The Stochastic Advance-Retreat Course: An Approach to Analyse Social-Economic Evolution

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Abstract. The paper presents the basic theory and conceptual model for advance-retreat course and provides the analytic model the stochastic advance-retreat course and the solving method of it, discusses the relations between the endogenous resistance and subject interests increase with the periodic vibration in an advance-retreat course, gets some results, like heightening appropriately the risk-free interest rate will be favorable to subject interests' increasing in stable, interests increasing in high-speed will result in the fast increase of resistance, the subject progress in a appropriate pace may bring the conclusion such as lasting interests increase and return with higher-level interests, etc. Finally, the empirical researches empirical, on data of USA GDP (chained) price index, has been made to the stochastic advance-retreat model, and the results show that the stochastic advance-retreat model can describe USA economic development process in recent 65 years.

Keywords. Economic process, advance-retreat course, the basic theory, analytic model

1 Introduction

Up till now, there are already a lot of remarkable achievements in the research of economic development, such as such as the business cycle (Lucas, 1981), the real business cycle (Kydland and Prescott, 1982, and Long and Plosser, 1983), and new growth theory (Romer, 1986), etc. These theories has availably propeled forward the economic development of world. In recent years, the economists pay their more attention to make the efficiency of economic development higher (Collier and Dollar, 2004), evaluate the government's aidding to national or regional economy and the real effects are better or not (Guillaumont and Chauvet, 2001), insure the economic growth to be continuous by establishing and choosing the policies and perfecting the system of economic education (Graziella, 2001 and 2004), etc.

Also, the economists study the economic growth in many of other ways, such as the economic dynamics from the points of view of overlapping generations or bionomics (Croix and Michel, 2002; Gandolfo. 2003; Barro and Sala-i-Martin, 1995), the rational choice on the economic policies (Colman and Hines, 1998; Brams, 2004), game theory for analyzing economic behaviour (Fudenberg and Tirole, 1991; Osborne, 2003; Brams, 2004), etc.

All of these works have studied the economic development in the points of view of theory, industry, process and selection, and are worth praising very much. However, the important problem that should also be discussed is the problem of relation between behavior of mankind full of wisdom and the economic environment with non-intelligent, for example, the economic development and environmental pollution, the enterprises management and their development environment, the policies establishment and the implement environment of them, studying knowledge and the application environment of them, the use of energy and shortage in energy, development of traffic and transportation and the shortage in road facility, etc. In the problems like these, human can be free to choose and carry out the economic tactics independently as the behavioral subject full of wisdom, and the environments, as the non-intelligent behavioral object, will be unable to choose and carry out the tactics independently, can only accept human choice and produce resistance passively to the subjects in its economic behavior. The higher the degree of economic development and the faster the development pace is, the larger these kinds of resistances are. All of these problems can be described as the dynamic games between the economic development and the environmental resistance. In fact, many things in society and social evolution are similar to those in characteristic.

The problems above are called the advance-retreat course problem. Because the human behaviors are varied and resistances of environments are changeful, this paper will focus the discussion on the random advance-retreat course. The results here are fit for the general advance-retreat course problem in some drgree.

2 The Basic Models of the Advance-Retreat Course

Any thing includes the two basic movements in its growth and evolution, i.e., going forward and going backward. It advances by means of motivity, and retreats because of resistance. This advance–retreat problem seems to be simple. However, there are many examples in reality (F.Dai, 2006) and ample intension in theory. The purpose of advance-retreat analysis is to research deeply the advance –retreat problem.

2.1 The concept model of advance-retreat course

The advance-retreat course is the process in which something is growing or declining. The problems for advance-retreat course are called the advance-retreat problems, and the analysis for advance-retreat course is called the advance-retreat analysis.

2.1.1 The basic components of advance-retreat problem

An advance-retreat problem includes two pairs of components, the subject and object in progress process, as well as the motivity and resistance to which the subject will face. Also, the asset of subject needs to be considered.

Definition 1 (subject and object) The subject is the one who can choose actively one or more tactics and strategies and carry out them. Object is the things existed in the impersonal environment in which subject is going forward.

The basic behavior of subject is going in the direction of progressing and developing. Though there may be something in object that urge subject to progress passively, the main behavior of object is passively retarding subject to progress or let the progressing subject retreat in the reverse direction.

The subjects include all the doers who are able to do anything actively. As saying above, subjects include the society, economy, firm or person, etc. The objects include all the environment factors which may influence subjects to go forward or make the trouble to them, such as, the problems in policy, management, science and technology, production, market, energy, transportation, service and other problems in socio- economic field; the storm, tsunami, drought and waterlogging, temperature and other problems in natural environment; disease, pollution, lack of energy, species reduced, natural disaster and other problems in living environment.

Definition 2 (subject motivity) Subject motivity is the power to impel subject to go forward. The subject power is also called the subject energy, and motivity for short.

The motivity includes the endogenous motivity and exogenous one. The endogenous motivity is one which is caused by subject itself, such as the subject's needs to exist and to develop, the expectation to obtain the interests, the management inside the subject system, and change in environment and the corresponding investment, etc. The exogenous motivity is caused by the changes in external environment of subject, it is power to push the subject to go forward, such as the progress in sciences and technologies, the development in production, the changes in the market, etc.

The subject motivity is brought from the needs to exist and to develop, the expectation to gain the asset income, change in environment and the corresponding investment.

Definition 3 (object resistance) Object resistance is the force which is come from object and will retard subject to progress. Object resistance is also called the environment resistance or pressure, and resistance for short. The resistances include the endogenous resistance and exogenous one. Endogenous resistance is the force caused by the change in internal environmental of the subject system, and exogenous resistance is the force caused by the change in external environmental of the subject system.

The resistance is generally caused by the environment pressures, like the unsuitable policies, confused management, backward technology, shortage in energy, obstructed transportation, prevalent disease, serious pollution, natural disasters, etc.

Definition 4 (interests) The interests of subject are all of assets owned by subject. The subject interests include the basic interest and the real interest. Basic interest is the amount of assets owned by subject at the beginning of advance-retreat course, and real interest is the amount of assets owned by subject at the current time.

The subject interests include the asset, cash, resource, service, etc.

2.1.2 The basic rules and assumptions for advance-retreat problem

In order to discuss the advance-retreat problems, we need to have some the basic rules and general assumptions as follow.

The basic rules. There are four basic rules to advance-retreat problem:

- Overcoming resistance and progressing hard will obtain the income.
- Only going forward persistently subject can make his interest growing continuously.
- Coming to a halt or retreating will undergo the economic losses.
- The subject will wither away if his interests is exhausted and unable to pay for losses.

The basic assumptions. The basic assumptions for advance-retreat problems are composed of three categories and nine items:

● The assumption on goal. The goal of subject is the maximum in its comprehensive interests. Here the comprehensive interests include economic income, social benefit and environment efficiency.

● The assumptions on condition. The subject's interests and route to advance satisfy meet the following conditions:

(ⅰ) The initial asset (the cost interest) of subject is larger than zero. The subject needs some capital for startup payout.

(ⅱ) The subject's interests can not be overdrawn. Subject is able to get the power to go forward when his interests (stock assets) is larger than zero. Subject will be withered away automatically if his interests are equal to or less than zero. If obtaining the loan, subject's interest is regarded to be larger than zero.

(ⅲ) There is no branch of route in the process. In general, there may be branch in progressing route. In the following discussion, we suppose that there is no branch of route.

● The assumptions on behavior. In the course, subject and object have the different characters in their status and behavior.

(ⅰ) The initiative of subject. Subject has the initiative to choose any strategies and carry out them voluntarily to increase his interests.

(ⅱ) The flexibility of subject. Subject can use his interests to make an investment in order to strengthen the power to go forward or to reduce the resistance from object, or can choose retreat by losing his interests to avoid the larger resistance.

(iii) The passivity of object. Object will not choose any strategies on his own initiative, and will make a passive response to any choice of subject by the resistances (or pressures). These resistances will slow down the growth in interests of subject or decrease the subject interests.

(iv) The memory of object. The object's memory will be presented as the resistance to subject becomes larger and larger when the subject goes forward far and far and obtains the income more and more, i.e. it is more and more difficult for subject to go forward. This memory appears to be accumulated gradually in some period of time, and to be released in another period of time.

(v) Controllability for resistance. The endogenous resistances could be controlled in a larger degree because they are produced from the internal environment of the subject system, and the exogenous resistances could be controlled in a smaller degree or even not be controlled because they are brought from external environment of subject system. The main way to control resistance is the investment.

2.1.3 The main characters of advance-retreat course and the basic game model

As we see, under the rules and assumptions described above, the advance-retreat course has the main characters and the basic game as follow.

The main characters. The three main characters are:

● The deterministic structure. It is a basic trend that the subject (especially to the macroeconomy) goes forward, and the total amount of the interests will be increasing along with the trend. The larger the range of going forward is, the greater the resistance faced to subject is.

● The stochastic environment. In the course of advance-retreat, some of resistances may occur stochastically or unexpectedly, those will influence the subject to go forward, and advancing motivity will change along with environment, so the subject may make a stochastic decision to go ahead or back.

● The uncertain outcome. Subject needs to get rid of various resistances and overcome a lot of difficulties in order to obtain income. Not all subjects can succeed and most of them will be eliminated. So it is hard to know who will be the victor. Correspondingly, the uncertain characteristics are appeared in the interest's level at some period of time.

The description of basic games in the course. In order to establish the game model of the advance-retreat course, we will use the following notations:

u : The whole advancing motivity, motivity for short, $u=u(\sigma, \alpha\kappa)\geq 0$.

$\overline{u}$: The whole resistance faced to subject while he is advancing, resistance for short, $\overline{u}=\overline{u}(\delta\sigma, \beta\kappa)\geq 0$.

U : The net motivity to push subject to advance, $U=U(u,\overline{u})$. $U=U(u,\overline{u})>0$ if $u>\overline{u}$; $U=U(u,\overline{u})=0$ if $u=\overline{u}$; $U=U(u,\overline{u})<0$ if $u<\overline{u}$.

L : The net interests of subject, $L=L(U)=L(u,\overline{u})$.

Then, the strategies of subject in the course can be described as:

● If $u-\overline{u}>0$, there is a net motivity for subject to go forward. At this moment, subject will determine to advance, and his interests $L(u,\overline{u})$ will increase.

● If $u-\overline{u}=0$, there is no power for subject to go forward or backward. At this moment, subject may come to a halt at the current place, and interests $L(u,\overline{u})$ does not vary (the cost for staying is neglected). Subject can make an investment to strengthen his motivity or reduce the resistance, and keep on progressing.

● If $u-\overline{u}<0$, there is a net resistance for subject to go backward. At this moment, subject may determine to retreat, and his interests $L(u,\bar{u})$ will decrease. Subject can also take measures voluntarily, such as to make an investment, to strengthen his motivity or reduce the resistance, and continue to advance.

Therefore, if we call the traditional game theory (all the attendee are able to make decision voluntarily, their behaviors are based on intelligence instead of pure natural selection) the micro-game theory, advance-retreat analyze can be called the macro-game theory.

2.2 The basic theory of the advance-retreat course

When the motivity and resistance are all time-variant, i.e., $U(t)=U[u(t),\bar{u}(t)]$, $L=L[U(t)]$, where $U(0)>0$ and $L=L[U(0)]>0$, then we have the mathematical definition of advance-retreat course:

Definition 5 (advance-retreat course) if existing time $\bar{t}>0$ (or $\bar{t}=+\infty$), make $L[U(\bar{t})]=0$ $L[U(t)]>0$ ($t\in[0,\bar{t})$ and $L[U(t)]<0$ ($t\in(\bar{t},\infty]$), then $\{L[U(t)], t\in[0,\bar{t}]\}$ is called a simple advance-retreat course. If time $\bar{t}$ (>0) exists, and make $U(\bar{t})<0$ and $L[U(t)]>L[U(\bar{t})]$ when $t>\bar{t}$, then call $\{L[U(t)], t\in[0,\bar{t}]\}$ a complex advance-retreat course. Both the simple and complex advance-retreat course is called the advance-retreat course, and course for short. If having $T\in(0,\bar{t})$ and $L[U(T)]=\max\limits_{0<t<\bar{t}}\{L[U(t)]\}$, then $L[U(T)]$ is called a solution of $L[U(t)]$.

In the following discussion, the course $L[U(t)]$ is as $L(U)$ for short.

Definition 6 (the course series) Let n be a positive integer, and $L_i(U_i)$ be a advance-retreat course ($0\le\bar{t}_{i-1}<t_i\le\bar{t}_i$) for integer $i\in[1, n]$, then $\{L_i(U_i), i\in[1, n]\}$is called a finite course series, $L_{i\le n}(U_i)$ for short. If the positive integer n in finite course series $L_{i\le n}(U_i)$ could be large arbitrarily, this course series is called a infinite course series, and $L_{i<\infty}(U_i)$ for short. Both infinite course series and finite series are called the course series. If time T_i (>0) exists, $T_i\in[\bar{t}_{i-1}, \bar{t}_i)$, and make $L_i[U_i(T_i)]=\max\limits_{\bar{t}_{i-1}<t<\bar{t}_i}\{L_i[U_i(t_i)]\}$, then $\boldsymbol{U(T)}=[U_1(T_1),\cdots,U_m(T_m)]^{\mathrm{T}}$ is called the solution of course series $L_{i\le n}(U_i)$.

Theorem 1 If subject course $L[U(t)]$ ($t\in[0, \bar{t}]$) is continuous about t, then its solution is existent.

Proof. Because $L[U(0)]>0$ (there is the initial asset for subject to go forward)and $L[U(\bar{t})]=0$, we have t^+ and $L[U(t^+)]=\max\limits_{0<t<\bar{t}}\{L[U(t)]\}$ according to the property of continue function, then the result as follows.

Theorem 2 If course series $L_{i\le n}(U_i(t))$ are continuous about t, $L_0=\lim\limits_{n\to\infty}L_n(U_n)$ is bounded above, thus the solution of infinite course series $L_{i<\infty}(U_i)$ is existing.

$L=\lim\limits_{n\to\infty}L_n(U_n)$ is finite means the stock income is bounded above finally, so all the advance-retreat course $L_i(U_i)$ in this series are continue and bounded above, then the solution of $L_{i<\infty}(U_i)$ is existent.

Definition 7 (the optimal route) Let Θ be the notations set of all the routes in which subject may go forward along, $U(\theta, t)$ is the motivity function with the route θ, $\theta\in\Theta$. $U(\theta, t)$ is differentiable about time t and function $F[U(\theta,t),U_t'(\theta,t),t)]$ is integrable, if $U(\theta^*, t)=\sup\{U(\theta, t)$: $\int_0^T F[U(\theta,t),U_t'(\theta,t),t)]dt$, $U(\theta, 0)=U_0$, $U(\theta, T)=0$, $\theta\in\Theta\}$ exists, then $U(\theta^*, t)$ is called the the optimal route about $F[U(\theta,t),U_t'(\theta,t),t)]$ on $[0,T]$, and the optimal route for short.

Definition 8 (the optimal course) For courses $L_h(U)$, $h\in H$, H is the notations set of all interests. If l ($l\in H$) is existing, and make $[u(\theta,t),\bar{u}(\theta,t)]\ge L_h[u(\theta,t),\bar{u}(\theta,t)]$($\forall$ $h\in H$), then $L_l(U)$ is called the optimal course on U under H, and the optimal course for short.

Theorem 3① If the second order derivative of net motivity $U(t)$ is continuous, the function $F[U(t),U'(t),t)]$ is differentiable for second order, $U(0)=U_0$, $U(T)=0$. $U(t)$ ($t\in[0, T]$) is the optimal route, thus $U(t)$ satisfies the following differential equation (1):

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① This is a problem to determine the extremum for functional with the fixed boundaries. The necessary condition is $F[U(t),U'(t),t)]$, according to the functional knowledge, follows the Eular equation (1).

Corollary 2 Suppose the first order derivatives of course $L(U)$ and motivity $U(t)$ are continuous, $t\in[0, T]$, $U(0)=U_0$, $U(T)=0$. If $L_h(U)$ is the optimal course on which both subject interests and motivity are maximum in their growth rates at the same time, then

$$L[U(t)]=L(U_0)\,e^{\int_0^t h(\tau)U(\tau)d\tau} \qquad (2)$$

where, $h(t)$ follows $\int_0^T h(t)U(t)dt=\ln\left[\frac{L(0)}{L(U_0)}\right]$ and $h(t)<1$.

Proof. Taking $F[U(t),U'(t),t]=\frac{L_t[U(t)]}{L[U(t)]U(t)}$ in theorem 3, we have $\frac{\partial F(U)}{\partial U}=0$ from differential equation (1), i.e. $\frac{L_t[U(t)]}{L[U(t)]U(t)}=h(t)$. Completing the integral, we obtain $L[U(t)]=C\,e^{\int_0^t h(\tau)U(\tau)d\tau}$. From $U(0)=U_0$, $U(T)=0$, we have the expression (2). Because $\frac{L_t[U(t)]}{L[U(t)]U(t)}$ can be expressed as $\frac{L'_U[U(t)]U'_t(t)}{L[U(t)]U(t)}$, this is just the product subject interest and motivity in their related growth rate, so the result follows.

In corollary 2, $h(t)<1$, so the $L[U(t)]=L(U_0)\,e^{\int_0^t U(\tau)d\tau}$ is the optimal course on $H(t)=\{h(t)$: $h(t)$ is continuous, $0\le h(t)\le 1$, $t\in[0, T]\}$.

2.3 The examples of advance-retreat course

Example 1. If the function $U(t)$ and $h(t)$ are continuous and integrable, where $U(0)=U_0>0$, $L[U(0)]>0$, $U(T)=0$. Thus

1) $L[U(t)]=L(U_0)\,e^{\int_0^t h(\tau)U(\tau)d\tau}$ is an advance-retreat course, and its solution is $L[U(T)]$.

2) $L[U(t)]= L(U_0)+\int_0^t h(\tau)U(\tau)d\tau$ is an advance-retreat course, and its solution is $L[U(T)]$.

Example 2. If function $\mu(t)$ and $U(t)$ the continuous and differentiable, h is a constant, where $U(0)=U_0>0$, $L[U(0)]>0$, then $L[U(t)]=\mu(t)+hU(t)$ is an advance-retreat course on $[0, +\infty)$. Let $\bar{t}$ be the root of equation $L[U(t)]=0$, and T make the following equations

$$\begin{cases} U(T)=0 \\ L'_t[U(T)]=\mu'(T)+hU'(T)=0 \end{cases}$$

come into existence, thus, $L[U(T)]=\max\limits_{t\ge 0} L[U(t)]$ is a solution of $L[U(t)]$.

3 The Partial Distribution and the Related Results

Though the partial distribution (F Dai, 2001) is the truncated Gaussian distribution, the two following results about partial distribution never be given in the discussions for the truncated Gaussian distribution. (Johnson, N. L., Kotz, S., Balakrishnan, N. 1994). So the partial distribution is still called.

3.1 The partial distribution and partial process

Definition 9 (The Partial Distribution, PD for short) Let X be a non-negative stochastic variable, and it follows the distribution of density

$$f(x)=\begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big/ \int_0^\infty e^{-\frac{(u-\mu)^2}{2\sigma^2}}du & x\ge 0 \\ 0 & x<0 \end{cases}$$

where $\mu \geq 0$ and $\sigma>0$. Then, X is called to follow a Partial Distribution, and note as $X \in P(\mu, \sigma^2)$.

Definition 10 (The Partial Process, PP for short) Let $\{X(t), 0 \leq t<\infty\}$ be a stochastic process, , if stochastic variable $X(t)$ satisfies Partial Distribution for any $t \geq 0$, i.e. $X(t) \in P(\mu(t), \sigma^2(t))$, then call $\{X(t), 0 \leq t<\infty\}$ to be a Partial Process, and $X(t) \in P(\mu(t), \sigma^2(t))$ for short, where $\mu=\mu(0)$ and $\sigma^2=\sigma^2(0)$.

Definition 11 (Rightward Partial Distribution, RPD for short) If X is a non-negative stochastic variable, and it has the probability density function as follows

$$f(x)=\begin{cases} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Big/ \int_a^{\infty} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du & x \geq a \\ 0 & x<a \end{cases}$$

where, the constant $a>0$, then X is called to follow the rightward Partial Process, and note as $X \in P_a(\mu, \sigma^2)$. When $\mu>\sigma$ and $a=\mu-\sigma$, X is called to follow the standard rightward partial distribution, SRPD for short, and note as $X \in F_a(\mu, \sigma^2)$. The standard rightward partial process is SRPP for short.

3.2 The basic results related to Partial Distribution②

We have the basic results about PD as follow (F. Dai, et al, 2001).

Theorem 4 For any $x \in [0,\infty)$, the following formulas are correct approximately:

(ⅰ) $\int_0^x e^{-\frac{t^2}{2}} dt = \sqrt{\frac{\pi}{2}(1-e^{-\frac{2}{\pi}x^2})}$;

(ⅱ) $\int_0^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{\frac{\pi}{2}}\sigma\left((\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}} + \operatorname{sgn}(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu}{\sigma})^2}}\right)$, where, $\operatorname{sgn}(x)=\begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$.

Corollary 3 For any $x \in [a,\infty]$, a, μ and σ are constant, a, $\mu \geq 0$, $\sigma>0$, then the following equations are correct approximately:

$$\int_a^x e^{-\frac{(v-\mu)^2}{2\sigma^2}} dv = \sqrt{\frac{\pi}{2}}\sigma\left(\operatorname{sgn}(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{x-\mu}{\sigma}\right)^2}} - \operatorname{sgn}(a-\mu)\sqrt{1-e^{-\frac{2}{\pi}\left(\frac{a-\mu}{\sigma}\right)^2}}\right)$$

where, sgn(x) is the same as in theorem 1.

Theorem 5 Stochastic variable X follow the partial distribution, i.e., $X \in P(\mu, \sigma^2)$, thus

(ⅰ) The expectation $E(X)$ is as follows

$$E(X)=\mu+\sqrt{\frac{2}{\pi}} \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{1-e^{-\frac{2}{\pi}(\frac{\mu}{\sigma})^2}}+1} \tag{3}$$

(ⅱ) The variance $D(X)$ is as follows

$$D(X)=\sigma^2+E(X)[\mu-E(X)] \tag{4}$$

If $X \in P_a(\mu, \sigma^2)$ and $a<\mu$, we have the following results from corollary 3

$$E(X)=\mu+\sqrt{\frac{2}{\pi}}\sigma \frac{e^{-\frac{(\mu-a)^2}{2\sigma^2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}\frac{(\mu-a)^2}{\sigma^2}}}}$$

$$D(X)=\sigma^2-[E(X)-a][E(X)-\mu]=\sigma^2-[E(X)-a]R(X)$$

② These results have never appeared in the discussions on the truncated Gaussian distribution.

If $X \in F_a(\mu, \sigma^2)$, $\mu \geq 0$, then the expectation of X is

$$E(X) = \mu + s\sigma \tag{5}$$

where, $s = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}}}{1+\sqrt{1-e^{-\frac{2}{\pi}}}}$ =0.2869947990, is called the average growth coefficient, and

$$D(Z) = [1-(1+b)b]\sigma^2 = c_0\sigma^2 \tag{6}$$

where, $c_0 = [1-(1+b)b] = 0.6306391865$.

Suppose $X \in P(\mu, \sigma^2)$, then $X^* = \frac{\mu + \sqrt{\mu^2 + 4\sigma^2}}{2}$ if the product of X and its appearing probability reaches maximum (F. Dai, et al, 2003), correspondingly, $D^* = D(X) + [E(X) - X^*]^2$.

Corollary 4 If $X \in F_a(\mu, \sigma^2)$, i.e., $a = \mu - \sigma$, thus

$$X^* = \mu + c\sigma \tag{7}$$

and corresponding variance is as follows

$$D^* = \{[1-(1+b)b] + [b-c]^2\}\sigma^2 = c_1\sigma^2 \tag{8}$$

where, $c = \frac{\sqrt{5}-1}{2}$ =0.618033989, c_1=0.7402261316, $\bar{c}^* = \sqrt{c_1}$ =0.8603639532.

4 The Stochastic Model for Advance-Retreat Course

4.1 Notations and assumption

Definition 12 (The consumed cost) The consumed cost is the decrement in subject interests caused by resistances. The consumed costs include the endogenous cost and the exogenous cost. The endogenous cost is the consumed interest caused by endogenous resistances, and the exogenous cost is the consumed interest caused by exogenous resistances.

We will use the notations and explains in following discussions:

μ	:	The basic interest of subject. Basic interest is the start-up asset for subject to go forward.
δ	:	The coefficient of endogenous resistance. $\delta\mu$ is the basic endogenous cost.
σ	:	The original motivity of subject himself, i.e. basic endogenous motivity which can be measured by the fluctuation range of the amount of basic interest. $\delta\sigma$ is basic endogenous resistance.
ϕ③	:	The basic exogenous cost, i.e. the amount of the basic interest of subject consumed by the exogenous resistance, $\phi \geq 0$.
κ	:	The integrated effort caused by the related external environment and can be measured by the fluctuation range of the amount of basic assets in related environments.
$\alpha\kappa$	:	The original exogenous motivity.
$\beta\kappa$	:	The original exogenous resistance.
r④	:	The coefficient of net resistance caused by exogenous environment, $r = \beta - \alpha$, $0 \leq r \leq 1$.

③ The sum of endogenous cost and exogenous cost are called the consumed cost.

④ Usually, $\alpha + \beta < 1$, this means the external environment includes the exogenous motivity, exogenous resistance, and some other factors. If $\alpha + \beta = 1$, the external environment is composed of exogenous motivity and exogenous resistance, no other part. We have $\alpha < \beta$ generally, this means resistance factors are more than motivity factors in the external environment.

u		The original motivity for subject to go forward, $u=u(\sigma, \alpha\kappa)\geq 0$.
$\bar{u}$	:	The original resistance faced to subject, $\bar{u}=\bar{u}(\delta\sigma, \beta\kappa)\geq 0$.
U	:	The net motivity to push subject to advance, $U=U(u,\bar{u})$. $U=U(u,\bar{u})>0$ if $u>\bar{u}$; $U=U(u,\bar{u})=0$ if $u=\bar{u}$; $U=U(u,\bar{u})<0$ if $u<\bar{u}$.
L		The net interests of subject, $L=L(U)=L(u,\bar{u})$.
X	:	The real subject interest, the whole asset (or property) at the present time.
$\bar{X}$	:	The real consumed cost, the whole increment in interest caused by resistances at the present time.
G	:	The real exogenous cost at the present time.
μ_L	:	The basic net interest of subject.
σ_L		The basic net motivity.

Assumption 1 The assumptions on interest properties:

● All the values of interests and consumed costs are non-negative, and All the values of motivities and resistances are positive.

● Basic interest and basic motivity are the basic elements to determine the real interest. Similarly, basic consumed cost and basic resistances are the basic elements to determine the real consumed cost.

● As the real interest (or the real consumed cost) departs from the basic interest further and further, the happening probability will be smaller and smaller. So do the consumed cost.

From assumption 1 and references (Dai, et al, 2001 and 2005), we could suppose $G\in P(\phi, (\beta\kappa)^2)$, $X\in P(\mu, (\sigma+\alpha\kappa)^2)$ and $\bar{X}\in P(\delta\mu+\phi, (\delta\sigma+\beta\kappa)^2)$.

Because all the real interest, real consumed cost and real net interest will be influenced by many random factors, they are supposed to be stochastic variables, and supposed to be stochastic processes if they change along with time.

4.2 The model of stochastic variable for advance-retreat course and related analysis

Based on assumption 1, we will establish the model of stochastic variable for advance-retreat course and make some related discussions.

4.2.1 The stochastic variable models

Let $U=u-\bar{u}$, $u=\sigma+\alpha\kappa$, $\bar{u}=\delta\sigma+\beta\kappa$. Remaining the generality, we suppose real interest $X\in P(\mu, (\sigma+\alpha\kappa)^2)$ and real consumed cost $\bar{X}\in P(\delta\mu+\phi, (\delta\sigma+\beta\kappa)^2)$. The basic net interest is

$$\mu_L=\mu-(\delta\mu+\phi) \tag{9}$$

Because the resistance is in the inverse direction to the motivity, the correlation coefficient between resistance and motivity is ρ=-1, then basic net motivity is

$$\sigma_L^2=(\sigma+\alpha\kappa)^2+(\delta\sigma+\beta\kappa)^2+2\rho(\sigma+\alpha\kappa)(\delta\sigma+\beta\kappa), \text{ i.e.,}$$

$$\sigma_L=\sigma-(\delta\sigma+r\kappa) \tag{10}$$

In general, the exogenous net resistance coefficient $r=\beta-\alpha>0$. Otherwise, subject interest will increase forever, and this is few to happen in reality.

Proposition 1 From the equation (10), we have obtained the following results:

(i) If $\delta + r\dfrac{\kappa}{\sigma} < 1$, i.e., $\sigma_L=\sigma-(\delta\sigma+r\kappa)>0$, subject gets its motivity to go forward. This indicates that subject will be pushed to go forward and develop by its motivity while both the endogenous resistance and the net exogenous resistance are smaller to a certain degree.

(ii) If $\delta + r\dfrac{\kappa}{\sigma} = 1$, i.e. $\sigma_L=0$, subject gets no motivity to go forward or backward. This indicates that subject will lose its direction to go while the endogenous resistance and the net exogenous resistance reaches a certain critical value.

(iii) If $\delta + r\dfrac{\kappa}{\sigma} > 1$, i.e. $\sigma_L<0$, the resistance will push subject to retreat. This indicates that subject will go backward while the endogenous resistance and the net exogenous resistance are larger to a certain degree.

4.2.2 The average mode and high-speed mode

Suppose the real net interest follows SRPD, i.e., $L\in F_a(\mu_L, \sigma_L^2)$, where, $\mu_L=\mu-(\delta\mu+\phi)$, $\sigma_L=(1-\delta)\sigma-r\kappa$. According to expressions (5), (6), (7), (8), we can define the average mode and high-speed mode as follow.

The average mode. The real net interest of subject is called to be in average mode in growth if it and the real net motivity could be computed separately by the ways of (11).

$$\begin{cases} E(L) = \mu_L + 0.2869947990\sigma_L \\ \sqrt{D(L)} = 0.7941279409\sigma_L \end{cases} \tag{11}$$

where, $R(L)=0.2869947990\sigma_L$ means the increment in net interest caused by the basic net motivity in average mode.

The high-speed mode. The real net interest of subject is called to be in high-speed mode in growth if it and the real net motivity could be computed separately by the ways of (12).

$$\begin{cases} E_*(L) = \mu_L + 0.618033989\sigma_L \\ \sqrt{D_*(L)} = 0.8603639532\sigma_L \end{cases} \tag{12}$$

where, $R_*(L)=0.618033989\sigma_L$ means the increment in net interest caused by the basic net motivity in high-speed mode.

The 0.2869947990 in expression (11) and 0.618033989 in (12) are called the growth coefficients. According to (11) and (12), $D(L)=0$ when $\sigma_L=0$. This means, subject gets no real net motivity if having no basic net motivity.

4.3 The model of stochastic process for advance-retreat course and the analysis on it

Based on the following assumptions, we will establish the the model of stochastic process for advance-retreat course and make some discussions on the model.

4.3.1 The assumptions for the dynamic properties about motivity and resistance

In the following discussion, the endogenous factors will be separate from the exogenous factors.

Assumption 2 Suppose the original real interest X follows the partial distribution, i.e. $X\in P(\mu, \sigma^2)$. In general, the advance-retreat course is related to time t, so we have the assumptions for subject interest as follow

(i) Because the subject will hope to gain more and more interest in its advance process, the motivity will accumulate and increase naturally along with the time. Thus, suppose the motivity accumulate and increase in the way of $[\sigma(t)]^2=\sigma^2(1+bt)$ ($b>0$), this is a linear way. If like this, the real interest follows the stochastic process $X(t)\in P[\mu, \sigma^2(1+bt)]$ ($t\geq 0$), where, b is a accumulated coefficient for motivity, $b>0$.

(ii) Taking the factors in consideration such as risk-free rate of interest, etc., the real interest will increase naturally along with time. Finally, the real interest follows the stochastic process as:

$$X(t)\in P[\mu e^{\lambda t}, (\sigma e^{\lambda t})^2(1+bt)]$$

where, λ the natural growth rate of real interest, $\lambda>0$.

Assumption 3 Suppose the original real exogenous cost $G\in P(\phi, (\beta\kappa)^2)$. In general, it will increase along with time t, so we have the assumptions as follow.

（i）Caused by many factors (technology market, etc) in social-economic environment, the real exogenous resistance will increase naturally along with the subject going forward. Thus, suppose real exogenous resistance will increase in the way of $[\beta\kappa(t)]^2=(\beta\kappa)^2(1+bt)^\theta$, this is a nonlinear way. If like this, the real exogenous cost follows the stochastic process $G(t)\in P(\phi, (\beta\kappa)^2(1+bt)^\theta)$, where θ is the accumulated index of exogenous resistance, also the route in definition 7, $\theta>1$. $(1+bt)^\theta$ means the memory of exogenous resistance to accumulated motivity.

（ii）Because the real exogenous cost will increase along with the real interest growth, the real exogenous cost should follow the stochastic process as:

$$G(t)\in P(\phi e^{\psi t}, (\kappa e^{\psi t})^2(1+bt)^\theta)$$

where, ψ is the growth rate of real exogenous cost, $\psi>0$.

4.3.2 The model of stochastic process for advance-retreat course and the related analysis

According to assumption 1～assumption 2 and expression (9) and (10), we suppose the net real interest follows SRPP, namely,

$$L(t)\in F_a[\mu_L(t),\ \sigma_L^2(t)]$$

where,

$$\mu_L(t)=(1-\delta)\mu(t)-\phi(t)=(1-\delta)\mu e^{\lambda t}-\phi e^{\psi t}$$

and

$$\sigma_L(t)=(1-\delta)\sigma e^{\lambda t}\sqrt{1+bt}-r\kappa e^{\psi t}(1+bt)^{\frac{\theta}{2}} \tag{13}$$

Because $a=\mu_L(t)-\sigma_L(t)\geq 0$ in SRPP, i.e., $\mu_L(t)\geq\sigma_L(t)$, this means net motivity not exceed its basis existed (i.e. the basic net interest).

From the expressions (11) or (12), we have $E_s[L(t)]=\mu_L(t)+s\sigma_L(t)$, where, s=0.2869947990 or s=0.618033989. Because $E_s[L(t)]$ is a single peak function about t, $E_s[L(t^+)]=\max\limits_t\{E_s[L(t)]\}$ is reached at t^+ if the derivative $\{E_s[L(t^+)]\}'=0$, i.e.

$$\{E_s[L(t^+)]\}'=\mu_L'(t^+)+s\,\sigma_L'(t^+)=0$$

where, $\mu_L'=(1-\delta)\mu\lambda e^{\lambda t}-\psi\phi e^{\psi t}$ and $\sigma_L'=(1-\delta)\sigma e^{\lambda t}\sqrt{1+bt}\left[\frac{b}{2(1+bt)}+\lambda\right]-r\kappa e^{\psi t}(1+bt)^{\frac{\theta}{2}}\left[\frac{b\theta}{2(1+bt)}+\psi\right]$.

In expression (13), if existing T to make $\sigma_L(T)=0$, then $T=t^+$ according to the definition 5, i.e., $\sigma_L(T)$ is the solution of the course $L(t)$ in the meaning of $E_s[L(t)]$. So the T satisfies the following equations

$$\begin{cases} (1-\delta)\sigma e^{\lambda T}\sqrt{1+bT}-r\kappa e^{\psi T}(1+bT)^{\frac{\theta}{2}}=0 \\ (1-\delta)\mu\lambda e^{\lambda T}-\psi\phi e^{\psi T}+s\left[(1-\delta)\sigma e^{\lambda T}\sqrt{1+bT}\left(\lambda+\frac{b}{2(1+bT)}\right)-r\kappa e^{\psi T}(1+bT)^{\frac{\theta}{2}}\left(\psi+\frac{b\theta}{2(1+bT)}\right)\right]=0 \end{cases} \tag{14}$$

Solving the group of equations (14), obtain

$$T=\frac{1}{\psi-\lambda}\ln\left[\frac{1-\delta}{\psi\phi}\left(\mu\lambda+s\sigma\sqrt{1+bT}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT)}\right)\right)\right] \tag{15}$$

$$r=\frac{(1-\delta)\sigma}{\kappa}e^{(\lambda-\psi)T}(1+bT)^{\frac{1-\theta}{2}} \tag{16}$$

In expression (16), r=r(T)=β-α. Solving the simultaneous equations of r=β-α and α+β =1, obtaining both the values of α and β which could be applied to analyze the influence of external environmental systems to subject, and to compare the exogenous motivity and exogenous resistance.

Remark 1 From expression (16), we see

(ⅰ) If the duration for subject interests' growing is longer, i.e., T is greater, the coefficient of net exogenous resistance need to be smaller, i.e., the exogenous motivity is near to the exogenous resistance.

(ⅱ) If the duration for subject interests' growing, T, is a constant, the smaller endogenous resistance (δ) or the larger endogenous motivity (σ) or the smaller environmental change (κ) will be able to tolerate the larger exogenous resistance.

Proposition 2 If the real numbers μ, σ, λ, ψ, δ, s, b, θ are all positive, ψ>λ, θ>1, δ<1, and the following algorithm (17), determined by expression (15), is of convergence

$$T_{i+1}=\frac{1}{\psi-\lambda}\ \ln\left[\frac{1-\delta}{\psi\phi}\left(\mu\lambda+s\sigma\sqrt{1+bT_i}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT_i)}\right)\right)\right] \tag{17}$$

Thus,

$$0<\delta<1-\frac{\psi\phi}{\mu\lambda-s\sigma\sqrt{2b(\psi-\lambda)(\theta-1)}} \tag{18}$$

and in algorithm (17), the original value of T_0 can generally be $T_0=\frac{1}{b}\left[\left(\frac{\sqrt{c^2+2b(\lambda-\psi)(1-\theta)}-c}{2(\psi-\lambda)}\right)^2-1\right]$-1,

where, $c=\frac{1}{s\sigma}\left(\frac{\psi\phi}{1-\delta}-\mu\lambda\right)$.

Proof. In expression (15), let $\frac{1-\delta}{\psi\phi}\left(\mu\lambda+s\sigma\sqrt{1+bT}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT)}\right)\right)$>1, have

$$(\psi-\lambda)\,u^2+\frac{1}{s\sigma}\left(\frac{\psi\phi}{1-\delta}-\mu\lambda\right)u+\frac{b(\theta-1)}{2}<0 \text{ (where, } c=\frac{1}{s\sigma}\left(\frac{\psi\phi}{1-\delta}-\mu\lambda\right),\ u=\sqrt{1+bT}\text{)}$$

Solving the inequation above, we get $u=\sqrt{1+bT}<\frac{\sqrt{c^2-2b(\psi-\lambda)(\theta-1)}-c}{2(\psi-\lambda)}$, i.e.

$$T<\frac{1}{b}\left[\left(\frac{\sqrt{c^2+2b(\lambda-\psi)(1-\theta)}-c}{2(\psi-\lambda)}\right)^2-1\right]$$

and $c^2-2b(\psi-\lambda)(\theta-1)$>0, i.e., $\frac{1}{s\sigma}\left(\mu\lambda-\frac{\psi\phi}{1-\delta}\right)-\sqrt{2b(\psi-\lambda)(\theta-1)}$>0, thus the inequation (18) is as follows.

There are two important results in remark 2 from proposition 2.

Remark 2 If the subject parameters (μ, b) and the environmental parameters (ϕ, ψ, θ) are all constant, we have

(ⅰ) When both the basic endogenous motivity σ and the natural growth rate λ are fixed and there are only one T making $U(T)=\sigma_L(T)=0$ (i.e. subject is going forward in a advance-retreat course instead of the series of advance-retreat course), the endogenous resistance must be smaller than a certain value, i.e., the expression (18) is correct. Otherwise, have

$$T=\frac{1}{\psi-\lambda}\ \ln\left|\frac{1-\delta}{\psi\phi}\left(\mu\lambda+s\sigma\sqrt{1+bT}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT)}\right)\right)\right|+\frac{(2m+1)\pi}{\psi-\lambda}i$$

where, i (i^2= -1) is the unit of complex number, m=0,1, · · ·. If like this, T is a series of multi-values, i.e. a lot of time T make $U(T)=\sigma_L(T)=0$. This means the process of subject's going forward is a course series, and subject's interest may change in the cycle way. There is no the exogenous resistance $r\kappa$ in inequation (18), so we could think that the periodic phenomenon in motivity and interest is caused by the larger endogenous resistance.

(ii) When basic endogenous resistance coefficient δ is fixed, the smaller natural growth rate λ will cause $\dfrac{\psi\phi}{\mu\lambda - s\sigma\sqrt{2b(\psi-\lambda)(\theta-1)}}$ to be larger than a certain value, this may arose motivity and interest to change in a cycle way. So we get an important conclusion that it will be able to stabilize the economic process to key up properly the risk-free rate of interest. Also, the larger basic endogenous motivity σ may arose motivity and interest to change in a cycle way. Thus, another important conclusion is too larger basic endogenous motivity σ is not benefit to stabilize the economic process.

Because the motivity and resistance are always existing at the same time and blending one with another, the original data gained directly are generally the basic net interest $\mu_L(0)=(1-\delta)\mu-\phi$ and the basic net motivity $\sigma_L(0)=(1-\delta)\sigma-r\kappa$. According to expression (16), $r=\dfrac{\sigma_L(0)+r\kappa}{\kappa}e^{(\lambda-\psi)T}(1+bT)^{\frac{1-\theta}{2}}$, i.e.,

$$r=\frac{\sigma_L(0)}{\kappa\left[e^{(\psi-\lambda)T}(1+bT)^{\frac{\theta-1}{2}}-1\right]} \tag{19}$$

Taking the expression (19) into the second equation in expression (14), obtain

$$T=\frac{1}{\psi-\lambda}\ln\left[\frac{1}{\psi\phi}\left((\mu_L(0)+\phi)\lambda\left(1-(1+bT)^{\frac{1-\theta}{2}}e^{-(\psi-\lambda)T}\right)+\psi\phi(1+bT)^{\frac{1-\theta}{2}}+s\sigma_L(0)\sqrt{1+bT}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT)}\right)\right)\right]$$

Corresponding iterative algorithm is as follows:

$$T_{i+1}=\frac{1}{\psi-\lambda}\ln\left[\frac{1}{\psi\phi}\left((\mu_L(0)+\phi)\lambda\left(1-(1+bT_i)^{\frac{1-\theta}{2}}e^{-(\psi-\lambda)T_i}\right)+\psi\phi(1+bT_i)^{\frac{1-\theta}{2}}+s\sigma_L(0)\sqrt{1+bT_i}\left(\lambda-\psi+\frac{b(1-\theta)}{2(1+bT_i)}\right)\right)\right] \tag{20}$$

Generally, we could let $T_0=1$ in iterative algorithm (20).

4.4 The analytical example for advance-retreat course

Here we give an example to explain the rationality of analytical results above.

Suppose the real subject interest $X(t)\in P[\mu e^{\lambda t}, (\sigma e^{\lambda t})^2(1+bt)]$, The real exogenous cost $G(t)\in P(\phi e^{\psi t}, (\kappa e^{\psi t})^2 t^{\theta})$, The real net interest $L(t)\in F_a[\mu_L(t), \sigma_L^2(t)]$, where, $\mu_L(t)=(1-\delta)\mu e^{\lambda t}-\phi e^{\psi t}$, $\sigma_L(t)=(1-\delta)\sigma e^{\lambda t}\sqrt{1+bt}-r\kappa e^{\psi t}(1+bt)^{\frac{\theta}{2}}$. And let $\mu=2.0$, $\sigma=1.5$, $\lambda=0.07$, $\phi=0.10$, $\kappa=0.9$, $\psi=0.09$, $b=0.9$, $\theta=1.5$, $\delta=0.3$.

(i) When the real net interest follows the average mode in growth, i.e. $E[L(t)]=\mu_L(t)+0.2869947990\sigma_L(t)$, T and r are computed by expression (14) as follow:

$T=94.94233335$, $r=0.07945084821$

$E[L(T)]=\max\limits_{t\geq 0}\{E[L(t)]\}$, i.e., $\sigma_L(T)=0$ is the solution of advance-retreat course $L(t)$ in the average mode.

Letting $X(t)=E[L(t)]$ and by use of the algorithm in appendix, we obtain $T_1=106.9626686$ and $E[L(T_1)]=0$, i.e., $T_1=106.9626686$ is the time when advance-retreat course finishes and subject withers away. This process is drawn in figure 1.

(ii) When the real net interest follows the high-speed mode in growth, i.e. $E_*[L(t)]=\mu_L(t)+0.618033989\sigma_L(t)$, T

and r are computed by expression (14) as follow:

$$T=58.60146975,\ r=0.1848242048$$

$E_*[L(T)]=\max_{t\geq 0}\{E_*[L(t)]\}$, i.e., $\sigma_L(T)=0$ is the solution of advance-retreat course $L(t)$ in the high-speed mode.

Letting $X(t)=E[L(t)]$ and by use of the algorithm in appendix, we obtain $T_1=70.22191871$ and $E_*[L(T_1)]=0$, i.e., $T_1=70.22191871$ is the time when advance-retreat course finishes and subject withers away. This process is drawn in figure 2.

We use the year as the unit of time in figure 1 and figure 2, then have the remark 3.

Remark 3 From figure 1 and figure 2 and the related computing results, we see the two important results as follows:

(ⅰ) In the early 35 years, interest in high-speed mode increases faster than one in the average mode notably, i.e. the real interest under the high-speed mode are higher. But, the stamina of interest's increasing in the average mode appears gradually after 40 years, not only its time for development is longer, but also the final interest is much higher. This means fast growth may bring high interest in short-term, the corresponding resistances in progressing will also increase fast ($r=0.1825799392$), and the ability to keep on going forward is not great. On the contrary, though the slower growth will not bring high interest in short-term, corresponding resistance in progressing will not increase fast ($r=0.07945084821$), therefore the time in interest's increasing is longer, and the ability to keep on going forward is great.

(ⅱ) Generally speaking, the change in the motivity can be divided into three basic stages. At initial stage, net motivity increase slowly because the subject motivity is not very larger. At middle stage, the motivity is much lager than the resistance, and this results the net motivity to come into a stronger and faster growth. At the last stage, the net motivity decreases rapidly while the resistance becomes larger and larger. Once the resistance exceeds the motivity, subject interest reduces rapidly until it becomes nothing, and the subject is withered away from then on. The change in subject interest is similar to the process in motivity just the interest can not smaller than zero.

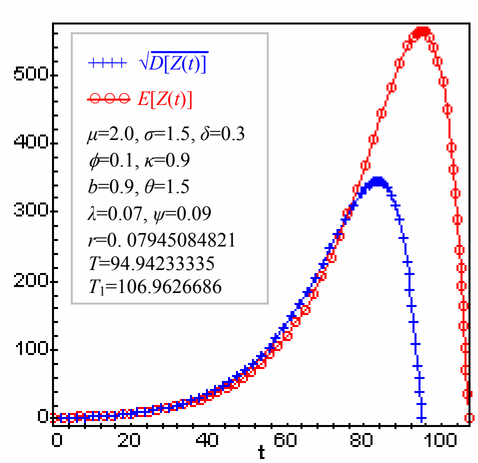

Figure 1 The interest and motivity changed in the way of average mode

The subject interest $E[L(t)]=\mu_L(t)+0.2869947990\sigma_L(t)$ reaches its maximum at time $T=94.94233335$. At this time, the motivity is equal to zero, i.e. $\sigma_L(T)=0$. And at the time $T_1=106.9626686$, the interest $E[L(T_1)]=0$, this means advance-retreat course finishes and subject withers away.

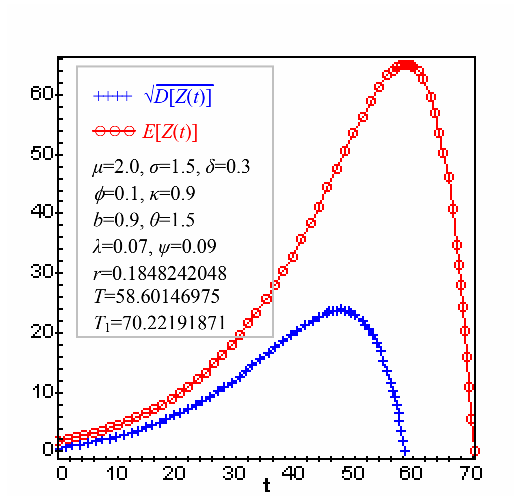

Figure 2 The interest and motivity changed in the way of high-speed mode

The subject interest $E[L(t)]=\mu_L(t)+0.618033989\sigma_L(t)$ reaches its maximum at time $T=58.60146975$. At this time, the motivity is equal to zero, i.e. $\sigma_L(T)=0$. And at the time $T_1=70.22191871$, the interest $E[L(T_1)]=0$, this means advance-retreat course finishes and subject withers away.

This kind of process is represented in practice as the developing process of many industries in socio-economic field. At the beginning, there is a period of time for suiting and groping, and many of difficulties will be faced in this period. If unremitting, a regular and rapidly developing stage will come to being. All the works will be systematically done in this stage. Finally, many of new problems, which can not be solved by the current methods, will appear, and the enormous resistance causes the advancing motivity to exhaust. In this stage, if not adopting the positive measures, making the innovations and looking for the new developing way, the end may be occurring now. To sum up

● Any of new industries or group of enterprises will suffer a difficult process to venture carve out the way.

● An industry or group of enterprises will come into a faster recession after developing rapidly if not make more efforts to increase motivity for advancing.

This process could be applied to analyze the many things in their evolution, including the social and economic evolution.

5 The Empirical Researches

In the following empirical researches, we take US GDP (chained) price index⑤ (the GDP index for short) in the period of 1940-2005 as the data samples. $G(t)$ is the real interest (measured by the value of GDP index) at the year t, t=1940, 1941, · · ·, 2005. The error is valuated by the following formula:

$$\varepsilon=\sqrt{\frac{\sum_{t=1941}^{2005}\left[F(t)-G(t)\right]^2}{2005-1940}}$$, where $F(t)$ is the estimated value.

Because $G(1940)=0.0978$, $G(1941)=0.1014$, the initial values is determined as $\mu_L(0)=G(1941)=0.1014$, $\sigma_L(0)=G(1941)-G(1940)=0.0036$⑥. Suppose $\delta=0.3$, $\phi=0.02$, $\kappa=0.01$, $b=0.5$, $\theta=2.0$. From the section 4.3.2 and 4.3.2, we have $\mu=\frac{\mu_L(0)+\phi}{1-\delta}$, $\sigma=\frac{\sigma_L(0)+r\kappa}{1-\delta}$, $\mu_L(t)=(1-\delta)\mu e^{\lambda t}-\phi e^{\psi t}$ and $\sigma_L(t)=(1-\delta)\sigma e^{\lambda t}\sqrt{1+bt}-r\kappa e^{\psi t}(1+bt)^{\frac{\theta}{2}}$.

5.1 The fitting analysis on advance-retreat course in average mode

For years t=1941, · · ·, 2005, according to the average mode, the real net interest is $E[L(t)]=\mu_L(t)+0.2869947990\sigma_L(t)$ and the real motivity is $\sqrt{D[L(t)]}=0.7941279409\sigma_L(t)$.

Based on the iterative algorithm (20) and by debugging the parameter λ and ψ to make the error ε minimum, we obtain:

The natural growth rate $\boldsymbol{\lambda}$=0.048, and the growth rate of real exogenous cost $\boldsymbol{\psi}$=0.0675.

The reverse time of real net interest (The time from increasing to decreasing) $\boldsymbol{T}$=2014.455458.

The coefficient of net resistance $\boldsymbol{r}$=0.01455840745.

The estimated error $\boldsymbol{\varepsilon}$=0.04636972318.

The basic interest $\boldsymbol{\mu}$=0.1734285714 and basic motivity σ=0.005350834393.

Letting $X(t)=E[L(t)]=\mu_L(t)+0.2869947990\sigma_L(t)$ and by use of the algorithm in appendix, we obtain T_1=2031.740771 and $E[L(T_1)]=0$, i.e., T_1=2031.740771is the time when advance-retreat course finishes. The results of fitting and related computation are drawn in figure 3.

Remark 4 From the fitting analysis on advance-retreat course in average mode, we have the following results:

⑤ The source of US GDP (chained) price index (Fiscal Year 2000 = 1.000) is Web, http://www.whitehouse.gov.

⑥ $\sigma(0)$—The initial fluctuating range of GDP, $\sigma(0)=|\mu(1)-\mu(0)|$. Because the interval unit of GDP data is one year, GDP data can reflect stably the economic status, so the $\sigma(0)=|\mu(1)-\mu(0)|$ can describe nicely economic fluctuation.

(i) $\sigma_L(T)=\sigma_L(2014.455458)=0$ is the solution of stochastic course $L(t)$ in the average mode. T=2014.455458 indicates the course may turn at 2014, i.e. US economy may turn from economic growth to economic recession. In order to avoid the possible economic recession, US government should put the effective reforms in practice in social and economic fields before the year 2014.

(ii) The data of US GDP index have contained all the motivities and resistances. According to the computing results, US GDP index with resistance-free should be μ=0.1734285714 under the related assumptions, and the real consumed cost, the value of GDP index consumed by resistances, should be μ-$\mu_L(0)$=0.0720285714. Similarly, the US economic motivity with resistance-free should be σ=0.005350834393, and the real resistances should be σ-$\sigma_L(0)$=0.001750834393.

(iii) If the external environments around economy include motivity and resistance only, i.e., $\alpha+\beta=1$, and appended to $r=\alpha-\beta$=0.01455840745, we have α=0.4927207963, β=0.5072792035. This means the resistance factors exceed 50%. The growth rate of US economy may be higher if minishing the resistances effectively.

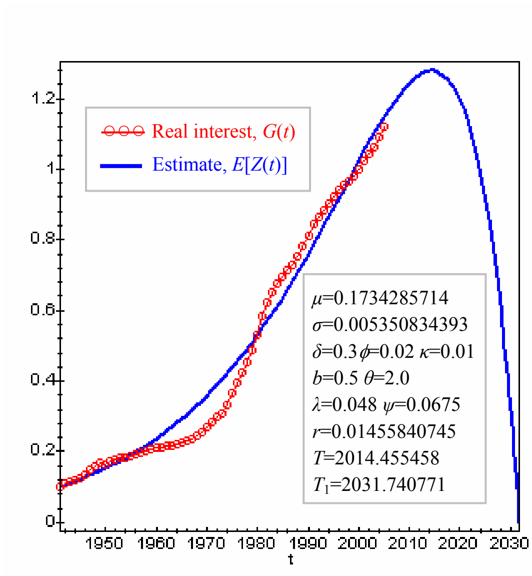

Figure 3 Fitting US GDP by use of average mode

To fit US GDP index by use of the average mode, $E[L(t)]=\mu_L(t)+0.2869947990\sigma_L(t)$, The estimated error is ε=0.04636972318. The results indicate that US economy may reach its top at the metaphase of year 2014.

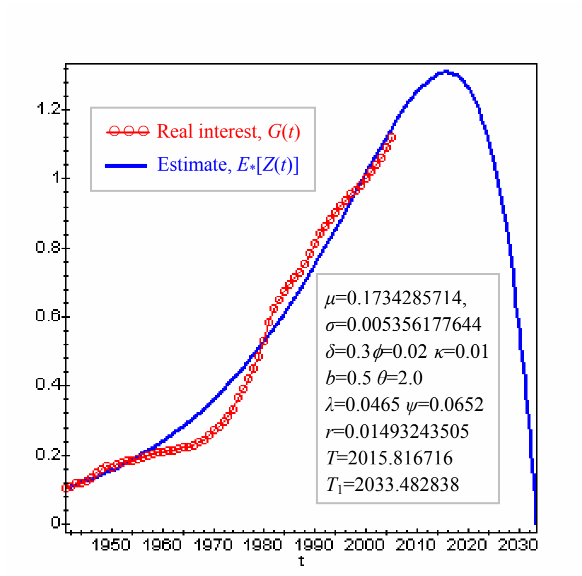

Figure 4 Fiting US GDP by use of high-speed mode

To fit US GDP index by use of the high speed mode, $E_*[L(t)]=\mu_L(t)+0.618033989\sigma_L(t)$, The estimated error is ε=0.04756066471. The results indicate that US economy may reach its top at the metaphase of year 2015.

5.2 The fitting analysis on advance-retreat course in high-speed mode

For years t=1941, · · ·, 2005, according to the high-speed mode, the real net interest is $E_*[L(t)]=\mu_L(t)+0.618033989\sigma_L(t)$ and the real motivity is $\sqrt{D_*[L(t)]}=0.8603639530\sigma_L(t)$.

Based on the iterative algorithm (20) and by debugging the parameter λ and ψ to make the error ε minimum, we obtain:

The natural growth rate $\boldsymbol{\lambda}$=0.0465, and the growth rate of real exogenous cost $\boldsymbol{\psi}$=0.0652.

The reverse time of real net interest (The time from increasing to decreasing) $\boldsymbol{T}$=2015.816716.

The coefficient of net resistance $\boldsymbol{r}$=0.01493243505.

The estimated error $\boldsymbol{\varepsilon}$=0.04756066471.

The basic interest $\boldsymbol{\mu}$=0.1734285714 and basic motivity σ=0.005356177644.

Letting $X(t)=E_*[L(t)]=\mu_L(t)+\ 0.618033989\sigma_L(t)$ and by use of the algorithm in appendix, we obtain T_1=2033.482838 when advance-retreat course finishes. The results of fitting and related computation are drawn in figure 4.

Similar to remark 4, have

Remark 5 From the fitting analysis on advance-retreat course in average mode, we have the following results:

(i) $\sigma_L(T)=\sigma_L(2015.816716)=0$ is the solution of stochastic course $L(t)$ in the high-speed mode. T=2015.816716 indicates US economy may turn from economic growth to economic recession at 2015. In order to avoid the possible economic recession, US government should put the effective reforms in practice in social and economic fields before the year 2015.

(ii) The data of US GDP index have contained all the motivities and resistances. According to the computing results, US GDP index with resistance-free should be μ=0.1734285714 under the related assumptions, and the real consumed cost, the value of GDP index consumed by resistances, should be $\mu-\mu_L(0)$=0.0720285714. Similarly, the US economic motivity with resistance-free should be σ=0.005356177644, and the real resistances should be $\sigma-\sigma_L(0)$=0.001756177644.

(iii) If the external economic environments include only the motivity and the resistance, i.e., $\alpha+\beta=1$, and appended to $r=\alpha-\beta$=0.01493243505, so we know the exogenous economic resistance is 1.493243505% larger than the exogenous economic motivity in US.

5.3 The comparison analysis between two fitting results

Comparing the computing data between modes in average and high-speed, we get the table 1.

Table 1 Comparing the data between the two modes

	Expressions estimated		Basic parameters			Time parameters (year)		
	Net interest	Net motivity	Growth index	Accumulated parameters	Net resistance	Turning time	End time of course	error
Average	$E[L(t)]=\mu_L(t)+0.28700\sigma_L(t)$	$\sqrt{D_*[L(t)]}=0.79413\sigma_L(t)$	λ=0.0480 ψ=0.0675	b=0.5 θ=2.0	r=0.01456	T= 2014.45546	T_1= 2031.74077	ε=0.04637
High-speed	$E_*[L(t)]=\mu_L(t)+0.61803\sigma_L(t)$	$\sqrt{D_*[L(t)]}=0.86036\sigma_L(t)$	λ=0.0465 ψ=0.0652	b=0.5 θ=2.0	r=0.01493	T= 2015.81672	T_1= 2033.48284	ε=0.04756

Explanation: The data is exact to five decimal places.

From table 1, the following comparing results are obtained as:

- The growth coefficient of high-speed mode, 0.618033989, is larger than the The growth coefficient of average mode, 0.2869947990. This allows the natural growth rates of high-speed mode (λ=0.0465, ψ=0.0652) are smaller than these of average mode (λ=0.0480, ψ=0.0675).
- The net resistance coefficient of high-speed mode, r=0.01493243505, is larger than that of average mode, r=0.01455840745. This is caused by the quicker growing of interest in high-speed mode.
- In high-speed mode, the turning time of interest growth, T=2015.816716, and the end time of course, T_1=2033.482838, are larger than these in average mode, T=2014.455458 and T_1=2031.740771. This indicates that if the basic net interest $\mu_L(0)$ and basic net motivity $\sigma_L(0)$ of high-speed mode are separately equal to these of average mode, interest growth will keep on for a longer time in high-speed mode because the interest in high-speed is growing quicker than that in average mode.

● The fitting error in high-speed mode (ε=0.04756066471) is larger than that in average mode (ε=0.04636972318), this means US economic process may be closer to the average mode than the high-speed. Because λ=0.0480 under the average mode, so we could deduce the average risk-free interest rate of US is about 4.8% from 1941 to 2005.

● It is the key period to US in its economic growth round about 2015. At this time, the US economic process may change from growth to recession.

6 The Conclusions

This paper has done the works as follow:

● Based on the works (Dai, 2006), the basic theory and conceptual model of advance-retreat analysis have been completed further (section 2.1).

● The basic concepts of the optimal route and the optimal course have been produced (section 2.2).

● The stochastic process model and its the solving method for advance-retreat course are given (section 4.3).

● The empirical researches has been carried out (section 5).

The basic and important conclusions are obtained as:

● The necessary condition under which the optimal route and the optimal course are existing (theorem 3 and corollary 2).

● The basic condition to determine the subject to go forward or backward (proposition 1).

● The smaller the net exogenous resistance is, the longer the duration for subject interests' growing is (remark 1).

● All the larger endogenous resistance, the smaller growth rates of interest or the larger basic endogenous motivity may cause periodic fluctuation in motivity and interest (remark 2). So, it is benefit to economic growth to increase properly the risk-free rate of interest or make the basic endogenous motivity smaller properly.

● The quicker increasing in interest will shorten the time for interest to increase (remark 3). So the stable and not too quick increase in interest will bring the longer and higher return.

● The quicker the increase in interest is, the quicker the increase in resistance is (remark 3).

In the empirical analysis, we see that the US economic status reflected by GDP index is the result in which the developing motivity and the resistance are blent and mixed each other, and it is the difference between resistance-free GDP index and the part consumed by resistances. Therefore, we can analyse and calculate the interest level without resistance and the consumed cost based on the actual data and the methods given in this paper. Further more, we can find how many the resistances are and where the resistances are. All of these are very useful to invest and for government to promulgate policies.

In reality, a lot of subjects existed in one or more industry at the same time, and their growth cycle and recession cycle are different interweaved each other. Though the growth or recession of the single subject can not determine these of the whole trade or industry, but its influence exists. In the larger span of time, the trade or industry takes on the same characters in its course as one subject to a certain extent. So, in order to keep on growing in interest, it is important for single subject or whole industry to choose appropriate opportunity to reform, innovate and invest. If like this, single subject or whole industry will get effectively the developing motivity again. In this meaning, the time, on which is the solution of an advance-retreat course, will be more important and worth to be paid attention to.

This paper has only discussed the stochastic advance-retreat course with a single stage and no branch route. Also, it is worth to study the advance-retreat course with many stages, many branch routes and investment.

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Appendix: The algorithm for computing the root of equation $\boldsymbol{X}(\boldsymbol{t})$=0.

Let $t_0=0$, choose a larger $t_1>0$ to make $X(t_1)<0$, and give a smaller $\varepsilon>0$.

1° Evaluating: $T \leftarrow t_1$.

2° Computing: $X(T)=0$.

3° If $|X(T)-0|<\varepsilon$, end. Otherwise, let $t_1=T$ if $X(T)<0$ or let $t_0=T$ if $X(T)>0$.

4° Evaluating: $T \leftarrow \frac{t_0+t_1}{2}$, go to 2°.