_{14}U_{34} +B_3K_2U_{12}U_{14}U_{34} -B_3K_2U_{14}^2U_{23} +B_4K_2U_{13}U_{14}U_{23} -B_1K_3U_{12}U_{24}U_{34} \\
&-B_1K_3U_{13}U_{24}^2 +B_1K_3U_{14}U_{23}U_{24} +B_2K_3U_{12}U_{14}U_{34} -B_4K_3U_{12}^2U_{34} -B_3K_3U_{12}U_{14}U_{24} +B_4K_3U_{12}U_{13}U_{24} \\
&+B_1K_4U_{12}U_{23}U_{34} +B_1K_4U_{13}U_{23}U_{24} -B_1K_4U_{14}U_{23}^2 +B_2K_4U_{12}U_{13}U_{34} -B_3K_4U_{12}^2U_{34} +B_3K_4U_{12}U_{14}U_{23} \\
&-B_4K_4U_{12}U_{13}U_{23} -B_2K_4U_{13}^2U_{24} +B_2K_4U_{13}U_{14}U_{23} +B_3K_4U_{12}U_{13}U_{24} -B_4K_4U_{12}U_{13}U_{23}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&=-\frac{1}{|J|}\Big\{-2p_1k_1b_1^3b_2b_3b_4 \quad -p_2k_1b_1^2b_2^2b_3b_4 \quad +p_4k_1b_1^2b_2b_3b_4^2 \quad +p_3k_1b_1^2b_2b_3^2b_4 \quad +p_3k_1b_1^2b_2^2b_3b_4 \quad -p_3k_1b_1^2b_2b_3^2b_4 \\
&+p_4k_1b_1^2b_2b_3b_4^2 \quad +p_2k_1b_1^2b_2^2b_3b_4 \quad +p_3k_1b_1^2b_2b_3^2b_4 \quad -p_4k_1b_1^2b_2b_3b_4^2 \quad -p_1k_2b_1^2b_2^2b_3b_4 \quad +2p_1k_2b_1^2b_2^2b_3b_4 \\
&-p_2k_2b_1b_2^3b_3b_4 \quad -p_4k_2b_1b_2^2b_3b_4^2 \quad +p_3k_2b_1b_2^2b_3^2b_4 \quad -p_4k_2b_1b_2^2b_3b_4^2 \quad -p_2k_2b_1b_2^3b_3b_4 \quad +p_3k_2b_1b_2^2b_3^2b_4 \\
&-p_3k_2b_1b_2^2b_3^2b_4 \quad +p_4k_2b_1b_2^2b_3b_4^2 \quad -p_1k_3b_1^2b_2b_3^2b_4 \quad -p_1k_3b_1^2b_2b_3^2b_4 \quad +p_1k_3b_1^2b_2b_3^2b_4 \quad +p_2k_3b_1b_2^2b_3^2b_4 \\
&-p_4k_3b_1b_2b_3^2b_4^2 \quad -p_3k_3b_1b_2b_3^3b_4 \quad +p_4k_3b_1b_2b_3^2b_4^2 \quad +p_1k_4b_1^2b_2b_3b_4^2 \quad +p_1k_4b_1^2b_2b_3b_4^2 \quad -p_1k_4b_1^2b_2b_3b_4^2 \\
&+p_2k_4b_1b_2^2b_3b_4^2 \quad -p_3k_4b_1b_2b_3^2b_4^2 \quad +p_3k_4b_1b_2b_3^2b_4^2 \quad -p_4k_4b_1b_2b_3b_4^3 \quad -p_2k_4b_1b_2^2b_3b_4^2 \quad +p_2k_4b_1b_2^2b_3b_4^2 \\
&+p_3k_4b_1b_2b_3^2b_4^2-p_4k_4b_1b_2b_3b_4^3\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\lambda_2}{\partial B}&=-\frac{1}{|J|}\Big\{-2p_1k_1b_1^3b_2b_3b_4 \quad -2p_2k_2b_1b_2^3b_3b_4 \quad -p_3k_3b_1b_2b_3^3b_4 \quad -2p_4k_4b_1b_2b_3b_4^3 \quad +\left(p_4k_1+p_1k_4\right)b_1^2b_2b_3b_4^2 \\
&+p_1k_2b_1^2b_2^2b_3b_4+p_3k_1b_1^2b_2b_3^2b_4+\left(p_3k_2+p_2k_3\right)b_1b_2^2b_3^2b_4+p_3k_4b_1b_2b_3^2b_4^2\Big\}. \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\lambda_2}{\partial B}=-\frac{b_1b_2b_3b_4}{|J|}\Big\{&-2p_1k_1b_1^2-2p_2k_2b_2^2-p_3k_3b_3^2-2p_4k_4b_4^2+\left(p_4k_1+p_1k_4\right)b_1b_4+p_1k_2b_1b_2+p_3k_1b_1b_3 \\
&+\left(p_3k_2+p_2k_3\right)b_2b_3+p_3k_4b_3b_4\Big\}. \qquad (22)
\end{aligned}
$$

Now we use $p_3 = p_1$, and $p_4 = p_2$ where pair of prices are same, and $k_3 = k_1$, and $k_4 = k_2$, i.e., two types of coupon numbers are same, $|J| = |H| = -2p_1p_2k_1k_2$. Now we use $b_3 = b_1$, and $b_4 = b_2$, then (22) becomes;

$$
\frac{\partial\lambda_2}{\partial B}=\frac{b_1^2b_2^2}{2p_1p_2k_1k_2}\left\{-2p_1k_1b_1^2-4p_2k_2b_2^2+2\left(p_2k_1+2p_1k_2\right)b_1b_2\right\}. \qquad (23)
$$

We put $b_1 = b_2 = 1$ then (23) becomes [Mohajan & Mohajan, 2022b];

$$
\frac{\partial\lambda_2}{\partial B}=\frac{1}{2p_1p_2k_1k_2}\left\{-2p_1k_1-4p_2k_2+2p_2k_1+4p_1k_2\right\}. \qquad (24)
$$

Now we use, $k_1 = k_2 = k$ in (24), and then we get,

$$
\frac{\partial\lambda_2}{\partial B}=\frac{1}{p_1p_2k}\left(p_1-p_2\right). \qquad (25)
$$

where $p_1, p_2, k > 0$. Now if $p_1 > p_2$ in (25) we get,

$$
\frac{\partial\lambda_2}{\partial B}>0. \qquad (26)
$$

Inequality (26) indicates that if the total budget of the consumers' increases, the level of marginal utility will also increase. Therefore, in this situation the consumers will find that, $p_1 > p_2$, in the commodity market. The organization should take attempts to increase the production level, depending on the consumers' demand.

Now if $p_1 < p_2$ in (25) we get,

$$\frac{\partial \lambda_2}{\partial B} < 0. \tag{27}$$

Inequality (27) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, in this situation the consumers will find that, $p_1 < p_2$, in the commodity market. The organization should take attempts to decrease the production level, depending on the consumers' demand.

From (25) we see that, $\frac{\partial \lambda_2}{\partial B} \neq 0$, so that, $p_1 \neq p_2$, i.e., the prices of two commodities b_1 and b_2 are not equal. It seems that these are different goods.

6. Conclusions

In this study we have tried to discuss sensitivity analysis between Lagrange multipliers and total budget during utility maximization investigation. We have applied four commodity variables and we have tried to run the mathematical calculations efficiently using two constraints: budget constraint and coupon constraint. In this study we have observed that the Lagrange multipliers are very useful both for the consumers and producers.

References

Baxley, J. V., & Moorhouse, J. C. (1984). Lagrange Multiplier Problems in Economics. *The American Mathematical Monthly*, 91(7), 404-412.

Bentham, J. (1780). *An Introduction to the Principles of Morals and Legislation*. CreateSpace Independent Publishing Platform.

Carter, M. (2001). *Foundations of Mathematical Economics*. MIT Press, Cambridge, London.

Ferdous, J., & Mohajan, H. K. (2022). Maximum Profit Ensured for Industry Sustainability. *Annals of Spiru Haret University. Economic Series*, 22(3), 317-337.

Fishburn, P. C. (1970). *Utility Theory for Decision Making*. Huntington, NY: Robert E. Krieger.

Gauthier, D. (1975). Reason and Maximization. *Canadian Journal of Philosophy*, 4(3), 411-433.

Goddard, W., & Melville, S. (2001). *Research Methodology: An Introduction* (2nd Ed.). Lansdowne: Juta & Co. Ltd.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2009a). Preference of Social Choice in Mathematical Economics. *Indus Journal of Management & Social Sciences*, 3(1), 17-38.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2009b). Political Economy and Social Welfare with Voting Procedure. *KASBIT Business Journal*, 2(1), 42-66.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2010). Utility Maximization Subject to Multiple Constraints. *Indus Journal of Management & Social Sciences*, 4(1), 15-29.

Islam, J.N.; Mohajan, H. K., & Moolio, P. (2011), Output Maximization Subject to a Nonlinear Constraint, *KASBIT Business Journal*, 4(2), 104-120.

Islam, J. N., Mohajan, H. K., & Datta, R. (2012a). Aspects of Microfinance System of Grameen Bank of Bangladesh. *International Journal of Economics and Research*, 3(4), 76-96.

Islam, J. N., Mohajan, H. K., & Datta, R. (2012b). Stress Management Policy Analysis: A Preventative Approach. *International Journal of Economics and Research*, 3(6), 1-17.

Islam, J. N., Mohajan, H. K., & Moolio, P. (2012c). Borda Voting is Non-manipulable but Cloning Manipulation is Possible. *International Journal of Development Research and Quantitative Techniques*, 2(1), 28-37.

Legesse, B. (2014). Research Methods in Agribusiness and Value Chains. School of Agricultural Economics and Agribusiness, Haramaya University.

Mohajan, D., & Mohajan, H. K. (2022a). Mathematical Analysis of SEIR Model to Prevent COVID-19 Pandemic. *Journal of Economic Development, Environment and People*, 11(4), 5-30.

Mohajan, D., & Mohajan, H. K. (2022b). Utility Maximization Analysis of an Emerging Firm: A Bordered Hessian Approach. *Annals of Spiru Haret University. Economic Series*, 22(4), 292-308.

Mohajan, D., & Mohajan, H. K. (2022c). Sensitivity Analysis among Commodities and Coupons during Utility Maximization. *Frontiers in Management Science*, 1(3), 13-28.

Mohajan, D., & Mohajan, H. K. (2022d). Importance of Total Coupon in Utility Maximization: A Sensitivity Analysis. *Law and Economy*, 1(5), 65-67.

Mohajan, D., & Mohajan, H. K. (2022e). Development of Grounded Theory in Social Sciences: A Qualitative Approach. *Studies in Social Science & Humanities*, 1(5), 13-24.

Mohajan, D., & Mohajan, H. K. (2022f). Exploration of Coding in Qualitative Data Analysis: Grounded Theory Perspective. *Research and Advances in Education*, 1(6), 50-60.

Mohajan, D., & Mohajan, H. K. (2022g). Memo Writing Procedures in Grounded Theory Research Methodology. *Studies in Social Science & Humanities*, 1(4), 10-18.

Mohajan, D., & Mohajan, H. K. (2022h). Constructivist Grounded Theory: A New Research Approach in Social Science. *Research and Advances in Education*, 1(4), 8-16.

Mohajan, D., & Mohajan, H. K. (2022i). Feminism and Feminist Grounded Theory: A Comprehensive Research Analysis. *Journal of Economic Development, Environment and People*, 11(3), 49-61.

Mohajan, D., & Mohajan, H. K. (2022j). Profit Maximization Strategy in an Industry: A Sustainable Procedure. *Law and Economy*, 1(3), 17-43.

Mohajan, D., & Mohajan, H. K. (2023a). Sensitivity Analysis among Commodities and Prices: Utility Maximization Perceptions. *Law and Economy*, 2(2), 1-16.

Mohajan, D., & Mohajan, H. K. (2023b). Straussian Grounded Theory: An Evolved Variant in Qualitative Research. *Studies in Social Science & Humanities*, 2(2), 33-40.

Mohajan, D., & Mohajan, H. K. (2023c). Sensitivity Analysis between Lagrange Multipliers and Consumer Coupon: Utility Maximization Perspective. *Frontiers in Management Science*, 2(1), 14-25.

Mohajan, D., & Mohajan, H. K. (2023d). Utility Maximization Analysis of an Organization: A Mathematical Economic Procedure. *Law and Economy*, 2(1), 1-15.

Mohajan, D., & Mohajan, H. K. (2023e). Classic Grounded Theory: A Qualitative Research on Human Behavior. *Studies in Social Science & Humanities*, 2(1), 1-7.

Mohajan, D., & Mohajan, H. K. (2023f). Sensitivity Analysis between Commodity and Budget: Utility Maximization Case. *Law and Economy*, 2(3), 10-21.

Mohajan, D., & Mohajan, H. K. (2023g). Sensitivity Analysis for Profit Maximization with Respect to Per Unit Cost of Subsidiary Raw Materials. *Frontiers in Management Science*, 2(2), 13-27.

Mohajan, D., & Mohajan, H. K. (2023h). Families of Grounded Theory: A Theoretical Structure for Novel Researchers. *Studies in Social Science & Humanities*, 2(1), 56-65.

Mohajan, D., & Mohajan, H. K. (2023i). Broca Index: A Simple Tool to Measure Ideal Body Weight. *Innovation in Science and Technology*, 2(2), 21–24.

Mohajan, D., & Mohajan, H. K. (2023j). Obesity and Its Related Diseases: A New Escalating Alarming in Global Health. *Journal of Innovations in Medical Research*, 2(3), 12–23.

Mohajan, D., & Mohajan, H. K. (2023k). Body Mass Index (BMI) is a Popular Anthropometric Tool to Measure Obesity among Adults. Unpublished Manuscript.

Mohajan, D., & Mohajan, H. K. (2023*l*). A Study on Body Fat Percentage for Physical Fitness and Prevention of Obesity: A Two Compartment Model. Unpublished Manuscript.

Mohajan, D., & Mohajan, H. K. (2023m). Ponderal Index: An Important Anthropometric Indicator for Physical Growth. Unpublished Manuscript.

Mohajan, D., & Mohajan, H. K. (2023n). Long-Term Regular Exercise Increases $\dot{V}O_2$max for Cardiorespiratory Fitness. Unpublished Manuscript.

Mohajan, D., & Mohajan, H. K. (2023*o*). Economic Situations of Lagrange Multiplier When Cost of Various Inputs Increases for Nonlinear Budget Constraint. *Studies in Social Science & Humanities*, 2(4), 40-64.

Mohajan, D., & Mohajan, H. K. (2023p). Glaserian Grounded Theory and Straussian Grounded Theory: Two Standard Qualitative Research Approaches in Social Science. *Journal of Economic Development, Environment and People*, 12(1), 72-81.

Mohajan, D., & Mohajan, H. K. (2023q). Sensitivity Analysis for Utility Maximization: A Study on Lagrange Multipliers and Commodity Coupons. *Journal of Economic Development, Environment, and People*, 12(1), 25-40.

Mohajan, H. K. (2011a). Greenhouse Gas Emissions Increase Global Warming. *International Journal of Economic and Political Integration*, 1(2), 21-34.

Mohajan, H. K. (2011b). The NNP and Sustainability in Open Economy: Highlights on Recent World Economy and on Open Economy of Bangladesh. *KASBIT Business Journal*, 4(2), 32-47.

Mohajan, H. K. (2012a). Green Marketing is a Sustainable Marketing System in the Twenty First Century. *International Journal of Management and Transformation*, 6(2), 23-39.

Mohajan, H. K. (2012b). Aspects of Green Marketing: A Prospect for Bangladesh. *International Journal of Economics and Research*, 3(3), 1-11.

Mohajan, H. K. (2012c). *Importance of Green Marketing at Present and Future*. Lambert Academic Publishing, Germany.

Mohajan, H. K. (2012d). Greenhouse Gas Emissions of the USA. *Indus Journal of Management & Social Sciences*, 6(2), 132-148.

Mohajan, H. K. (2012e). Relation between Lease Finance and Purchase. *International Journal of Economics and Research*, 3(3), 146–158.

Mohajan, H. K. (2012f). Air Pollution Causes Health Effects and Net National Product of a Country Decreases: A Theoretical Framework. *International Journal of Development Research and Quantitative Techniques*, 2(2), 3–10.

Mohajan, H. K. (2013a). Economic Development of Bangladesh. *Journal of Business Management and Administration*, 1(4), 41-48.

Mohajan, H. K. (2013b). Ethiopia: A Socio-economic Study. *Journal of Business Management and Administration*, 1(5), 59-74.

Mohajan, H. K. (2013c). Friedmann, Robertson-Walker (FRW) Models in Cosmology. *Journal of Environmental Treatment Techniques*, 1(3), 158-164.

Mohajan, H. K. (2013d). *Global Greenhouse Gas Emissions and Climate Change*. Lambert Academic Publishing, Germany.

Mohajan, H. K. (2013e). Poverty and Economic Development of Kenya. *International Journal of Information Technology and Business Management*, 18(1), 72-82.

Mohajan, H. K. (2013f). Global Food Price Hike is a Burden to the Poor. *International Journal of Information Technology and Business Management*, 19(1), 1–15.

Mohajan, H. K. (2013g). Food, Agriculture and Economic Situation of Bangladesh. MPRA Paper No. 54240. https://mpra.ub.uni-muenchen.de/54240/

Mohajan, H. K. (2014a). Greenhouse Gas Emissions of China. *Journal of Environmental Treatment Techniques*, 1(4), 190–202.

Mohajan, H. K. (2014b). Chinese Sulphur Dioxide Emissions and Local Environment Pollution. *International Journal of Scientific Research in Knowledge*, 2(6), 265-276.

Mohajan, H. K. (2014c). The Most Fatal 2014 Outbreak of Ebolavirus Disease in Western Africa. *American Journal of Epidemiology and Infectious Disease*, 2(4), 101-108.

Mohajan, H. K. (2014d). Food and Nutrition of Bangladesh. *Peak Journal of Food Science and Technology*, 2(1), 1–17.

Mohajan, H. K. (2014e). Improvement of Health Sector in Kenya. *American Journal of Public Health Research*, 2(4), 159–169.

Mohajan, H. K. (2014f). Gravitational Collapse of a Massive Star and Black Hole Formation. *International Journal of Reciprocal Symmetry & Theoretical Physics*, 1(2), 125–140.

Mohajan, H. K. (2014g). General Upper Limit of the Age of the Universe. *ARPN Journal of Science and Technology*, 4(1), 4–12.

Mohajan, H. K. (2015a). Sustainable Development Policy of Global Economy. *American Journal of Environmental Protection*, 3(1), 12-29.

Mohajan, H. K. (2015b). Present and Future of Nestlé Bangladesh Limited. *American Journal of Food and Nutrition*, 3(2), 34-43.

Mohajan, H. K. (2015c). Basic Concepts of Differential Geometry and Fibre Bundles. *ABC Journal of Advanced Research*, 4(1), 57-73.

Mohajan, H. K. (2015d). Tuberculosis is a Fatal Disease among Some Developing Countries of the World. *American Journal of Infectious Diseases and Microbiology*, 3(1), 18-31.

Mohajan, H. K. (2015e). Generalization of Euler and Ramanujan's Partition Function. *Asian Journal of Applied Science and Engineering*, 4(3), 167–190.

Mohajan, H. K. (2016a). An Analysis of Knowledge Management for the Development of Global Health. *American Journal of Social Sciences*, 4(4), 38-57.

Mohajan, H. K. (2016b). Global Hyperbolicity in Space-time Manifold. *International Journal of Professional Studies*, 1(1), 14-30.

Mohajan, H. K. (2017a). Roles of Communities of Practice for the Development of the Society. *Journal of Economic Development, Environment and People*, 6(3), 27-46.

Mohajan, H. K. (2017b). Two Criteria for Good Measurements in Research: Validity and Reliability. *Annals of Spiru Haret University Economic Series*, 17(3), 58-82.

Mohajan, H. K. (2017c). Optimization Models in Mathematical Economics. *Journal of Scientific Achievements*, 2(5), 30-42.

Mohajan, H. K. (2017d). A Brief Analysis of de Sitter Universe in Relativistic Cosmology. *Journal of Scientific Achievements*, 2(11), 1–17.

Mohajan, H. K. (2017e). Development of Einstein's Static Cosmological Model of the Universe. *Journal of Scientific Achievements*, 2(7), 18–30.

Mohajan, H. K. (2018a). Qualitative Research Methodology in Social Sciences and Related Subjects. *Journal of Economic Development, Environment and People*, 2(1), 19-46.

Mohajan, H. K. (2018b). *Aspects of Mathematical Economics, Social Choice and Game Theory*. PhD Dissertation, Jamal Nazrul Islam Research Centre for Mathematical and Physical Sciences (JNIRCMPS), University of Chittagong, Chittagong, Bangladesh.

Mohajan, H. K. (2018c). The Rohingya Muslims in Myanmar are Victim of Genocide! *ABC Journal of Advanced Research*, 7(1), 59-72.

Mohajan, H. K. (2018d). Medical Errors Must be Reduced for the Welfare of the Global Health Sector. *International Journal of Public Health and Health Systems*, 3(5), 91-101.

Mohajan, H. K. (2018e). Analysis of Food Production and Poverty Reduction of Bangladesh. *Annals of Spiru Haret University Economic Series*, 18(1), 191-205.

Mohajan, H. K. (2020a). Quantitative Research: A Successful Investigation in Natural and Social Sciences. *Journal of Economic Development, Environment and People*, 9(4), 50–79.

Mohajan, H. K. (2020b). COVID-19–The Most Fatal Pandemic Outbreak: An Analysis of Economic Consequences. *Annals of Spiru Haret University Economic Series*, 20(2), 127-146.

Mohajan, H. K. (2020c). The COVID-19 in Italy: Remedies to Reduce the Infections and Deaths. *Malaysian Journal of Medical and Biological Research*, 7(2), 59-66.

Mohajan, H. K. (2020d). Most Fatal Pandemic COVID-19 Outbreak: An Analysis of Economic Consequences. *Annals of Spiru Haret University Economic Series*, 20(2), 127-146.

Mohajan, H. K. (2020e). Circular Economy can Provide a Sustainable Global Society. *Journal of Economic Development, Environment and People*, 9(3), 38–62.

Mohajan, H. K. (2021a). *Aspects of Global COVID-19 Pandemic*. Lambert Academic Publishing, Germany.

Mohajan, H. K. (2021b). Global COVID-19 Pandemic: Prevention and Protection Techniques. *Journal of Economic Development, Environment and People*, 10(1), 51-72.

Mohajan, H. K. (2021c). Estimation of Cost Minimization of Garments Sector by Cobb-Douglass Production Function: Bangladesh Perspective. *Annals of Spiru Haret University. Economic Series*, 21(2), 267-299.

Mohajan, H. K. (2021d). Product Maximization Techniques of a Factory of Bangladesh: A Sustainable Procedure. *American Journal of Economics, Finance and Management*, 5(2), 23–44.

Mohajan, H. K. (2022a). Four Waves of Feminism: A Blessing for Global Humanity. *Studies in Social Science & Humanities*, 1(2), 1-8.

Mohajan, H. K. (2022b). An Overview on the Feminism and Its Categories. *Research and Advances in Education*, 1(3), 11-26.

Mohajan, H. K. (2022c). Cost Minimization Analysis of a Running Firm with Economic Policy. *Annals of Spiru Haret University. Economic Series*, 22(3), 317-337.

Mohajan, H. K. (2022d). Mathematical Analysis of SIR Model for COVID-19 Transmission. *Journal of Innovations in Medical Research*, 1(2), 1-18.

Mohajan, H. K., Datta, R., & Das, A. K. (2012). Emerging Equity Market and Economic Development: Bangladesh Perspective. *International Journal of Economics and Research*, 3(3), 128–145.

Mohajan, H. K., Islam, J. N., & Moolio, P. (2013). *Optimization and Social Welfare in Economics*. Lambert Academic Publishing, Germany.

Moolio, P., Islam, J. N., & Mohajan, H. K. (2009). Output Maximization of an Agency. *Indus Journal of Management and Social Sciences*, 3(1), 39-51.

Polit, D. F., & Hungler, B. P. (2013). *Essentials of Nursing Research: Methods, Appraisal, and Utilization* (8th Ed.). Philadelphia: Wolters Kluwer/Lippincott Williams and Wilkins.

Rahman, M. M., & Mohajan, H. K. (2019). Rohingya-The Stateless Community Becoming the Lost Generation. *Journal of Economic Development, Environment and People*, 8(2), 24-36.

Remenyi, D. S. J., Swartz, E., Money, A., & Williams, B. (1998). Doing Research in Business and Management: An Introduction to Process and Method. SAGE Publications, London.

Roy, L., Molla, R., & Mohajan, H. K. (2021). Cost Minimization is Essential for the Sustainable Development of an Industry: A Mathematical Economic Model Approach. *Annals of Spiru Haret University. Economic Series*, 21(1), 37-69.

Samuelson, P. A. (1947). *Foundations of Economic Analysis*. Harvard University Press, Cambridge, MA.

Somekh, B., & Lewin, C. (2005). *Research Methods in the Social Sciences*. Sage Publications.

Zhao, Q., Zhang, Y., & Friedman, D. (2017). Multi-Product Utility Maximization for Economic Recommendation. *WSDM*, 435-443.