Estimation and Inference for High Dimensional Factor Model with Regime Switching

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Abstract

This paper proposes maximum (quasi)likelihood estimation for high dimensional factor models with regime switching in the loadings. The model parameters are estimated jointly by the EM (expectation maximization) algorithm, which in the current context only requires iteratively calculating regime probabilities and principal components of the weighted sample covariance matrix. When regime dynamics are taken into account, smoothed regime probabilities are calculated using a recursive algorithm. Consistency, convergence rates and limit distributions of the estimated loadings and the estimated factors are established under weak cross-sectional and temporal dependence as well as heteroscedasticity. It is worth noting that due to high dimension, regime switching can be identified consistently after the switching point with only one observation. Simulation results show good performance of the proposed method. An application to the FRED-MD dataset illustrates the potential of the proposed method for detection of business cycle turning points.

Keywords: Factor model, Regime switching, Maximum likelihood, High dimension, EM algorithm, Turning points

JEL Classification: C13, C38, C55

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1 Introduction

A great deal of attention has focused on the loading instability issue in high dimensional factor models. For empirical evidences of parameter instability in macroeconomic and financial time series, see for example, Banerjee, Marcellino and Masten (2008), Stock and Watson (2009) and Korobilis (2013). Several procedures are proposed to detect and/or estimate common abrupt breaks in the loadings, including Cheng, Liao and Shorfheide (2016), Baltagi, Kao and Wang (2017, 2021), Bai, Han and Shi (2020), and Ma and Su (2018), to mention a few. Other models of time varying loadings, such as i.i.d./random walk, smooth change, vector autoregression and threshold type, are studied in Bates, Plagborg-Moller, Stock and Watson (2013), Su and Wang (2017), Mikkelsen, Hillebrand and Urga (2019) and Massacci (2017), respectively.

An alternative approach of modeling loading instability is common regime switching. In business cycle analysis, several unobservable factors summarize the comovements of many economic variables and the loadings measure the importance of factors for each economic variable. The importance of each factor may be different depending on fiscal policy (expansionary, contractionary, neutral), or monetary policy (expansionary, contractionary), or the stage of the business cycle (peak, trough, expansion, contraction), hence the loadings may switch synchronously between several states under different scenarios. In stock return analysis, the loadings measure the impact of the factor return on the expected return of each individual stock, hence the loadings may switch synchronously depending on the stock market scenarios (bull versus bear markets, high versus low volatility), see for example Gu (2005) and Guidolin and Timmermann (2008) for related discussions. In bond return analysis, the yields of bonds with different maturities are well captured by the level factor, the slope factor and the curvature factor, see for example Cochrane and Piazzesi (2005) and Diebold and Li (2006). The importance of each factor could be different depending on the stock market volatility, or the stage of the business cycle, or the unemployment rate, hence the loadings may also switch synchronously according to these state variables. In general, large factor models with regime switching in the loadings could also be useful for other topics, such as tracking labor productivity.

There are only a few related results on large factor models with regime switching in the loadings. Liu and Chen (2016) proposes an iterative algorithm for estimating the model parameters and the hidden states based on eigen-decomposition and the Viterbi algorithm, however, the asymptotic properties of the estimated parameters are established only when the true states are known. Considering loadings as general functions of some recurrent states, Pelger and Xiong (2021) develops nonparametric kernel estimator for the loadings and the factors, and establishes the relevant asymptotic theory. However, Pelger and Xiong (2021) requires observable state variables. In general, state variables may be misspecified or unobservable.

This paper proposes maximum (quasi)likelihood estimation for high dimensional factor models with regime switching in the loadings when the state variables are unobservable. This paper also proposes new criteria to consistently determine the number of regimes and the number of factors in each regime. The model parameters are estimated jointly by the EM algorithm, which in the current context only requires calculating principal components iteratively.

More specifically, in the E-step, the probabilities of each regime at each time t are calculated based on the observed data and the parameter values at the current iteration using a recursive algorithm modified from Hamilton (1990), and then the joint (log)likelihood of the observed data and the unobserved states are averaged with respect to the calculated regime probabilities. In the M-step, the estimated loadings for each regime are the principal components of the weighted sample covariance matrix of the observed time series, where the weight on x_t (the observed time series at time t) equals the probability of that regime at time t. Since principal components can be easily calculated even when N (the dimension of time series) is large, our method is very easy to implement.

For the proposed algorithm, this paper establishes the convergence rates of the estimated loading spaces and the estimated factor spaces, the limit distributions of the estimated loadings and the estimated factors, the consistency of the estimated regime probabilities, and the consistency of the estimated transition probability matrix when the true state process is Markovian. Note that asymptotic analysis under the regime switching setup is more difficult than under the structural break setup, because the pattern of regimes for the latter is much simpler.

These asymptotic results are essential in many empirical contexts. First, the limit distributions of the estimated factors allow us to construct confidence intervals for the true factors, which represent economic indices in many applications. The result on the estimated factor spaces implies that if the estimated factors are used in factor-augmented forecasting (or factor-augmented VAR), the forecasting equation (or the VAR equation) would have induced regime switching in the model parameters. Second, for asset management, the estimated loadings of each regime allow us to construct portfolios according to each specific market scenario. For structural dynamic factor analysis, consistently estimated loadings are also crucial for recovering the impulse responses. Third, the consistency of the estimated regime probabilities implies that for each x_t, we can consistently identify which regime x_t belongs to as $N \to \infty$. For asset management, this allows us to consistently identify the current market scenario. For business cycle analysis, this allows us to consistently date turning points of the business cycle and detect new recessions or expansions, especially when high frequency (weekly, daily) data is utilized.

For cases with small N, various methods have been proposed for estimating factor models with regime switching. Kim (1994) proposes approximate Kalman filter for likelihood evaluation and uses nonlinear optimization for likelihood maximization. Kim and Yoo (1995) and Chauvet (1998) apply Kim (1994)'s method to a small number of economic series and obtain recession probabilities and turning points very close to the official NBER dates. Kim (1994) allows for regime switching in both the factor mean and the factor loadings, but when N is large, Kim (1994)'s method would be very time consuming and may have convergence problems[1]. Other methods, such as Diebold and Rudebusch (1996) and Kim and Nelson (1998), assume stable loadings and only focus on regime switching in the factor mean. If the loadings are unstable, these methods are not applicable. More importantly, if there is only regime switching in the factor mean, we can not consistently identify each regime even when

[1]This is because the number of parameters grows proportionally to N and the likelihood function is calculated numerically and maximized by nonlinear optimization algorithm.

N is large.

In contrast with Kim (1994), our method is fast and easy to implement even when N is very large. The crucial point behind our EM algorithm is to ignore factor dynamics[2] and integrate out the factors in the likelihood function. If factors dynamics are taken into account or factors are treated as parameters in the likelihood function, the estimated loadings would not be the principal components of the weighted sample covariance matrix, and consequently both the algorithm and the asymptotic analysis would become infeasible. On the other hand, the efficiency loss of ignoring factor dynamics is small when N is large.

This paper may also contribute to the literature on dating turning points of the business cycle. Currently there are two main approaches for dating business cycle using multiple time series. The first approach, aggregating then dating, is to date business cycle by focusing on a few highly aggregated time series such as GDP, industrial production and nonfarm employment. The second approach, dating then aggregating, is to date turning points in each disaggregated series and then aggregate these turning points in some appropriate way, see Burns and Mitchell (1946), Harding and Pagan (2006) and Chauvet and Piger (2008). These papers only use a small number of time series. Stock and Watson (2010, 2014) studies this issue using many time series. This paper shows that it is possible to consistently identify turning points if regime switching is synchronous and N is large enough. If N is small, consistency is not possible no matter how large T is. This paper also shows that if N is large, it is possible to consistently detect regime switching right after the turning point with only one observation, thus the speed of detection could be improved significantly. If N is small, we have to wait for enough observations from the new regime.

The rest of the paper is organized as follows. Section 2 introduces the model setup and the estimation procedures. Section 3 presents the assumptions and the asymptotic results. Section 4 proposes criteria for determining the number of regimes and the number of factors in each regime. Section 5 presents simulation results.

[2] Factor dynamics are still allowed for the data generating process.

Section 6 presents an empirical application of the proposed method to the FRED-MD dataset. Section 7 concludes. All proofs are relegated to the appendix.

Through out the paper, $(N, T) \to \infty$ denotes N and T going to infinity jointly, $\delta_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$. $\xrightarrow{p}$ and $\xrightarrow{d}$ denotes convergence in probability and convergence in distribution, respectively. For matrix A, let $\|A\|$, $\|A\|_F$, $\rho_{\max}(A)$ and $\rho_{\min}(A)$ denote its spectral norm, Frobenius norm, largest eigenvalue and smallest eigenvalue, respectively. Let $P_A = A(A'A)^{-1}A'$ denote the projection matrix and $M_A = I - P_A$. "w.p.a.1" denotes with probability approaching one.

2 Identification and Estimation

Consider the following factor model with regime switching: for $i = 1, ..., N$ and $t = 1, ..., T$,

$$x_{it} = f_t^{0\prime}\lambda_{ji}^0 + e_{it} \text{ if } z_t = j, \tag{1}$$

where λ_{ji}^0 is an r_j^0 dimensional vector of loadings for regime j, f_t^0 is an $r_{z_t}^0$ dimensional vector of factors, z_t is the state variable indicating which regime x_{it} belongs to, and e_{it} is the error term allowed to have cross-sectional and temporal dependence as well as heteroscedasticity. x_{it} is observable and all of the right hand side variables are unobservable. The number of regimes J^0 and the number of factors in each regime r_j^0 (could be different across j) are fixed as $(N, T) \to \infty$ and assumed to be known in this section and Section 3. How to consistently determine r_j^0 and J^0 will be studied in Section 4.

The factor process $\{f_t^0, t = 1, ..., T\}$ is allowed to be dynamic, and similar to the principal component estimator (PCE) in Bai (2003) and the maximum likelihood estimator (MLE) in Bai and Li (2012, 2016), factor dynamics are ignored when estimating the model parameters, thus there is no need to model factor dynamics.

For the state process $\{z_t, t = 1, ..., T\}$, the asymptotic results in Section 3.2 and Section 4 are valid as long as $\frac{1}{T}\sum_{t=1}^{T} 1_{z_t=j} \xrightarrow{p} q_j^0 > 0$ for $j = 1, ..., J^0$ ($q_j^0 = \Pr(z_t = j)$ is the unconditional probability of regime j and $1_{z_t=j} = 1$ if $z_t = j$ and 0 otherwise), and Assumptions 1-3 and 5-7 in Section 3.1 hold conditioning on $\{z_t, t = 1, ..., T\}$.

Thus $\{z_t, t = 1, ..., T\}$ is allowed to be correlated with f_s^0 and e_{is} for all i and s, and we do not need to know the true model of $\{z_t, t = 1, ..., T\}$.

In vector form, the model can be written as:

$$x_t = \Lambda_j^0 f_t^0 + e_t \text{ if } z_t = j, \text{ for } t = 1, ..., T, \tag{2}$$

where $\Lambda_j^0 = (\lambda_{j1}^0, ..., \lambda_{jN}^0)'$, $x_t = (x_{1t}, ..., x_{Nt})'$ and $e_t = (e_{1t}, ..., e_{Nt})'$. Let $\Lambda^0 = (\Lambda_1^0, ..., \Lambda_{J^0}^0)$ and let $E = (e_1, ..., e_T)'$ be the $T \times N$ matrix of errors. When there are no superscripts, Λ_j and Λ denote parameters as variables.

2.1 Identification

Since the factors are unobservable, regimes are defined in terms of the linear spaces spanned by the loadings. Two regimes are different if their loading spaces are different, and vice versa. More specifically, the identification condition is: for any j and k,

$$\min_t \frac{1}{N} \left\| M_{\Lambda_k^0} \Lambda_j^0 f_t^0 \right\|^2 = \min_t \frac{1}{N} f_t^{0\prime} \Lambda_j^{0\prime} M_{\Lambda_k^0} \Lambda_j^0 f_t^0 \geq C \text{ for some } C > 0. \tag{3}$$

A sufficient condition for (3) is:

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \Lambda_k^{0\prime} M_{\Lambda_j^0} \Lambda_k^0 \text{ is positive definite for any } j \text{ and } k, \\ &\text{and } \min_t \|f_t\| \text{ is nonzero.} \end{aligned} \tag{4}$$

Condition (4) requires $\lim\limits_{N\to\infty} \frac{1}{N}(\Lambda_j^0, \Lambda_k^0)'(\Lambda_j^0, \Lambda_k^0)$ to be full rank for any j and k. Thus Λ_j^0 and Λ_k^0 are not allowed to share some columns, and columns of Λ_j^0 could not be linear combination of Λ_k^0 and vice versa. An alternative sufficient condition for (3) is:

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \Lambda_k^{0\prime} M_{\Lambda_j^0} \Lambda_k^0 \neq 0 \text{ for any } j \text{ and } k, \\ &\text{and } \min_t \left| g_{jk}' f_t \right| \text{ is nonzero,} \end{aligned} \tag{5}$$

where g_{jk} is the eigenvector of $\lim\limits_{N\to\infty} \frac{1}{N} \Lambda_k^{0\prime} M_{\Lambda_j^0} \Lambda_k^0$ corresponding to nonzero eigenvalue. Condition (5) only requires that the linear spaces spanned by Λ_j^0 and Λ_k^0 are different. Thus Λ_k^0 and Λ_j^0 are allowed to share some columns, and some columns of Λ_k^0 are allowed to be linear combinations of the columns of Λ_j^0 and vice versa, but Λ_k^0 is not allowed to be a subset of Λ_j^0. For example, if there are two regimes with two factors in each regime and only the loadings of f_{2t} (the second factor) switch across the regimes, then condition (5) requires that $\min_t |f_{2t}|$ is nonzero.

Note that condition (4) does not rule out the possibility that any regime j can be further decomposed into multiple regimes. Suppose the true model is $x_t = \Lambda_j^0 f_t^0 + e_t$ if $z_t = j$, $j = 1, 2, 3$, and $(\Lambda_1^0, \Lambda_2^0, \Lambda_3^0)$ satisfies condition (4). If we consider Λ_1^0 as the first regime and $(\Lambda_2^0, \Lambda_3^0)$ as the second regime, the true model can be equivalently written as $x_t = \Lambda_1^0 f_t^0 + e_t$ if $z_t = 1$, and $x_t = (\Lambda_2^0, \Lambda_3^0) f_t^* + e_t$ if $z_t = 2$ *or* 3, where $f_t^* = (f_t^{0\prime}, 0')'$ if $z_t = 2$ and $f_t^* = (0', f_t^{0\prime})'$ if $z_t = 3$. The equivalent model also satisfies condition (4). However, while $plim \frac{1}{T} \sum_{z_t = 2 \ or \ 3} f_t^* f_t^{*\prime}$ is positive definite, $plim \frac{1}{T} \sum_{z_t = 2} f_t^* f_t^{*\prime}$ and $plim \frac{1}{T} \sum_{z_t = 3} f_t^* f_t^{*\prime}$ are not positive definite. To rule out the possibility that any regime j can be further decomposed, we assume that

$$plim \frac{1}{|A_j|} \sum_{t \in A_j} f_t^0 f_t^{0\prime} \text{ is positive definite,} \tag{6}$$

where A_j denotes any subset of $\{t : z_t = j\}$ with cardinality $|A_j|$ and $\lim \frac{|A_j|}{T} > 0$. If $\frac{1}{|A_j|} \sum_{t \in A_j} f_t^0 f_t^{0\prime}$ is not positive definite as $T \to \infty$ for some A_j, then A_j and $\{t : z_t = j, t \notin A_j\}$ are considered as two separate regimes.

2.2 First Order Conditions

Consider the following log-likelihood function for Gaussian mixture in covariance:

$$l(\Lambda, \sigma^2) = \log[\sum_{z_T=1}^{J^0} \dots \sum_{z_1=1}^{J^0} \prod_{t=1}^{T} L(x_t \left| z_t; \Lambda, \sigma^2 \right.) \Pr(z_1, \dots, z_T)], \tag{7}$$

where $\prod_{t=1}^{T} L(x_t \left| z_t; \Lambda, \sigma^2 \right.)$ is the density of $(x_1, \dots, x_T)$ conditioning on $(z_1, \dots, z_T)$, $\Pr(z_1, \dots, z_T)$ is the joint probability of $(z_1, \dots, z_T)$,

$$L(x_t \left| z_t = j; \Lambda_j, \sigma^2 \right.) = (2\pi)^{-\frac{N}{2}} \left| \Sigma_j \right|^{-\frac{1}{2}} e^{-\frac{1}{2} x_t' \Sigma_j{}^{-1} x_t}, \tag{8}$$

Σ_j is the covariance matrix of x_t for regime j, and

$$\Sigma_j = \Lambda_j \Lambda_j' + \sigma^2 I_N. \tag{9}$$

The above log-likelihood function avoids estimating the factors. If the factors are estimated jointly with the loadings, we would not have the analytical first order conditions presented below, and consequently the EM algorithm would become infeasible.

Equation (7) is a misspecified log-likelihood function. First, the state process $\{z_t, t = 1, ..., T\}$ is not specified yet, and the probability $\Pr(z_1, ..., z_T)$ depends on how we model the state process. Second, similar to the principal component estimator in Stock and Watson (2002) and Bai (2003), equation (9) ignores the cross-sectional and serial dependence and heteroscedasticity of the error term. We may also take into account the heteroscedasticity as Doz, Giannone and Reichlin (2012) and Bai and Li (2012, 2016). With regime switching, the algorithm and the asymptotic analysis would be much more complicated, but the results should be conceptually similar.

Third, the factor dynamics are ignored. As shown in Bai (2003) for PCE and in Bai and Li (2012, 2016) for MLE, when there is no regime switching, the asymptotic properties of the estimated factors and the estimated loadings are robust to the presence of the factor dynamics if both N and T are large. We shall show in Section 3 that when there is regime switching, the asymptotic results are also robust to the presence of the factor dynamics. More importantly, ignoring the factor dynamics greatly simplifies the computation algorithm for regime switching factor models. As shown below, with factor dynamics ignored, we just need to calculate principal components iteratively. If the factor dynamics are not ignored, Kim (1994)'s method would be very time consuming and may have convergence problems if N is large[3].

Fourth, equation (7) implicitly assumes that $\mathbb{E}(f_t^0) = 0$ and $\mathbb{E}(f_t^0 f_t^{0\prime})$ is stable within each regime, and $\mathbb{E}(f_t^0 f_t^{0\prime})$ is absorbed into $\Lambda_j \Lambda_j'$ in equation (9). This does

[3]When there is no regime switching, as suggested by Doz et al. (2012), large N factor model with factor dynamics can be calculated by the EM algorithm. However, when there are both regime switching and factor dynamics, the EM algorithm also fails. This is because in the E-step we need to calculate the likelihood for each possible state chain $z_1, ..., z_T$ and there are $(J^0)^T$ possibilities, and in the M-step numerical optimization is still needed.

not matter, since all results of this paper still hold when $\mathbb{E}(f_t^0) \neq 0$ and $\mathbb{E}(f_t^0 f_t^{0\prime})$ is unstable within regime, as long as Assumption 1 is satisfied.

First order conditions for Λ and σ^2

The parameters Λ and σ^2 are estimated by maximizing $l(\Lambda, \sigma^2)$. Define $x_{1:t} \equiv (x_1, ..., x_t)$ and $z_{1:t} \equiv (z_1, ..., z_t)$, and let $p_{tj|T} \equiv \Pr(z_t = j \,|x_{1:T}; \Lambda, \sigma^2)$ denote the probability of $z_t = j$ conditional on $x_{1:T}$. Based on equation (7), it can be easily verified that

$$\begin{aligned} \frac{\partial l(\Lambda, \sigma^2)}{\partial \Lambda_j} &= \sum\nolimits_{t=1}^{T} \frac{\partial \log L(x_t \,|z_t = j; \Lambda_j, \sigma^2)}{\partial \Lambda_j} p_{tj|T} \\ &= \sum\nolimits_{t=1}^{T} p_{tj|T} (-\Sigma_j^{-1}\Lambda_j + \Sigma_j^{-1} x_t x_t' \Sigma_j^{-1} \Lambda_j), && (10) \\ \frac{\partial l(\Lambda, \sigma^2)}{\partial \sigma^2} &= \sum\nolimits_{t=1}^{T} \sum\nolimits_{j=1}^{J^0} \frac{\partial \log L(x_t \,|z_t = j; \Lambda_j, \sigma^2)}{\partial \sigma^2} p_{tj|T}\,, && (11) \end{aligned}$$

where equation (10) follows from

$$\begin{aligned} \frac{\partial \log|\Sigma_j|}{\partial \Lambda_j} &= 2\Sigma_j^{-1}\Lambda_j, && (12) \\ \frac{\partial x_t' \Sigma_j^{-1} x_t}{\partial \Lambda_j} &= -2\Sigma_j^{-1} x_t x_t' \Sigma_j^{-1} \Lambda_j, && (13) \end{aligned}$$

see Chapter 14.3 in Andersen (2003) for the details on calculating these derivatives. Set $\frac{\partial l(\Lambda,\sigma^2)}{\partial \Lambda_j}$ to 0, we have

$$\begin{aligned} \Sigma_j^{-1}\Lambda_j &= \Sigma_j^{-1} S_j \Sigma_j^{-1} \Lambda_j, && (14) \\ \text{and } S_j &= \sum\nolimits_{t=1}^{T} p_{tj|T}\, x_t x_t' / \sum\nolimits_{t=1}^{T} p_{tj|T}\,. \end{aligned}$$

S_j can be considered as sample covariance matrix for Σ_j based on importance sampling. The weights $p_{tj|T} / \sum_{t=1}^{T} p_{tj|T}$ depend on the importance of the sample x_t for regime j, the larger $p_{tj|T}$ is, the more important x_t is for regime j.

From equation (9), we have $\Sigma_j \Lambda_j = \Lambda_j(\Lambda_j'\Lambda_j + \sigma^2 I_{r_j^0})$. Left multiply $S_j \Sigma_j^{-1}$ on both sides, we have $S_j \Lambda_j = S_j \Sigma_j^{-1} \Lambda_j (\Lambda_j'\Lambda_j + \sigma^2 I_{r_j^0})$. From equation (14), we have $\Lambda_j = S_j \Sigma_j^{-1} \Lambda_j$, thus

$$S_j \Lambda_j = \Lambda_j(\Lambda_j'\Lambda_j + \sigma^2 I_{r_j^0}). \qquad (15)$$

If Λ_j is a solution for equation (15) and Λ_j^* equals post-multiplying Λ_j by the eigenvector matrix of $\Lambda_j'\Lambda_j$, then Λ_j^* is also a solution for equation (15) and $\Lambda_j^{*\prime}\Lambda_j^*$ is diagonal. Thus we can directly choose the solution Λ_j with $\Lambda_j'\Lambda_j$ being diagonal. It follows that the solution Λ_j is the eigenvectors of S_j and $\Lambda_j'\Lambda_j + \sigma^2 I_{r_j^0}$ is the corresponding eigenvalues. We show in Appendix G that σ^2 satisfies the following condition:

$$\sigma^2 = \frac{1}{N} tr(\frac{1}{T}\sum\nolimits_{t=1}^{T} x_t x_t' - \sum\nolimits_{j=1}^{J^0} \frac{1}{T}\sum\nolimits_{t=1}^{T} p_{tj|T}\Lambda_j\Lambda_j'). \tag{16}$$

Note that we do not need to specify the state process $\{z_1, ..., z_T\}$ when deriving the first order conditions (15) and (16), and different models of $\{z_1, ..., z_T\}$ correspond to different ways of calculating $p_{tj|T}$. In the EM algorithm presented below, we consider $\{z_1, ..., z_T\}$ as a Markov process regardless of what the true process of $\{z_1, ..., z_T\}$ is.

2.3 EM Algorithm

Let $q^0 = (q_1^0, ..., q_{J^0}^0)'$ denote the unconditional regime probabilities, $\phi^0 = (\phi_1^0, ..., \phi_{J^0}^0)'$ denote the initial probabilities of z_1, Q^0 denote the $(J^0 \times J^0)$ matrix of transition probabilities and Q_{jk}^0 denote the probability of switching from state k to state j. If there are no superscripts, q, Q and ϕ denote parameters as variables.

For any given Q and ϕ, at the h-th iteration, let $\tilde{\Lambda}^{(h)}$ denote the estimated loadings, $\tilde{\sigma}^{2(h)}$ denote the estimated variance, and $\Pr(z_1, ..., z_T \left| x_{1:T}; \tilde{\theta}^{(h)}\right.)$ denote the probability of $z_{1:T}$ conditioning on $x_{1:T}$ and evaluated at $\tilde{\theta}^{(h)} = (\tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)}, Q, \phi)$. The EM algorithm maximizes the expectation of the log-likelihood of $(x_{1:T}, z_{1:T})$ with respect to $\Pr(z_1, ..., z_T \left| x_{1:T}; \tilde{\theta}^{(h)}\right.)$, i.e.,

$$\begin{aligned} l^{(h)}(\Lambda, \sigma^2, Q, \phi) \equiv & \sum\nolimits_{z_T=1}^{J^0} ... \sum\nolimits_{z_1=1}^{J^0} \log[\prod\nolimits_{t=1}^{T} L(x_t \left| z_t; \Lambda, \sigma^2\right.) \Pr(z_1, ..., z_T \left| Q, \phi\right.)] \\ & \times \Pr(z_1, ..., z_T \left| x_{1:T}; \tilde{\theta}^{(h)}\right.). \end{aligned}$$

Considering z_t as a Markov process, $\Pr(z_1, ..., z_T \left| Q, \phi\right.) = \Pr(z_1 \left| \phi\right.) \prod_{t=2}^{T} \Pr(z_t \left| z_{t-1}; Q\right.)$.

Thus

$$
\begin{aligned}
l^{(h)}(\Lambda,\sigma^2,Q,\phi) &= \sum_{z_T=1}^{J^0}\dots\sum_{z_1=1}^{J^0}[\sum_{t=1}^{T}\log L(x_t\,|z_t;\Lambda,\sigma^2\,) \\
&\quad+\sum_{t=2}^{T}\log\Pr(z_t\,|z_{t-1};Q\,)+\log\Pr(z_1\,|\phi)]\Pr(z_1,...,z_T\,\Big|x_{1:T};\tilde{\theta}^{(h)}) \\
&= \sum_{t=1}^{T}\sum_{j=1}^{J^0}\log L(x_t\,|z_t=j;\Lambda_j,\sigma^2\,)\tilde{p}^{(h)}_{tj|T} \\
&\quad+\sum_{t=2}^{T}\sum_{j=1}^{J^0}\sum_{k=1}^{J^0}\log Q_{jk}\tilde{p}^{(h)}_{tjk|T}+\sum_{k=1}^{J^0}\log\phi_k\tilde{p}^{(h)}_{1k|T}\,, \qquad (17)
\end{aligned}
$$

where $\tilde{p}^{(h)}_{tjk|T} = \Pr(z_t = j, z_{t-1} = k\,\Big|x_{1:T};\tilde{\theta}^{(h)})$ and $\tilde{p}^{(h)}_{tj|T} = \Pr(z_t = j\,\Big|x_{1:T};\tilde{\theta}^{(h)}) = \sum_{k=1}^{J^0}\tilde{p}^{(h)}_{tjk|T}$ are the smoothed probabilities based on $x_{1:T}$ and $\tilde{\theta}^{(h)}$. Appendix H presents a recursive algorithm for calculating $\tilde{p}^{(h)}_{tjk|T}$. From equations (12) and (13), we have $\frac{\partial\log L(x_t\,|z_t=j;\Lambda_j,\sigma^2\,)}{\partial\Lambda_j} = -\Sigma_j^{-1}\Lambda_j+\Sigma_j^{-1}x_tx_t'\Sigma_j^{-1}\Lambda_j$. Thus

$$
\frac{\partial\sum_{t=1}^{T}\log L(x_t\,|z_t=j;\Lambda_j,\sigma^2\,)\tilde{p}^{(h)}_{tj|T}}{\partial\Lambda_j}=\sum_{t=1}^{T}(-\Sigma_j^{-1}\Lambda_j+\Sigma_j^{-1}x_tx_t'\Sigma_j^{-1}\Lambda_j)\tilde{p}^{(h)}_{tj|T}=0,
$$

and it follows that

$$
\begin{aligned}
\Sigma_j^{-1}\Lambda_j &= \Sigma_j^{-1}\tilde{S}_j^{(h)}\Sigma_j^{-1}\Lambda_j, \qquad (18)\\
\text{where } \tilde{S}_j^{(h)} &= \sum_{t=1}^{T}\tilde{p}^{(h)}_{tj|T}x_tx_t'/\sum_{t=1}^{T}\tilde{p}^{(h)}_{tj|T}\,.
\end{aligned}
$$

Similar to equation (15), equation (18) implies that

$$
\tilde{S}_j^{(h)}\tilde{\Lambda}_j^{(h+1)}=\tilde{\Lambda}_j^{(h+1)}(\tilde{\Lambda}_j^{(h+1)\prime}\tilde{\Lambda}_j^{(h+1)}+\tilde{\sigma}^{2(h+1)}I_{r_j^0}), \qquad (19)
$$

thus the columns of $\tilde{\Lambda}_j^{(h+1)}$ are the eigenvectors of $\tilde{S}_j^{(h)}$ and the diagonal elements of $\tilde{\Lambda}_j^{(h+1)\prime}\tilde{\Lambda}_j^{(h+1)}+\tilde{\sigma}^{2(h+1)}I_{r_j^0}$ are the corresponding eigenvalues. To save space, we show in Appendix G that

$$
\tilde{\sigma}^{2(h+1)}=\frac{1}{N}tr(\frac{1}{T}\sum_{t=1}^{T}x_tx_t'-\sum_{j=1}^{J^0}\frac{1}{T}\sum_{t=1}^{T}\tilde{p}^{(h)}_{tj|T}\tilde{\Lambda}_j^{(h+1)}\tilde{\Lambda}_j^{(h+1)\prime}). \qquad (20)
$$

Remark 1 *The second equality of equation (17) is crucial. Since factor dynamics are ignored,* $L(x_{1:T}\,|z_{1:T};\Lambda,\sigma^2\,)=\prod_{t=1}^{T}L(x_t\,|z_t;\Lambda,\sigma^2\,)$*, thus we only need to calculate*

$\tilde{p}^{(h)}_{tj|T}$ *rather than the probability of the whole chain* $\Pr(z_1, ..., z_T \left| x_{1:T}; \tilde{\theta}^{(h)} \right.)$. *The latter requires* $(J^0)^T$ *calculations, which is hopeless when* T *is large. If factor dynamics are not ignored, then* $L(x_{1:T} | z_{1:T}; \Lambda, \sigma^2) = L(x_1 | z_{1:T}; \Lambda, \sigma^2) \prod_{t=2}^{T} L(x_t | x_{1:t-1}, z_{1:T}; \Lambda, \sigma^2)$. $L(x_t | x_{1:t-1}, z_{1:T}; \Lambda, \sigma^2)$ *depends on the chain* $(z_1, ..., z_T)$ *through* $z_{1:t}$, *thus we need to calculate* $\Pr(z_{1:t} \left| x_{1:T}; \tilde{\theta}^{(h)} \right.)$. *This requires* $(J^0)^t$ *calculations, which is hopeless when* t *is large.*

EM algorithm for Λ and σ^2

Choose any Q *and* ϕ *such that* $Q_{jk} > 0$ *for any* j *and* k *and* $\phi_k > 0$ *for all* k. *Start from randomly generated initial values of* $\tilde{\Lambda}^{(0)}$ *and* $\tilde{\sigma}^{2(0)} = 1$. *For* $h = 0, 1, ...,$

(E-step): calculate $\tilde{p}^{(h)}_{tjk|T}$ *using the algorithm in Appendix H, and calculate* $\tilde{S}^{(h)}_j = \sum_{t=1}^{T} \tilde{p}^{(h)}_{tj|T} x_t x_t' / \sum_{t=1}^{T} \tilde{p}^{(h)}_{tj|T}$ with $\tilde{p}^{(h)}_{tj|T} = \sum_{k=1}^{J^0} \tilde{p}^{(h)}_{tjk|T}$;

(M-step): given $\tilde{p}^{(h)}_{tjk|T}$ *and* $\tilde{S}^{(h)}_j$, *calculate* $\tilde{\Lambda}^{(h+1)}_j$ *as the eigenvectors of* $\tilde{S}^{(h)}_j$ *corresponding to the* r^0_j *largest eigenvalues, and then normalize* $\tilde{\Lambda}^{(h+1)}_j$ *such that* $\left\| \tilde{\Lambda}^{(h+1)}_{jl} \right\|^2 + \tilde{\sigma}^{2(h+1)}$ *equals the* l*-th largest eigenvalue of* $\tilde{S}^{(h)}_j$ *for* $l = 1, ..., r^0_j$ *and equation (20) is also satisfied, where* $\tilde{\Lambda}^{(h+1)}_{jl}$ *is the* l*-th column of* $\tilde{\Lambda}^{(h+1)}_j$. *Note that the computation of* $\left\| \tilde{\Lambda}^{(h+1)}_{jl} \right\|^2$ *and* $\tilde{\sigma}^{2(h+1)}$ *requires iteration between equations (19) and (20).*

Iterate the E-step and the M-step until converge. Let $\tilde{\Lambda}_j = (\tilde{\lambda}_{j1}, ..., \tilde{\lambda}_{jN})'$, $\tilde{\Lambda} = (\tilde{\Lambda}_1, ..., \tilde{\Lambda}_{J^0})$ *and* $\tilde{\sigma}^2$ *denote the estimated parameters, and let* $\tilde{p}_{tj|T}$ *and* $\tilde{p}_{tjk|T}$ *denote the smoothed probabilities based on* $x_{1:T}$ *and* $(\tilde{\Lambda}, \tilde{\sigma}^2, Q, \phi)$.

A special case of the above EM algorithm is when we choose $\phi = q$ and $Q = q1'_{J^0}$ (1_{J^0} denotes the $J^0 \times 1$ vector of ones), i.e., we consider $\{z_1, ..., z_T\}$ as an independent process. For this case, the computation of $\tilde{p}^{(h)}_{tj|T}$ is simplified because the unsmoothed regime probabilities can be calculated directly by

$$\tilde{p}^{(h)}_{tj|T} = q_j L(x_t \left| z_t = j; \tilde{\Lambda}^{(h)}_j, \tilde{\sigma}^{2(h)} \right.) / \sum_{k=1}^{J^0} q_k L(x_t \left| z_t = k; \tilde{\Lambda}^{(h)}_k, \tilde{\sigma}^{2(h)} \right.).$$

This case is preferable if we knew the true process of $\{z_1, ..., z_T\}$ is independent. In practice, since the state process of the business cycle/stock market is highly persistent, smoothed regime probabilities that capture the persistence should perform significantly better, especially when mixed frequency data or ragged edge data (data released at non-synchronized dates) are used. The asymptotic results in Section 3.2 and Section 4 hold for any Q and ϕ as long as $\phi_j > 0$ for any j and $Q_{jk} > 0$ for any j and k, i.e., they hold for both the smoothed algorithm and the unsmoothed algorithm.

If the true process of $\{z_1, ..., z_T\}$ is Markovian, Q^0_{jk} and ϕ^0_k can be estimated by

$$\tilde{Q}_{jk} = \sum\nolimits_{t=2}^{T} \tilde{p}_{tjk|T} / \sum\nolimits_{j=1}^{J^0} \sum\nolimits_{t=2}^{T} \tilde{p}_{tjk|T}, \tag{21}$$

$$\tilde{\phi}_k = \tilde{p}_{1k|T} = \sum\nolimits_{j=1}^{J^0} \tilde{p}_{2jk|T}. \tag{22}$$

We can also plug $\tilde{Q}_{jk}$ and $\tilde{\phi}_k$ back in the above EM algorithm and iterate between $(\tilde{\Lambda}, \tilde{\sigma}^2)$ and $(\tilde{Q}, \tilde{\phi})$ until convergence. This is the maximum likelihood estimator when (Q, ϕ) is estimated jointly with (Λ, σ^2), see Appendix G for details.

The asymptotic results in Section 3.2 and Section 4 also hold as long as $\tilde{\sigma}^2$ is bounded and bounded away from zero in probability. Consistency of $\tilde{\sigma}^2$ is not needed. We could restrict $\tilde{\sigma}^2$ in $[\frac{1}{C^2}, C^2]$ for some large C or simply fix down $\tilde{\sigma}^2 = 1$ to avoid the iteration between $\tilde{\Lambda}_j^{(h+1)}$ and $\tilde{\sigma}^{2(h+1)}$. This only affects the Euclidean norm of the columns of $\tilde{\Lambda}_j^{(h+1)}$.

Remark 2 *Pelger and Xiong (2021) also considers the model*[4] $x_t = \Lambda(z_t) f_t^0 + e_t$. *The state variable z_t is discrete and unobservable in this paper, while in Pelger and Xiong (2021) z_t is continuous and observable. Also, in this paper $\tilde{\Lambda}_j$ are eigenvectors of $\tilde{S}_j = \frac{1}{\sum_{t=1}^T \tilde{p}_{tj|T}} \sum_{t=1}^T \tilde{p}_{tj|T} x_t x_t'$, while in Pelger and Xiong (2021) $\hat{\Lambda}(s)$ are eigenvectors of $\frac{1}{\sum_{t=1}^T K_s(z_t)} \sum_{t=1}^T K_s(z_t) x_t x_t'$, where $K_s(z_t) = \frac{1}{h} K(\frac{z_t - s}{h})$ is the kernel function. The key difference is that the weight $K_s(z_t)$ is observable because z_t is observable in Pelger and Xiong (2021), but in this paper the weight $\tilde{p}_{tj|T}$ is unobservable and need to be estimated jointly with Λ_j.*

Remark 3 *We can take into account cross-sectional heteroscedasticity as Bai and Li (2012, 2016) by replacing equation (9) by $\Sigma_j = \Lambda_j \Lambda_j' + \Sigma_e$, where Σ_e is a $N \times N$*

[4] We changed Pelger and Xiong (2021)'s notation to our notation for better comparison.

diagonal matrix. We show in Appendix G that the first order conditions are

$$\Sigma_e^{-\frac{1}{2}} S_j \Sigma_e^{-1} \Lambda_j = \Sigma_e^{-\frac{1}{2}} \Lambda_j (\Lambda_j' \Sigma_e^{-1} \Lambda_j + I_{r_j^0}), \tag{23}$$

$$\Sigma_e = diag(\frac{1}{T} \sum\nolimits_{t=1}^{T} x_t x_t' - \sum\nolimits_{j=1}^{J^0} \frac{1}{T} \sum\nolimits_{t=1}^{T} p_{tj|T} \Lambda_j \Lambda_j'), \tag{24}$$

i.e., columns of $\Sigma_e^{-\frac{1}{2}} \Lambda_j$ *are the eigenvectors of* $\Sigma_e^{-\frac{1}{2}} S_j \Sigma_e^{-\frac{1}{2}}$ *and diagonal elements of* $\Lambda_j' \Sigma_e^{-1} \Lambda_j + I_{r_j^0}$ *are the corresponding eigenvalues. Accordingly, in the M-step of the EM algorithm, we iterate*

$$\tilde{\Sigma}_e^{-\frac{1}{2}(h)} \tilde{S}_j^{(h)} \tilde{\Sigma}_e^{-1(h)} \tilde{\Lambda}_j^{(h+1)} = \tilde{\Sigma}_e^{-\frac{1}{2}(h)} \tilde{\Lambda}_j^{(h+1)} (\tilde{\Lambda}_j^{(h+1)\prime} \tilde{\Sigma}_e^{-1(h)} \tilde{\Lambda}_j^{(h+1)} + I_{r_j^0}),$$

$$\text{and } \tilde{\Sigma}_e^{(h+1)} = diag(\frac{1}{T} \sum\nolimits_{t=1}^{T} x_t x_t' - \sum\nolimits_{j=1}^{J^0} \frac{1}{T} \sum\nolimits_{t=1}^{T} \tilde{p}_{tj|T}^{(h)} \tilde{\Lambda}_j^{(h+1)} \tilde{\Lambda}_j^{(h+1)\prime}).$$

The other steps of the EM algorithm remain unchanged. If we further take into account cross-sectional dependence, then $\Sigma_j = \Lambda_j \Lambda_j' + \Sigma_e$ *and* Σ_e *is non-diagonal. It can be verified that for this case equation (23) is still valid, but equation (24) is not. Since* Σ_e *is of dimension* $N \times N$ *and* $N \to \infty$ *jointly with* T*, certain sparsity condition has to be imposed on* Σ_e *to consistently estimate* Σ_e*. Results on this topic are very rare (if any) even for factor model with single regime.*

2.4 Estimate the Factors

If the factor dynamics are taken into account, the expectation of f_t conditioning on $x_{1:t}$ is

$$\sum\nolimits_{z_1=1}^{J^0} ... \sum\nolimits_{z_t=1}^{J^0} \mathbb{E}(f_t \Big| x_{1:t}, z_{1:t}; \tilde{\Lambda}, \tilde{\sigma}^2) \Pr(z_{1:t} \Big| x_{1:t}; \tilde{\Lambda}, \tilde{\sigma}^2, Q, \phi),$$

which is formidable since we need to calculate $\Pr(z_{1:t} \Big| x_{1:t}; \tilde{\Lambda}, \tilde{\sigma}^2, Q, \phi)$ for each possible $z_{1:t}$, i.e., we need to calculate $(J^0)^t$ probabilities. For large N, the benefit of considering factor dynamics is marginal and outweighed by the computational simplicity of ignoring factor dynamics. If the factor dynamics are ignored, the expectation of f_t conditioning on $x_{1:T}$ is

$$\begin{aligned}\tilde{f}_t &= \sum_{j=1}^{J^0} \mathbb{E}(f_t \Big| x_{1:T}, z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2)\tilde{p}_{tj|T} = \sum_{j=1}^{J^0} \mathbb{E}(f_t \Big| x_t, z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2)\tilde{p}_{tj|T} \\ &= \sum_{j=1}^{J^0} \tilde{\Lambda}_j'(\tilde{\Lambda}_j\tilde{\Lambda}_j' + \tilde{\sigma}^2 I_N)^{-1} x_t \tilde{p}_{tj|T}. \end{aligned} \tag{25}$$

Note that the dimension of $\tilde{\Lambda}_j'(\tilde{\Lambda}_j\tilde{\Lambda}_j' + \tilde{\sigma}^2 I_N)^{-1} x_t$ is different across j if r_j^0 is different across j. Here and also in the proof of Theorem 5, when we add two vectors of different dimensions, we implicitly augment the vector of smaller dimension with zeros to make the dimensions of these two vectors equal. Thus $\tilde{f}_t$ is a $\max r_j^0$ dimensional vector.

3 Asymptotic Results

3.1 Assumptions

We assume the following conditions hold as $(N, T) \to \infty$. These conditions are mainly Assumptions A-G in Bai (2003) adapted to the current regime switching setup.

Assumption 1 *(1) For $j = 1, ..., J^0$, $\frac{1}{Tq_j^0}\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j} \xrightarrow{p} \Sigma_{F_j}$ for some positive definite Σ_{F_j}, and $\text{plim}\frac{1}{|A_j|}\sum_{t\in A_j} f_t^0 f_t^{0\prime}$ is also positive definite, where A_j is defined in section 2.1.*

(2) For some $\alpha > 16$, there exists $M > 0$ such that $\mathbb{E}(\|f_t^0\|^\alpha) \leq M$ for all t.

Assumption 1 corresponds to Assumption A in Bai (2003). Assumption 1(1) rules out the possibility that for regime j, the subsample $\{t : z_t = j\}$ can be further decomposed into multiple regimes, see the discussion in Section 2.1. The factor process is allowed to be dynamic such that $C(L)f_t = \epsilon_t$. Assumption 1(2) assumes that the factors have bounded moments.

Assumption 2 *(1) For $j = 1, ..., J^0$, $\frac{1}{N}\Lambda_j^{0\prime}\Lambda_j^0 \to \Sigma_{\Lambda_j}$ for some positive definite Σ_{Λ_j} and $\left\|\lambda_{ji}^0\right\| \leq M$ for any $i = 1, ..., N$.*

(2) For any $j = 1, ..., J^0$ and $k = 1, ..., J^0$, $\min_t \frac{1}{N} f_t^{0\prime}\Lambda_j^{0\prime} M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \geq C$ for some $C > 0$.

Assumption 2(1) corresponds to Assumption B in Bai (2003). Assumption 2(1) ensures that each factor has a nontrivial contribution within each regime, and $\left\|\lambda_{ji}^0\right\|$ is assumed to be uniformly bounded over i. Assumption 2(2) is the identification condition for determining which regime each x_t belongs to, see Section 2.1 for details on the implication of this condition.

Assumption 3 *(1)* $\mathbb{E}(e_{it}) = 0$, $\mathbb{E}(e_{it}^{\alpha}) \leq M$ *for some* $\alpha > 16$.

(2) $\sum_{k=1}^{N} \tau_{ik} \leq M$ *for any* i*, where* $\mathbb{E}(e_{it}e_{kt}) = \tau_{ik,t}$ *with* $|\tau_{ik,t}| \leq \tau_{ik}$ *for some* $\tau_{ik} > 0$ *and for all* t*.*

(3) $\sum_{s=1}^{T} \gamma_{ts} \leq M$ *for all* t*, where* $E(e_{it}e_{is}) = \gamma_{i,ts}$ *with* $\left|\gamma_{i,ts}\right| \leq \gamma_{ts}$ *for some* $\gamma_{ts} > 0$ *and for all* i*.*

(4) $\mathbb{E}(\left\|\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (e_{it}e_{kt} - \mathbb{E}(e_{it}e_{kt}))1_{z_t=j}\right\|^2) \leq M$ *for all* $i = 1, ..., N$*,* $k = 1, ..., N$ *and* $j = 1, ..., J^0$*.*

Assumption 3 is modified slightly from Assumption C in Bai (2003). The error term is allowed to have limited cross-sectional and serial dependence as well as heteroscedasticity.

Assumption 4 *For* $j = 1, ..., J^0$*,* $\frac{1}{T} \sum_{t=1}^{T} 1_{z_t=j} \xrightarrow{p} q_j^0$ *and* $0 < q_j^0 < 1$*.*

The asymptotic results in Section 3.2 and Section 4 are valid as long as Assumption 4 holds and the other assumptions in this section hold conditioning on $\{z_t, t = 1, ..., T\}$. Thus the state process $\{z_t, t = 1, ..., T\}$ is allowed to be non-Markovian and correlated with f_s^0 and e_{is} for all i and s. Knowledge of the true state process is not needed.

Assumption 5 *(1)For some* $\beta \geq 2$*,* $\mathbb{E}(\left\|\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{ji}^0 e_{it}\right\|^{\beta}) \leq M$ *for all* $j = 1, ..., J^0$ *and all* t*.*

(2) $\mathbb{E}(\left\|\frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t^0 e_{it} 1_{z_t=j}\right\|^2) \leq M$ *for all* $j = 1, ..., J^0$ *and all* i*.*

Assumption 5 is modified slightly from Assumption D in Bai (2003). Assumption 5(1) assumes that the errors are weakly correlated across i for each t. When $\beta = 2$, Assumption 5(1) is implied by Assumptions 2(1), 3(1) and 3(2). Assumption 5(2) assumes that the errors are weakly correlated across t for each i. Assumption 5(2) is implied by Assumptions 1(2), 3(1) and 3(4) if we further assume the factors are nonrandom or independent with the errors.

Assumption 6 *For each $j = 1, ..., J^0$, the eigenvalues of $\Sigma_{\Lambda_j}^{\frac{1}{2}} \Sigma_{F_j} \Sigma_{\Lambda_j}^{\frac{1}{2}}$ are different.*

Assumption 6 corresponds to Assumption G in Bai (2003). With Assumption 6, the loadings and the factors are identifiable up to a rotation. For identification of the loading space and the factor space, Assumption 6 is not needed.

Assumption 7 *(1) $\mathbb{E}(\left\| \frac{1}{\sqrt{NT}} \sum_{k=1}^{N} \sum_{t=1}^{T} \lambda_i^0 (e_{it}e_{kt} - \mathbb{E}(e_{it}e_{kt})) 1_{z_t=j} \right\|^2) \leq M$ for all $i = 1, ..., N$ and $j = 1, ..., J^0$; and $\mathbb{E}(\left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} (e_{it}e_{is} - \mathbb{E}(e_{it}e_{is})) f_t^0 1_{z_t=j} \right\|^2) \leq M$ for all $s = 1, ..., T$ and $j = 1, ..., J^0$.*

(2) $\mathbb{E}(\left\| \frac{1}{\sqrt{NT}} \sum_{k=1}^{N} \sum_{t=1}^{T} \lambda_k^0 f_t^{0\prime} e_{kt} 1_{z_t=j} \right\|^2) \leq M$ for $j = 1, ..., J^0$.

(3) Define $\Phi_{ji} = plim \frac{1}{T} \sum_{s=1}^{T} \sum_{t=1}^{T} \mathbb{E}(f_t^0 f_s^{0\prime} e_{is} e_{it} 1_{z_s=j} 1_{z_t=j})$. For $j = 1, ..., J^0$, $\frac{1}{\sqrt{Tq_j^0}} \sum_{t=1}^{T} f_t^0 e_{it} 1_{z_t=j} \xrightarrow{d} \mathcal{N}(0, \Phi_{ji})$.

(4) Define $\Gamma_{jt} = lim \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{N} \lambda_{ji}^0 \lambda_{jk}^0 \mathbb{E}(e_{it}e_{kt})$. For $j = 1, ..., J^0$, $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{ji}^0 e_{it} \xrightarrow{d} \mathcal{N}(0, \Gamma_{jt})$.

Assumption 7 corresponds to Assumption F in Bai (2003). Part (3) and part (4) are just central limit theorems and will be used for deriving the limit distributions of the estimated factors and loadings.

3.2 Asymptotic Results

Consistency of the estimated loading space

Theorem 1 *Under Assumptions 1, 2(1), 3 and 4, $\frac{1}{N} \left\| M_{\tilde{\Lambda}_j} \Lambda_j^0 \right\|_F^2 = O_p(\frac{1}{\sqrt{\delta_{NT}}})$ for each j as $(N, T) \to \infty$.*

Theorem 1 shows that the estimated loading space is consistent without observing the state variable z_t. Note that the estimated loadings $\tilde{\Lambda}_j$ and the estimated regime probabilities $\tilde{p}_{tj|T}$ depend on each other, but the standard technique in Bai (2003) for analyzing $\tilde{\Lambda}_j$ is applicable only when $\tilde{p}_{tj|T} = 1_{z_t=j}$. This is the first technical difficulty we encounter in going from one regime to multiple regimes. The crucial point for Theorem 1 is that if the linear spaces spanned by $\tilde{\Lambda}_j$ and Λ_j^0 differ too much, as long as $\min \phi_j > 0$ and $\min Q_{jk} > 0$, the likelihood of $\tilde{\Lambda}$ would be smaller than the likelihood of Λ^0 uniformly over all possible $\{z_1, ..., z_T\}$, i.e.,

$$\begin{aligned} e^{l(\tilde{\Lambda},\tilde{\sigma}^2)} &\leq \sup_{\{z_1,...,z_T\}} \prod_{t=1}^{T} L(x_t \left| z_t; \tilde{\Lambda}, \tilde{\sigma}^2 \right.) \\ &< \sum_{z_T=1}^{J^0} ... \sum_{z_1=1}^{J^0} \prod_{t=1}^{T} L(x_t \left| z_t; \Lambda^0, \tilde{\sigma}^2 \right.) \Pr(z_1, ..., z_T) = e^{l(\Lambda_j^0,\tilde{\sigma}^2)}. \end{aligned}$$

This crucial point is due to large N, see the Appendix for the formal proof. Based on Theorem 1, we show that the estimated regime probabilities are consistent.

Consistency of the estimated regime probabilities

Theorem 2 *Under Assumptions 1-4 and 5(1), as $(N,T) \to \infty$, for each j and for any fixed $\eta > 0$,*

(1) $\sup_t \left| \tilde{p}_{tj|T} - 1_{z_t=j} \right| = o_p(\frac{1}{N^\eta})$ *if* $T^{\frac{16}{\alpha}}/N \to 0$ *and* $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$,

(2) $\left| \tilde{p}_{tj|T} - 1_{z_t=j} \right| = o_p(\frac{1}{N^\eta})$.

Note that η could be large but it is fixed as $(N,T) \to \infty$. α and β could also be large as long as Assumptions 1(2), 3(1) and 5(1) are satisfied. Theorem 2 shows that $\tilde{p}_{tj|T}$ is consistent as $N \to \infty$ and is uniformly consistent if T is relatively small compared to N. The proof utilizes the exponential likelihood ratio.

Theorem 2 implies that we can consistently identify which regime x_t belongs to for all t, if there is common regime switching in the loadings and the dimension of x_t tends to infinity. Theorem 2 also implies that we can consistently detect regime switching right after the turning point with only one observation, so that we do not need to wait for many observations of the time series from the new regime. This could improve the speed of detection of new turning points, especially when high frequency data is used.

An interesting special case is when the proposed algorithm is applied to factor models with common breaks in the loadings. Various methods are proposed recently for estimating the break points, Theorem 2 implies that we can also consistently estimate the break points using the proposed EM algorithm.

Convergence rate of the estimated loading space

If the true states z_t were known, asymptotic properties of the estimated loadings and factors are straightforward. Based on Theorem 2, we shall show that using estimated regime probabilities does not affect the asymptotic results. Define $W_{jNT} = \frac{1}{N}(\tilde{\Lambda}_j'\tilde{\Lambda}_j + \tilde{\sigma}^2 I_{r_j^0})(\frac{1}{T}\sum_{t=1}^{T}\tilde{p}_{tj|T})$ and $H_j = \frac{\sum_{t=1}^{T} f_t^0 f_t^{0\prime} 1_{z_t=j}}{T}\frac{\Lambda_j^{0\prime}\tilde{\Lambda}_j}{N}W_{jNT}^{-1}$, then we have:

Proposition 1 *Let V_j be a $r_j^0 \times r_j^0$ diagonal matrix consisting of eigenvalues of $\Sigma_{\Lambda_j}^{\frac{1}{2}}\Sigma_{F_j}\Sigma_{\Lambda_j}^{\frac{1}{2}}$ in descending order and Υ_j be the corresponding eigenvectors. Under Assumptions 1-6, and assume $T^{\frac{16}{\alpha}}/N \to 0$ and $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$, as $(N,T) \to \infty$,*

(1) $W_{jNT} \xrightarrow{p} q_j^0 V_j$ for each j,

(2) $H_j \xrightarrow{p} \Sigma_{\Lambda_j}^{-\frac{1}{2}}\Upsilon_j V_j^{\frac{1}{2}}$ for each j.[5]

Proposition 1 is an important auxiliary result, and part (1) and part (2) corresponds to Lemma A.3 and Proposition 1 in Bai (2003), respectively. Lemma A.3 in Bai (2003) is based on the fact that the estimated factors are $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues[6] of XX' and consequently $\tilde{\Lambda}'\tilde{\Lambda}$ is a diagonal matrix consisting of the r largest eigenvalues of $\frac{1}{T}\sum_{t=1}^{T} x_t x_t'$. However, here the first order condition (15) only tells us the columns of $\tilde{\Lambda}_j$ are the eigenvectors of S_j and $\tilde{\Lambda}_j'\tilde{\Lambda}_j + \tilde{\sigma}^2 I_{r_j^0}$ are the corresponding eigenvalues. Condition (15) does not tells us whether these eigenvalues are the r_j^0 largest eigenvalues of S_j or not. This is the second technical difficulty we encounter in going from one regime to multiple regimes. Our proof strategy of Proposition 1 utilizes Theorem 1 and is totally different from Bai (2003)'s proof for his Proposition 1.

Theorem 3 *Under Assumptions 1-6, and assume $T^{\frac{16}{\alpha}}/N \to 0$ and $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$, as $(N,T) \to \infty$, $\frac{1}{N}\left\|\tilde{\Lambda}_j - \Lambda_j^0 H_j\right\|_F^2 = O_p(\frac{1}{\delta_{NT}^2})$ for each j.*

[5] H_j corresponds to $(H^{-1})'$ for the rotation matrix H in Bai (2003).

[6] r denotes the number of factors in Bai (2003).

Theorem 3 establishes the convergence rate of the estimated loading space for each regime. This could help us study the effect of using estimated loadings on subsequent applications. For example, if the estimated loadings are used to construct portfolios, Theorem 3 could help us calculate how the estimation error contained in $\tilde{\Lambda}_j$ would affect the performance of these portfolios.

Limit distributions of the estimated loadings

Theorem 4 *Under Assumptions 1-7, and assume* $\sqrt{T}/N \to 0$, $T^{\frac{16}{\alpha}}/N \to 0$ *and* $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$, *as* $(N,T) \to \infty$, $\sqrt{Tq_j^0}(\tilde{\lambda}_{ji} - H_j'\lambda_{ji}^0) \xrightarrow{d} \mathcal{N}(0, V_j^{-\frac{1}{2}}\Upsilon_j'\Sigma_{\Lambda_j}^{\frac{1}{2}}\Phi_{ji}\Sigma_{\Lambda_j}^{\frac{1}{2}}\Upsilon_j V_j^{-\frac{1}{2}})$ *for each* j.

Theorem 4 shows that for each j and i, $\tilde{\lambda}_{ji}$ has a limiting normal distribution. This allows us to construct confidence interval for the estimated loadings. Also note that the rotation matrix H_j is different for different regime.

Remark 4 *We can also prove the consistency and limit distribution of* $\tilde{\sigma}^2$ *(the probability limit of* $\tilde{\sigma}^2$ *is* $\lim\limits_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \sigma_i^2$*), we omit it since this is not our focus.*

Asymptotic properties of the estimated factors

Theorem 5 *Under Assumptions 1-7, and assume* $\sqrt{N}/T \to 0$, $T^{\frac{16}{\alpha}}/N \to 0$ *and* $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$, *as* $(N,T) \to \infty$,

(1) $\frac{1}{T}\sum_{t=1}^T \left\| \tilde{f}_t - [(H_{z_t}^{-1} f_t^0)', 0'_{\max r_j^0 - r_{z_t}^0}]' \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$,

(2) $\sqrt{N}(\tilde{f}_t - [(H_{z_t}^{-1} f_t^0)', 0'_{\max r_j^0 - r_{z_t}^0}]')$

$$\xrightarrow{d} \mathcal{N}(0, \begin{bmatrix} V_{z_t}^{-\frac{1}{2}}\Upsilon_{z_t}'\Sigma_{\Lambda_{z_t}}^{-\frac{1}{2}}\Gamma_{z_t t}\Sigma_{\Lambda_{z_t}}^{-\frac{1}{2}}\Upsilon_{z_t}V_{z_t}^{-\frac{1}{2}} & 0_{r_{z_t}^0 \times (\max r_j^0 - r_{z_t}^0)} \\ 0_{(\max r_j^0 - r_{z_t}^0)\times r_{z_t}^0} & 0_{(\max r_j^0 - r_{z_t}^0)\times(\max r_j^0 - r_{z_t}^0)} \end{bmatrix}).$$

Theorem 5(2) shows that the limit distribution of $\tilde{f}_t$ is mixed normal, since the rotation matrix $H_{z_t}^{-1}$ and the asymptotic variance depend on the state variable z_t. Theorem 5(1) establishes the convergence rate of the estimated factor space. Note that if $\{\tilde{f}_t, t = 1, ...T\}$ is used as proxies for the true factors in factor-augmented forecasting (or factor-augmented VAR), the forecasting equation (or the VAR equation) would have induced regime switching in the model parameters, because $H_{z_t}^{-1}$ depends on z_t. For illustration, consider the following h-period ahead forecasting model using factors and some other observable variables W_t: $y_{t+h} = a' f_t^0 + b' W_t + u_{t+h}$. If $\tilde{f}_t$ is used as proxies for f_t^0, the model can be written as

$$y_{t+h} = -a' H_{z_t} (\tilde{f}_t - H_{z_t}^{-1} f_t^0) + a' H_{z_t} \tilde{f}_t + b' W_t + u_{t+h}.$$

The first term on the right hand side is asymptotically negligible. It is easy to see that the coefficient $a' H_{z_t}$ depends on z_t and this need to be taken into account when we estimate the forecasting equation. Finally, we show that the estimated transition probability matrix is also consistent when $\{z_1, ..., z_T\}$ is a Markov process.

Theorem 6 *Assume that $\{z_1, ..., z_T\}$ is a Markov process, under Assumptions 1-4 and 5(1), $\tilde{Q}_{jk} \xrightarrow{p} Q_{jk}^0$ for each j and k as $(N, T) \to \infty$ if $T^{\frac{16}{\alpha}}/N \to 0$ and $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$.*

4 Determine the Number of Factors and the Number of Regimes

Given the number of factors $(r_1, ..., r_J)$ and the number of regimes J, let $(\tilde{\Lambda}_{1,r_1}, ..., \tilde{\Lambda}_{J,r_J})$ be the solution for maximizing the log-likelihood $l(\Lambda_{1,r_1}, ..., \Lambda_{J,r_J}, \sigma^2, Q, \phi)$. Here we use Λ_{j,r_j} to emphasize that Λ_{j,r_j} is of dimension $N \times r_j$. The criterion we propose for model selection is:

$$PC(r_1, ..., r_J) = \frac{1}{NT} l(\tilde{\Lambda}_{1,r_1}, ..., \tilde{\Lambda}_{J,r_J}, \sigma^2, Q, \phi) - \sum\nolimits_{j=1}^{J} (g(N,T))^{b(r_j)}, \tag{26}$$

where $g(N,T)$ is a penalty function depending on both N and T, and $b(\cdot)$ is a positive and decreasing function with $b(1) = 1$, e.g., $b(r_j) = \frac{1}{r_j}$. For each J, the numbers of factors are estimated by

$$(\tilde{r}_1, ..., \tilde{r}_J) = \arg\max\nolimits_{r_j \le \bar{r}, j=1,...,J} PC(r_1, ..., r_J), \tag{27}$$

and then the number of regimes is estimated by

$$\tilde{J} = \arg\max_{J \leq \bar{J}} PC(\tilde{r}_1, ..., \tilde{r}_J), \tag{28}$$

where $\bar{r}$ is the maximal number of factors in each regime and $\bar{J}$ is the maximal number of regimes. In the following theorem we show that $(\tilde{r}_1, ..., \tilde{r}_J)$ and $\tilde{J}$ are consistent.

Theorem 7 *Under Assumptions 1, 2(1), 3, 4 and assume* $\lim_{N\to\infty} \frac{1}{N}\Lambda_k^{0\prime} M_{\Lambda_j^0}\Lambda_k^0 \neq 0$ *for any* j *and* k*, we have* $\Pr(\tilde{J} = J^0$ *and* $\tilde{r}_j = r_j^0$ *for all* $j) \to 1$ *as* $(N,T) \to \infty$ *if (i)* $g(N,T) \to 0$*, (ii)* $\delta_{NT} g(N,T) \to \infty$*, and (iii)* $b(\cdot)$ *is a positive and decreasing function with* $b(1) = 1$*.*

Note that the condition $\lim_{N\to\infty} \frac{1}{N}\Lambda_k^{0\prime} M_{\Lambda_j^0}\Lambda_k^0 \neq 0$ allows Λ_k^0 and Λ_j^0 to share some columns, i.e., Theorem 7 holds for the case where the loadings of some (but not all) factors remain the same across different regimes.

The basic idea behind Theorem 7 is similar to Theorem 2 of Bai and Ng (2002), i.e., add a penalty term that converges to zero but slowly enough so that underparameterized models and overparameterized models will not be chosen. Here the penalty $(g(N,T))^{b(r_j)}$ converges to zero because $g(N,T) \to 0$ and $b(r_j)$ is positive, and $\delta_{NT}(g(N,T))^{b(r_j)} \to \infty$ because $\delta_{NT} g(N,T) \to \infty$ and $b(r_j)$ is a decreasing function of r_j with $b(1) = 1$. Compared to Bai and Ng (2002), the difference and difficulty here is that the number of regimes is unknown and the number of factors in each regime may be different. For example, suppose the true model is $(r_1 = 2, r_2 = 1, J = 2)$ and the two columns in Λ_1^0 are linearly independent with Λ_2^0. This model can be equivalently written as

$$x_t = (\Lambda_1^0, \Lambda_2^0)\begin{pmatrix} f_{1t}^0 \\ f_{2t}^0 \\ 0 \end{pmatrix} + e_t \text{ if } z_t = 1, \text{ and } x_t = (\Lambda_1^0, \Lambda_2^0)\begin{pmatrix} 0 \\ 0 \\ f_{1t}^0 \end{pmatrix} + e_t \text{ if } z_t = 2,$$

i.e., there is only one regime and there are three factors in this regime. The difference between the log-likelihood of the true model $(r_1 = 2, r_2 = 1, J = 2)$ and the log-likelihood of the equivalent model $(r_1 = 3, J = 1)$ is negligible and clearly Bai and Ng (2002) is not applicable to this example.

Our solution is to add a penalty term for each regime and let the penalty term of different regime have different asymptotic order, so that overestimating the number of factors in one regime can not be compensated by underestimating the number of factors in another regime. For example, the penalty for the equivalent model is $(g(N,T))^{b(3)}$ while the penalty for the true model is $(g(N,T))^{b(2)} + (g(N,T))^{b(1)}$. Since $\frac{(g(N,T))^{b(3)}}{(g(N,T))^{b(2)}+(g(N,T))^{b(1)}} \to \infty$ as $(N,T) \to \infty$, the true model would be chosen with probability approaching one as $(N,T) \to \infty$. The formal proof of Theorem 7 is provided in the Appendix.

Our method can also be used to consistently determine the number of factors and the number of breaks for factor models with multiple common breaks in the loadings. If we replace $l(\tilde{\Lambda}_{1,r_1}, ..., \tilde{\Lambda}_{J,r_J}, \sigma^2, Q, \phi)$ in expression (26) by minus the minimum of the least squares over all possible break points and calculate $(\tilde{r}_1, ..., \tilde{r}_J, \tilde{J})$ as expressions (27)-(28), then it is not difficult to prove that we still have $\Pr(\tilde{J} = J^0$ and $\tilde{r}_j = r_j^0$ for all $j) \to 1$ as $(N,T) \to \infty$. As we discussed in the Introduction, recently the literature on the factor loading instability issues developed quite a lot, but as far as we know, there are very few (if any) consistent model selection procedures that allow r_j to be different across j and allow Λ_k^0 and Λ_j^0 to share some columns.

5 Simulations

In this section, we perform simulations to confirm the theoretical results and examine the finite sample performance of our methods under various empirically relevant scenarios.

5.1 Simulation Design

The data is generated as follows:

$$x_{it} = \begin{cases} f_t^{0\prime}\lambda_{1i}^0 + e_{it} \text{ if } z_t = 1, \\ f_t^{0\prime}\lambda_{2i}^0 + e_{it} \text{ if } z_t = 2, \end{cases} \text{ for } i = 1, ..., N \text{ and } t = 1, ..., T,$$

i.e., we consider two regimes. For the factors and the loadings, we consider four data generating processes (DGP) as listed below:

DGP 1: There are two factors in both regimes and the loadings of both factors have regime switching.

DGP 2: There are two factors in both regimes and only the loadings of the second factor have regime switching.

DGP 3: There is one factor in both regimes and its loadings have regime switching.

For DGP1 - DGP3, the factors are generated as follows:

$$f^0_{t,p} = \rho f^0_{t-1,p} + \epsilon_{t,p} \text{ for } t = 2, ..., T \text{ and } p = 1, ..., r^0.$$

$\epsilon_{t,p}$ is i.i.d. $N(0, 1)$, and $f^0_{1,p}$ is i.i.d. $N(0, \frac{1}{1-\rho^2})$ so that the distributions of the factors are stationary. Serial correlation of the factors is controlled by the scalar ρ.

DGP 4: The loadings are generated in the same way as DGP2. $f^0_{t,1}$ *is generated as i.i.d.* $N(0, 1)$ *and* $f^0_{t,2}$ *is generated as uniform* $(0.5, 1.5)$.

The errors are generated as follows:

$$e_{it} = \zeta e_{i,t-1} + v_{it} \text{ for } i = 1, ..., N \text{ and } t = 2, ..., T,$$

where $v_t = (v_{1,t}, ..., v_{N,t})'$ is i.i.d. $N(0, \Omega)$ for $t = 2, ..., T$ and $(e_{1,1}, ..., e_{N,1})'$ is $N(0, \frac{1}{1-\zeta^2}\Omega)$ so that the distributions of the errors are stationary. Serial correlation of the errors is controlled by the scalar ζ. For Ω, we set $\Omega_{ij} = \xi^{|i-j|}$ for some ξ between 0 and 1, thus cross-sectional dependence of the errors is controlled by ξ. In addition, the processes $\{\epsilon_{t,p}\}$ and $\{v_{it}\}$ are mutually independent for all p and i.

The loadings are generated as follows: For DGP1, both λ^0_{1i} and λ^0_{2i} are generated as i.i.d. $N(0, \frac{1-\rho^2}{1-\zeta^2}\frac{2R^2}{1-R^2}I_2)$ across i, and λ^0_{1i} and λ^0_{2i} are also independent with each other. For DGP2, λ^0_{1i} and the second element of λ^0_{2i} are generated as i.i.d. $N(0, \frac{1-\rho^2}{1-\zeta^2}\frac{2R^2}{1-R^2}I_3)$ across i. For DGP3, both λ^0_{1i} and λ^0_{2i} are generated as i.i.d. $N(0, \frac{1-\rho^2}{1-\zeta^2}\frac{R^2}{1-R^2})$ across i, and λ^0_{1i} and λ^0_{2i} are also independent with each other. All loadings are independent of the factors and the errors. The variance $\frac{1-\rho^2}{1-\zeta^2}\frac{2R^2}{1-R^2}$ guarantees that the regression R-square of each series i is equal to R^2, this controls the signal-noise ratio. Following the literature, we set $R^2 = 0.5$.

For the state process $\{z_t, t = 1, ..., T\}$, we consider four cases as listed below:

Regime Pattern 1: US business cycle 1945Q2-2020Q1

Regime Pattern 2: single common break at $t = T/2$

Regime Pattern 3: two common breaks at $t = T/3$ and $t = 2T/3$, *and the loadings switch back after the second break*

Regime Pattern 4: a randomly generated Markov process

Regime pattern 1 is based on the US business cycle from 1945 Quarter 2 to 2020 Quarter 1, as determined by the NBER business cycle dating committee. There are 75 years (300 quarters) in total, thus we have $T = 300$. For $t = 1, ..., 300$, $z_t = 1$ if the US economy at time t is in expansion and $z_t = 2$ if the US economy at time t is in recession. The transition probabilities of the state process calibrated to the US business cycle is $Q_{11}^0 = 0.95$ and $Q_{22}^0 = 0.72$ (average duration of expansion is $1/(1 - Q_{11}^0) = 20$ and average duration of recession is $1/(1 - Q_{22}^0) \approx 3.5$).

Regime patterns 2 and 3 correspond to the case where loadings have single common break and multiple common breaks, respectively. Regime patterns 3 is especially interesting since the case where there are multiple breaks and the loadings switch back to their original values after the second break is rarely studied in the literature. Various methods are proposed in the literature recently for estimating the break points, here we perform simulations for regime patterns 2 and 3 to evaluate the finite sample performance of our method when it is applied to these interesting cases.

Regime pattern 4 is a Markov process randomly generated with transition probabilities $Q_{11}^0 = 0.95$ and $Q_{22}^0 = 0.72$, and $\{z_t, t = 1, ..., T\}$ is independent with f_s^0 and e_{is} for all i and s. Regime patterns 1-3 are prespecified and are not necessarily Markov processes, thus here we consider regime pattern 4 to evaluate the performance of our method when applied to a Markov state process.

We study both the unsmoothed algorithm and the smoothed algorithm. The key difference is that in the E-step, the former uses unsmoothed regime probabilities while the latter uses smoothed regime probabilities. Both algorithms start from randomly generated initial values of the loadings and iterate between the E-step and the M-step until convergence. To search for the global maximum of the likelihood function, we generate initial values randomly for many times and take the one with the largest likelihood. For other parameters, we set $\sigma^2 = 1$, $q_j = 0.5$ for $j = 1, 2$, $\phi_k = 0.5$ for $k = 1, 2$, $Q_{11} = 0.95$ and $Q_{22} = 0.72$. Q_{11} and Q_{22} are calibrated to regime pattern 1.

Once we get the estimated regime probabilities and the estimated loadings, $\tilde{Q}_{11}$ and $\tilde{Q}_{22}$ are estimated by equation (21), and the factors are estimated by equation (25).

5.2 Simulation Results

Figure 1 displays the smoothed probabilities of regime 2 for DGP 1 with $(N, T) = (100, 300)$ and $(\rho, \zeta, \xi) = (0, 0, 0)$. Subfigures 1-4 of Figure 1 correspond to regime patterns 1-4, respectively. It is easy to see that in all subfigures when the true regime is regime 1, the smoothed probabilities stay at zero with only a few short and mild spikes. At the beginning of each shaded region, the smoothed probabilities increase to one instantly, and at the end of each shaded region, the smoothed probabilities instantly decrease to zero. Figure 2 displays the unsmoothed probabilities of regime 2 for DGP1 under the four regime patterns with $(N, T) = (100, 300)$ and $(\rho, \zeta, \xi) = (0, 0, 0)$. The estimated probabilities still stay at zero when it's regime 1 and instantly increase to one (decrease to zero) when there is regime switching, but compared to Figure 1, Figure 2 shows more and sharper spikes (upward or downward). These spikes are false positives in detecting regime switching. Figure 3 and Figure 4 display the smoothed and the unsmoothed probabilities of regime 2 for DGP2, respectively. The performance of the estimated probabilities deteriorates since for DGP2 only one factor has regime switching in its loadings. Overall, Figures 1-4 confirm the theoretical results that turning points (break points) can be identified consistently if N is large.

Comparing Figure 2 to Figure 1 and Figure 4 to Figure 3, it is obvious that the smoothed probabilities performs much better than the unsmoothed probabilities. Many false positives in Figure 3 and Figure 4 are eliminated by the smoother. This is because for each t, regimes at $t-1$ and $t+1$ contains information for detecting the regime at period t. Comparing subfigures 2-3 to subfigures 1 and 4 in Figures 1-4, we can see that the performance of the estimated probabilities under regime patterns 2-3 is better than the performance under patterns 1 and 4. This is also because regimes at the neighborhood periods provide information for the current regime. Roughly speaking, the performance is better when the regime pattern is relatively simple. In addition, we can also see that the performance under regime pattern 1 is slightly better than the performance under regime pattern 4. This is because the subsample size of regime 2 under pattern 4 is larger than the subsample size under pattern 1 (72 vs 45). In general, we find that to guarantee good performance, the subsample size for each regime should be not less than 40.

Figure 5 focuses on regime pattern 1 and displays the estimated probabilities of regime 2 for DGP1 and DGP2 with $N = 200$. Comparing the subfigure 3 and subfigure 4 of Figure 5 to subfigure 1 of Figure 3 and subfigure 1 of Figure 4, it is easy to see that $N = 200$ improves the performance of the estimated probabilities. Figure 6 also focuses on regime pattern 1 and displays the smoothed probabilities of regime 2 for DGP1 and DGP2 with $(\rho, \zeta, \xi) = (0.5, 0, 0)$ or $(0, 0.5, 0.5)$. Comparing to subfigure 1 of Figure 1 and subfigure 1 of Figure 3, it seems that the value of (ρ, ζ, ξ) does not affect the performance too much if they were far away from 1.

Figure 7 displays the smoothed and unsmoothed probabilities for regime pattern 4 (a randomly generated Markov process) under DGP1 with $(N, T) = (50, 500)$ or $(N, T) = (500, 50)$. The smoothed probabilities still perform well even under such extreme case, especially when $T = 50$, the subsample size of regime 2 is just 13. However, the unsmoothed probabilities in subfigures 3-4 deteriorate obviously, compared to subfigure 4 of Figure 2.

Figure 8 displays the smoothed and unsmoothed probabilities under DGP4 for regime patterns 1 and 2 with $(N, T) = (100, 300)$. DGP4 modifies DGP2 so that the second factor stays away from zero. Comparing subfigures 1-2 to subfigures 1-2 of Figure 3 and subfigures 3-4 to subfigures 1-2 of Figure 4, we can see that the performance improvement is quite significant. This is because under DGP2, the second factor $f_{t,2}^0$ is likely to be close to zero and it is difficult to identify the regime of x_t when $f_{t,2}^0$ is close to zero. Thus Figures 3-4 reflect more of the identification problem when the factors equal zeros.

Finally, to access the adequacy of the asymptotic distributions of the estimated loadings and factors in approximating their finite sample counterparts, we display in Figures 9-12 the histograms of the standardized estimated factors for $t = T/2$ and the standardized estimated loadings for $i = N/2$ under DGP3. The number of simulations is 1000. The histograms are normalized to be a density function and the standard normal density curve is overlaid on them for comparison. It is easy to see that in all subfigures of Figures 9-12, the standard normal density curve provides good approximation to the normalized histograms. The histograms of the estimated factors in Figure 9 are slightly fat-tailed because of bad initial values. Comparing the four rows in each of Figures 9-12, we can see that the estimated loadings and factors using the smoothed algorithm perform better than using the unsmoothed algorithm, $(\rho, \zeta, \xi) = (0.5, 0.5, 0.5)$ does not matter too much, and $N = 200$ significantly improves the performance.

The number of initial value trials also significantly affect the performance. We find that for regime pattern 1, normally 5 trials are enough, but to guarantee good performance in all of 1000 replications, 30 trials are needed. For regime pattern 4, normally 2 trials are enough and 15 trials are needed to guarantee good performance in all replications. For regime patterns 2-3, 5 trials are enough for all replications. In general, more trials are needed when the regime pattern is complex and the subsample size is small.

In addition, we also present in Table 1 the average R^2 of the estimated loadings of regime 1 and regime 2 projecting on the true loadings, the average R^2 of the estimated factors, and the average absolute error of the estimated transition probabilities. It is easy to see that in Table 1, R^2_{l1} and R^2_{l2} are always close to one. R^2_{Hf} is always close to one but R^2_f is much smaller than R^2_{Hf}. This is because R^2_{Hf} considers the regime specific rotation matrix, as shown in Theorem 5(1). In summary, results in Figures 1-12 and Table 1 lend strong support to the theoretical results and illustrate the usefulness of the proposed EM algorithms.

6 Empirical Application

In this section we apply the proposed method to detect turning points of US business cycle from 02/1980 to 01/2023 in real-time using the FRED-MD (Federal Reserve Economic Data - Monthly Data) data set. The FRED database is maintained by the Research division of the Federal Reserve Bank of St. Louis, and is publicly accessible and updated in real-time. The 02/2023 vintage of the FRED-MD data set contains

128 unbalanced monthly time series from 01/1959 to 01/2023, including eight groups (output and income, labor market, housing, consumption and inventories, money and credit, prices, stock market). After removing those series with missing values and data transformation[7], we have 106 balanced monthly series ranging from 03/1959 to 01/2023. Finally, the data is demeaned and standardized.

For each month from 02/1980 to 01/2023 (516 months in total), we use the data from 03/1959 to that month for calculating the probability of recession of that month, i.e., we behave as if we were standing at that month[8]. More specifically, we apply the EM algorithm in Section 2.3 to the data from 03/1959 to the previous month to estimate the model parameters[9], and then use the estimated parameters and the data from 03/1959 to that month to calculate the filtered probability of recession for that month. Since the data of that month is available at the end of that month or the beginning of the next month, new recession or expansion starting from the beginning of that month could only be detected with at least one month delay.

To convert the recession probability of each month into a binary variable that indicates the state of the economy in that month, we compare the estimated recession probability to a prespecified threshold. More specifically, if the previous turning point is a trough and the recession probability of month t exceeds 0.8 for the first time after the previous turning point, month t would be considered as a new turning point from expansion to recession. Similarly, if the previous turning point is a peak and the recession probability of month t falls below 0.2 for the first time after the previous turning point, month t would be considered as a new turning point from recession to expansion. For robustness check, we also consider $(0.9, 0.1)$ as the threshold, the results are quite similar.

We consider the turning points determined by the NBER BCDC (business cycle

[7]See the Appendix of McCracken and Ng (2016) for the details of data description and transformation.

[8]For simplicity, we do not use the vintage data of that month. Compared to the vintage data, the data we use contains revision in some series if more accurate observations were available after that month, but previous studies on business cycle dating show that data revisions have little effects on the results.

[9]US business cycle from 03/1959 to the previous month as determined by NBER is used as the initial values for probabilities.

dating committee) as the benchmark for comparison and we mainly focus on the accuracy and speed of the proposed method in detecting turning points. The proposed method is applied to both the whole panel and a subset of the whole panel which consists of only the first 50 series among all 106 series. The results of using only the first 50 series are better. We conjecture that this is mainly because not all 106 series had regime switching in the factor loadings at each turning point determined by the NBER BCDC[10], or some series had regime switching in their loadings at time periods that are different from the NBER BCDC turning points. Thus we may further improve the performance of the proposed method by selecting series that are most relevant to and synchronous with the business cycle. A careful selection is out of the scope of this paper.

Table 2 presents the real-time results of 02/1980-02/2020 using the first 50 series. The number of factors in each regime is set to be six. mm/yyyy in the second and the seventh row indicate the starting month of each recession and expansion. The row corresponds to "NBER BCDC", "Chauvet Piger" and "This paper" shows the number of months it takes the NBER BCDC, Chauvet and Piger (2008) and this paper to detect each recession and expansion, respectively. For example, the recession starting from the beginning of February 1980 would be detected by the NBER BCDC at the beginning of June 1980, by Chauvet and Piger (2008) at the beginning of August 1980, and by this paper at the beginning of May 1980, respectively. Overall, it is easy to see that this paper detects turning points much faster than NBER BCDC and slightly faster than Chauvet and Piger (2008). On average, this paper detects recessions with 6.25 months delay and expansions with 5.4 months delay, NBER BCDC detects recessions with 7.4 months delay and expansions with 14.8 months delay, and Chauvet and Piger (2008) detects recessions with 8.6 months delay and expansions with 6.2 months delay.

We also detect two recessions after the Covid-19 pandemic, one from 03/2020 to 08/2020 and the second from 02/2021 to 05/2021, so we would have detected the

[10]The NBER BCDC mainly focuses on four series, (1) non-farm payroll employment, (2) industrial production, (3) real manufacturing and trade sales, and (4) real personal income excluding transfer payments.

03/2020-08/2020 recession in 04/2020 because the data for 03/2020 is available with one month delay. This is quite interesting, given that our method detects recessions with 6.25 months delay on average during the period 02/1980-02/2020.

Table 2 shows that using more series could improve the speed of turning points detection. However, using more series could also bring in false positives (turning points detected by the proposed method using many series but not detected by NBER BCDC), because the extra series may not be synchronous with the NBER business cycle. Here we detect eight false recessions: 09/1983-11/1983, 10/1986-02/1987, 07/1989-10/1989, 01/1993-02/1993, 01/1995-03/1995, 08/1998, 05/2000-08/2000, 06/2010-10/2010, and one false expansion: 02/1982. While these false positives should not be ignored, most of them only last for a very short periods and would have little effect on macroeconomic policy. Overall, our results illustrate the potential of using a large number of series and factor models with common loading switching for real-time detection of the business cycle turning points.

To get rid of those false recessions, one possible solution is to select time series that are synchronous with the NBER business cycle, and another promising solution is to extend our results to the case where regime switching in the loadings is approximately synchronous rather than exactly synchronous. In fact, Stock and Watson (2014) mainly focuses on how to combine different turning points of many individual series (determined by the Bry-Boschan algorithm) into a single point.

7 Conclusions

The exposure of economic time series to common factors may switch depending on state variables such as fiscal policy, monetary policy, business cycle stage, stock market volatility, technology and so on. For consistent estimation of the factor structure, it is crucial to take into account such regime switching phenomena. This paper considers maximum likelihood estimation for large factor models with common regime switching in the loadings and proposes EM algorithm for computation, which is easy to implement and runs fast even when N is large. Convergence rates and limit distributions of the estimated loadings and the estimated factors are established under the approximate factor model setup. This paper also shows that when N is large, regime switching can be identified consistently and only one observation after the switching point is needed. This allows us to detect regime switching at very early times. Monte Carlo simulations confirm the theoretical results and good performance of our method. An application to the FRED-MD dataset demonstrates the potential of using many time series with our method for detection of the business cycle turning points.

Some related topics are worth further study. First, it would be interesting to see the performance of the portfolio constructed using regime specific loadings, and how the identified regime is related to exogenous variables such as market volatility and money growth. Second, our results imply that the forecasting equation would have induced regime switching if the estimated factors are used for forecasting, so we want to know whether it indeed matters. Finally, a selection of time series that are most synchronous with or related to business cycle could improve the speed and accuracy of our method for turning points detection, so we would like to see how much we can achieve after careful selection.

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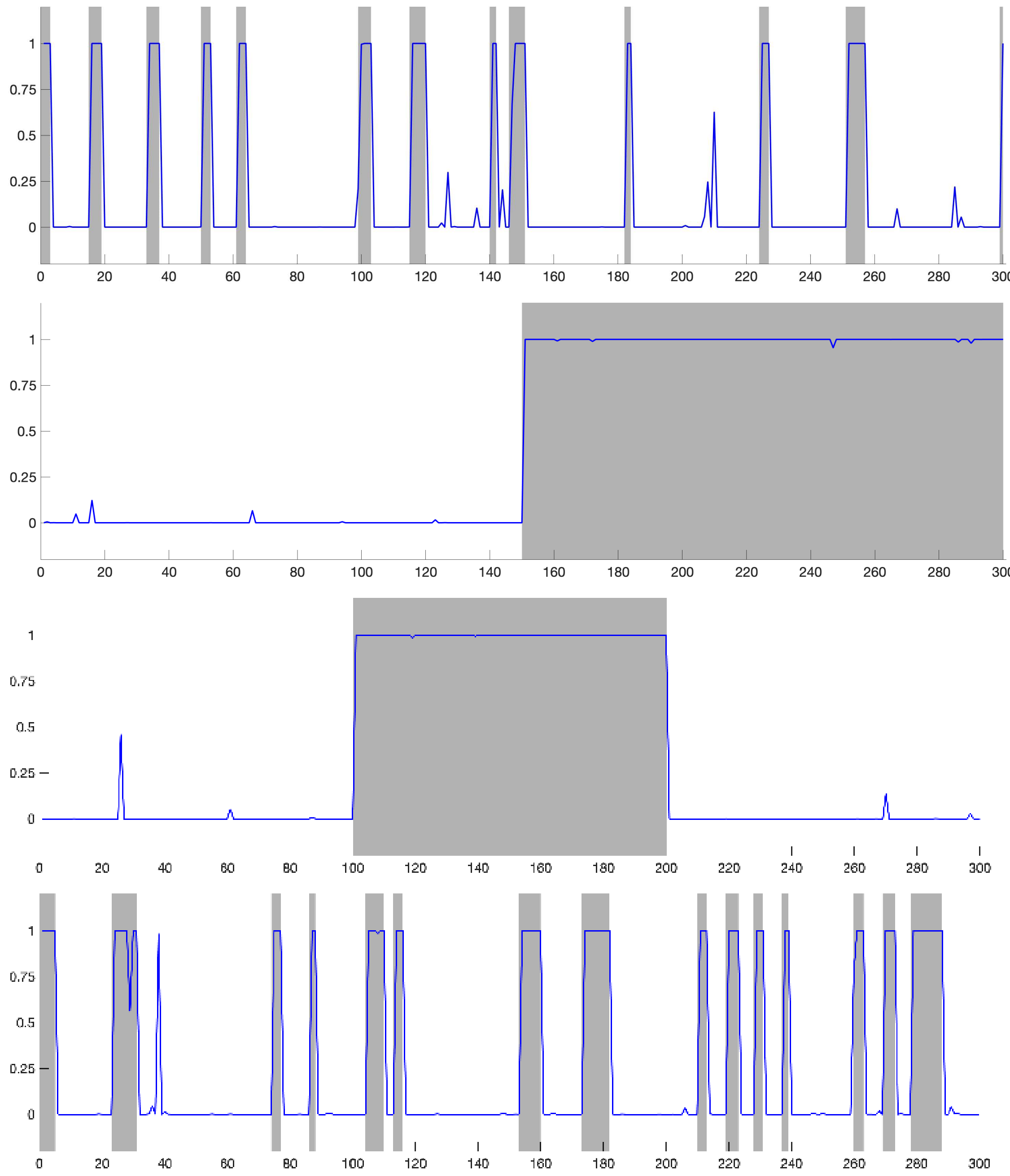
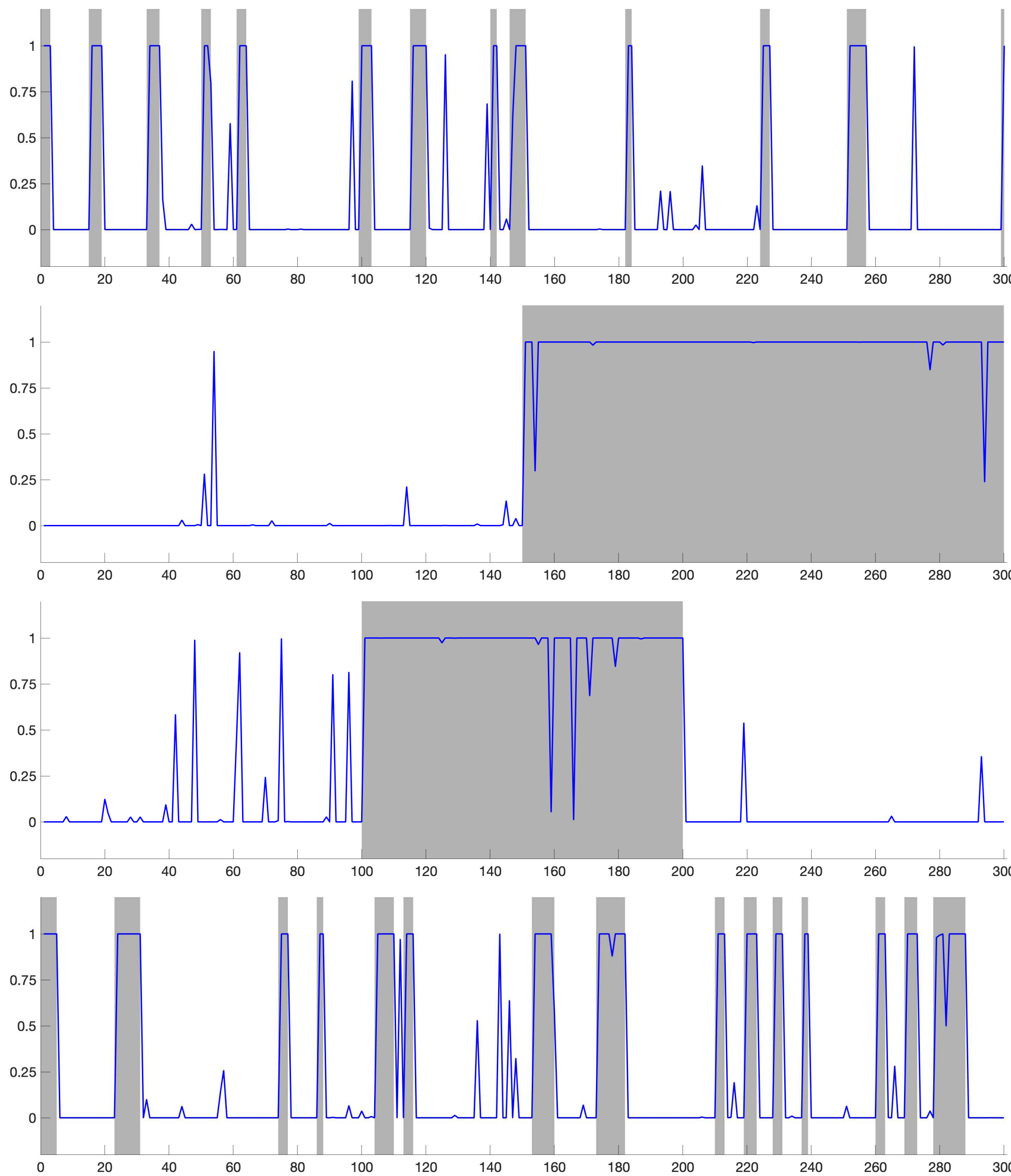
Figure 1: Smoothed Probabilities of Regime 2 for DGP 1



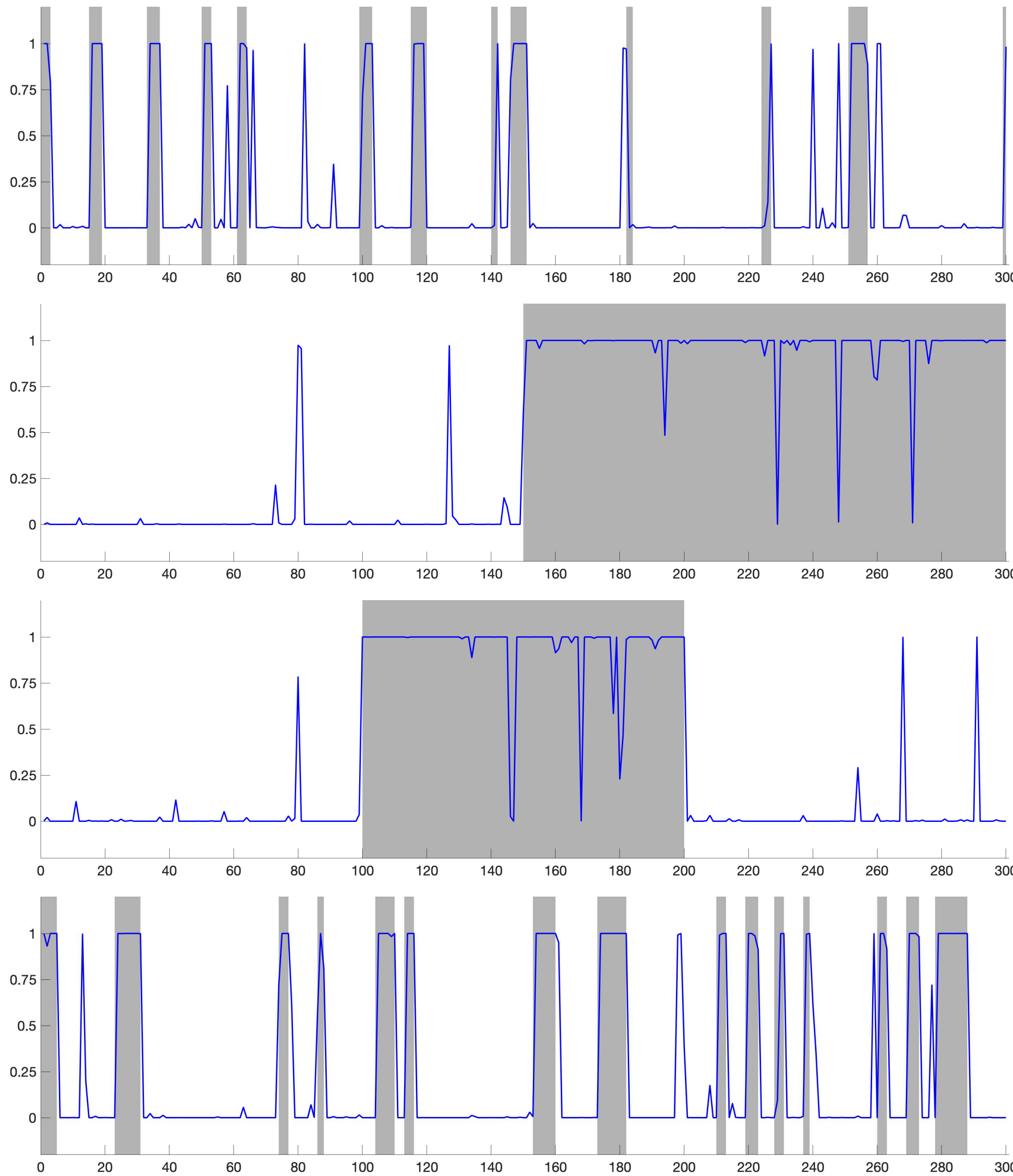
Notes: Subfigures 1-4 correspond to regime pattern 1-4, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2. $(N, T) = (100, 300)$ and $(\rho,\zeta,\xi) = (0, 0, 0)$.

Figure 2: Unsmoothed Probabilities of Regime 2 for DGP 1



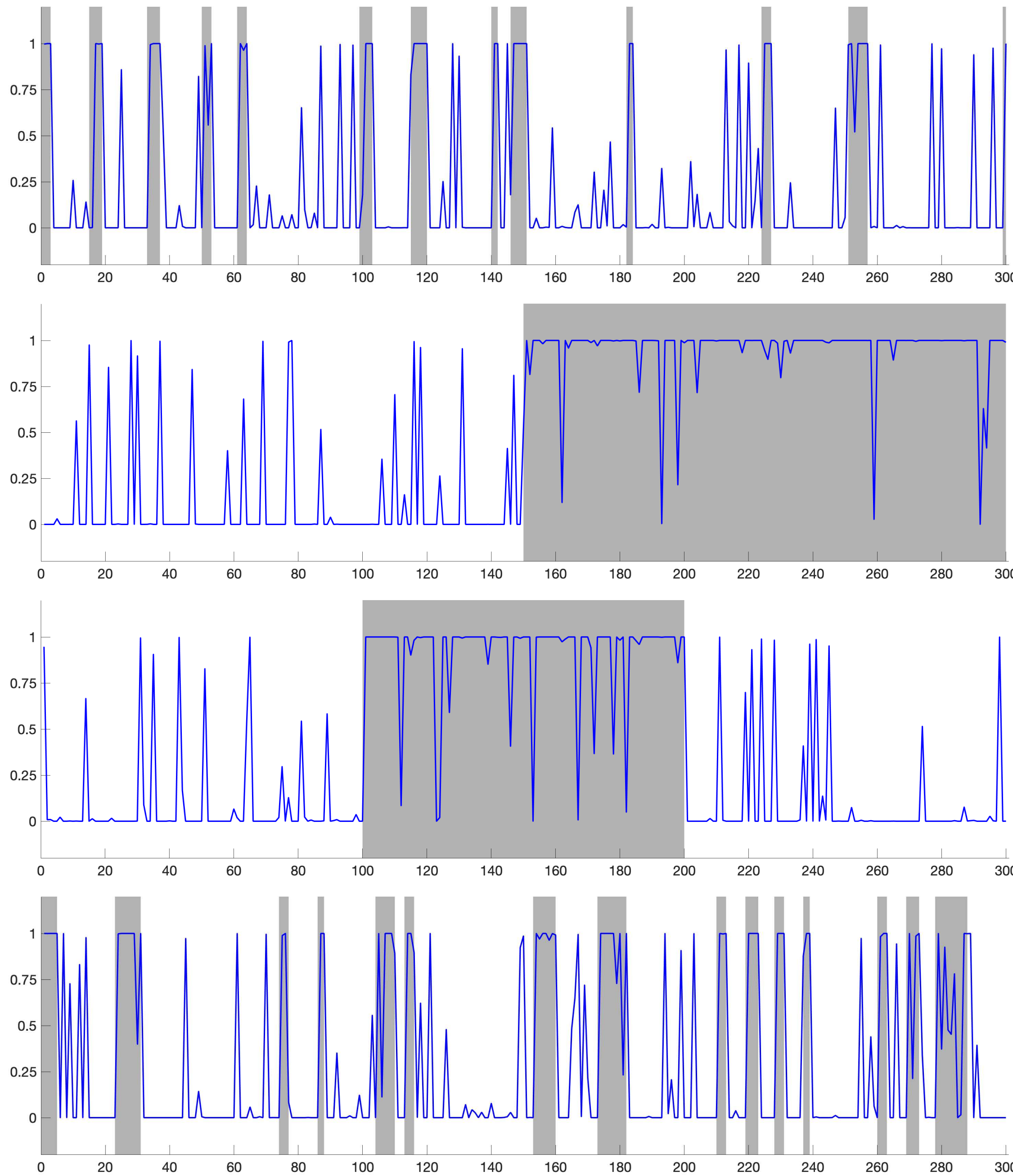
Notes: Subfigures 1-4 correspond to regime pattern 1-4, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2. $(N, T) = (100, 300)$ and $(\rho,\zeta,\xi) = (0, 0, 0)$.

Figure 3: Smoothed Probabilities of Regime 2 for DGP 2



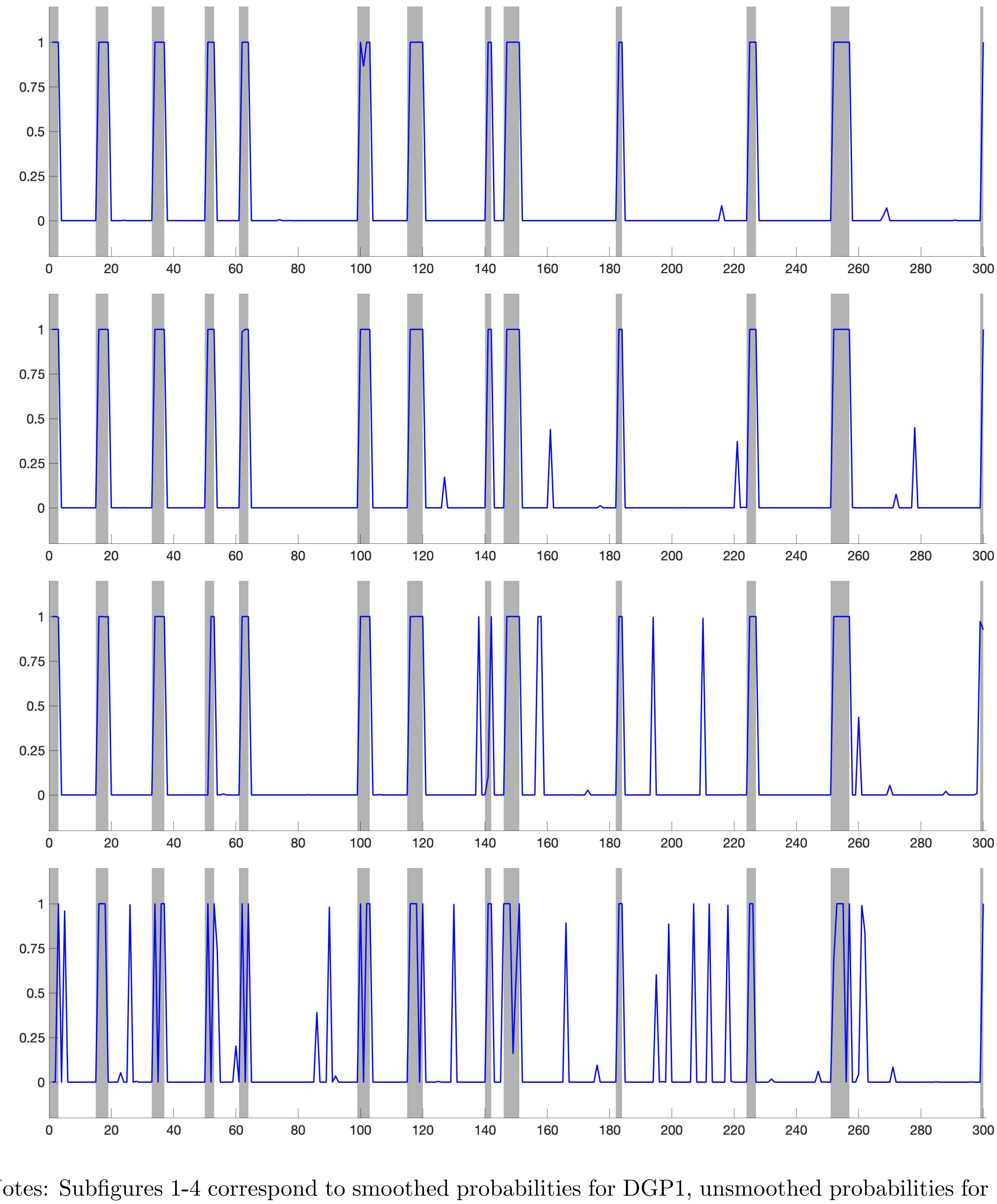
Notes: Subfigures 1-4 correspond to regime pattern 1-4, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2. $(N, T) = (100, 300)$ and $(\rho, \zeta, \xi) = (0, 0, 0)$.

Figure 4: Unsmoothed Probabilities of Regime 2 for DGP 2



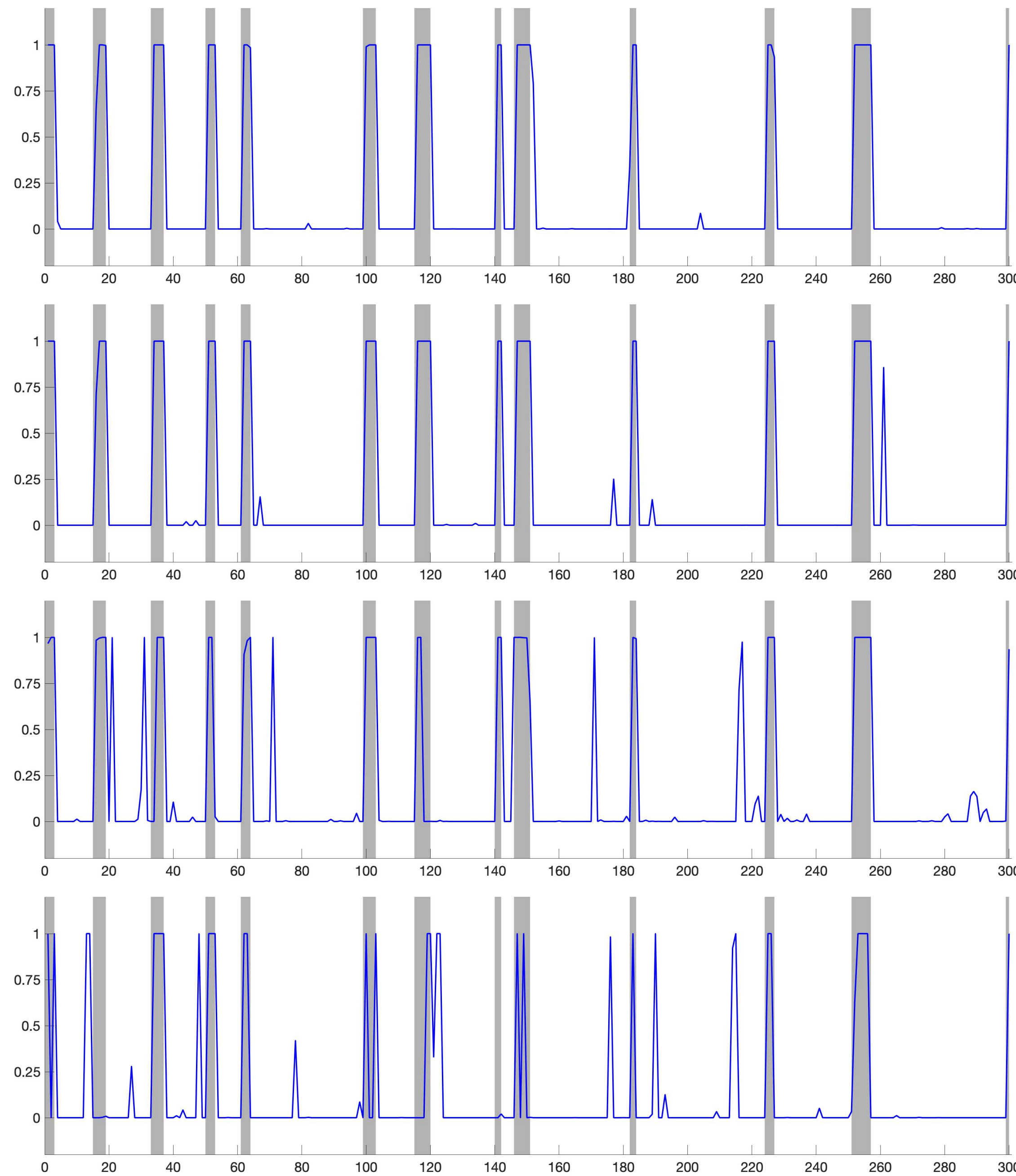
Notes: Subfigures 1-4 correspond to regime pattern 1-4, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2. $(N, T) = (100, 300)$ and $(\rho,\zeta,\xi) = (0, 0, 0)$.

Figure 5: Smoothed and Unsmoothed Probabilities of Regime 2 for Regime Pattern 1, $(N, T) = (200, 300)$ and $(\rho, \alpha, \beta) = (0, 0, 0)$



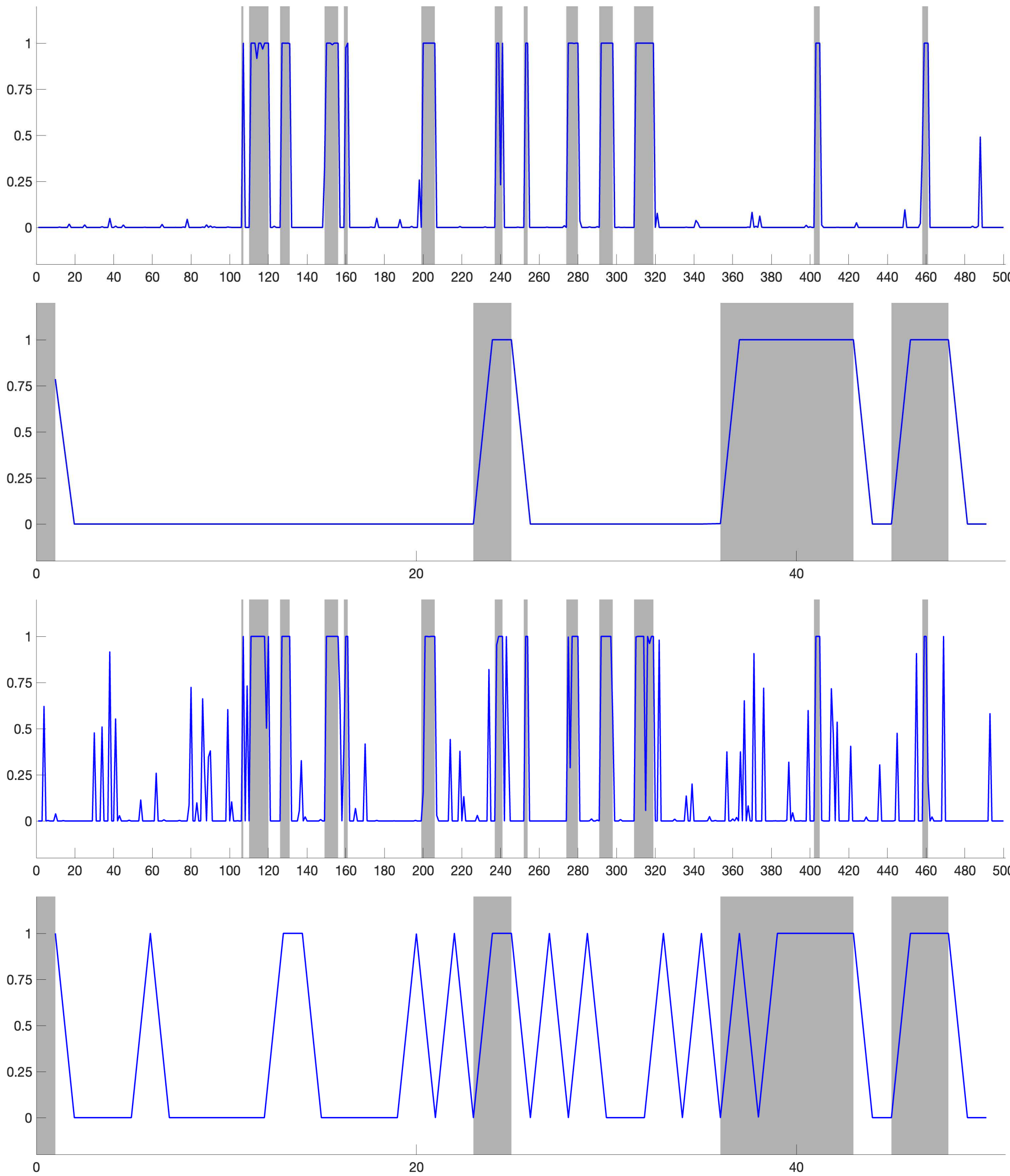
Notes: Subfigures 1-4 correspond to smoothed probabilities for DGP1, unsmoothed probabilities for DGP1, smoothed probabilities for DGP2 and unsmoothed probabilities for DGP2, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2.

Figure 6: Smoothed Probabilities of Regime 2 for Regime Pattern 1, $(N, T) = (100, 300)$ and $(\rho, \alpha, \beta) = (0.5, 0, 0)$ or $(\rho, \alpha, \beta) = (0, 0.5, 0.5)$



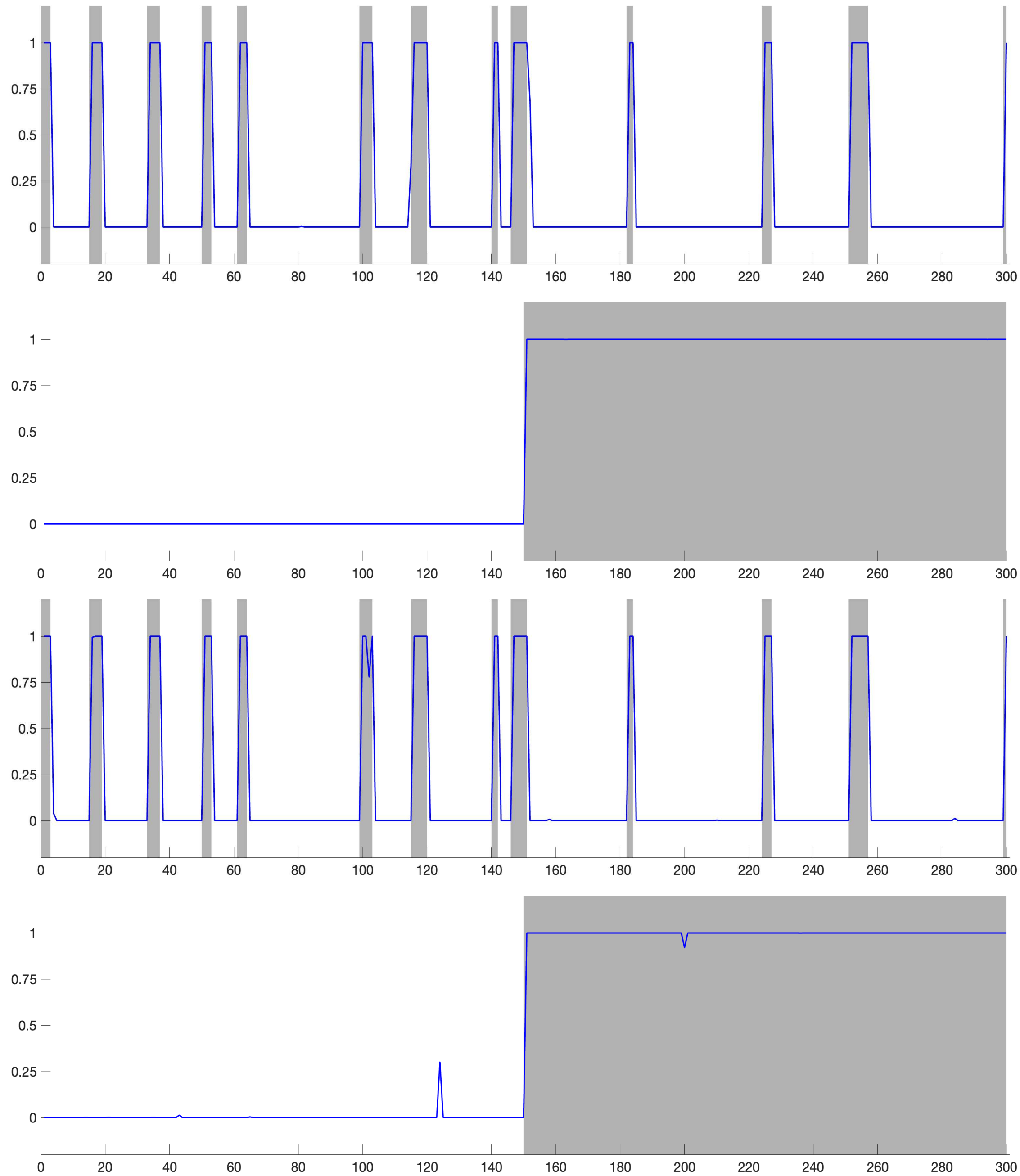
Notes: Subfigures 1-4 correspond to smoothed probabilities for DGP1 with $(\rho,\zeta,\xi) = (0.5, 0, 0)$, DGP1 with $(\rho,\zeta,\xi) = (0, 0.5, 0.5)$, DGP2 with $(\rho,\zeta,\xi) = (0.5, 0, 0)$ and DGP2 with $(\rho,\zeta,\xi) = (0, 0.5, 0.5)$, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2.

Figure 7: Smoothed and Unsmoothed Probabilities of Regime 2 for Regime Pattern 1, $(N, T) = (50, 500)$ and $(N, T) = (500, 50)$



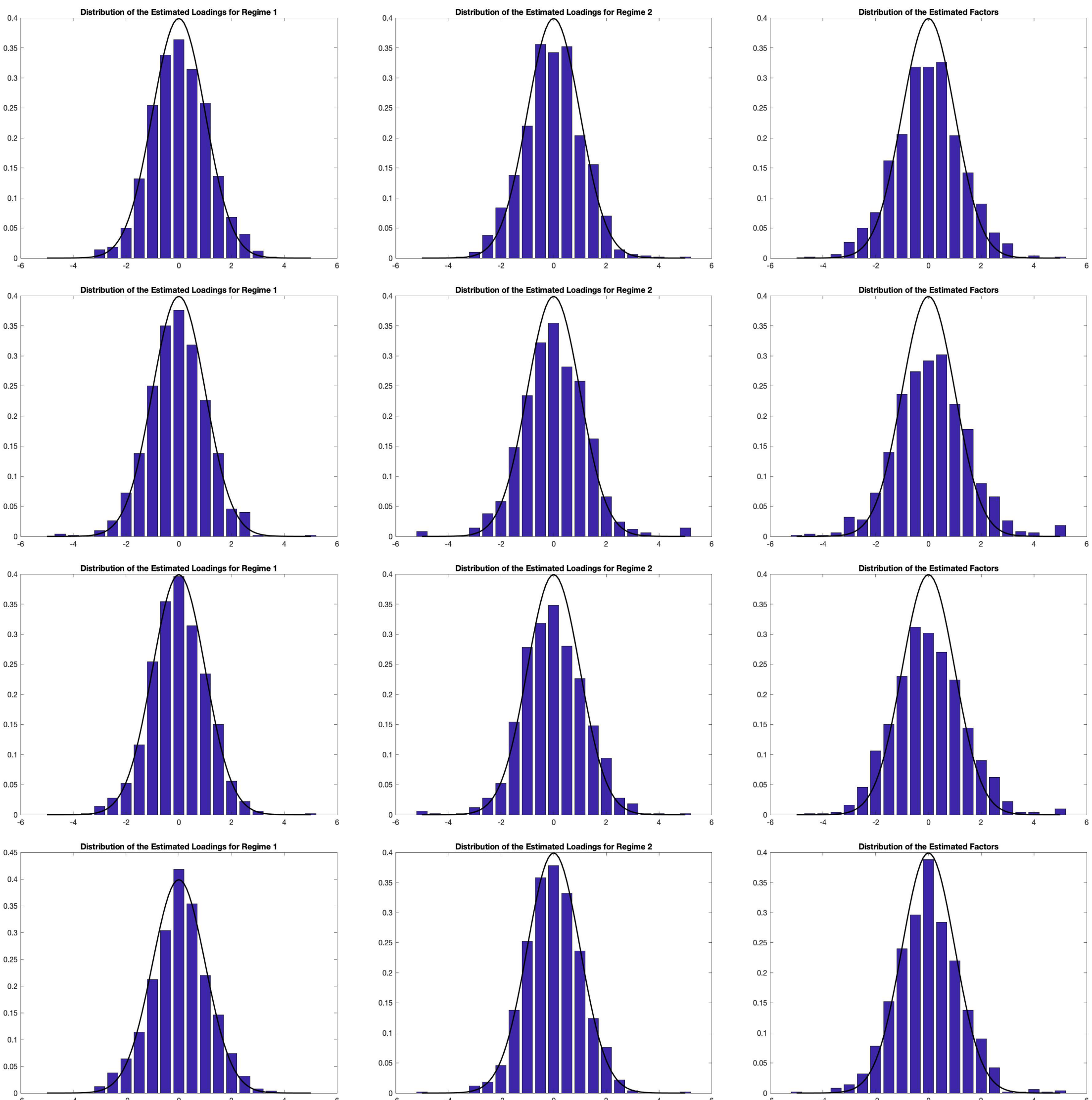
Notes: Subfigures 1-2 correspond to smoothed probabilities under regime pattern 4 and DGP1 with $(N, T) = (50, 500)$ or $(500, 50)$, respectively. Subfigures 3-4 correspond to unsmoothed probabilities under regime pattern 4 and DGP1 with $(N, T) = (500, 50)$ or $(500, 50)$, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2.

Figure 8: Smoothed and Unsmoothed Probabilities of Regime 2 for DGP 4



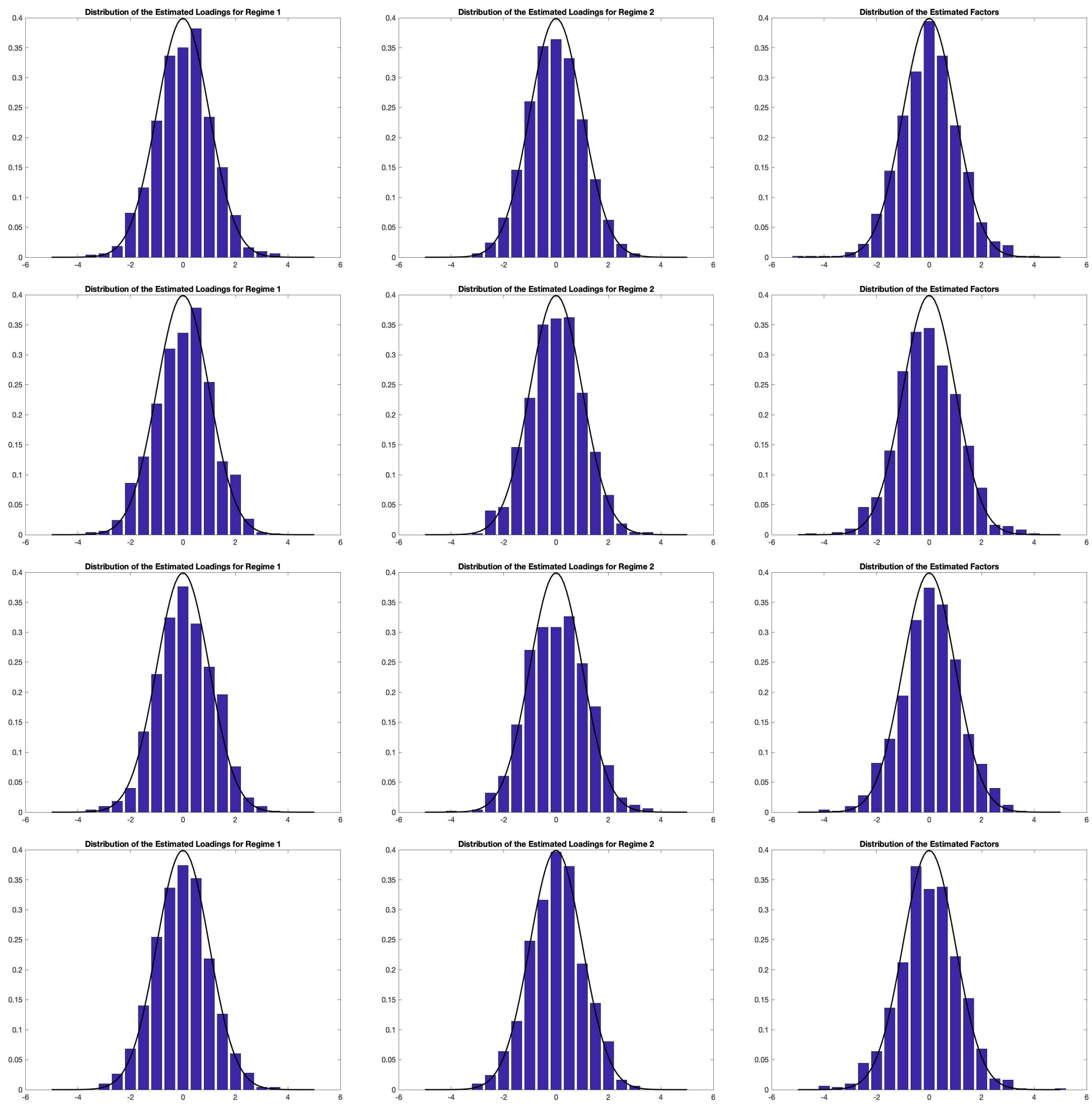
Notes: Subfigures 1-2 correspond to smoothed probabilities for DGP4 under regime patterns 1 and 2, respectively. Subfigures 3-4 correspond to unsmoothed probabilities for DGP4 under regime patterns 1 and 2, respectively. The x-axis is time and the y-axis is the probability. The shaded regions correspond to regime 2.

Figure 9: Histograms of the Estimated Loadings and the Estimated Factors for Regime Pattern 1



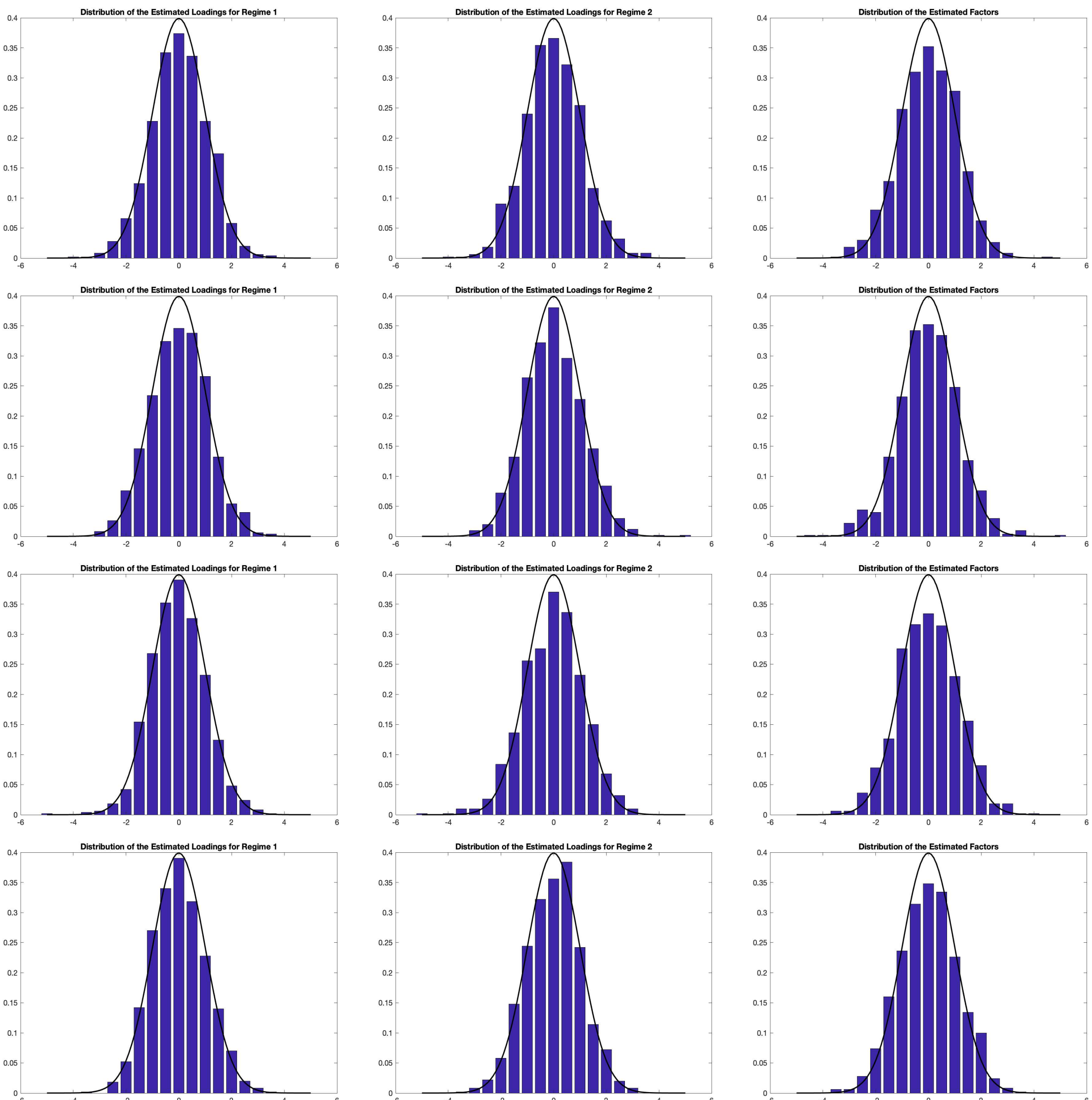
Notes: Subfigures in the first to the fourth row correspond to the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the unsmoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the smoothed algorithm with $\rho = \zeta = \xi = 0.5$ and $(N, T) = (100, 300)$, and the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (200, 300)$, respectively. Subfigures in the first to the third column correspond to the estimated loadings for regime 1, the estimated loadings for regime 2 and the estimated factors, respectively. The curve overlaid on the histograms is the standard normal density function.

Figure 10: Histograms of the Estimated Loadings and the Estimated Factors for Regime Pattern 2



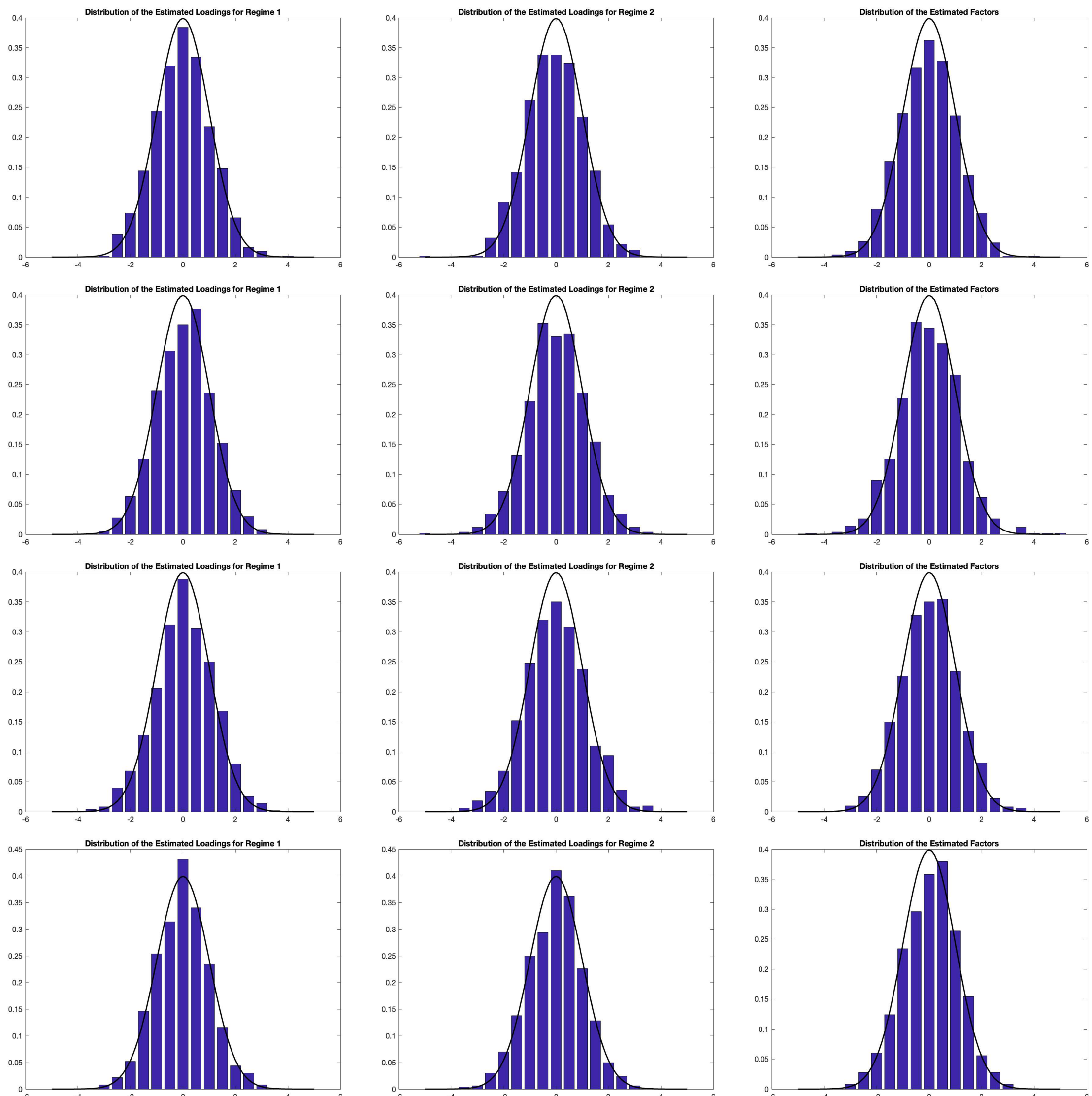
Notes: Subfigures in the first to the fourth row correspond to the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the unsmoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the smoothed algorithm with $\rho = \zeta = \xi = 0.5$ and $(N, T) = (100, 300)$, and the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (200, 300)$, respectively. Subfigures in the first to the third column correspond to the estimated loadings for regime 1, the estimated loadings for regime 2 and the estimated factors, respectively. The curve overlaid on the histograms is the standard normal density function.

Figure 11: Histograms of the Estimated Loadings and the Estimated Factors for Regime Pattern 3



Notes: Subfigures in the first to the fourth row correspond to the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the unsmoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the smoothed algorithm with $\rho = \zeta = \xi = 0.5$ and $(N, T) = (100, 300)$, and the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (200, 300)$, respectively. Subfigures in the first to the third column correspond to the estimated loadings for regime 1, the estimated loadings for regime 2 and the estimated factors, respectively. The curve overlaid on the histograms is the standard normal density function.

Figure 12: Histograms of the Estimated Loadings and the Estimated Factors for Regime Pattern 4



Notes: Subfigures in the first to the fourth row correspond to the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the unsmoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (100, 300)$, the smoothed algorithm with $\rho = \zeta = \xi = 0.5$ and $(N, T) = (100, 300)$, and the smoothed algorithm with $\rho = \zeta = \xi = 0$ and $(N, T) = (200, 300)$, respectively. Subfigures in the first to the third column correspond to the estimated loadings for regime 1, the estimated loadings for regime 2 and the estimated factors, respectively. The curve overlaid on the histograms is the standard normal density function.

Table 1: Average R^2 of the Estimated Loading Space, Average R^2 of the Estimated Factor Space, and Average Absolute Error of the Estimated Transition Probabilities

	R^2_{l1}	R^2_{l2}	R^2_f	R^2_{Hf}	$\tilde{Q}_{11}$	$\tilde{Q}_{22}$
Smoothed with $(\rho, \zeta, \xi) = (0, 0, 0)$ and $(N, T) = (100, 300)$						
Pattern 1	0.996	0.9762	0.7337	0.9889	0.0028	0.013
Pattern 2	0.9931	0.9932	0.5155	0.9896	N.A.	N.A.
Pattern 3	0.9949	0.9895	0.541	0.9894	N.A.	N.A.
Pattern 4	0.9955	0.9854	0.6256	0.9892	0.0216	0.0378
Unsmoothed with $(\rho, \zeta, \xi) = (0, 0, 0)$ and $(N, T) = (100, 300)$						
Pattern 1	0.9959	0.9678	0.6786	0.9782	N.A.	N.A.
Pattern 2	0.9931	0.9932	0.4855	0.9885	N.A.	N.A.
Pattern 3	0.9949	0.9892	0.5157	0.988	N.A.	N.A.
Pattern 4	0.9955	0.9853	0.6127	0.9875	N.A.	N.A.
Smoothed with $(\rho, \zeta, \xi) = (0.5, 0.5, 0.5)$ and $(N, T) = (100, 300)$						
Pattern 1	0.9933	0.9631	0.7255	0.9849	0.0053	0.0137
Pattern 2	0.9928	0.9929	0.4797	0.9891	N.A.	N.A.
Pattern 3	0.9915	0.9827	0.5458	0.9889	N.A.	N.A.
Pattern 4	0.9927	0.9782	0.6285	0.9886	0.0239	0.0328
Smoothed with $(\rho, \zeta, \xi) = (0, 0, 0)$ and $(N, T) = (200, 300)$						
Pattern 1	0.996	0.9756	0.7408	0.9936	0.0017	0.0151
Pattern 2	0.9933	0.9933	0.5183	0.9949	N.A.	N.A.
Pattern 3	0.995	0.9898	0.5611	0.9949	N.A.	N.A.
Pattern 4	0.9956	0.9856	0.6227	0.9947	0.019	0.024

Notes: The column under R^2_{l1} shows the average R^2 of the estimated loadings of regime 1 projecting on the true loadings of regime 1. The column under R^2_{l2} shows the average R^2 of the estimated loadings of regime 2 projecting on the true loadings of regime 2. The column under R^2_f shows the average R^2 of the estimated factors projecting on the true factors. The column under R^2_{Hf} shows the average R^2 of the estimated factors projecting on the factors rotated by the regime dependent rotation matrix H_{z_t}. "N.A." means not available.

Table 2: Out of Sample Turning Points Detection

	Recession 02/1980	Expansion 08/1980	Recession 08/1981	Expansion 12/1982	Recession 08/1990
NBER BCDC	4	11	5	7	9
Chauvet Piger	6	5	7	6	7
This paper	3	2	3	7	N.A.
	Expansion 04/1991	Recession 04/2001	Expansion 12/2001	Recession 01/2008	Expansion 07/2009
NBER BCDC	21	8	20	11	15
Chauvet Piger	6	10	7	13	7
This paper	1	8	7	11	10

Notes: mm/yyyy in the second and the seventh row indicate the starting month of each recession and expansion. The row corresponds to "NBER BCDC", "Chauvet Piger" and "This paper" shows the number of months it takes the NBER BCDC, Chauvet and Piger (2008) and this paper to detect each recession and expansion, respectively. "N.A." means not available.

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APPENDIX

A Details for Theorem 1

Lemma 1 *Under Assumption 3(2) and 3(4),* $\|E\| = O_p(N^{\frac{1}{4}}T^{\frac{1}{2}} + N^{\frac{1}{2}}T^{\frac{1}{4}})$.

Proof. We shall show $\mathbb{E}\,\|E\|^4 = O(NT^2 + N^2T)$. First note that

$$\|E\|^4 = \|E'E\|^2 \leq \|E'E\|_F^2 = \sum_{i=1}^{N}\sum_{k=1}^{N}(\sum_{t=1}^{T} e_{it}e_{kt})^2.$$

$\mathbb{E}(\sum_{t=1}^{T} e_{it}e_{kt})^2$ is not larger than the sum of $2\mathbb{E}(\sum_{t=1}^{T} e_{it}e_{kt} - \sum_{t=1}^{T}\mathbb{E}(e_{it}e_{kt}))^2$ and $2(\sum_{t=1}^{T}\mathbb{E}(e_{it}e_{kt}))^2$. The sum of the former over i and k is not larger than N^2TM since by Assumption 3(4), $\mathbb{E}(\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(e_{it}e_{kt} - \mathbb{E}(e_{it}e_{kt}))\right\|^2) \leq M$. The sum of the latter over i and k is not larger than NT^2M under Assumption 3(2). ■

Proof of Theorem 1

Proof. Step (1): Since z_t follows a Markov process,

$$l(\Lambda,\sigma^2,Q,\phi) = \log[\sum_{z_T=1}^{J^0}...\sum_{z_1=1}^{J^0}\prod_{t=1}^{T} L(x_t\left|z_t;\Lambda,\sigma^2\right.)\Pr(z_1\left|\phi\right.)\prod_{t=2}^{T}\Pr(z_t\left|z_{t-1};Q\right.)].$$

For $(\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)$, let $m_t = \arg\max_j\{(2\pi)^{-\frac{N}{2}}\left|\tilde{\Lambda}_j\tilde{\Lambda}_j' + \tilde{\sigma}^2 I_N\right|^{-\frac{1}{2}} e^{-\frac{1}{2}x_t'(\tilde{\Lambda}_j\tilde{\Lambda}_j'+\tilde{\sigma}^2 I_N)^{-1}x_t}\}$, i.e, $L(x_t\left|z_t = j;\tilde{\Lambda},\tilde{\sigma}^2\right.)$ takes maximum when $j = m_t$. Since $\sum_{z_t=1}^{J^0}\Pr(z_t\left|z_{t-1};Q\right.) = 1$ for any z_{t-1}, $\sum_{z_t=1}^{J^0} L(x_t\left|z_t;\tilde{\Lambda},\tilde{\sigma}^2\right.)\Pr(z_t\left|z_{t-1};Q\right.) \leq L(x_t\left|z_t = m_t;\tilde{\Lambda},\tilde{\sigma}^2\right.)$, thus

$$\begin{aligned}
& l(\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi) \\
= \;& \log\{\sum_{z_{T-1}=1}^{J^0}...\sum_{z_1=1}^{J^0}\prod_{t=1}^{T-1} L(x_t\left|z_t;\tilde{\Lambda},\tilde{\sigma}^2\right.)\Pr(z_1\left|\phi\right.)\prod_{t=2}^{T-1}\Pr(z_t\left|z_{t-1};Q\right.) \\
& [\sum_{z_T=1}^{J^0} L(x_T\left|z_T;\tilde{\Lambda},\tilde{\sigma}^2\right.)\Pr(z_T\left|z_{T-1};Q\right.)]\} \\
\leq \;& \log\{\sum_{z_{T-1}=1}^{J^0}...\sum_{z_1=1}^{J^0}\prod_{t=1}^{T-1} L(x_t\left|z_t;\tilde{\Lambda},\tilde{\sigma}^2\right.)\Pr(z_1\left|\phi\right.)\prod_{t=2}^{T-1}\Pr(z_t\left|z_{t-1};Q\right.) \\
& L(x_T\left|z_T = m_T;\tilde{\Lambda},\tilde{\sigma}^2\right.)\} \\
\leq \;& ... \leq \sum_{t=1}^{T}\log L(x_t\left|z_t = m_t;\tilde{\Lambda},\tilde{\sigma}^2\right.). \qquad (29)
\end{aligned}$$

It follows that

$$
\begin{aligned}
l(\tilde{\Lambda}, \tilde{\sigma}^2, Q, \phi) &\leq \sum\nolimits_{t=1}^{T} \log[(2\pi)^{-\frac{N}{2}} \left|\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N\right|^{-\frac{1}{2}} e^{-\frac{1}{2}x'_t(\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t}+\tilde{\sigma}^2 I_N)^{-1}x_t}] \\
&= -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum\nolimits_{t=1}^{T} \log\left|\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N\right| \\
&\quad -\frac{1}{2}\sum\nolimits_{t=1}^{T} x'_t(\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N)^{-1}x_t. \qquad (30)
\end{aligned}
$$

Consider the last term on the right hand side of equation (30). By Woodbury identity, $(\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N)^{-1} = \tilde{\sigma}^{-2}I_N - \tilde{\sigma}^{-2}\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2 I_{r^0_{m_t}} + \tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}'_{m_t}$. Thus

$$
\begin{aligned}
&\sum\nolimits_{t=1}^{T} x'_t(\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N)^{-1}x_t \\
= \;& \tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T} x'_t x_t - \tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T} x'_t\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2 I_{r^0_{m_t}} + \tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}'_{m_t}x_t.
\end{aligned}
$$

Since $(\tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1} - (\tilde{\sigma}^2 I_{r^0_{m_t}} + \tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1} = (\tilde{\sigma}^2 I_{r^0_{m_t}} + \tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}\tilde{\sigma}^2(\tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}$,

$$
\begin{aligned}
&\sum\nolimits_{t=1}^{T} x'_t(\tilde{\Lambda}_{m_t}\tilde{\Lambda}'_{m_t} + \tilde{\sigma}^2 I_N)^{-1}x_t \\
= \;& \tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T} \left\|M_{\tilde{\Lambda}_{m_t}}x_t\right\|^2 + \sum\nolimits_{t=1}^{T} x'_t\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2 I_{r^0_{m_t}} + \tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}'_{m_t}\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}'_{m_t}x_t. \qquad (31)
\end{aligned}
$$

Step (2): Now consider $l(\Lambda^0, \tilde{\sigma}^2, Q, \phi)$. Since $\Pr(z_t\,|z_{t-1}; Q) \geq \min_{j,k} Q_{jk}$,

$$
\sum\nolimits_{z_t=1}^{J^0} L(x_t\,\middle|z_t; \Lambda^0, \tilde{\sigma}^2)\Pr(z_t\,|z_{t-1}; Q) \geq L(x_t\,\middle|z_t; \Lambda^0, \tilde{\sigma}^2)\min_{j,k} Q_{jk}.
$$

Note that z_t denotes the true state on the right hand side. The left hand side has J^0 terms in the summation, and the inequality follows from throwing away all the other $J^0 - 1$ terms. Thus similar to inequality (29),

$$
\begin{aligned}
l(\Lambda^0, \tilde{\sigma}^2, Q, \phi) &\geq \sum\nolimits_{t=1}^{T} \log L(x_t\,|z_t; \Lambda^0, \tilde{\sigma}^2)\min_{j,k} Q_{jk} \\
&= T\log\min_{j,k} Q_{jk} - \frac{NT}{2}\log 2\pi - \frac{1}{2}\sum\nolimits_{t=1}^{T}\log\left|\Lambda^0_{z_t}\Lambda^{0\prime}_{z_t} + \tilde{\sigma}^2 I_N\right| \\
&\quad -\frac{1}{2}\sum\nolimits_{t=1}^{T} x'_t(\Lambda^0_{z_t}\Lambda^{0\prime}_{z_t} + \tilde{\sigma}^2 I_N)^{-1}x_t, \qquad (32)
\end{aligned}
$$

and similar to equation (31),

$$\begin{aligned}
&\sum\nolimits_{t=1}^{T} x_t'(\Lambda_{z_t}^0\Lambda_{z_t}^{0\prime}+\tilde{\sigma}^2 I_N)^{-1}x_t \\
=\ & \tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T}\left\|M_{\Lambda_{z_t}^0}x_t\right\|^2+\sum\nolimits_{t=1}^{T}x_t'\Lambda_{z_t}^0(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}(\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}\Lambda_{z_t}^{0\prime}x_t. \quad (33)
\end{aligned}$$

Step (3): $l(\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)-l(\Lambda^0,\tilde{\sigma}^2,Q,\phi)\geq 0$. Thus by equations (30)-(33), we have

$$\begin{aligned}
&\frac{1}{2}[\tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}x_t\right\|^2-\tilde{\sigma}^{-2}\sum\nolimits_{t=1}^{T}\left\|M_{\Lambda_{z_t}^0}x_t\right\|^2] \\
\leq\ & -T\log\min_{j,k}Q_{jk}-\frac{1}{2}\sum\nolimits_{t=1}^{T}\log\frac{\left|\tilde{\Lambda}_{m_t}\tilde{\Lambda}_{m_t}'+\tilde{\sigma}^2 I_N\right|}{\left|\Lambda_{z_t}^0\Lambda_{z_t}^{0\prime}+\tilde{\sigma}^2 I_N\right|} \\
& -\frac{1}{2}\sum\nolimits_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2 I_{r_{m_t}^0}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'x_t \\
& +\frac{1}{2}\sum\nolimits_{t=1}^{T}x_t'\Lambda_{z_t}^0(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}(\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}\Lambda_{z_t}^{0\prime}x_t. \quad (34)
\end{aligned}$$

(3.1) The first term on the right hand side is $O(T)$ since $\min_{j,k}Q_{jk}>0$.

(3.2) The second term on the right hand side equals $-\frac{1}{2}\sum_{t=1}^{T}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t}+I_{r_{m_t}^0}\right|+\frac{1}{2}\sum_{t=1}^{T}\log\left|\frac{1}{\tilde{\sigma}^2}\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0+I_{r_{z_t}^0}\right|$ since

$$\begin{aligned}
\left|\Lambda_{z_t}^0\Lambda_{z_t}^{0\prime}+\tilde{\sigma}^2 I_N\right| &= \tilde{\sigma}^{2N}\left|\frac{1}{\tilde{\sigma}^2}\Lambda_{z_t}^0\Lambda_{z_t}^{0\prime}+I_N\right|=\tilde{\sigma}^{2N}\left|\frac{1}{\tilde{\sigma}^2}\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0+I_{r_{z_t}^0}\right|, \quad (35)\\
\text{and }\left|\tilde{\Lambda}_{m_t}\tilde{\Lambda}_{m_t}'+\tilde{\sigma}^2 I_N\right| &= \tilde{\sigma}^{2N}\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t}+I_{r_{m_t}^0}\right|. \quad (36)
\end{aligned}$$

$-\frac{1}{2}\sum_{t=1}^{T}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t}+I_{r_{m_t}^0}\right|$ is negative, thus inequality (34) still holds when this term is thrown away. By Assumption 2(1), $\left|\frac{1}{\tilde{\sigma}^2}\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0+I_{r_{z_t}^0}\right|\leq c(\frac{N}{\tilde{\sigma}^2})^{r_{z_t}^0}$ for some $c>0$, thus $\frac{1}{2}\sum_{t=1}^{T}\log\left|\frac{1}{\tilde{\sigma}^2}\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0+I_{r_{z_t}^0}\right|$ is $O_p(T\log N)$.

(3.3) The third term on the right hand side is negative, thus inequality (34) still holds when this term is thrown away.

(3.4) The fourth term is bounded by $\frac{1}{2}\sum_{t=1}^{T}\|x_t\|^2\left\|(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}\right\|$ since $\left\|(\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-\frac{1}{2}}\Lambda_{z_t}^{0\prime}x_t\right\|=\left\|P_{\Lambda_{z_t}^0}x_t\right\|\leq\|x_t\|$ and $(\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{\frac{1}{2}}(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}(\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-\frac{1}{2}}=(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}$. The latter is because $\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0$ and $\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0$ have the same eigenvectors. By Assumption 2(1), $\left\|(\tilde{\sigma}^2 I_{r_{z_t}^0}+\Lambda_{z_t}^{0\prime}\Lambda_{z_t}^0)^{-1}\right\|\leq\sup_j\left\|(\Lambda_j^{0\prime}\Lambda_j^0)^{-1}\right\|=O_p(\frac{1}{N})$.

By Assumptions 1(2), 2(1) and 3(1), $\sum_{t=1}^{T}\|x_t\|^2 = O_p(NT)$. Thus the fourth term is $O_p(T)$.

(3.5) Now consider the left hand side of expression (34). Since $x_t = \Lambda^0_{z_t} f^0_t + e_t$ and $M_{\Lambda^0_{z_t}} \Lambda^0_{z_t} f^0_t = 0$, it is easy to verify that the left hand side equals

$$\frac{1}{2}\tilde{\sigma}^{-2}[\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t\right\|^2 + 2\sum_{t=1}^{T} e_t' M_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t + \sum_{t=1}^{T}\left\|P_{\Lambda^0_{z_t}} e_t\right\|^2 - \sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_{m_t}} e_t\right\|^2]. \tag{37}$$

For the fourth term of expression (37), we have

$$\begin{aligned}\sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_{m_t}} e_t\right\|^2 &\leq \sum_{j=1}^{J^0}\sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_j} e_t\right\|^2 = \sum_{j=1}^{J^0}\sum_{t=1}^{T} e_t'\tilde{\Lambda}_j(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j' e_t \\ &= \sum_{j=1}^{J^0} tr[(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-\frac{1}{2}}\tilde{\Lambda}_j'(\sum_{t=1}^{T} e_t e_t')\tilde{\Lambda}_j(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-\frac{1}{2}}] \\ &\leq \sum_{j=1}^{J^0} r^0_j \rho_{\max}(\sum_{t=1}^{T} e_t e_t') = \sum_{j=1}^{J^0} r^0_j \|E'E\| \\ &= O_p(N^{\frac{1}{2}}T + NT^{\frac{1}{2}}). \end{aligned} \tag{38}$$

The last equality follows from Lemma 1. Similarly, the third term of expression (37) is $O_p(N^{\frac{1}{2}}T + NT^{\frac{1}{2}})$. The second term of expression (37) equals $2\sum_{t=1}^{T} e_t'\Lambda^0_{z_t}f^0_t - 2\sum_{t=1}^{T} e_t' P_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t$. Since $\mathbb{E}(\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda^0_{ji}e_{it}\right\|^2) \leq M$ for all j and t by Assumptions 2(1), 3(1) and 3(2), and $\sum_{t=1}^{T}\|f^0_t\|^2 = O_p(T)$ by Assumption 1(2), we have

$$\left\|\sum_{t=1}^{T} e_t'\Lambda^0_{z_t}f^0_t\right\| \leq (\sum_{t=1}^{T}\|e_t'\Lambda^0_{z_t}\|^2)^{\frac{1}{2}}(\sum_{t=1}^{T}\|f^0_t\|^2)^{\frac{1}{2}} = O_p(N^{\frac{1}{2}}T).$$

By expression (38), Assumption 1(2) and Assumption 2(1), we have

$$\left\|\sum_{t=1}^{T} e_t' P_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t\right\| \leq (\sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_{m_t}} e_t\right\|^2\sum_{t=1}^{T}\|f^0_t\|^2)^{\frac{1}{2}}\sup_j\left\|\Lambda^0_j\right\| = O_p(N^{\frac{3}{4}}T + NT^{\frac{3}{4}}).$$

Thus the second term of expression (37) is $O_p(N^{\frac{3}{4}}T + NT^{\frac{3}{4}})$.

(3.6) Move the second to the fourth term of expression (37) to the right hand side of equation (34), and take the results (3.1)-(3.5) together, we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\tilde{\sigma}^{-2}\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t\right\|^2 \leq O(T) + O_p(T\log N) \\ &\quad + O(T) + O_p(N^{\frac{1}{2}}T + NT^{\frac{1}{2}}) + O_p(N^{\frac{3}{4}}T + NT^{\frac{3}{4}}). \end{aligned}$$

Thus $\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda_{z_t}^{0}f_t^{0}\right\|^{2}$ is $O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}})$. In the summation, there are q_1^0T terms[11] with $\Lambda_{z_t}^0=\Lambda_1^0$, since q_1^0 is the unconditional probability of $z_t=1$. For each t with $z_t=1$, $\Lambda_1^0f_t^0$ are projected on one of $\tilde{\Lambda}_j, j=1,...,J^0$, thus there exists one certain $\tilde{\Lambda}_j$ such that $\Lambda_1^0f_t^0$ is projected on $\tilde{\Lambda}_j$ at least $\frac{q_1^0T}{J^0}$ times. Define this $\tilde{\Lambda}_j$ as $\tilde{\Lambda}_1$, then $\sum_{t=1}^{T}1_{m_t=1}1_{z_t=1}\geq\frac{q_1^0T}{J^0}$. Thus by Assumption 1(1),

$$\rho_{\min}\left(\frac{1}{\sum_{t=1}^{T}1_{m_t=1}1_{z_t=1}}\sum_{t=1}^{T}f_t^0f_t^{0\prime}1_{m_t=1}1_{z_t=1}\right)\geq c$$

for some $c>0$ w.p.a.1. Since $\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda_{z_t}^{0}f_t^{0}\right\|$ is positive for any z_t and m_t, we have

$$\begin{aligned}
O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}}) &= \sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda_{z_t}^{0}f_t^{0}\right\|^{2}\geq\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_1}\Lambda_1^0f_t^0\right\|^21_{m_t=1}1_{z_t=1}\\
&= tr(\Lambda_1^{0\prime}M_{\tilde{\Lambda}_1}\Lambda_1^0\sum_{t=1}^{T}f_t^0f_t^{0\prime}1_{m_t=1}1_{z_t=1})\\
&\geq tr(\Lambda_1^{0\prime}M_{\tilde{\Lambda}_1}\Lambda_1^0)\rho_{\min}(\sum_{t=1}^{T}f_t^0f_t^{0\prime}1_{m_t=1}1_{z_t=1})\\
&\geq tr(\Lambda_1^{0\prime}M_{\tilde{\Lambda}_1}\Lambda_1^0)\frac{Tq_1^0}{J^0}c\text{ w.p.a.1},
\end{aligned}$$

thus $\frac{1}{N}\left\|M_{\tilde{\Lambda}_1}\Lambda_1^0\right\|_F^2=\frac{1}{N}tr(\Lambda_1^{0\prime}M_{\tilde{\Lambda}_1}\Lambda_1^0)=O_p(\frac{1}{\sqrt{\delta_{NT}}})$. Similarly, for $j=2,...,J^0$, we also have $\frac{1}{N}\left\|M_{\tilde{\Lambda}_j}\Lambda_j^0\right\|_F^2=O_p(\frac{1}{\sqrt{\delta_{NT}}})$. ■

B Details for Theorem 2

Lemma 2 *Under the assumptions of Theorem 1,* $\frac{1}{\tilde{\sigma}^2+\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl}}=O_p(\frac{1}{\sqrt{\delta_{NT}}})$ *for each* $j=1,...,J^0$ *and each* $l=1,...,r_j^0$, *where* $\tilde{\Lambda}_{jl}$ *denotes the* l*-th column of* $\tilde{\Lambda}_j$.

Proof. (1) Consider expression (34). In step (3.1), (3.2) and (3.4) of proof of Theorem 1, we have shown that the first, the second, and the fourth term on the right hand side of expression (34) is $O_p(T)$, $O_p(T\log N)$ and $O_p(T)$ respectively. In step (3.5) we have shown that the left hand side of expression (34) equals expression (37), and the last three terms of expression (37) together is $O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}})$. Move the last three terms of expression (37) to the right hand side of expression (34), and move the third

[11] Rigorously speaking, there are $\sum_{t=1}^{T}1_{z_t=1}$ terms, but $\frac{1}{T}\sum_{t=1}^{T}1_{z_t=1}\xrightarrow{p}q_1^0$ as $T\to\infty$.

term on the right hand side of expression (34) to the left hand side, then we have

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\tilde{\sigma}^2}[\sum\nolimits_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t}}\Lambda^0_{z_t}f^0_t\right\|^2+\frac{1}{2}\sum\nolimits_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'x_t\\
=\;& O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}}). \qquad (39)
\end{aligned}$$

The two terms on the left hand side of (39) are nonnegative, thus $\sum_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'x_t=O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}})$.

(2)

$$\begin{aligned}
&\left\|\sum\nolimits_{t=1}^{T}e_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'\Lambda^0_{z_t}f^0_t\right\|\\
\leq\;&(\sum\nolimits_{t=1}^{T}e_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-2}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'e_t)^{\frac{1}{2}}(\sum\nolimits_{t=1}^{T}\left\|\Lambda^0_{z_t}f^0_t\right\|^2)^{\frac{1}{2}}\\
\leq\;&\frac{1}{\tilde{\sigma}^4}(\sum\nolimits_{t=1}^{T}e_t'P_{\tilde{\Lambda}_{m_t}}e_t)^{\frac{1}{2}}(\sum\nolimits_{t=1}^{T}\left\|f^0_t\right\|^2)^{\frac{1}{2}}\sup_j\left\|\Lambda^0_j\right\|=O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}}),
\end{aligned}$$

where the first inequality follows from Cauchy-Schwarz inequality, the second inequality follows from the fact that $\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t}$ is diagonal and all diagonal elements of $\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t}$ are larger than $\tilde{\sigma}^2$, and the equality follows from Assumption 1(2), Assumption 2(1) and expression (38) in step (3.5) of proof of Theorem 1.

(3) It follows from (1) and (2) that

$$\begin{aligned}
&\sum\nolimits_{t=1}^{T}f^{0\prime}_t\Lambda^{0\prime}_{z_t}\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'\Lambda^0_{z_t}f^0_t\\
&+\sum\nolimits_{t=1}^{T}e_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'e_t\\
=\;&\sum\nolimits_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'x_t\\
&-2\sum\nolimits_{t=1}^{T}e_t'\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'\Lambda^0_{z_t}f^0_t\\
=\;&O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}}). \qquad (40)
\end{aligned}$$

The two terms on the left hand side of (40) are nonnegative, thus $\sum_{t=1}^{T}f^{0\prime}_t\Lambda^{0\prime}_{z_t}\tilde{\Lambda}_{m_t}(\tilde{\sigma}^2I_{r^0_{m_t}}+\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}(\tilde{\Lambda}_{m_t}'\tilde{\Lambda}_{m_t})^{-1}\tilde{\Lambda}_{m_t}'\Lambda^0_{z_t}f^0_t=O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}})$. Since each term in the summation is nonnegative, we have $\sum_{t=1}^{T}f^{0\prime}_t\Lambda^{0\prime}_j\tilde{\Lambda}_j(\tilde{\sigma}^2I_{r^0_j}+\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j'\Lambda^0_jf^0_t1_{z_t=j}1_{m_t=j}=O_p(N^{\frac{3}{4}}T+NT^{\frac{3}{4}})$ for each j.

As explained in step (3.6) of proof of Theorem 1, $\sum_{t=1}^{T}1_{m_t=j}1_{z_t=j}\geq\frac{q^0_jT}{J^0}$, and by Assumption 1(1), $\rho_{\min}(\frac{1}{\sum_{t=1}^{T} 1_{m_t=j}1_{z_t=j}} \sum_{t=1}^{T} f_t^0 f_t^{0\prime} 1_{m_t=j} 1_{z_t=j}) \geq c$ for some $c > 0$ w.p.a.1. Thus we have

$$\begin{aligned}
O_p(N^{\frac{3}{4}}T + NT^{\frac{3}{4}}) &= \sum_{t=1}^{T} f_t^{0\prime} \Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0 f_t^0 1_{z_t=j} 1_{m_t=j} \\
&= tr(\Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0 \sum_{t=1}^{T} f_t^0 f_t^{0\prime} 1_{z_t=j} 1_{m_t=j}) \\
&\geq tr(\Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0) \rho_{\min}(\sum_{t=1}^{T} f_t^0 f_t^{0\prime} 1_{m_t=j} 1_{z_t=j}) \\
&\geq tr(\Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0) \frac{T q_j^0}{J^0} c \text{ w.p.a.1.}
\end{aligned}$$

Thus $tr(\Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0) = O_p(\frac{N}{\sqrt{\delta_{NT}}})$ for each j.

(4) Noting that $\tilde{\Lambda}_{jl}$ is orthogonal to $\tilde{\Lambda}_{jl'}$ for $l \neq l'$, we have

$$\sum_{l=1}^{r_j^0} \left\| P_{\tilde{\Lambda}_{jl}} \Lambda_j^0 \right\|_F^2 = \left\| P_{\tilde{\Lambda}_j} \Lambda_j^0 \right\|_F^2 = \left\| \Lambda_j^0 \right\|_F^2 - \left\| M_{\tilde{\Lambda}_j} \Lambda_j^0 \right\|_F^2, \tag{41}$$

$$\sum_{l=1}^{r_j^0} \frac{1}{\tilde{\sigma}^2 + \tilde{\Lambda}_{jl}' \tilde{\Lambda}_{jl}} \left\| P_{\tilde{\Lambda}_{jl}} \Lambda_j^0 \right\|_F^2 = tr(\Lambda_j^{0\prime} \tilde{\Lambda}_j (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} (\tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_j^0). \tag{42}$$

Each term in the summation on the left hand side of equation (42) is nonnegative, thus

$$\frac{1}{\tilde{\sigma}^2 + \tilde{\Lambda}_{jl}' \tilde{\Lambda}_{jl}} \left\| P_{\tilde{\Lambda}_{jl}} \Lambda_j^0 \right\|_F^2 = O_p(\frac{N}{\sqrt{\delta_{NT}}}) \text{ for each } j \text{ and } l. \tag{43}$$

Now consider $\left\| P_{\tilde{\Lambda}_{j1}} \Lambda_j^0 \right\|_F^2$. Let $\tilde{\Lambda}_{j,-1} = (\tilde{\Lambda}_{j2}, ..., \tilde{\Lambda}_{jr_j^0})$, we have

$$\begin{aligned}
\sum_{l \neq 1} \left\| P_{\tilde{\Lambda}_{jl}} \Lambda_j^0 \right\|_F^2 &= \left\| P_{\tilde{\Lambda}_{j,-1}} \Lambda_j^0 \right\|_F^2 = tr(\Lambda_j^{0\prime} P_{\tilde{\Lambda}_{j,-1}} \Lambda_j^0) \\
&= tr[(\tilde{\Lambda}_{j,-1}' \tilde{\Lambda}_{j,-1})^{-\frac{1}{2}} \tilde{\Lambda}_{j,-1}' \Lambda_j^0 \Lambda_j^{0\prime} \tilde{\Lambda}_{j,-1} (\tilde{\Lambda}_{j,-1}' \tilde{\Lambda}_{j,-1})^{-\frac{1}{2}}] \\
&\leq \left\| \Lambda_j^0 \right\|_F^2 - \rho_{\min}(\Lambda_j^0 \Lambda_j^{0\prime}).
\end{aligned} \tag{44}$$

The inequality in expression (44) becomes equality when $\tilde{\Lambda}_{j,-1}(\tilde{\Lambda}_{j,-1}' \tilde{\Lambda}_{j,-1})^{-\frac{1}{2}}$ are eigenvectors of $\Lambda_j^0 \Lambda_j^{0\prime}$ corresponding to the largest $r_j^0 - 1$ eigenvalues. Expressions (41) and (44) together implies that $\left\| P_{\tilde{\Lambda}_{j1}} \Lambda_j^0 \right\|_F^2 \geq \rho_{\min}(\Lambda_j^0 \Lambda_j^{0\prime}) - \left\| M_{\tilde{\Lambda}_j} \Lambda_j^0 \right\|_F^2$, thus by Assumption 2(1) and Theorem 1, $\frac{1}{N} \left\| P_{\tilde{\Lambda}_{j1}} \Lambda_j^0 \right\|_F^2$ is bounded away from zero in probability. This together with expression (43) implies that $\frac{1}{\tilde{\sigma}^2 + \tilde{\Lambda}_{j1}' \tilde{\Lambda}_{j1}} = O_p(\frac{1}{\sqrt{\delta_{NT}}})$. Similarly,

$\frac{1}{\tilde{\sigma}^2+\tilde{\Lambda}'_{jl}\tilde{\Lambda}_{jl}} = O_p(\frac{1}{\sqrt{\delta_{NT}}})$ for $l = 2, ..., r_j^0$. ■

Proof of Theorem 2

Proof. Part (1):

Step (1): We first show $\left|\tilde{p}_{tj|t} - 1_{z_t=j}\right| = o_p(\frac{1}{N^\eta})$.

When $z_t = j$, since $\tilde{p}_{tj|t} = \frac{\tilde{p}_{tj|t-1} L(x_t \left| z_t=j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.)}{\sum_{k=1}^{J^0} \tilde{p}_{tk|t-1} L(x_t \left| z_t=k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.)}$, we have

$$\begin{aligned}
\left|\tilde{p}_{tj|t} - 1_{z_t=j}\right| &= \frac{\sum_{k\neq j} \tilde{p}_{tk|t-1} L(x_t \left| z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.)}{\sum_{k=1}^{J^0} \tilde{p}_{tk|t-1} L(x_t \left| z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.)} \\
&\leq \sum_{k\neq j} \frac{\tilde{p}_{tk|t-1}}{\tilde{p}_{tj|t-1}} e^{\log L(x_t \left| z_t=k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.) - \log L(x_t \left| z_t=j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.)}.
\end{aligned}$$

When $z_t = h \neq j$, since $\sum_{k=1}^{J^0} \tilde{p}_{tk|t} = 1$, we have $\tilde{p}_{tj|t} - 1_{z_t=j} = \tilde{p}_{tj|t} \leq 1 - \tilde{p}_{th|t}$, thus it suffices to show $\left|\tilde{p}_{tj|t} - 1_{z_t=j}\right| = o_p(\frac{1}{N^\eta})$ when $z_t = j$. Since $\tilde{p}_{tj|t-1} = Q_{j\cdot}\tilde{p}_{t-1|t-1} \geq \min_k Q_{jk} > 0$ for all j ($Q_{j\cdot}$ denotes the j-th row of Q), it suffices to show $\sup_t e^{\log L(x_t \left| z_t=k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.) - \log L(x_t \left| z_t=j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.)} = o_p(\frac{1}{N^\eta})$ for any $k \neq j$, i.e., it suffices to show for any fixed $M > 0$,

$$\begin{aligned}
\Pr(\sup_t[\log L(x_t \left| z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.) - \log L(x_t \left| z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.)] &\geq \log \frac{M}{N^\eta}) \to 0, \text{ or} \\
\Pr(\min_t[\log L(x_t \left| z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.) - \log L(x_t \left| z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.)] &\leq \eta \log N - \log M) \to 0. \quad (45)
\end{aligned}$$

Similar to equation (33),

$$\begin{aligned}
\log L(x_t \left| z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2 \right.) &= -\frac{N}{2}\log 2\pi - \frac{1}{2}\log\left|\tilde{\Lambda}_j\tilde{\Lambda}'_j + \tilde{\sigma}^2 I_N\right| - \frac{1}{2}\tilde{\sigma}^{-2}\left\|M_{\tilde{\Lambda}_j}x_t\right\|^2 \\
&\quad -\frac{1}{2}x'_t\tilde{\Lambda}_j(\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}'_j\tilde{\Lambda}_j)^{-1}(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}'_j x_t, \\
\log L(x_t \left| z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2 \right.) &= -\frac{N}{2}\log 2\pi - \frac{1}{2}\log\left|\tilde{\Lambda}_k\tilde{\Lambda}'_k + \tilde{\sigma}^2 I_N\right| - \frac{1}{2}\tilde{\sigma}^{-2}\left\|M_{\tilde{\Lambda}_k}x_t\right\|^2 \\
&\quad -\frac{1}{2}x'_t\tilde{\Lambda}_k(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}'_k\tilde{\Lambda}_k)^{-1}(\tilde{\Lambda}'_k\tilde{\Lambda}_k)^{-1}\tilde{\Lambda}'_k x_t,
\end{aligned}$$

and similar to equations (35) and (36), $\frac{\left|\tilde{\Lambda}_j\tilde{\Lambda}_j'+\tilde{\sigma}^2 I_N\right|}{\left|\tilde{\Lambda}_k\tilde{\Lambda}_k'+\tilde{\sigma}^2 I_N\right|} = \frac{\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_j'\tilde{\Lambda}_j+I_{r_j^0}\right|}{\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_k'\tilde{\Lambda}_k+I_{r_k^0}\right|}$. Thus

$$
\begin{aligned}
& \log L(x_t \,\Big|\, z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2) - \log L(x_t \,\Big|\, z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2) \\
= \; & -\frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_j'\tilde{\Lambda}_j + I_{r_j^0}\right| + \frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_k'\tilde{\Lambda}_k + I_{r_k^0}\right| \\
& +\frac{1}{2}\tilde{\sigma}^{-2}(\left\|M_{\tilde{\Lambda}_k}x_t\right\|^2 - \left\|M_{\Lambda_k^0}x_t\right\|^2 + \left\|M_{\Lambda_j^0}x_t\right\|^2 - \left\|M_{\tilde{\Lambda}_j}x_t\right\|^2 + \left\|M_{\Lambda_k^0}x_t\right\|^2 - \left\|M_{\Lambda_j^0}x_t\right\|^2) \\
& -\frac{1}{2}x_t'\tilde{\Lambda}_j(\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j'x_t + \frac{1}{2}x_t'\tilde{\Lambda}_k(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k'\tilde{\Lambda}_k)^{-1}(\tilde{\Lambda}_k'\tilde{\Lambda}_k)^{-1}\tilde{\Lambda}_k'x_t \\
\geq \; & -\frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_j'\tilde{\Lambda}_j + I_{r_j^0}\right| - \frac{1}{2}x_t'\tilde{\Lambda}_j(\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j'x_t \\
& +\frac{1}{2}\tilde{\sigma}^{-2}(\left\|M_{\tilde{\Lambda}_k}x_t\right\|^2 - \left\|M_{\Lambda_k^0}x_t\right\|^2) + \frac{1}{2}\tilde{\sigma}^{-2}(\left\|M_{\Lambda_j^0}x_t\right\|^2 - \left\|M_{\tilde{\Lambda}_j}x_t\right\|^2) \\
& -\frac{1}{2}\tilde{\sigma}^{-2}e_t'P_{\Lambda_k^0}e_t + \tilde{\sigma}^{-2}e_t'M_{\Lambda_k^0}\Lambda_j^0 f_t^0 + \frac{1}{2}\tilde{\sigma}^{-2}f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0,
\end{aligned} \tag{46}
$$

where the inequality follows from throwing away $\frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_k'\tilde{\Lambda}_k + I_{r_k^0}\right|$, $\frac{1}{2}x_t'\tilde{\Lambda}_k(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k'\tilde{\Lambda}_k)^{-1}(\tilde{\Lambda}_k'\tilde{\Lambda}_k)^{-1}\tilde{\Lambda}_k'x_t$ and $e_t'P_{\Lambda_j^0}e_t$. It follows that

$$
\begin{aligned}
& \min_t[\log L(x_t \,\Big|\, z_t = j; \tilde{\Lambda}_j, \tilde{\sigma}^2) - \log L(x_t \,\Big|\, z_t = k; \tilde{\Lambda}_k, \tilde{\sigma}^2)] \\
\geq \; & -\frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_j'\tilde{\Lambda}_j + I_{r_j^0}\right| - \frac{1}{2}\sup_t x_t'\tilde{\Lambda}_j(\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j'x_t \\
& -\frac{1}{2}\tilde{\sigma}^{-2}\sup_t\left|\left\|M_{\tilde{\Lambda}_k}x_t\right\|^2 - \left\|M_{\Lambda_k^0}x_t\right\|^2\right| - \frac{1}{2}\tilde{\sigma}^{-2}\sup_t\left|\left\|M_{\Lambda_j^0}x_t\right\|^2 - \left\|M_{\tilde{\Lambda}_j}x_t\right\|^2\right| \\
& -\frac{1}{2}\tilde{\sigma}^{-2}\sup_t e_t'P_{\Lambda_k^0}e_t - \tilde{\sigma}^{-2}\sup_t\left|e_t'M_{\Lambda_k^0}\Lambda_j^0 f_t^0\right| + \frac{1}{2}\tilde{\sigma}^{-2}\min_t f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \\
\equiv \; & -(A_1 + A_2 + A_3 + A_4 + A_5 + A_6) + \frac{1}{2}\tilde{\sigma}^{-2}\min_t f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0.
\end{aligned} \tag{47}
$$

Thus for expression (45), it suffices to show

$$
\Pr(\frac{1}{2}\tilde{\sigma}^{-2}\min_t f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \leq A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + \eta\log N) \to 0. \tag{48}
$$

By Assumption 2(2), $\min_t f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \geq NC$ for some $C > 0$. Thus it suffices to show that $A_1, ..., A_6$ are all $o_p(N)$ when $T^{\frac{16}{\alpha}}/N \to 0$ and $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$.

Term A_1: As shown in equation (15), $\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl} + \tilde{\sigma}^2$ is an eigenvalue of $\tilde{S}_j =$

$\sum_{t=1}^{T}\tilde{p}_{tj|T}\,x_t x_t'/\sum_{t=1}^{T}\tilde{p}_{tj|T}$, which is bounded by $\sup_t\|x_t\|^2$. We next show that $\sup_t\|x_t\| = O_p(N^{\frac{1}{2}}T^{\frac{1}{\alpha}})$. By Assumption 2(1) and 1(2), $\sup_t\left\|\Lambda_{z_t}^0 f_t^0\right\|^{\alpha} \leq \sup_j\left\|\Lambda_j^0\right\|^{\alpha}\sum_{t=1}^{T}\|f_t^0\|^{\alpha} = N^{\frac{\alpha}{2}}T$. By Holder inequality, $\|e_t\|^2 = \sum_{i=1}^{N}e_{it}^2 \leq (\sum_{i=1}^{N}e_{it}^{\alpha})^{\frac{2}{\alpha}}N^{1-\frac{2}{\alpha}}$, thus $\sup_t\|e_t\|^{\alpha} \leq N^{\frac{\alpha}{2}-1}\sup_t(\sum_{i=1}^{N}e_{it}^{\alpha}) \leq N^{\frac{\alpha}{2}-1}\sum_{t=1}^{T}\sum_{i=1}^{N}e_{it}^{\alpha} = O_p(N^{\frac{\alpha}{2}}T)$ by Assumption 3(1). It follows that $\sup_t\|x_t\| \leq \sup_t\left\|\Lambda_{z_t}^0 f_t^0\right\| + \sup_t\|e_t\| = O_p(N^{\frac{1}{2}}T^{\frac{1}{\alpha}})$. Thus

$$
\begin{aligned}
A_1 &= \frac{1}{2}\log\left|\frac{1}{\tilde{\sigma}^2}\tilde{\Lambda}_j'\tilde{\Lambda}_j + I_{r_j^0}\right| = \frac{1}{2}\sum_{l=1}^{r_j^0}\log\frac{\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl}+\tilde{\sigma}^2}{\tilde{\sigma}^2} \\
&\leq \frac{1}{2}r_j^0\log\frac{\sup_t\|x_t\|^2}{\tilde{\sigma}^2} = O_p(\log NT^{\frac{2}{\alpha}}) = o_p(N) \text{ when } \frac{\log T}{N}\to 0.
\end{aligned}
$$

Term A_2: By Lemma 2, $\frac{1}{\tilde{\sigma}^2+\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl}} = O_p(\frac{1}{\sqrt{\delta_{NT}}})$ for each j and each l. We have shown for term A_1 that $\sup_t\|x_t\| = O_p(N^{\frac{1}{2}}T^{\frac{1}{\alpha}})$. Thus

$$
\begin{aligned}
A_2 &\leq \frac{1}{2}\sup_t(x_t'P_{\tilde{\Lambda}_j}x_t\sup_l\frac{1}{\tilde{\sigma}^2+\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl}}) \leq \frac{1}{2}\sup_t\|x_t\|^2\sup_l\frac{1}{\tilde{\sigma}^2+\tilde{\Lambda}_{jl}'\tilde{\Lambda}_{jl}} \\
&= O_p(NT^{\frac{2}{\alpha}})O_p(\frac{1}{\sqrt{\delta_{NT}}}) = o_p(N) \text{ when } T^{\frac{8}{\alpha}}/N\to 0 \text{ and } \alpha > 8.
\end{aligned}
$$

Term A_3:

$$
\begin{aligned}
\left\|P_{\Lambda_k^0} - P_{\tilde{\Lambda}_k}\right\|^2 &\leq \left\|P_{\Lambda_k^0} - P_{\tilde{\Lambda}_k}\right\|_F^2 = tr[(P_{\Lambda_k^0} - P_{\tilde{\Lambda}_k})^2] \\
&= 2tr(I_{r_k^0} - P_{\Lambda_k^0}P_{\tilde{\Lambda}_k}) = 2\left\|M_{\tilde{\Lambda}_k}\Lambda_k^0(\Lambda_k^{0\prime}\Lambda_k^0)^{-\frac{1}{2}}\right\|_F^2 \\
&\leq 2\frac{1}{N}\left\|M_{\tilde{\Lambda}_k}\Lambda_k^0\right\|_F^2\left\|(\frac{1}{N}\Lambda_k^{0\prime}\Lambda_k^0)^{-\frac{1}{2}}\right\|_F^2 = O_p(\frac{1}{\sqrt{\delta_{NT}}}), \qquad (49)
\end{aligned}
$$

where the last equality follows from Theorem 1 and Assumption 2(1). We have shown for term A_1 that $\sup_t\|x_t\| = O_p(N^{\frac{1}{2}}T^{\frac{1}{\alpha}})$. Thus

$$
\begin{aligned}
A_3 &= \frac{1}{2}\tilde{\sigma}^{-2}\sup_t\left|x_t'(P_{\Lambda_k^0} - P_{\tilde{\Lambda}_k})x_t\right| \leq \frac{1}{2}\tilde{\sigma}^{-2}\left\|P_{\Lambda_k^0} - P_{\tilde{\Lambda}_k}\right\|\sup_t\|x_t\|^2 \\
&= O_p(\delta_{NT}^{-\frac{1}{4}})NT^{\frac{2}{\alpha}} = o_p(N) \text{ when } T^{\frac{16}{\alpha}}/N\to 0 \text{ and } \alpha > 16.
\end{aligned}
$$

Similar to term A_3, Term A_4 is also $o_p(N)$ when $T^{\frac{16}{\alpha}}/N\to 0$ and $\alpha > 16$.

Term A_5: By Assumption 5(1), $\sup_t \left\|\frac{\Lambda_k^{0\prime}e_t}{\sqrt{N}}\right\|^\beta \leq \sum_{t=1}^T \left\|\frac{\Lambda_k^{0\prime}e_t}{\sqrt{N}}\right\|^\beta = O_p(T)$. Thus

$$A_5 \leq \frac{1}{2}\tilde{\sigma}^{-2}\left\|(\frac{1}{N}\Lambda_k^{0\prime}\Lambda_k^0)^{-1}\right\| \sup_t \left\|\frac{\Lambda_k^{0\prime}e_t}{\sqrt{N}}\right\|^2 = O_p(T^{\frac{2}{\beta}}) = o_p(N) \text{ when } T^{\frac{2}{\beta}}/N \to 0.$$

Term A_6: By Assumption 1(2), $\sup_t \|f_t^0\|^\alpha \leq \sum_{t=1}^T \|f_t^0\|^\alpha = O_p(T)$, thus $\sup_t \|f_t^0\| = O_p(T^{\frac{1}{\alpha}})$. We have shown for term A_5 that $\sup_t \left\|\frac{\Lambda_j^{0\prime}e_t}{\sqrt{N}}\right\| = O_p(T^{\frac{1}{\beta}})$. Thus

$$\begin{aligned} A_6 &\leq \tilde{\sigma}^{-2}\sup_t \left|e_t'\Lambda_j^0 f_t^0\right| + \tilde{\sigma}^{-2}\sup_t \left|e_t'\Lambda_k^0(\frac{\Lambda_k^{0\prime}\Lambda_k^0}{N})^{-1}\frac{\Lambda_k^{0\prime}\Lambda_j^0}{N}f_t^0\right| \\ &\leq \tilde{\sigma}^{-2}\sup_t \left\|e_t'\Lambda_j^0\right\| \sup_t \left\|f_t^0\right\| (1 + \left\|(\frac{\Lambda_k^{0\prime}\Lambda_k^0}{N})^{-1}\right\| \left\|\frac{\Lambda_k^{0\prime}\Lambda_j^0}{N}\right\|) \\ &= O_p(N^{\frac{1}{2}}T^{\frac{1}{\alpha}+\frac{1}{\beta}}) = o_p(N) \text{ when } T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0. \end{aligned}$$

Step (2): We next prove $\tilde{p}_{tk|T} = o_p(\frac{1}{N^\eta})$ for $k \neq j$ when the true state is $z_t = j$. Let $Q_{\cdot k}$ denote the k-th column of Q and "$\div$" denotes element-wise division for two vectors.

$$\begin{aligned} \tilde{p}_{tk|T} &= \tilde{p}_{tk|t} \times Q_{\cdot k}'(\tilde{p}_{t+1|T} \div \tilde{p}_{t+1|t}) = \tilde{p}_{tk|t} \times \tilde{p}_{t+1|T}'(Q_{\cdot k} \div \tilde{p}_{t+1|t}) \\ &\leq \tilde{p}_{tk|t}\max_l \frac{Q_{lk}}{Q_{lj}}\frac{1}{\tilde{p}_{tj|t}} = o_p(\frac{1}{N^\eta}), \end{aligned}$$

where the inequality is due to the fact that each element of $\tilde{p}_{t+1|t} = Q\tilde{p}_{t|t}$ is not smaller than $Q_{\cdot j}\tilde{p}_{tj|t}$ and the last equality follows from step (1) and $\min_l Q_{lj} > 0$.

Part (2):

Similar to expression (48), it suffices to show

$$\Pr(\frac{1}{2}\tilde{\sigma}^{-2}f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \leq A_1' + A_2' + A_3' + A_4' + A_5' + A_6' + \eta\log N) \to 0, \quad (50)$$

where $A_1', ..., A_6'$ equals $A_1, ..., A_6$ without taking supremum with respect to t. Given the calculation of terms $A_1, ..., A_6$, it is not difficult to see that without taking supremum, $A_1', ..., A_6'$ becomes $O_p(\log N)$, $O_p(\frac{N}{\sqrt{\delta_{NT}}})$, $O_p(\frac{N}{\delta_{NT}^{\frac{1}{4}}})$, $O_p(\frac{N}{\delta_{NT}^{\frac{1}{4}}})$, $O_p(1)$ and $O_p(N^{\frac{1}{2}})$ respectively. Since $f_t^{0\prime}\Lambda_j^{0\prime}M_{\Lambda_k^0}\Lambda_j^0 f_t^0 \geq NC$ for some $C > 0$, $A_1', ..., A_6'$ are all dominated by this term. ■

C Details for Theorem 3 and Theorem 4

Proof of Proposition 1

Proof. (1) Let V_{jNT} be an $r_j^0 \times r_j^0$ diagonal matrix consisting of eigenvalues of $\frac{(\Lambda_j^{0\prime}\Lambda_j^0)^{\frac{1}{2}}(\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j})(\Lambda_j^{0\prime}\Lambda_j^0)^{\frac{1}{2}}}{NTq_j^0}$ in descending order and Υ_{jNT} be the corresponding eigenvectors. Let $\bar{\Lambda}_j^0 = \Lambda_j^0(\Lambda_j^{0\prime}\Lambda_j^0)^{-\frac{1}{2}}\Upsilon_{jNT}$ be the normalized version of Λ_j^0, then $\bar{\Lambda}_j^{0\prime}\bar{\Lambda}_j^0 = I_{r_j^0}$. Let $\check{\Lambda}_j = \tilde{\Lambda}_j(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{-\frac{1}{2}}$ be the normalized version of $\tilde{\Lambda}_j$, then $\check{\Lambda}_j'\check{\Lambda}_j = I_{r_j^0}$.

From equation (15), we have $\check{\Lambda}_j W_{jNT} = (\frac{1}{NT}\sum_{t=1}^T \tilde{p}_{tj|T} x_t x_t')\check{\Lambda}_j$. The left hand side equals $P_{\bar{\Lambda}_j^0}\check{\Lambda}_j W_{jNT} + M_{\bar{\Lambda}_j^0}\check{\Lambda}_j W_{jNT} = \bar{\Lambda}_j^0\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} + M_{\bar{\Lambda}_j^0}\check{\Lambda}_j W_{jNT}$. The right hand side equals

$$
\begin{aligned}
& \Lambda_j^0 \frac{(\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j})\Lambda_j^{0\prime}\check{\Lambda}_j}{NT} + \frac{\sum_{t=1}^T \mathbb{E}(e_t e_t') 1_{z_t=j}\check{\Lambda}_j}{NT} + \frac{\sum_{t=1}^T (e_t e_t' - \mathbb{E}(e_t e_t')) 1_{z_t=j}\check{\Lambda}_j}{NT} \\
& + \frac{\sum_{t=1}^T e_t f_t^{0\prime} 1_{z_t=j}\Lambda_j^{0\prime}\check{\Lambda}_j}{NT} + \frac{\Lambda_j^0 \sum_{t=1}^T f_t^0 e_t' 1_{z_t=j}\check{\Lambda}_j}{NT} + \frac{\sum_{t=1}^T (\tilde{p}_{tj|T} - 1_{z_t=j}) x_t x_t'}{NT}\check{\Lambda}_j \\
\equiv \; & \Lambda_j^0 \frac{(\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j})\Lambda_j^{0\prime}\check{\Lambda}_j}{NT} + I + II + III + IV + D. \qquad (51)
\end{aligned}
$$

Note that $\Lambda_j^0 \frac{(\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j})\Lambda_j^{0\prime}\check{\Lambda}_j}{NT} = \bar{\Lambda}_j^0 q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$, thus we have

$$
\bar{\Lambda}_j^0(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j) + M_{\bar{\Lambda}_j^0}\check{\Lambda}_j W_{jNT} = I + II + III + IV + D \qquad (52)
$$

Terms $I, ..., IV$ correspond to the right hand of equation (A.1) in Bai (2003). By Assumption 3(2), $\|I\|_F^2 = O_p(\frac{1}{N})$. By Assumption 3(4), $\|II\|_F^2 = O_p(\frac{1}{T})$. By Assumptions 5(2) and 2(1), $\|III\|_F^2$ and $\|IV\|_F^2$ are $O_p(\frac{1}{T})$. The detailed calculation is similar to the proof of Theorem 1 in Bai and Ng (2002), hence omitted here. Now consider term D. Since $\left\|\frac{\sum_{t=1}^T(\tilde{p}_{tj|T} - 1_{z_t=j})x_t x_t'}{NT}\right\| \leq \frac{\sum_{t=1}^T \left|\tilde{p}_{tj|T} - 1_{z_t=j}\right| \|x_t\|^2}{NT} \leq \sup_t \left|\tilde{p}_{tj|T} - 1_{z_t=j}\right| \frac{\sum_{t=1}^T \|x_t\|^2}{NT}$,

$$
\begin{aligned}
\|D\|_F & \leq \left\|\frac{\sum_{t=1}^T (\tilde{p}_{tj|T} - 1_{z_t=j}) x_t x_t'}{NT}\right\| \left\|\check{\Lambda}_j\right\|_F \\
& \leq \sqrt{r_j^0} \sup_t \left|\tilde{p}_{tj|T} - 1_{z_t=j}\right| \frac{\sum_{t=1}^T \|x_t\|^2}{NT} = o_p(\frac{1}{N^\eta}). \qquad (53)
\end{aligned}
$$

The last equality follows from Theorem 2 and $\frac{\sum_{t=1}^{T}\|x_t\|^2}{NT} = O_p(1)$, which can be easily shown using Assumptions 1(2), 2(1) and 3(1). In summary, the right hand side of equation (52) is $O_p(\frac{1}{\delta_{NT}})$. The two terms on the left hand side[12] are orthogonal to each other, thus both $\left\|M_{\bar{\Lambda}_j^0}\check{\Lambda}_j W_{jNT}\right\|_F$ and $\left\|\bar{\Lambda}_j^0(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)\right\|_F$ are $O_p(\frac{1}{\delta_{NT}})$. Since $\|A\|_F^2 = tr(A'A)$ for any matrix A and $\bar{\Lambda}_j^0$ is orthonormal,

$$\left\|\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j\right\|_F = \left\|\bar{\Lambda}_j^0(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)\right\|_F = o_p(1). \quad (54)$$

We next show that equation (54) implies that $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j \xrightarrow{p} I_{r_j^0}$ and $W_{jNT} \xrightarrow{p} q_j^0 V_j$.

First, the Euclidean norm of each column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ converges in probability to 1 and the inner product of different columns converges in probability to 0, because

$$\begin{aligned} \left\|I_{r_j^0} - \check{\Lambda}_j'\bar{\Lambda}_j^0\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j\right\|_F &\leq \sqrt{r_j^0}\left\|I_{r_j^0} - \check{\Lambda}_j'\bar{\Lambda}_j^0\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j\right\| \leq \sqrt{r_j^0}tr(I_{r_j^0} - \check{\Lambda}_j'\bar{\Lambda}_j^0\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j) \\ &= \sqrt{r_j^0}\left\|M_{\bar{\Lambda}_j^0}\check{\Lambda}_j\right\|_F^2 = \sqrt{r_j^0}\left\|M_{\check{\Lambda}_j}\bar{\Lambda}_j^0\right\|_F^2 = o_p(1). \end{aligned} \quad (55)$$

The second inequality follows from the fact that $I_{r_j^0} - \check{\Lambda}_j'\bar{\Lambda}_j^0\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ is positive semi-definite. The second to last equality follows from the fact that both $\bar{\Lambda}_j^0$ and $\check{\Lambda}_j$ are orthonormal. The last equality follows from Theorem 1.

Let $V_{jNT,i}$, $W_{jNT,1}$ and $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}$ denote the i-th diagonal element of V_{jNT}, the 1st diagonal element of W_{jNT} and the $(i,1)$-th element of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$, then the first column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ is $(W_{jNT,1} - q_j^0 V_{jNT,i})(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}$, $i = 1, ..., r_j^0$. Equation (54) implies that for all $i = 1, ..., r_j^0$, $(W_{jNT,1} - q_j^0 V_{jNT,i})(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}$ is $o_p(1)$. We have shown through expression (55) that $\sum_{i=1}^{r_j^0}(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}^2 \xrightarrow{p} 1$, thus there exists at least one certain i such that $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}$ is bounded away from zero in probability. Without loss of generality, suppose $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{11}$ is bounded away from zero in probability. Since $(W_{jNT,1} - q_j^0 V_{jNT,1})(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{11}$ is $o_p(1)$, we must have $W_{jNT,1} - q_j^0 V_{jNT,1} = o_p(1)$. This implies that $W_{jNT,1} - q_j^0 V_{jNT,i}$ is bounded away from zero in probability for $i \neq 1$ because by Assumption 6, $V_{jNT,i} \neq V_{jNT,1}$ w.p.a.1. Since $(W_{jNT,1} - q_j^0 V_{jNT,i})(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}$ is $o_p(1)$ for all i, we must have $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1} = o_p(1)$ for $i \neq 1$. This together with

[12] The left hand side of equation (52) corresponds to a further decomposition of the left hand side of equation (A.1) in Bai (2003).

$\sum_{i=1}^{r_j^0}(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i1}^2 \xrightarrow{p} 1$ implies that $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{11} \xrightarrow{p} 1$. In summary, we have shown that the first column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ converges in probability to $(1,0,...,0)$.

Similarly, for the second column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j W_{jNT} - q_j^0 V_{jNT}\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$, we can also show that one element converges in probability to 1 and the other elements are $o_p(1)$. Since the inner product of the first column and the second column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ is $o_p(1)$, $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{12}$ must be $o_p(1)$. Thus $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i2} \xrightarrow{p} 1$ for certain $i \neq 1$ and $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i2} = o_p(1)$ for all the other i. Without loss of generality, suppose $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{22} \xrightarrow{p} 1$ and $(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i2} = o_p(1)$ for $i \neq 2$. Since $(W_{jNT,2} - q_j^0 V_{jNT,i})(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)_{i2}$ is $o_p(1)$ for all i, we must have $W_{jNT,2} - q_j^0 V_{jNT,2} = o_p(1)$.

Similarly, the third column of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$ converges in probability to $(0,0,1,...,0)$ and $W_{jNT,3} - q_j^0 V_{jNT,3} = o_p(1)$. Repeat the argument for all columns of $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j$, we have $\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j \xrightarrow{p} I_{r_j^0}$ and $W_{jNT} - q_j^0 V_{jNT} = o_p(1)$. Since $V_{jNT} \xrightarrow{p} V_j$, we have $W_{jNT} \xrightarrow{p} q_j^0 V_j$.

(2) By Theorem 2(1), $\left|\frac{1}{T}\sum_{t=1}^T(\tilde{p}_{tj|T} - 1_{z_t=j})\right| \leq \sup_t \left|\tilde{p}_{tj|T} - 1_{z_t=j}\right| = o_p(\frac{1}{N^\eta})$. By Assumption 4, $\frac{1}{T}\sum_{t=1}^T 1_{z_t=j} \xrightarrow{p} q_j^0$. Thus $\frac{1}{T}\sum_{t=1}^T \tilde{p}_{tj|T} \xrightarrow{p} q_j^0$. We have shown that $W_{jNT} \xrightarrow{p} q_j^0 V_j$, thus $\frac{\tilde{\Lambda}_j'\tilde{\Lambda}_j}{N} = W_{jNT}/\frac{1}{T}\sum_{t=1}^T \tilde{p}_{tj|T} - \frac{\tilde{\sigma}^2}{N} I_{r_j^0} \xrightarrow{p} V_j$. It follows that

$$
\begin{aligned}
H_j &= \frac{\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j}}{T}\frac{\Lambda_j^{0\prime}\tilde{\Lambda}_j}{N}W_{jNT}^{-1} \\
&= (\Lambda_j^{0\prime}\Lambda_j^0)^{-\frac{1}{2}}\frac{(\Lambda_j^{0\prime}\Lambda_j^0)^{\frac{1}{2}}(\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{z_t=j})(\Lambda_j^{0\prime}\Lambda_j^0)^{\frac{1}{2}}}{NT}(\Lambda_j^{0\prime}\Lambda_j^0)^{-\frac{1}{2}}\Lambda_j^{0\prime}\check{\Lambda}_j(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}W_{jNT}^{-1} \\
&= (\frac{\Lambda_j^{0\prime}\Lambda_j^0}{N})^{-\frac{1}{2}}\Upsilon_{jNT}V_{jNT}(\bar{\Lambda}_j^{0\prime}\check{\Lambda}_j)(\frac{\tilde{\Lambda}_j'\tilde{\Lambda}_j}{N})^{\frac{1}{2}}W_{jNT}^{-1}q_j^0 \xrightarrow{p} \Sigma_{\Lambda_j}^{-\frac{1}{2}}\Upsilon_j V_j^{\frac{1}{2}}. \qquad (56)
\end{aligned}
$$

■

Proof of Theorem 3

Proof. From equation (51), we have $\tilde{\Lambda}_j W_{jNT} = \Lambda_j^0\frac{(\sum_{t=1}^T f_t^0 f_t^{0\prime}1_{z_t=j})\Lambda_j^{0\prime}\tilde{\Lambda}_j}{NT} + (I + II + III + IV + D)(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}$, i.e.,

$$\tilde{\Lambda}_j - \Lambda_j^0 H_j = (I + II + III + IV + D)(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}W_{jNT}^{-1}. \qquad (57)$$

We have shown in Proposition 1 that $\|I + II + III + IV + D\|_F^2 = O_p(\frac{1}{\delta_{NT}^2})$, $W_{jNT} \xrightarrow{p} q_j^0 V_j$ and $\frac{\tilde{\Lambda}_j'\tilde{\Lambda}_j}{N} \xrightarrow{p} V_j$. Thus $\frac{1}{N}\left\|\tilde{\Lambda}_j - \Lambda_j^0 H_j\right\|_F^2 = O_p(\frac{1}{\delta_{NT}^2})$. ■

Proof of Theorem 4

Proof. Let I_i, II_i, III_i, IV_i and D_i denote the i-th row of I, II, III, IV and D respectively. From equation (57), we have

$$\tilde{\lambda}'_{ji} - \lambda^{0\prime}_{ji} H_j = (I_i + II_i + III_i + IV_i + D_i)(\tilde{\Lambda}'_j \tilde{\Lambda}_j)^{\frac{1}{2}} W^{-1}_{jNT}.$$

By Assumptions 2(1) and 3(2) and Theorem 3, $I_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}} = O_p(\frac{1}{\sqrt{N}\delta_{NT}})$. By Assumptions 3(4) and 7(1) and Theorem 3, $II_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}} = O_p(\frac{1}{\sqrt{T}\delta_{NT}})$. By Assumption 3(2) and Theorem 3, $III_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}} = \frac{\sum_{t=1}^T e_{it} f_t^{0\prime} 1_{z_t=j} \Lambda_j^{0\prime}\Lambda_j^0 H_j}{NT} + O_p(\frac{1}{\sqrt{T}\delta_{NT}})$. By Assumptions 3(2) and 7(2) and Theorem 3, $IV_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}} = O_p(\frac{1}{\sqrt{T}\delta_{NT}})$. The detailed calculation of these four terms is similar to the proof of Lemma A.2 in Bai (2003), hence omitted here. For the term $D_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}}$, we have

$$\begin{aligned}
\left\| D_i(\tilde{\Lambda}'_j\tilde{\Lambda}_j)^{\frac{1}{2}} \right\|^2 &= \left\| \frac{1}{NT} \sum\nolimits_{t=1}^T (\tilde{p}_{tj|T} - 1_{z_t=j}) x_{it} x'_t \tilde{\Lambda}_j \right\|^2 \\
&\leq \frac{1}{N^2T^2} \sum\nolimits_{t=1}^T (\tilde{p}_{tj|T} - 1_{z_t=j})^2 x_{it}^2 \sum\nolimits_{t=1}^T \|x_t\|^2 \left\| \tilde{\Lambda}_j \right\|_F^2 \\
&\leq \sup_t \left| \tilde{p}_{tj|T} - 1_{z_t=j} \right|^2 \frac{\sum_{t=1}^T x_{it}^2}{T} \frac{\sum_{t=1}^T \|x_t\|^2}{NT} \frac{\left\| \tilde{\Lambda}_j \right\|_F^2}{N} = o_p(\frac{1}{N^{2\eta}}),
\end{aligned}$$

where the last equality follows from Theorem 2. We have shown in Proposition 1 that $W_{jNT} \xrightarrow{p} q_j^0 V_j$, thus $W^{-1}_{jNT} = O_p(1)$. It follows that

$$\sqrt{Tq_j^0}(\tilde{\lambda}_{ji} - H'_j \lambda^0_{ji}) = q_j^0 W^{-1}_{jNT} H'_j \frac{\Lambda_j^{0\prime}\Lambda_j^0}{N} \frac{\sum_{t=1}^T f_t^0 e_{it} 1_{z_t=j}}{\sqrt{Tq_j^0}} + O_p(\frac{\sqrt{T}}{N}) + o_p(1).$$

Thus by Proposition 1 and Assumption 7(3),

$$\sqrt{Tq_j^0}(\tilde{\lambda}_{ji} - H'_j\lambda^0_{ji}) \xrightarrow{d} \mathcal{N}(0, V_j^{-\frac{1}{2}} \Upsilon'_j \Sigma_{\Lambda_j}^{\frac{1}{2}} \Phi_{ji} \Sigma_{\Lambda_j}^{\frac{1}{2}} \Upsilon_j V_j^{-\frac{1}{2}}) \text{ when } \sqrt{T}/N \to 0.$$

■

D Details for Theorem 5

Lemma 3 *Under Assumptions 1-7, and assume* $T^{\frac{16}{\alpha}}/N \to 0$ *and* $T^{\frac{2}{\alpha}+\frac{2}{\beta}}/N \to 0$,

(1) $\frac{1}{N}e_t'(\tilde{\Lambda}_j - \Lambda_j^0 H_j) = O_p(\frac{1}{\delta_{NT}^2})$ *for each* j *and* t,

(2) $\frac{1}{N}\Lambda_j^{0\prime}(\tilde{\Lambda}_j - \Lambda_j^0 H_j) = O_p(\frac{1}{\delta_{NT}^2})$ *for each* j.

Proof. Part (1): From equation (57), we have $\frac{1}{N}e_t'(\tilde{\Lambda}_j - \Lambda_j^0 H_j) = \frac{1}{N}e_t'(I + II + III + IV + D)(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}} W_{jNT}^{-1}$. Consider each term one by one.

$$\frac{e_t' I(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}}{N} = e_t' \frac{\sum_{t=1}^T \mathbb{E}(e_t e_t') 1_{z_t=j}}{N^2 T} \Lambda_j^0 H_j + e_t' \frac{\sum_{t=1}^T \mathbb{E}(e_t e_t') 1_{z_t=j}}{N^2 T}(\tilde{\Lambda}_j - \Lambda_j^0 H_j).$$

By Assumption 3(1) and 3(2), the first term is $O_p(\frac{1}{N})$. By Assumption 3(2) and Theorem 3, the second term is $O_p(\frac{1}{\sqrt{N}\delta_{NT}})$.

$$\frac{e_t' II(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}}{N} = e_t' \frac{\sum_{t=1}^T (e_t e_t' - \mathbb{E}(e_t e_t')) 1_{z_t=j}}{N^2 T} \Lambda_j^0 H_j + e_t' \frac{\sum_{t=1}^T (e_t e_t' - \mathbb{E}(e_t e_t')) 1_{z_t=j}}{N^2 T}(\tilde{\Lambda}_j - \Lambda_j^0 H_j).$$

By Assumption 7(1), the first term is $O_p(\frac{1}{\sqrt{NT}})$. By Assumption 3(4) and Theorem 3, the second term is $O_p(\frac{1}{\sqrt{T}\delta_{NT}})$.

$$\frac{e_t' III(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}}{N} = e_t' \frac{\sum_{t=1}^T e_t f_t^{0\prime} 1_{z_t=j} \Lambda_j^{0\prime}}{N^2 T} \Lambda_j^0 H_j + e_t' \frac{\sum_{t=1}^T e_t f_t^{0\prime} 1_{z_t=j} \Lambda_j^{0\prime}}{N^2 T}(\tilde{\Lambda}_j - \Lambda_j^0 H_j).$$

The first term is $O_p(\frac{1}{\sqrt{NT}}) + O_p(\frac{1}{T})$ since $\frac{1}{NT}e_t' \sum_{t=1}^T e_t f_t^{0\prime} 1_{z_t=j} = O_p(\frac{1}{\sqrt{NT}}) + O_p(\frac{1}{T})$, which follows from Assumptions 3(3), 7(1) and $e_{it}e_{is} = \gamma_{i,ts} + (e_{it}e_{is} - \gamma_{i,ts})$. By Theorem 3, the second term is $O_p(\frac{1}{\sqrt{T}\delta_{NT}})$.

$$\frac{e_t' IV(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}}{N} = e_t' \frac{\Lambda_j^0 \sum_{t=1}^T f_t^0 e_t' 1_{z_t=j}}{N^2 T} \Lambda_j^0 H_j + e_t' \frac{\Lambda_j^0 \sum_{t=1}^T f_t^0 e_t' 1_{z_t=j}}{N^2 T}(\tilde{\Lambda}_j - \Lambda_j^0 H_j).$$

The first term is $O_p(\frac{1}{\sqrt{NT}})$ since by Assumption 7(2), $\frac{1}{NT}\sum_{t=1}^T f_t^0 e_t' 1_{z_t=j} \Lambda_j^0 = O_p(\frac{1}{\sqrt{NT}})$. By Theorem 3, the second term is $O_p(\frac{1}{\sqrt{T}\delta_{NT}})$.

$$\left\| \frac{e_t' D(\tilde{\Lambda}_j'\tilde{\Lambda}_j)^{\frac{1}{2}}}{N} \right\| \le \left\| \frac{e_t'}{\sqrt{N}} \right\| \|D\|_F \left\| (\frac{\tilde{\Lambda}_j'\tilde{\Lambda}_j}{N})^{\frac{1}{2}} \right\|.$$

Thus from equation (53), this term is $o_p(\frac{1}{N^\eta})$. Finally, note that $W_{jNT}^{-1} \xrightarrow{d} \frac{1}{q_j^0} V_j^{-1}$, part (1) is proved.

Part (2) can be proved similarly. ■

Proof of Theorem 5

Proof. First, by Woodbury identity,

$$\begin{aligned}
\tilde{f}_t &= \sum_{j=1}^{J^0} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' x_t \tilde{p}_{tj|T} \\
&= \sum_{k=1}^{J^0} \sum_{j=1}^{J^0} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_k^0 f_t^0 1_{z_t=k} + \sum_{j=1}^{J^0} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' e_t.
\end{aligned}$$

When $z_t = k$, we have

$$\begin{aligned}
& \sum_{j=1}^{J^0} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_k^0 f_t^0 \\
= & (\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\Lambda}_k' \Lambda_k^0 f_t^0 \\
& + (\tilde{p}_{tk|T} - 1)(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\Lambda}_k' \Lambda_k^0 f_t^0 + \sum_{j \neq k} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_k^0 f_t^0 \\
= & H_k^{-1} f_t^0 + (\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\Lambda}_k' (\Lambda_k^0 H_k - \tilde{\Lambda}_k) H_k^{-1} f_t^0 - (\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\sigma}^2 H_k^{-1} f_t^0 \\
& + (\tilde{p}_{tk|T} - 1)(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\Lambda}_k' \Lambda_k^0 f_t^0 + \sum_{j \neq k} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' \Lambda_k^0 f_t^0 \\
\equiv & H_k^{-1} f_t^0 + B_{k1t} - B_{k2t} + B_{k3t} + B_{k4t},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=1}^{J^0} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' e_t \\
= & (\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} (\tilde{\Lambda}_k - \Lambda_k^0 H_k)' e_t + (\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} H_k' \Lambda_k^{0\prime} e_t \\
& + (\tilde{p}_{tk|T} - 1)(\tilde{\sigma}^2 I_{r_k^0} + \tilde{\Lambda}_k' \tilde{\Lambda}_k)^{-1} \tilde{\Lambda}_k' e_t + \sum_{j \neq k}^{J^0} \tilde{p}_{tj|T} (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' e_t \\
\equiv & C_{k1t} + C_{k2t} + C_{k3t} + C_{k4t}.
\end{aligned}$$

It follows that $\tilde{f}_t - H_{z_t}^{-1} f_t^0 = B_{z_t1t} - B_{z_t2t} + B_{z_t3t} + B_{z_t4t} + C_{z_t1t} + C_{z_t2t} + C_{z_t3t} + C_{z_t4t}$.

Proof of part (1): First consider B_{z_t1t}.

$$\frac{\sum_{t=1}^T \|B_{z_t1t}\|^2}{T} \leq \sum_{j=1}^{J^0} \left\| (\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j' \tilde{\Lambda}_j)^{-1} \tilde{\Lambda}_j' (\Lambda_j^0 H_j - \tilde{\Lambda}_j) H_j^{-1} \right\|^2 \frac{\sum_{t=1}^T \|f_t^0\|^2}{T} = O_p(\frac{1}{\delta_{NT}^2}),$$

where the equality is due to the following facts:

(1) By Proposition 1(1), $\frac{1}{N}(\tilde{\sigma}^2 I_{r_j^0} + \tilde{\Lambda}_j'\tilde{\Lambda}_j) = W_{jNT}/\frac{1}{T}\sum_{t=1}^T \tilde{p}_{tj|T} \xrightarrow{p} V_j$ for all j.

(2) $\left\|\frac{1}{\sqrt{N}}\tilde{\Lambda}_j'\right\|_F = \sqrt{tr(\frac{1}{N}\tilde{\Lambda}_j'\tilde{\Lambda}_j)} \xrightarrow{p} \sqrt{tr(V_j)}$ for all j.

(3) By Theorem 3, $\left\|\frac{1}{\sqrt{N}}(\Lambda_j^0 H_j - \tilde{\Lambda}_j)\right\| = O_p(\frac{1}{\delta_{NT}})$ for all j.

(4) By Proposition 1(1), $\left\|H_j^{-1}\right\| = O_p(1)$ for all j.

(5) $\frac{1}{T}\sum_{t=1}^T \|f_t^0\|^2 = O_p(1)$ by Assumption 1.

It is easy to see that $\frac{1}{T}\sum_{t=1}^T \|B_{z_t 2t}\|^2 = O_p(\frac{1}{N^2})$. For $B_{z_t 3t}$, we have

$$\frac{\sum_{t=1}^T \|B_{z_t 3t}\|^2}{T} \leq \sup_t \left\|\tilde{p}_{tz_t|T} - 1\right\|^2 \left\|(\tilde{\sigma}^2 I_{r_{z_t}^0} + \tilde{\Lambda}_{z_t}'\tilde{\Lambda}_{z_t})^{-1}\tilde{\Lambda}_{z_t}'\Lambda_{z_t}^0\right\|^2 \frac{\sum_{t=1}^T \|f_t^0\|^2}{T} = o_p(\frac{1}{N^{2\eta}}),$$

where the equality is due to $\sup_t \left\|\tilde{p}_{tz_t|T} - 1\right\| \leq \sup_j \sup_t \left\|\tilde{p}_{tj|T} - 1_{z_t=j}\right\| = o_p(\frac{1}{N^\eta})$ by Theorem 2. It is easy to see that $\frac{1}{T}\sum_{t=1}^T \|B_{z_t 4t}\|^2 = o_p(\frac{1}{N^{2\eta}})$. Similarly, we can show that $\frac{\sum_{t=1}^T \left\|C_{z_t 1t}\right\|^2}{T}$ is $O_p(\frac{1}{\delta_{NT}^2})$, $\frac{\sum_{t=1}^T \left\|C_{z_t 2t}\right\|^2}{T}$ is $O_p(\frac{1}{N})$, and both $\frac{\sum_{t=1}^T \left\|C_{z_t 3t}\right\|^2}{T}$ and $\frac{\sum_{t=1}^T \left\|C_{z_t 4t}\right\|^2}{T}$ are $o_p(\frac{1}{N^{2\eta}})(O_p(\frac{1}{N}) + O_p(\frac{1}{\delta_{NT}^2}))$. Thus $\frac{1}{T}\sum_{t=1}^T \left\|\tilde{f}_t - H_{z_t}^{-1} f_t^0\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$.

Proof of part (2): By Theorem 3 and Lemma 3(2), $\tilde{\Lambda}_k'(\Lambda_k^0 H_k - \tilde{\Lambda}_k) = O_p(\frac{1}{\delta_{NT}^2})$ for any k. This together with fact (1) and fact (4) listed above implies $B_{z_t 1t} = O_p(\frac{1}{\delta_{NT}^2})$. Similarly, it is easy to see that $B_{z_t 2t} = O_p(\frac{1}{N})$, $B_{z_t 3t} = o_p(\frac{1}{N^\eta})$, $B_{z_t 4t} = o_p(\frac{1}{N^\eta})$, $C_{z_t 1t} = O_p(\frac{1}{\delta_{NT}^2})$, $C_{z_t 2t} = O_p(\frac{1}{\sqrt{N}})$, $C_{z_t 3t} = o_p(\frac{1}{N^\eta})$ and $C_{z_t 4t} = o_p(\frac{1}{N^\eta})$. The leading term is $C_{z_t 2t}$. Since $\frac{\tilde{\Lambda}_{z_t}'\tilde{\Lambda}_{z_t}}{N} \xrightarrow{p} V_{z_t}$, $H_{z_t} \xrightarrow{p} \Sigma_{\Lambda_{z_t}}^{-\frac{1}{2}}\Upsilon_{z_t} V_{z_t}^{\frac{1}{2}}$ and $\frac{1}{\sqrt{N}}\Lambda_{z_t}^{0\prime} e_t \xrightarrow{d} \mathcal{N}(0, \Gamma_{z_t t})$ by Assumption 7(4), we have $\sqrt{N}(\tilde{f}_t - H_{z_t}^{-1} f_t^0) \xrightarrow{d} \mathcal{N}(0, V_{z_t}^{-\frac{1}{2}}\Upsilon_{z_t}'\Sigma_{\Lambda_{z_t}}^{-\frac{1}{2}}\Gamma_{z_t t}\Sigma_{\Lambda_{z_t}}^{-\frac{1}{2}}\Upsilon_{z_t} V_{z_t}^{-\frac{1}{2}})$. ■

E Proof of Theorem 6

Proof. First, $\tilde{Q}_{jk} = \sum_{t=2}^T \tilde{p}_{tjk|T} / \sum_{j=1}^{J^0}\sum_{t=2}^T \tilde{p}_{tjk|T} = \frac{1}{T-1}\sum_{t=2}^T \tilde{p}_{tjk|T} / \frac{1}{T-1}\sum_{t=1}^{T-1}\tilde{p}_{tk|T}$. For the denominator, by Theorem 2, we have

$$\frac{1}{T-1}\sum_{t=1}^{T-1}\tilde{p}_{tk|T} = \frac{1}{T-1}\sum_{t=1}^{T-1} 1_{z_t=k} + o_p(\frac{1}{N^\eta}) \xrightarrow{p} q_k^0. \tag{58}$$

For the numerator, we have

$$\frac{1}{T-1}\sum_{t=2}^{T}\tilde{p}_{tjk|T} = \frac{1}{T-1}\sum_{t=2}^{T}\tilde{p}_{tj|T}\Pr(z_{t-1}=k \Big| z_t=j, x_{1:T};\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)$$
$$= \frac{1}{T-1}\sum_{t=2}^{T}[1_{z_t=j}+o_p(\frac{1}{N^\eta})][1_{z_{t-1}=k}+o_p(\frac{1}{N^\eta})]. \quad (59)$$

The second equality of (59) follows from: (1) $\tilde{p}_{tj|T} = 1_{z_t=j} + o_p(\frac{1}{N^\eta})$ by Theorem 2,

$$(2)\ \Pr(z_{t-1} = k \Big| z_t=j, x_{1:T};\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi) = \Pr(z_{t-1}=k\Big| z_t=j, x_{1:t-1};\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)$$
$$= \frac{\Pr(z_{t-1}=k, z_t=j \Big| x_{1:t-1};\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)}{\Pr(z_t=j\Big| x_{1:t-1};\tilde{\Lambda},\tilde{\sigma}^2,Q,\phi)} = \frac{Q_{jk}\tilde{p}_{t-1,k|t-1}}{\sum_{h=1}^{J^0}Q_{jh}\tilde{p}_{t-1,h|t-1}}$$
$$= 1_{z_{t-1}=k}+o_p(\frac{1}{N^\eta}),$$

where the last equality follows from Theorem 2. Since z_t follows a Markov process,

$$\frac{1}{T-1}\sum_{t=2}^{T}1_{z_t=j}1_{z_{t-1}=k} \xrightarrow{p} \mathbb{E}(1_{z_t=j}1_{z_{t-1}=k}) = \mathbb{E}[\mathbb{E}(1_{z_t=j}1_{z_{t-1}=k}\Big|1_{z_{t-1}=k})] = q_k^0 Q_{jk}^0. \quad (60)$$

Take equations (58)-(60) together, we have shown $\tilde{Q}_{jk} \xrightarrow{p} Q_{jk}^0$. ■

F Proof of Theorem 7

Proof. First, for notational convenience, let the $\bar{J}$-dimensional vector $(r_1,...,r_{\bar{J}})$ denote the numbers of factors. When the number of regime is less than $\bar{J}$, some elements of this vector are zeros. Rank $(r_1,...,r_{\bar{J}})$ in descending order and denote it as $(r_{(1)},...,r_{(\bar{J})})$. Similarly, let $(r^0_{(1)},...,r^0_{(J^0)},0,...,0)$ denote the true numbers of factors, then it suffices to show that $\Pr(\tilde{r}_{(j)} = r^0_{(j)}$ for $j=1,...,\bar{J}) \to 1$.

Step (1):

$$\Pr(PC(r_{(1)},...,r_{(\bar{J})}) > PC(r^0_{(1)},...,r^0_{(J^0)})) \to 0 \text{ if } r_{(1)} > r^0_{(1)}, \quad (61)$$
$$\Pr(PC(r_{(1)},...,r_{(\bar{J})}) > PC(r^0_{(1)},...,r^0_{(J^0)})) \to 0 \text{ if } r_{(1)} < r^0_{(1)}. \quad (62)$$

Proof of expression (61): From expressions (30) and (31),

$$l(\tilde{\Lambda}_{1,r_{(1)}},...,\tilde{\Lambda}_{\bar{J},r_{(\bar{J})}},\sigma^2,Q,\phi) \leq$$
$$-\frac{NT\log 2\pi}{2} - \frac{1}{2}\sum_{t=1}^{T}\log\left|\tilde{\Lambda}_{m_t,r_{(m_t)}}\tilde{\Lambda}'_{m_t,r_{(m_t)}} + \sigma^2 I_N\right| - \frac{\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}x_t\right\|^2}{2\sigma^2}$$
$$-\frac{1}{2}\sum_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t,r_{(m_t)}}(\sigma^2 I_{r_{(m_t)}} + \tilde{\Lambda}'_{m_t,r_{(m_t)}}\tilde{\Lambda}_{m_t,r_{(m_t)}})^{-1}(\tilde{\Lambda}'_{m_t,r_{(m_t)}}\tilde{\Lambda}_{m_t,r_{(m_t)}})^{-1}\tilde{\Lambda}'_{m_t,r_{(m_t)}}x_t,$$

where $\tilde{\Lambda}_{j,r_{(j)}}$ is $N \times r_{(j)}$ and the subscript $r_{(j)}$ is suppressed if $r_{(j)} = r^0_{(j)}$. If $r_{(j)} = 0$, then $\tilde{\Lambda}_{j,r_{(j)}} = 0$. From expressions (32) and (33), we have

$$\begin{aligned} & l(\Lambda^0_1,...,\Lambda^0_{J^0},\sigma^2,Q,\phi) \\ \geq\; & -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left|\Lambda^0_{z_t}\Lambda^{0\prime}_{z_t} + \sigma^2 I_N\right| - T\log\min_{j,k} Q_{jk} - \\ & \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left\|M_{\Lambda^0_{z_t}}x_t\right\|^2 - \frac{1}{2}\sum_{t=1}^{T}x_t'\Lambda^0_{z_t}(\sigma^2 I_{r^0_{z_t}} + \Lambda^{0\prime}_{z_t}\Lambda^0_{z_t})^{-1}(\Lambda^{0\prime}_{z_t}\Lambda^0_{z_t})^{-1}\Lambda^{0\prime}_{z_t}x_t. \end{aligned}$$

It follows that

$$\begin{aligned} & l(\tilde{\Lambda}_{1,r_{(1)}},...,\tilde{\Lambda}_{\bar{J},r_{(\bar{J})}},\sigma^2,Q,\phi) - l(\Lambda^0_1,...,\Lambda^0_{J^0},\sigma^2,Q,\phi) \leq \\ & -\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left\|M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}\Lambda^0_{z_t}f^0_t\right\|^2 - \frac{1}{\sigma^2}\sum_{t=1}^{T}e_t'M_{\tilde{\Lambda}_{m_t,,r_{(m_t)}}}\Lambda^0_{z_t}f^0_t \\ & -\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left\|P_{\Lambda^0_{z_t}}e_t\right\|^2 + \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}e_t\right\|^2 \\ & +T\log\min_{j,k} Q_{jk} - \frac{1}{2}\sum_{t=1}^{T}\log\frac{\left|\tilde{\Lambda}_{m_t,r_{(m_t)}}\tilde{\Lambda}'_{m_t,r_{(m_t)}} + \sigma^2 I_N\right|}{\left|\Lambda^0_{z_t}\Lambda^{0\prime}_{z_t} + \sigma^2 I_N\right|} \\ & -\frac{1}{2}\sum_{t=1}^{T}x_t'\tilde{\Lambda}_{m_t,r_{(m_t)}}(\sigma^2 I_{r_{(m_t)}} + \tilde{\Lambda}'_{m_t,r_{(m_t)}}\tilde{\Lambda}_{m_t,r_{(m_t)}})^{-1}(\tilde{\Lambda}'_{m_t,r_{(m_t)}}\tilde{\Lambda}_{m_t,r_{(m_t)}})^{-1}\tilde{\Lambda}'_{m_t,r_{(m_t)}}x_t \\ & +\frac{1}{2}\sum_{t=1}^{T}x_t'\Lambda^0_{z_t}(\sigma^2 I_{r^0_{z_t}} + \Lambda^{0\prime}_{z_t}\Lambda^0_{z_t})^{-1}(\Lambda^{0\prime}_{z_t}\Lambda^0_{z_t})^{-1}\Lambda^{0\prime}_{z_t}x_t \end{aligned} \tag{63}$$

The first, the third and the seventh term on the right hand side of expression (63) are negative, thus the inequality still holds when these terms are thrown away. Steps (3.1), (3.2) and (3.4) in the proof of Theorem 1 show that the fifth term is $O(T)$, the sixth term is $O_p(T\log N)$ and the eighth term is $O_p(T)$. For the fourth term, by expression (38) and Lemma 1, we have $\sum_{t=1}^{T}\left\|P_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}e_t\right\|^2 \leq \sum_{j=1}^{J^0} r_{(j)}\left\|E'E\right\| = O_p(\frac{NT}{\delta_{NT}})$.

The second term is $O_p(\frac{NT}{\delta_{NT}})$ because step (3.5) in the proof of Theorem 1 shows $\sum_{t=1}^T e_t'\Lambda_{z_t}^0 f_t^0 = O_p(N^{\frac{1}{2}}T)$ and $\sum_{t=1}^T e_t' P_{\tilde{\Lambda}_{m_t,r_{m_t}}}\Lambda_{z_t}^0 f_t^0 = O_p(N^{\frac{1}{2}}T^{\frac{1}{2}}(\sum_{t=1}^T \left\| P_{\tilde{\Lambda}_{m_t,r_{m_t}}} e_t \right\|^2)^{\frac{1}{2}})$. In summary, we have

$$l(\tilde{\Lambda}_{1,r_{(1)}},...,\tilde{\Lambda}_{\bar{J},r_{(\bar{J})}},\sigma^2,Q,\phi) \leq l(\Lambda_1^0,...,\Lambda_{J^0}^0,\sigma^2,Q,\phi) + O_p(\frac{NT}{\delta_{NT}}). \tag{64}$$

Note that expression (64) holds no matter what the values of $r_{(1)},...,r_{(\bar{J})}$ are. When $r_{(1)},...,r_{(\bar{J})}$ equal the true values, $l(\Lambda_1^0,...,\Lambda_{J^0}^0,\sigma^2,Q,\phi) \leq l(\tilde{\Lambda}_1,...,\tilde{\Lambda}_{J^0},\sigma^2,Q,\phi)$. Thus

$$\begin{aligned} & PC(r_{(1)},...,r_{(\bar{J})}) - PC(r_{(1)}^0,...,r_{(J^0)}^0) \\ \leq\ & O_p(\frac{1}{\delta_{NT}}) - \sum_{j=1}^{\bar{J}} (g(N,T))^{b(r_{(j)})} + \sum_{j=1}^{J^0} (g(N,T))^{b(r_{(j)}^0)}. \end{aligned}$$

If $r_{(1)} > r_{(1)}^0$, $\Pr(PC(r_{(1)},...,r_{(\bar{J})}) \leq PC(r_{(1)}^0,...,r_{(J^0)}^0)) \to 1$, i.e., $(g(N,T))^{b(r_{(1)})}$ would dominate if $r_{(1)} > r_{(1)}^0$, no matter what the values of $r_{(2)},...,r_{(\bar{J})}$ are.

Proof of expression (62): Since $l(\Lambda_1^0,...,\Lambda_{J^0}^0,\sigma^2,Q,\phi) \leq l(\tilde{\Lambda}_1,...,\tilde{\Lambda}_{J^0},\sigma^2,Q,\phi)$,

$$\begin{aligned} & l(\tilde{\Lambda}_{1,r_{(1)}},...,\tilde{\Lambda}_{\bar{J},r_{(\bar{J})}},\sigma^2,Q,\phi) - l(\tilde{\Lambda}_1,...,\tilde{\Lambda}_{J^0},\sigma^2,Q,\phi) \\ \leq\ & l(\tilde{\Lambda}_{1,r_{(1)}},...,\tilde{\Lambda}_{\bar{J},r_{(\bar{J})}},\sigma^2,Q,\phi) - l(\Lambda_1^0,...,\Lambda_{J^0}^0,\sigma^2,Q,\phi) \\ \leq\ & \text{the right hand side of expression (63)} \\ \leq\ & -\frac{1}{2\sigma^2}\sum_{t=1}^T \left\| M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}\Lambda_{z_t}^0 f_t^0 \right\|^2 + O_p(\frac{NT}{\delta_{NT}}) \\ \leq\ & -\frac{1}{2\sigma^2}\sum_{t:z_t=(1)} \left\| M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}}\Lambda_{(1)}^0 f_t^0 \right\|^2 + O_p(\frac{NT}{\delta_{NT}}), \end{aligned} \tag{65}$$

where $\sum_{t:z_t=(1)}$ denotes summation over the regime corresponding to $r_{(1)}^0$, and there are $q_{(1)}^0 T$ terms in this summation since $q_{(1)}^0$ is the unconditional probability of $z_t = (1)$. For each t with $z_t = (1)$, $\Lambda_{(1)}^0 f_t^0$ is projected on one of $\tilde{\Lambda}_{j,r_{(j)}}, j = 1,...,\bar{J}$, thus there exists $\tilde{\Lambda}_{k,r_{(k)}}$ such that $\Lambda_{(1)}^0 f_t^0$ is projected on $\tilde{\Lambda}_{k,r_{(k)}}$ at least $\frac{q_{(1)}^0 T}{\bar{J}}$ times, i.e., $\sum_{t=1}^T 1_{m_t=(k)}1_{z_t=(1)} \geq \frac{q_{(1)}^0 T}{\bar{J}}$. Thus by Assumption 1(1), $\rho_{\min}(\frac{\sum_{t=1}^T f_t^0 f_t^{0\prime} 1_{m_t=(k)}1_{z_t=(1)}}{\sum_{t=1}^T 1_{m_t=(k)}1_{z_t=(1)}}) \geq c$

for some $c > 0$ w.p.a.1. It follows that

$$\begin{aligned}\sum\nolimits_{t:z_t=(1)} \left\| M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}} \Lambda^0_{(1)} f^0_t \right\|^2 &\geq \sum\nolimits_{t=1}^{T} \left\| M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)} f^0_t \right\|^2 1_{m_t=(k)} 1_{z_t=(1)} \\ &= tr(\Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)} \sum\nolimits_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(k)} 1_{z_t=(1)}) \\ &\geq tr(\Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)}) \rho_{\min}(\sum\nolimits_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(k)} 1_{z_t=(1)}) \\ &\geq tr(\Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)}) \frac{q^0_{(1)} T}{\bar{J}} c \text{ w.p.a.1.} \end{aligned} \tag{66}$$

If $r_{(1)} < r^0_{(1)}$, then $r_{(k)} \leq r_{(1)} < r^0_{(1)}$, and consequently $\frac{1}{N} tr(\Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)}) > c$ for some $c > 0$. Since $(g(N,T))^{b(r_{(j)})} \to 0$ for $1 \leq r_{(j)} \leq \bar{r}$, we have $\Pr(PC(r_{(1)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)})) \to 0$ if $r_{(1)} < r^0_{(1)}$.

Step (2):

$$\Pr(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)})) \to 0 \text{ if } r_{(2)} > r^0_{(2)}, \tag{67}$$

$$\Pr(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)})) \to 0 \text{ if } r_{(2)} < r^0_{(2)}. \tag{68}$$

Proof of expression (67): In the proof of expression (61) we have shown that expression (64) holds no matter what the values of $r_{(1)}, ..., r_{(\bar{J})}$ are, thus (64) also holds here for $(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})})$. It follows that

$$\begin{aligned}&PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) - PC(r^0_{(1)}, ..., r^0_{(J^0)}) \\ \leq\ &O_p(\frac{1}{\delta_{NT}}) - \sum\nolimits_{j=2}^{\bar{J}} (g(N,T))^{b(r_{(j)})} + \sum\nolimits_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})}.\end{aligned}$$

If $r_{(2)} > r^0_{(2)}$, $\Pr(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) \leq PC(r^0_{(1)}, ..., r^0_{(J^0)})) \to 1$, i.e., $(g(N,T))^{b(r_{(2)})}$ would dominate if $r_{(2)} > r^0_{(2)}$, no matter what the values of $r_{(3)}, ..., r_{(\bar{J})}$ are.

Proof of expression (68): Similar to expressions (65) and (66), we have

$$\begin{aligned}&l(\tilde{\Lambda}_{1,r^0_{(1)}}, \tilde{\Lambda}_{2,r_{(2)}}, ..., \tilde{\Lambda}_{\bar{J},r_{(\bar{J})}}, \sigma^2, Q, \phi) - l(\tilde{\Lambda}_1, ..., \tilde{\Lambda}_{J^0}, \sigma^2, Q, \phi) \\ \leq\ &-\frac{1}{2\sigma^2} \sum\nolimits_{t:z_t=(1)} \left\| M_{\tilde{\Lambda}_{m_t,r_{(m_t)}}} \Lambda^0_{(1)} f^0_t \right\|^2 + O_p(\frac{NT}{\delta_{NT}}) \\ \leq\ &-\frac{1}{2\sigma^2} tr(\Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)}) \rho_{\min}(\sum\nolimits_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(k)} 1_{z_t=(1)}) + O_p(\frac{NT}{\delta_{NT}}),\end{aligned}$$

and $\sum_{t=1}^{T} 1_{m_t=(k)} 1_{z_t=(1)} \geq \frac{q^0_{(1)} T}{\bar{J}}$. The event $PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)})$ implies

$$\frac{l(\tilde{\Lambda}_{1,r^0_{(1)}}, \tilde{\Lambda}_{2,r_{(2)}}, ..., \tilde{\Lambda}_{\bar{J},r_{(\bar{J})}}, \sigma^2, Q, \phi) - l(\tilde{\Lambda}_1, ..., \tilde{\Lambda}_{J^0}, \sigma^2, Q, \phi)}{NT} + \sum_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})} > 0,$$

i.e.,

$$tr(\frac{1}{N} \Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{k,r_{(k)}}} \Lambda^0_{(1)}) < \frac{2\sigma^2 (O_p(\frac{1}{\delta_{NT}}) + \sum_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})})}{\rho_{\min}(\frac{1}{T}\sum_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(k)} 1_{z_t=(1)})}.$$

If $r_{(2)} < r^0_{(2)}$, then $r_{(k)} < r^0_{(2)}$ for $k = 2, ..., \bar{J}$. Thus if $r_{(2)} < r^0_{(2)}$,

$$\Pr(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)}), k \neq 1) \to 0.$$

Similarly, the event $PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)})$ also implies that

$$tr(\frac{1}{N} \Lambda^{0\prime}_{(2)} M_{\tilde{\Lambda}_{k',r_{(k')}}} \Lambda^0_{(2)}) < \frac{2\sigma^2 (O_p(\frac{1}{\delta_{NT}}) + \sum_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})})}{\rho_{\min}(\frac{1}{T}\sum_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(k')} 1_{z_t=(2)})},$$

with $\sum_{t=1}^{T} 1_{m_t=(k)} 1_{z_t=(1)} \geq \frac{q^0_{(1)} T}{\bar{J}}$. If $r_{(2)} < r^0_{(2)}$, then $r_{(k')} < r^0_{(2)}$ for $k' = 2, ..., \bar{J}$, thus

$$\Pr(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)}), k' \neq 1) \to 0.$$

Finally, the event $(PC(r^0_{(1)}, r_{(2)}, ..., r_{(\bar{J})}) > PC(r^0_{(1)}, ..., r^0_{(J^0)}), k = 1, k' = 1)$ implies

$$tr(\frac{1}{N} \Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{1,r^0_{(1)}}} \Lambda^0_{(1)}) < \frac{2\sigma^2 (O_p(\frac{1}{\delta_{NT}}) + \sum_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})})}{\rho_{\min}(\frac{1}{T}\sum_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(1)} 1_{z_t=(1)})}, \quad (69)$$

$$tr(\frac{1}{N} \Lambda^{0\prime}_{(2)} M_{\tilde{\Lambda}_{1,r^0_{(1)}}} \Lambda^0_{(2)}) < \frac{2\sigma^2 (O_p(\frac{1}{\delta_{NT}}) + \sum_{j=2}^{J^0} (g(N,T))^{b(r^0_{(j)})})}{\rho_{\min}(\frac{1}{T}\sum_{t=1}^{T} f^0_t f^{0\prime}_t 1_{m_t=(1)} 1_{z_t=(2)})}, \quad (70)$$

with $\sum_{t=1}^{T} 1_{m_t=(1)} 1_{z_t=(1)} \geq \frac{q^0_{(1)} T}{\bar{J}}$ and $\sum_{t=1}^{T} 1_{m_t=(1)} 1_{z_t=(2)} \geq \frac{q^0_{(2)} T}{\bar{J}}$. From expression (49), we have

$$\left\| P_{\Lambda^0_{(1)}} - P_{\tilde{\Lambda}_{1,r^0_{(1)}}} \right\|^2 \leq 2tr(\frac{1}{N} \Lambda^{0\prime}_{(1)} M_{\tilde{\Lambda}_{1,r^0_{(1)}}} \Lambda^0_{(1)}) \left\| (\frac{1}{N} \Lambda^{0\prime}_{(1)} \Lambda^0_{(1)})^{-\frac{1}{2}} \right\|_F^2,$$

and it follows that

$$
\begin{aligned}
& tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\tilde{\Lambda}_{1,r_{(1)}^0}}\Lambda_{(2)}^0) \\
> \; & tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\Lambda_{(1)}^0}\Lambda_{(2)}^0) - \left| tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}(P_{\Lambda_{(1)}^0} - P_{\tilde{\Lambda}_{1,r_{(1)}^0}})\Lambda_{(2)}^0) \right| \\
\geq \; & tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\Lambda_{(1)}^0}\Lambda_{(2)}^0) - \sqrt{2tr(\frac{1}{N}\Lambda_{(1)}^{0\prime}M_{\tilde{\Lambda}_{1,r_{(1)}^0}}\Lambda_{(1)}^0)} \left\| (\frac{\Lambda_{(1)}^{0\prime}\Lambda_{(1)}^0}{N})^{-\frac{1}{2}} \right\|_F \left\| \frac{\Lambda_{(2)}^0}{\sqrt{N}} \right\|_F^2 \quad (71)
\end{aligned}
$$

Expressions (69)-(71) together imply that $tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\Lambda_{(1)}^0}\Lambda_{(2)}^0) \leq o_p(1)$. Since we assume $\lim \frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\Lambda_{(1)}^0}\Lambda_{(2)}^0 \neq 0$, we have $\Pr(tr(\frac{1}{N}\Lambda_{(2)}^{0\prime}M_{\Lambda_{(1)}^0}\Lambda_{(2)}^0) \leq o_p(1)) \to 0$, thus

$$
\Pr(PC(r_{(1)}^0, r_{(2)}, ..., r_{(\bar{J})}) > PC(r_{(1)}^0, ..., r_{(J^0)}^0), k = 1, k' \neq 1) \to 0.
$$

In summary, we have proved expression (68). Similarly, we can continue to prove that for $j = 3, ..., \bar{J}$, $\Pr(PC(r_{(1)}^0, ...r_{(j-1)}^0, r_{(j)}, ..., r_{(\bar{J})}) > PC(r_{(1)}^0, ..., r_{(J^0)}^0)) \to 0$ if $r_{(j)} \neq r_{(j)}^0$. ■

G Details on First Order Conditions

First order condition of σ^2:

$$
\begin{aligned}
& \frac{\partial \sum_{t=1}^T \sum_{j=1}^{J^0} \log L(x_t | z_t = j; \Lambda_j, \sigma^2) p_{tj|T}}{\partial \sigma^2} \\
= \; & \sum_{t=1}^T \sum_{j=1}^{J^0} p_{tj|T} \frac{\partial(-\frac{1}{2}\log|\Sigma_j| - \frac{1}{2}x_t'\Sigma_j^{-1}x_t)}{\partial \sigma^2} \\
= \; & -\frac{1}{2}\sum_{t=1}^T \sum_{j=1}^{J^0} p_{tj|T}\, tr(\Sigma_j^{-1} - \Sigma_j^{-1}x_t x_t'\Sigma_j^{-1}) \\
= \; & -\frac{1}{2}\sum_{j=1}^{J^0} tr(\sum_{t=1}^T p_{tj|T}\Sigma_j^{-1} - \Sigma_j^{-1}\sum_{t=1}^T p_{tj|T}\, x_t x_t'\Sigma_j^{-1}) \\
= \; & -\frac{1}{2}\sum_{j=1}^{J^0} (\sum_{t=1}^T p_{tj|T}) tr(\Sigma_j^{-1} - \Sigma_j^{-1}S_j\Sigma_j^{-1}) \\
= \; & -\frac{1}{2\sigma^4}\sum_{j=1}^{J^0} (\sum_{t=1}^T p_{tj|T}) tr(\Sigma_j - S_j) \\
= \; & -\frac{1}{2\sigma^4} tr(\sum_{j=1}^{J^0}\sum_{t=1}^T p_{tj|T}\Lambda_j\Lambda_j' + T\sigma^2 I_N - \sum_{t=1}^T x_t x_t') = 0.
\end{aligned}
$$

The second equality is due to

$$\frac{\partial \log |\Sigma_j|}{\partial \sigma^2} = tr(\Sigma_j^{-1}), \tag{72}$$

$$\frac{\partial x_t' \Sigma_j^{-1} x_t}{\partial \sigma^2} = -tr(\Sigma_j^{-1} x_t x_t' \Sigma_j^{-1}), \tag{73}$$

and the fifth equality is due to

$$\Sigma_j(\Sigma_j - S_j)\Sigma_j = (\Lambda_j\Lambda_j' + \sigma^2 I_N)(\Sigma_j - S_j)(\Lambda_j\Lambda_j' + \sigma^2 I_N) = \sigma^4(\Sigma_j - S_j), \tag{74}$$

since $(\Sigma_j - S_j)\Lambda_j\Lambda_j' = (\Lambda_j\Lambda_j' + \sigma^2 I_N - S_j)\Lambda_j\Lambda_j' = 0$ by equation (15). Thus we have $\sigma^2 = \frac{1}{N} tr(\frac{1}{T}\sum_{t=1}^T x_t x_t' - \sum_{j=1}^{J^0} \frac{1}{T}\sum_{t=1}^T p_{tj|T}\Lambda_j\Lambda_j')$. The proof of expression (20) is the same, with $p_{tj|T}$ replaced by $\tilde{p}_{tj|T}^{(h)}$ and S_j replaced by $\tilde{S}_j^{(h)}$.

First order condition of Λ_j and Σ_e:

When equation (9) is replaced by $\Sigma_j = \Lambda_j\Lambda_j' + \Sigma_e$, equations (10) and (14) are still valid, i.e., $\Sigma_j^{-1}\Lambda_j = \Sigma_j^{-1}S_j\Sigma_j^{-1}\Lambda_j$. Right multiple Σ_j by $\Sigma_e^{-1}\Lambda_j$, we have $\Sigma_j\Sigma_e^{-1}\Lambda_j = \Lambda_j(\Lambda_j'\Sigma_e^{-1}\Lambda_j + I_{r_j^0})$. Left multiply $S_j\Sigma_j^{-1}$ on both sides of this equation, we have $S_j\Sigma_e^{-1}\Lambda_j = S_j\Sigma_j^{-1}\Lambda_j(\Lambda_j'\Sigma_e^{-1}\Lambda_j + I_{r_j^0})$. From equation (14), we have $\Lambda_j = S_j\Sigma_j^{-1}\Lambda_j$, thus $S_j\Sigma_e^{-1}\Lambda_j = \Lambda_j(\Lambda_j'\Sigma_e^{-1}\Lambda_j + I_{r_j^0})$, i.e., $\Sigma_e^{-\frac{1}{2}}\Lambda_j$ is the eigenvectors of $\Sigma_e^{-\frac{1}{2}}S_j\Sigma_e^{-\frac{1}{2}}$ and $\Lambda_j'\Sigma_e^{-1}\Lambda_j + I_{r_j^0}$ is corresponding eigenvalues.

$$\begin{aligned}
& \frac{\partial \sum_{t=1}^T \sum_{j=1}^{J^0} \log L(x_t | z_t = j; \Lambda_j, \sigma^2) p_{tj|T}}{\partial diag(\Sigma_e)} \\
= \; & \sum_{t=1}^T \sum_{j=1}^{J^0} p_{tj|T} \frac{\partial(-\frac{1}{2}\log|\Sigma_j| - \frac{1}{2}x_t'\Sigma_j^{-1}x_t)}{\partial diag(\Sigma_e)} \\
= \; & -\frac{1}{2}\sum_{t=1}^T \sum_{j=1}^{J^0} p_{tj|T}\, diag(\Sigma_j^{-1} - \Sigma_j^{-1}x_t x_t'\Sigma_j^{-1}) \\
= \; & -\frac{1}{2}\sum_{j=1}^{J^0} diag(\sum_{t=1}^T p_{tj|T}\Sigma_j^{-1} - \Sigma_j^{-1}\sum_{t=1}^T p_{tj|T}\, x_t x_t'\Sigma_j^{-1}) \\
= \; & -\frac{1}{2}\sum_{j=1}^{J^0} (\sum_{t=1}^T p_{tj|T}) diag(\Sigma_j^{-1} - \Sigma_j^{-1}S_j\Sigma_j^{-1}) \\
= \; & \Sigma_e^{-1}[-\frac{1}{2}\sum_{j=1}^{J^0}(\sum_{t=1}^T p_{tj|T}) diag(\Sigma_j - S_j)]\Sigma_e^{-1} \\
= \; & \Sigma_e^{-1}[-\frac{1}{2} diag(\sum_{j=1}^{J^0}\sum_{t=1}^T p_{tj|T}\Lambda_j\Lambda_j' + T\Sigma_e - \sum_{t=1}^T x_t x_t')]\Sigma_e^{-1} = 0.
\end{aligned}$$

The second equality is due to

$$\frac{\partial \log|\Sigma_j|}{\partial diag(\Sigma_e)} = diag(\Sigma_j^{-1}), \quad (75)$$

$$\frac{\partial x_t'\Sigma_j^{-1}x_t}{\partial diag(\Sigma_e)} = -diag(\Sigma_j^{-1}x_t x_t'\Sigma_j^{-1}), \quad (76)$$

and the fifth equality is due to

$$\Sigma_j\Sigma_e^{-1}(\Sigma_j - S_j)\Sigma_e^{-1}\Sigma_j = (\Lambda_j\Lambda_j' + \Sigma_e)\Sigma_e^{-1}(\Sigma_j - S_j)\Sigma_e^{-1}(\Lambda_j\Lambda_j' + \Sigma_e) = (\Sigma_j - S_j), \quad (77)$$

since $(\Sigma_j - S_j)\Sigma_e^{-1}\Lambda_j\Lambda_j' = 0$, which follows from $S_j\Sigma_e^{-1}\Lambda_j = \Lambda_j(\Lambda_j'\Sigma_e^{-1}\Lambda_j + I_{r_j^0})$. Thus we have $\Sigma_e = diag(\frac{1}{T}\sum_{t=1}^T x_t x_t' - \sum_{j=1}^{J^0}\frac{1}{T}\sum_{t=1}^T p_{tj|T}\Lambda_j\Lambda_j')$.

First order condition of Q:

Since $\sum_{j=1}^{J^0} Q_{jk} = 1$, the Lagrangean is $\sum_{t=2}^T\sum_{j=1}^{J^0}\sum_{k=1}^{J^0}\log Q_{jk}p_{tjk|T} + \sum_{k=1}^{J^0} w_k(1 - Q_{1k} - Q_{2k} - ... - Q_{J^0k})$. The first order derivative of the Lagrangean with respect to Q_{jk} is $\frac{1}{Q_{jk}}\sum_{t=2}^T p_{tjk|T} - w_k$. Set it to zero, we have $\sum_{t=2}^T p_{tjk|T} = Q_{jk}w_k$. Take sum over j, we have $\sum_{j=1}^{J^0}\sum_{t=2}^T p_{tjk|T} = \sum_{j=1}^{J^0} Q_{jk}w_k = w_k$. Thus $Q_{jk} = \sum_{t=2}^T p_{tjk|T} / \sum_{j=1}^{J^0}\sum_{t=2}^T p_{tjk|T}$.

First order condition of ϕ:

Since $\sum_{k=1}^{J^0}\phi_k = 1$, the Lagrangean is $\sum_{k=1}^{J^0}\log\phi_k p_{1k|T} + w(1 - \phi_1 - \phi_2 - ... - \phi_{J^0})$. The first order derivative of the Lagrangean with respect to ϕ_k is $\frac{1}{\phi_k}p_{1k|T} - w$. Set it to zero, we have $p_{1k|T} = \phi_k w$. Take sum over k, we have $1 = \sum_{k=1}^{J^0} p_{1k|T} = \sum_{k=1}^{J^0}\phi_k w = w$, thus $\phi_k = p_{1k|T} = \sum_{j=1}^{J^0} p_{2jk|T}$.

H Smoother Algorithm for $\tilde{p}_{tjk|T}^{(h)}$

Step (1): Calculate conditional likelihoods $L(x_t \,\big|\, x_{1:t-1}; \tilde{\theta}^{(h)})$ and filtered estimates $\tilde{p}_{tjk|t}^{(h)}$ for $t = 2, ..., T$.

$$\begin{aligned}\tilde{p}_{tjk|t}^{(h)} &= \Pr(z_t = j, z_{t-1} = k \,\big|\, x_{1:t}; \tilde{\theta}^{(h)}) = L(x_t \,\big|\, z_t = j; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)})) \\ \times \Pr(z_t &= j \,|\, z_{t-1} = k; Q) \Pr(z_{t-1} = k \,\big|\, x_{1:t-1}; \tilde{\theta}^{(h)}) / L(x_t \,\big|\, x_{1:t-1}; \tilde{\theta}^{(h)}),\end{aligned}$$

where $\Pr(z_1 = k \left| x_1; \tilde{\theta}^{(h)} \right.) = \frac{L(x_1 \left| z_1 = k; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)} \right.)\phi_k}{\sum_{j=1}^{J^0} L(x_1 \left| z_1 = j; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)} \right.)\phi_j}$ and $\Pr(z_{t-1} = k \left| x_{1:t-1}; \tilde{\theta}^{(h)} \right.) = \sum_{z_{t-2}=1}^{J^0} \Pr(z_{t-1} = k, z_{t-2} \left| x_{1:t-1}; \tilde{\theta}^{(h)} \right.)$. The denominator $L(x_t \left| x_{1:t-1}; \tilde{\theta}^{(h)} \right.)$ equals the sum of the numerator with respect to z_t and z_{t-1}.

Step (2): Fix down $z_t = j, z_{t-1} = k$, for all z_{t+1},

$$\begin{aligned} \Pr(z_{t+1}, z_t &= j, z_{t-1} = k \left| x_{1:t+1}; \tilde{\theta}^{(h)} \right.) = L(x_{t+1} \left| z_{t+1}; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)} \right.)) \Pr(z_{t+1} \left| z_t = j; Q \right.) \\ \times \Pr(z_t &= j, z_{t-1} = k \left| x_{1:t}; \tilde{\theta}^{(h)} \right.) / L(x_{t+1} \left| x_{1:t}; \tilde{\theta}^{(h)} \right.), \end{aligned}$$

for all z_{t+1} and z_{t+2},

$$\begin{aligned} \Pr(z_{t+2}, z_{t+1}, z_t &= j, z_{t-1} = k \left| x_{1:t+2}; \tilde{\theta}^{(h)} \right.) = L(x_{t+2} \left| z_{t+2}; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)} \right.)) \Pr(z_{t+2} \left| z_{t+1}; Q \right.) \\ \times \Pr(z_{t+1}, z_t &= j, z_{t-1} = k \left| x_{1:t+1}; \tilde{\theta}^{(h)} \right.) / L(x_{t+2} \left| x_{1:t+1}; \tilde{\theta}^{(h)} \right.), \end{aligned}$$

and for $\tau = t+3, ..., T$, for all z_τ and $z_{\tau-1}$,

$$\begin{aligned} \Pr(z_\tau, z_{\tau-1}, z_t &= j, z_{t-1} = k \left| x_{1:\tau}; \tilde{\theta}^{(h)} \right.) = L(x_\tau \left| z_\tau; \tilde{\Lambda}^{(h)}, \tilde{\sigma}^{2(h)} \right.)) \Pr(z_\tau \left| z_{\tau-1}; Q \right.) \\ \times \Pr(z_{\tau-1}, z_t &= j, z_{t-1} = k \left| x_{1:\tau-1}; \tilde{\theta}^{(h)} \right.) / L(x_\tau \left| x_{1:\tau-1}; \tilde{\theta}^{(h)} \right.), \end{aligned}$$

where $\Pr(z_{\tau-1}, z_t = j, z_{t-1} = k \left| x_{1:\tau-1}; \tilde{\theta}^{(h)} \right.) = \sum_{z_{\tau-2}=1}^{J^0} \Pr(z_{\tau-1}, z_{\tau-2}, z_t = j, z_{t-1} = k \left| x_{1:\tau-1}; \tilde{\theta}^{(h)} \right.)$.

Step (3): Calculate $\tilde{p}_{tjk|T}^{(h)} = \sum_{z_T=1}^{J^0} \sum_{z_{T-1}=1}^{J^0} \Pr(z_T, z_{T-1}, z_t = j, z_{t-1} = k \left| x_{1:T}; \tilde{\theta}^{(h)} \right.)$.

Repeat steps (1)-(3) for all j and k.