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Kukushkin, Nikolai S.

Russian Academy of Sciences, Dorodnicyn Computing Center

20 November 2008

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Nikolai S. Kukushkin†

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Abstract

Two new properties of a finite strategic game, strong and weak BR-dominance solvability, are introduced. The first property holds, e.g., if the game is strongly dominance solvable or if it is weakly dominance solvable and all best responses are unique. It ensures that every simultaneous best response adjustment path, as well as every non-discriminatory individual best response improvement path, reaches a Nash equilibrium in a finite number of steps. The second property holds, e.g., if the game is weakly dominance solvable; it ensures that every strategy profile can be connected to a Nash equilibrium with a simultaneous best response path and with an individual best response path (if there are more than two players, switches from one best response to another may be needed). In a two person game, weak BR-dominance solvability is necessary for the acyclicity of simultaneous best response adjustment paths; if the set of Nash equilibria is rectangular, it is also necessary for the acyclicity of best response improvement paths.

Journal of Economic Literature Classification Number: C 72.

Key words: Dominance solvability; Best response dynamics; Potential game

*Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-2982.2008.1), the Russian Foundation for Basic Research (project 08-07-00158), the Russian Foundation for Humanities (project 08-02-00347), and the Spanish Ministry of Education and Science (project SEJ 2007-67135) is acknowledged. I thank Hervé Moulin for helpful comment on draft versions as well as for his kind invitation and hospitality during my visit to the Department of Economics of the Rice University. I also thank Francisco Marhuenda, without whose generously exerted effort the visit could not have taken place.

†Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru

1 Introduction

The two strands of game theory referred to in the title have two things in common. First, some dynamic notions are involved in both cases. Second, both can be developed in a purely ordinal framework although are equally applicable to mixed extensions. They radically differ in their assumptions about the rationality of the players.

Dominance solvability presupposes a high degree of sophistication. Each player is able to analyze the whole game and anticipate the results of similar analyses by the others. The term is due to Moulin (1979) although the origins of the notion itself can be traced back to Luce and Raiffa (1957). Actually, there are two versions of the property, strong and weak ones. The elimination of strongly dominated strategies does not change, say, the set of Nash equilibria. The elimination of weakly dominated strategies is not at all innocuous (Samuelson, 1992), but, nonetheless, is often regarded as legitimate.

Individual myopic adaptation, on the contrary, is natural when the players' rationality is bounded and they have to rely on "local" considerations. The study of best response dynamics by A.-A. Cournot predated the very term "game theory" by about a century. Similar processes in various contexts were investigated by Topkis (1979), Bernheim (1984), Vives (1990), Milgrom and Roberts (1990).

In the light of this difference, even opposition, it is very interesting to know whether a game nice from one viewpoint may be nasty from the other. This question was addressed by Moulin (1984), who found that dominance solvability usually implies the convergence of Cournot tatonnement; in a rather special case, an equivalence was established. Dominance was weak although the assumption of unique best responses made it "not so weak." Two scenarios of tatonnement were considered: simultaneous and sequential (with a fixed order of the players).

In a sense, this paper returns to the same subject with a newer toolbox. Although none of the results is strikingly dissimilar to those of Moulin (1984), a much more detailed picture of "what depends on what" is obtained. For technical convenience, we only consider finite games, where we can essentially restrict ourselves to finite improvement (or adjustment) paths; in a continuous game, this would be insufficient. Similarly, in a finite game dominated strategies can be eliminated one at a time, which gives considerable technical freedom; in a continuous game, we have to delete strategies *en mass*, and even then cannot expect a finite number of eliminations to be sufficient. Iterative elimination of dominated strategies in infinite games raises quite a few complicated questions (Gilboa et al., 1990; Marx and Swinkels, 1997; Dufwenberg and Stegeman, 2002).

Concerning adaptive dynamics, we consider both (best response) improvements as defined by Monderer and Shapley (1996) and Milchtaich (1996), and simultaneous best response adjustments. The former include sequential tatonnement of Moulin (1984). In a broader approach to learning in strategic games (Fudenberg and Levine, 1998), more sophisticated scenarios of adaptation or evolution are often considered, which involve random moves and conscious use of mixed strategies. We work in a purely ordinal framework; however, the basic properties of improvement paths to be studied here are relevant to the convergence of more complicated processes (Kalai and Schmeidler, 1977; Young, 1993; Kandori and Rob, 1995; Milchtaich, 1996; Friedman and Mezzetti, 2001). The language of binary relations, suggested in Kukushkin (1999), proves useful.

Since dominance solvability seems to have no implications for *better* reply dynamics anyway, we introduce an apparently new notion of BR-dominance solvability. A strategy is called strongly BR-dominated if it is not among the best responses to any profile of strategies of the partners. A strategy is weakly BR-dominated if it is not indispensable for providing the best responses to all profiles of strategies of the partners; to be more precise, we consider three different versions of the property.

A game is called strongly (weakly) BR-dominance solvable if iterative elimination of strongly (weakly) BR-dominated strategies produces a game where all strategy profiles are Nash equilibria. Clearly, a strongly (weakly) dominance solvable game is strongly (weakly) BR-dominance solvable; both converse statements are wrong.

The iterative elimination of strongly BR-dominated strategies can be viewed as an ordinal analogue of the rationalizability concept (Bernheim, 1984; Pearce, 1984). Admittedly, there is a serious difference between the two situations: If a pure strategy is not a best response to any probability distribution on the strategies of the partners, then it is dominated by a mixed strategy, hence the latter provides a justification for the elimination of the former. When only pure strategies are allowed, the fact that a strategy is not a best response to any profile of strategies of the partners does not make it inferior to any other strategy.

An ordinal version of rationalizability was developed by Borges (1993), but its departure from conventional notions of dominance was less radical than here. Actually, the question of which strategies are not needed by a player can only be resolved with a particular scenario (or a list of scenarios) in view; e.g., the Stackelberg solution of a two person game may well include the choice of a strongly dominated strategy by the leader. And it is easy to see that the elimination of strongly BR-dominated strategies does not change the set of Nash equilibria.

A very interesting feature of Moulin (1984) is an equivalence result (Corollary of Lemmas 1 and 2), even though obtained in a rather special case. From our current viewpoint, that result is just a fortunate coincidence: when all best responses are unique, our four levels of BR-dominance solvability become equivalent. Generally, strong BR-dominance solvability is sufficient for nice best response dynamics, whereas weak BR-dominance solvability is necessary when there are two players. The latter is only sufficient for the possibility to reach a Nash equilibrium from every strategy profile with a tatonnement path. There seems to be no necessity result for more than two players.

Section 2 contains the basic definitions and facts about improvement dynamics in strategic games; a new version of the acyclicity of improvements in a strategic game is introduced, "finite inclusive best response improvement property." In Section 3, standard notions of (strong and weak) dominance solvability are reproduced, and their "best response" modifications are defined; the section also contains auxiliary results about the new concepts. Implications of strong BR-dominance solvability, Theorems 4.4–4.6, are given in Section 4: every simultaneous best response adjustment path reaches a Nash equilibrium in a finite number of steps; every individual best response improvement path does the same unless a player is never given an opportunity to adapt. Weak BR-dominance solvability also has some "positive" implications, especially in the case of two players; they are in Section 5. Theorems 6.2, 6.4 and 6.5 about the necessity of weak BR-dominance solvability are in Section 6; a plausible Hypothesis 6.6 remains unproven. The last Section 7 consists of examples showing the impossibility of easy extensions of the results.

2 Improvement paths in strategic games

Our basic model is a *strategic game with ordinal preferences*. It is defined by a finite set of players N, and strategy sets X_i and preference relations on $X_N = \prod_{i \in N} X_i$ for all $i \in N$. We always assume that each X_i is finite and preferences are described with ordinal utility functions $u_i \colon X_N \to \mathbb{R}$. For notational simplicity, we assume $X_i \cap X_j = \emptyset$ whenever $i \neq j$, and denote $\mathcal{X} = \bigcup_{i \in N} X_i$. For each $i \in N$, we denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ and

$$R_i(x_{-i}) = \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$$

for every $x_{-i} \in X_{-i}$ (the *best response correspondence*); if $\#N = 2$, then $-i$ refers to the partner of player i.

We introduce the *individual improvement* relation $\rhd^{\mathrm{Ind}}$ and *best response improvement* relation $\rhd^{\mathrm{BR}}$ on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$\begin{aligned}
y_N \rhd^{\mathrm{Ind}}{}_i\, x_N &\rightleftharpoons [y_{-i} = x_{-i} \;\&\; u_i(y_N) > u_i(x_N)],\\
y_N \rhd^{\mathrm{Ind}}\, x_N &\rightleftharpoons \exists i \in N\, [y_N \rhd^{\mathrm{Ind}}{}_i\, x_N];\\
y_N \rhd^{\mathrm{BR}}{}_i\, x_N &\rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin R_i(x_{-i}) \ni y_i],\\
y_N \rhd^{\mathrm{BR}}\, x_N &\rightleftharpoons \exists i \in N\, [y_N \rhd^{\mathrm{BR}}{}_i\, x_N].
\end{aligned}$$

By definition, a strategy profile $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a *maximizer* of $\rhd^{\mathrm{Ind}}$, i.e., if $y_N \rhd^{\mathrm{Ind}} x_N$ is impossible for any $y_N \in X_N$. In a finite game, $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a maximizer of $\rhd^{\mathrm{BR}}$.

A (*best response*) *improvement path* is a finite or infinite sequence $\{x_N^k\}_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ ($x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$) whenever $k \geq 0$ and x_N^{k+1} is defined; henceforth, we call such k *admissible* (for a given path).

As in Kukushkin et al. (2005), we combine the terminology of Monderer and Shapley (1996), Milchtaich (1996), and Friedman and Mezzetti (2001). A game has the *finite improvement property* (*FIP*) if it admits no infinite improvement path. A game has the *finite best response improvement property* (*FBRP*) if it admits no infinite best response improvement path. FIP (FBRP) means that every (best response) improvement path reaches a Nash equilibrium in a finite number of steps. A game has the *weak FIP* (*weak FBRP*) if, for every $x_N \in X_N$, there exists a finite (best response) improvement path $\{x_N^0, \ldots, x_N^m\}$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium. Clearly, FIP $\Rightarrow$ FBRP $\Rightarrow$ weak FBRP $\Rightarrow$ weak FIP.

A *Cournot potential* is a strict order (irreflexive and transitive binary relation) $\succ$ on X_N such that $y_N \succ x_N$ whenever $y_N \rhd^{\mathrm{BR}} x_N$; a *weak Cournot potential* is a strict order $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \rhd^{\mathrm{BR}} x_N$ and $y_N \succ x_N$. By Propositions 6.1 and 6.2 from Kukushkin (2004), a finite game has the (weak) FBRP if and only if it admits a (weak) Cournot potential. Henceforth, best response improvement paths will be called just *Cournot paths*; clearly, the FBRP is equivalent to the absence of *Cournot cycles*, i.e., Cournot paths $x_N^0, x_N^1, \dots, x_N^m$ such that $m > 0$ and $x_N^0 = x_N^m$.

A property intermediate between the FBRP and weak FBRP deserves attention. We say that a player $i \in N$ *is involved* in a Cournot path $\{x_N^k\}_{k=0,1,\dots}$ if for each admissible $m \in \mathbb{N}$ there is an admissible $k \geq m$ such that $x_i^k \in R_i(x_{-i}^k)$. A Cournot path is *inclusive* if each player $i \in N$ is involved in it; a Cournot cycle is *complete* if for each player $i \in N$ there is $k \leq m$ such that $x_i^k \in R_i(x_{-i}^k)$.

A game has the *finite inclusive best response improvement property* (*FIBRP*) if it admits no infinite inclusive Cournot path. It is immediately clear that the FIBRP implies, in particular, the convergence of the sequential tatonnement process as defined by Moulin (1984, p. 87) in a finite number of steps.

A *preorder* is a reflexive and transitive binary relation; with every preorder $\succeq$, a strict order $\succ$ and an equivalence relation $\sim$ are naturally associated. A *Cournot quasipotential* is a preorder $\succeq$ on X_N such that for every $x_N \in X_N$ there exists a subset $M(x_N) \subseteq N$ satisfying

$$y_N \rhd^{\mathrm{BR}} x_N \Rightarrow \big[y_N \succ x_N \ \text{ or } \ [y_N \sim x_N \ \& \ M(y_N) = M(x_N) \neq \emptyset]\big]; \tag{1a}$$

$$i \in M(x_N) \Rightarrow x_i \notin R_i(x_{-i}). \tag{1b}$$

It immediately follows that $y_N \succ x_N$ whenever $y_N \rhd^{\mathrm{BR}}{}_i\, x_N$ and $i \in M(x_N)$. If $\succ$ is a Cournot potential, then its reflexive closure $\succeq$ is a Cournot quasipotential with $M(x_N) = \emptyset$ for all $x_N \in X_N$. If $\succeq$ is a Cournot quasipotential, then its asymmetric component $\succ$ is a weak Cournot potential.

Proposition 2.1. *For every finite strategic game Γ, the following statements are equivalent:*

1. *Γ has the FIBRP;*

2. *Γ admits no complete Cournot cycle;*

3. *Γ admits a Cournot quasipotential.*

Proof. Infinite repetition of a complete Cournot cycle generates an infinite inclusive Cournot path, hence Statement 1 implies Statement 2.

Let Statement 2 hold. To verify Statement 3, we denote $\succeq$ the reflexive and transitive closure of $\rhd^{\mathrm{BR}}$: $y_N \succeq x_N$ if and only if there is a finite Cournot path $x_N^0, x_N^1, \dots, x_N^m$ such that $x_N^0 = x_N$ and $x_N^m = y_N$ ($m \geq 0$). Let $Y \subseteq X_N$ be an equivalence class of $\sim$ with $\#Y > 1$; we denote $D(Y) = \{i \in N \mid \forall x_N \in Y\, [x_i \notin R_i(x_{-i})]\}$. Since all $x_N \in Y$ can be arranged into a single Cournot cycle and that cycle cannot be complete, $D(Y) \neq \emptyset$. Now we define $M(x_N) = D(Y)$ if x_N belongs to a non-singleton equivalence class Y, and $M(x_N) = \emptyset$ otherwise. The conditions (1) are checked easily.

Finally, let $\succeq$ be a Cournot quasipotential and $\{x_N^k\}_{k=0,1,\dots}$ be an infinite Cournot path; we have to show that a player $i \in N$ is not involved in the path. Since X_N is finite, at least one strategy profile $\bar{x}_N$ must enter into the path an infinite number of times. Let $x_N^m = \bar{x}_N$ for the first time; clearly, we must have $x_N^{k+1} \sim x_N^k$ for all $k \geq m$. By (1a), $M(x_N^{k+1}) = M(x_N^k) = M^0 \neq \emptyset$ for all $k \geq m$. By (1b), we have $x_i^k \notin R_i(x_{-i}^k)$ for all $i \in M^0$ and $k \geq m$. Thus, each player $i \in M^0$ is not involved. □

Corollary. *If a finite two person game* Γ *has the FIBRP, then it has the FBRP.*

Proof. By Proposition 2.1, Γ admits no complete Cournot cycle; on the other hand, best response improvements by one player cannot form a cycle in any game. □

Remark. In the proof of Theorem 3 of Kukushkin (2004), the FBRP was derived from the presence of a "quasipotential" in an even weaker sense than (1). The point is that whenever a game satisfies the conditions of that theorem, so do all its reduced games. Generally, we only obtain FIBRP. In particular, dominance solvability (in any sense) need not be inherited by the reduced games, hence Theorem 4.4 below also asserts only FIBRP.

We introduce the *simultaneous best response adjustment* relation $\rhd^{*\mathrm{BR}}$ on X_N (y_N, $x_N \in X_N$):

$$y_N \rhd^{*\mathrm{BR}} x_N \rightleftharpoons \big[\forall i \in N\, [y_i = x_i \in R_i(x_{-i}) \text{ or } x_i \notin R_i(x_{-i}) \ni y_i] \;\&\; y_N \neq x_N \big].$$

In a finite game, $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a maximizer of $\rhd^{*\mathrm{BR}}$. A *simultaneous Cournot path* is a finite or infinite sequence $\{x_N^k\}_{k=0,1,\dots}$ such that $x_N^{k+1} \rhd^{*\mathrm{BR}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined.

Remark. We do not use the term "improvement" here because $y_N \rhd^{*\mathrm{BR}} x_N$ is compatible with $u_i(y_N) < u_i(x_N)$ for all $i \in N$.

A game has the *finite simultaneous best response adjustment property* (*FSP*) if there exists no infinite simultaneous Cournot path. FSP implies that every simultaneous Cournot path eventually leads to a Nash equilibrium. A game has the *weak FSP* if, for every $x_N \in X_N$, there exists a finite simultaneous Cournot path $\{x_N^0, \dots, x_N^m\}$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium.

A *simultaneous Cournot potential* is a strict order $\succ$ on X_N such that $y_N \succ x_N$ whenever $y_N \rhd^{*\mathrm{BR}} x_N$; a *weak simultaneous Cournot potential* is a strict order $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \rhd^{*\mathrm{BR}} x_N$ and $y_N \succ x_N$. By Propositions 6.1 and 6.2 from Kukushkin (2004), a finite game has the (weak) FSP if and only if it admits a (weak) simultaneous Cournot potential.

Proposition 2.2. *If a finite two person game* Γ *has the* (*weak*) *FSP, then it has the* (*weak*) *FBRP.*

Proof. For every $x_N \in X_N$, we define

$$\nu(x_N) = \#\{i \in N \mid x_i \in R_i(x_{-i})\}. \tag{2}$$

If $\nu(x_N) = 2$, then x_N is a Nash equilibrium. If $y_N \rhd^{\mathrm{BR}} x_N$, then $\nu(y_N) \geq 1$. If $x_N^0, \dots, x_N^m = x_N^0$ ($m > 0$) is a Cournot cycle, then $\nu(x_N^k) = 1$ for all k. If $\nu(x_N) = 1$, then $y_N \rhd^{*\mathrm{BR}} x_N$ is equivalent to $y_N \rhd^{\mathrm{BR}} x_N$. Therefore, every Cournot cycle is a simultaneous Cournot cycle, hence FSP implies FBRP.

Let Γ have the weak FSP and $x_N^0 \in X_N$; then there is a simultaneous Cournot path $x_N^0, \dots, x_N^m$ such that x_N^m is a Nash equilibrium. If $\nu(x_N^0) = 1$, then $\nu(x_N^k) = 1$ as well for all $k < m$, hence the path is also a Cournot path. Let $\nu(x_N^0) = 0$ and $\nu(x_N^k) \geq 1$ for the first time when $k = \bar{k}$ $(0 < \bar{k} \leq m)$. Without restricting generality, we may assume $x_1^{\bar{k}} \in R_1(x_2^{\bar{k}})$. We denote $y_N^{\bar{k}+1} = x_N^{\bar{k}}$, $y_N^0 = x_N^0$, $y_N^{\bar{k}-2h} = (x_1^{\bar{k}-2h}, x_2^{\bar{k}-2h-1})$ $(h = 0, 1, \dots, 2h+1 \leq \bar{k})$, and $y_N^{\bar{k}-2h-1} = (x_1^{\bar{k}-2h-2}, x_2^{\bar{k}-2h-1})$ $(h = 0, 1, \dots, 2h+1 < \bar{k})$. It is immediately clear from the definitions that $y_1^{\bar{k}-2h} \in R_1(y_2^{\bar{k}-2h-1})$, $y_2^{\bar{k}-2h} = y_2^{\bar{k}-2h-1}$, $y_2^{\bar{k}-2h-1} \in R_2(y_1^{\bar{k}-2h-2})$, and $y_1^{\bar{k}-2h-1} = y_1^{\bar{k}-2h-2}$ for all admissible h. (If $\bar{k}$ is odd, then player 1 moves from $x_N^0 = y_N^0$ to y_N^1; if $\bar{k}$ is even, it is player 2.) For every $k = 0, 1, \dots, \bar{k}$, either $y_N^{k+1} \rhd^{\mathrm{BR}} y_N^k$ or y_N^k is a Nash equilibrium. Therefore, we have obtained a Cournot path starting at $x_N^0 = y_N^0$ and ending either at a Nash equilibrium or at $x_N^{\bar{k}}$ with $\nu(x_N^{\bar{k}}) = 1$. In the first case, we are home immediately; in the second, we recall that $x_N^{\bar{k}}, \dots, x_N^m$ is a Cournot path. $\square$

When there are more than two players, there seems to be no relation between the convergence of Cournot paths and simultaneous Cournot paths (see Moulin, 1986).

Proposition 2.3. *For every finite two person game Γ where $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, the weak FSP (FBRP) implies the FSP (FBRP).*

Proof. No more than one simultaneous Cournot path can be started from any x_N. Therefore, if there were a simultaneous Cournot cycle, no equilibrium could be reached from any strategy profile belonging to the cycle. Similarly, no more than one Cournot path can be started from x_N such that $x_i \in R_i(x_{-i})$ for at least one $i \in N$, and every Cournot cycle must consist of such profiles. $\square$

3 Elimination of dominated strategies

Let Γ be a strategic game, $i \in N$, and $x_i, y_i \in X_i$. We call y_i and x_i *equivalent*, $y_i \approx x_i$, if $u_i(y_i, x_{-i}) = u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. We say that y_i *strongly dominates* x_i, $y_i \ggg x_i$, if for every $x_{-i} \in X_{-i}$, there holds $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$. We say that y_i *weakly dominates* x_i, $y_i \gg x_i$, if $u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for every $x_{-i} \in X_{-i}$, while $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ for some $x_{-i} \in X_{-i}$. A strategy $y_i \in X_i$ is *strongly* (*weakly*) *dominant* if $y_i \ggg x_i$ $(y_i \gg x_i)$ for any $x_i \neq y_i$. A strategy $x_i \in X_i$ is *strongly* (*weakly*) *dominated* if there exists $y_i \in X_i$ such that $y_i \ggg x_i$ $(y_i \gg x_i)$.

A *fragment* Γ' of Γ is a strategic game with the same set of players N, nonempty subsets $\emptyset \neq X_i' \subseteq X_i$ for all $i \in N$, and the restrictions of the same utility functions to $X_N' = \prod_{i \in N} X_i'$. Let X_i' contain both y_i and x_i. Then the relations $y_i \approx x_i$ or $y_i \ggg x_i$ in Γ imply the same relations in Γ'; if $y_i \gg x_i$ in Γ, then either $y_i \approx x_i$ or $y_i \gg x_i$ in Γ'.

Given a strategic game Γ, an *elimination scheme* of the *length* $m > 0$ is a mapping $\xi\colon \{1, \dots, m\} \to \mathcal{X}$; we associate with the scheme a sequence of fragments Γ^k of Γ: $\Gamma^0 = \Gamma$;

$X_i^k = X_i \setminus \xi(\{1,\dots,k\})$ for each $k \in \{1,\dots,m\}$ and $i \in N$. It is convenient to allow an elimination scheme of the length 0, which means just taking $\Gamma^0 = \Gamma$. An elimination scheme of the length $m \geq 0$ is *perfect* if $y_i \approx x_i$ in Γ^m for every $i \in N$ and $y_i, x_i \in X_i^m$ (hence every $x_N \in X_N^m$ is a Nash equilibrium in Γ^m).

A game Γ is *strongly dominance solvable* if it admits a perfect elimination scheme such that, for each $k \in \{1,\dots,m\}$, the deleted strategy $\xi(k)$ is strongly dominated in Γ^{k-1}. A game Γ is *weakly dominance solvable* if it admits a perfect elimination scheme such that, for each $k \in \{1,\dots,m\}$, there is $\varkappa(k) < k$ such that the deleted strategy $\xi(k)$ is weakly dominated in $\Gamma^{\varkappa(k)}$.

Remark. When strongly dominated strategies are iteratively deleted, the result does not depend on the details of the process. The latter may very much matter in the case of the elimination of weakly dominated strategies; the presence of $\varkappa(k)$ in our definition allows for both simultaneous and sequential elimination. The more usual requirement is that each player should become indifferent between all *outcomes* when the elimination process is completed; our perfect schemes do not ensure that. However, our weaker condition is sufficient for all "nice" conclusions.

With a slight abuse, we denote $R_i^{-1}(x_i) = \{x_{-i} \in X_{-i} \mid x_i \in R_i(x_{-i})\}$. A strategy $x_i \in X_i$ is *strongly BR-dominated* if $R_i^{-1}(x_i) = \emptyset$. Let $x_i, y_i \in X_i$; we say that y_i *weakly* (*"not so weakly"*) *BR-dominates* x_i, denoting the fact $y_i \succeq\!\!\succ x_i$ ($y_i \succ\!\!\succ x_i$), if $y_i \neq x_i$ and $R_i^{-1}(x_i) \subseteq R_i^{-1}(y_i)$ ($R_i^{-1}(x_i) \subset R_i^{-1}(y_i)$); note that $\succ\!\!\succ$ is the asymmetric component of $\succeq\!\!\succ$. It is immediately clear that a strongly (weakly) dominated strategy is strongly (weakly) BR-dominated, while a strongly BR-dominated strategy is weakly BR-dominated by any other, and "not so weakly" BR-dominated by any strategy which is not strongly BR-dominated itself. A strategy $x_i \in X_i$ is *very weakly BR-dominated* if $R_i(x_{-i}) \setminus \{x_i\} \neq \emptyset$ for every $x_{-i} \in X_{-i}$. Every weakly BR-dominated strategy is very weakly BR-dominated.

An S-scheme (W$^+$-scheme, W-scheme, W$^-$-scheme) is an elimination scheme ξ of the length m such that, for every $k \in \{1,\dots,m\}$, the deleted strategy $\xi(k)$ is strongly ("not so weakly", weakly, or very weakly) BR-dominated in Γ^{k-1}. We call Γ *strongly* (*weakly, etc.*) *BR-dominance solvable* if it admits a perfect S-scheme (W-scheme, etc.). Since equivalent strategies weakly BR-dominate each other, the elimination of (very) weakly BR-dominated strategies can be continued until each X_i^m is a singleton; however, it is technically more convenient to have all definitions as similar to one another as possible.

Since BR-dominance solvability seems to have never been studied in the literature, we provide detailed proofs of familiar results in the new context. Four implications are obvious: a strongly dominance solvable game is strongly BR-dominance solvable with the same elimination scheme; a strongly BR-dominance solvable game is "not so weakly" BR-dominance solvable with the same elimination scheme; and similarly for ("not so") weak BR-dominance solvability.

Proposition 3.1. *If Γ is weakly dominance solvable, then Γ is weakly BR-dominance solvable with the same elimination scheme.*

Proof. At every step k, the deleted strategy $\xi(k) \in X_i^{k-1}$ is weakly dominated in $\Gamma^{\varkappa(k)}$: $y_i \gg \xi(k)$ with $y_i \in X_i^{\varkappa(k)}$. The strategy y_i need not belong to X_i^{k-1}, but the transitivity of $\gg$ implies that there is $k' < k$ and $y_i' \in X_i^{k-1}$ such that $y_i' \gg \xi(k)$ in $\Gamma^{k'}$. Clearly, $y_i' \neq \xi(k)$ and either $y_i' \gg \xi(k)$ or $y_i' \approx \xi(k)$ in Γ^{k-1}; therefore, $y_i' \not\succeq \xi(k)$ in Γ^{k-1}, i.e., $\xi(k)$ is weakly BR-dominated in Γ^{k-1}. □

Proposition 3.2. *If x_N is a Nash equilibrium in Γ and ξ is an S-scheme of the length m, then $x_N \in X_N^m$.*

Proof. Supposing the contrary, let k be the first step when $x_N \notin X_N^k$; then $x_i = \xi(k)$ and $x_{-i} \in X_{-i}^{k-1}$ for some $i \in N$. On the other hand, $x_i \in R_i(x_{-i})$ in Γ, hence it cannot be BR-dominated in Γ^{k-1}: a contradiction. □

Lemma 3.3. *Let ξ be a W^--scheme of the length m; then $R_i(x_{-i}) \cap X_i^k \neq \emptyset$ whenever $i \in N$, $k \leq m$, and $x_{-i} \in X_{-i}^k$.*

Proof. Supposing the contrary, let $h \geq 0$ be the first step when $R_i(x_{-i}) \cap X_i^{h+1} = \emptyset$. Then $\xi(h+1) \in R_i(x_{-i})$, hence $R_i(x_{-i}) = R_i^h(x_{-i})$. By definition, there is $y_i \in R_i^h(x_{-i})$ such that $y_i \neq x_i$. Clearly, $y_i \in R_i(x_{-i}) \cap X_i^{h+1}$, which contradicts the definition of h. □

Proposition 3.4. *If Γ is very weakly BR-dominance solvable and $x_N \in X_N^m$, then x_N is a Nash equilibrium in Γ.*

Proof. For each $i \in N$, we apply Lemma 3.3 to $x_{-i} \in X_{-i}^m$ and pick $y_i \in R_i(x_{-i}) \cap X_i^m$. By definition, $y_i \approx x_i$ in Γ^m, hence $x_i \in R_i(x_{-i})$ as well. □

Propositions 3.2 and 3.4 immediately imply that the set of Nash equilibria in a strongly BR-dominance solvable game is rectangular, and all perfect S-schemes eliminate the strategies not participating in the equilibria.

4 Strong BR-dominance solvability

First, we show that weak and strong BR-dominance solvability are equivalent under the uniqueness of best responses as assumed in Moulin (1984).

Lemma 4.1. *If $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, then every W^--scheme is an S-scheme.*

Proof. Supposing the contrary, we must have a stage k ($1 \leq k \leq m$) when the deleted, very weakly BR-dominated strategy $\xi(k) \in X_i$ is not strongly BR-dominated in Γ^{k-1}, i.e., is a best response to $x_{-i} \in X_{-i}^{k-1}$. Let $R_i(x_{-i}) = \{y_i\}$; applying Lemma 3.3, we obtain $y_i \in X_i^{k-1}$, hence $R_i^{k-1}(x_{-i}) = R_i(x_{-i}) = \{y_i\}$. Therefore, $\xi(k) = y_i$, while $R_i^{k-1}(x_{-i}) \setminus \{y_i\} = \emptyset$, i.e., y_i is not very weakly BR-dominated in Γ^{k-1}. □

Proposition 4.2. *If Γ is very weakly BR-dominance solvable and $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, then Γ is strongly BR-dominance solvable.*

Proof. The statement immediately follows from Lemma 4.1. □

Let us introduce some useful notations and an auxiliary result. Given an elimination scheme ξ of the length m, we define $\mu\colon \mathcal{X} \to \{1, \dots, m+1\}$ by

$$\mu(\xi(k)) = k; \tag{3a}$$

$$\mu(x_i) = m + 1 \text{ if } x_i \notin \xi(\{1, \dots, m\}). \tag{3b}$$

We also define $\mu^-\colon X_N \to \{1, \dots, m+1\}$ by

$$\mu^-(x_N) = \min_{i \in N} \mu(x_i). \tag{3c}$$

As long as $\mu(x_i) \le m$, μ is injective, hence $\operatorname{Argmin}_{i \in N} \mu(x_i)$ is a singleton whenever $\mu^-(x_N) \le m$.

Lemma 4.3. *Let ξ be an S-scheme of the length m and $x_N \in X_N$ be such that $\mu^-(x_N) \le m$; then for every $i \in N$ and $y_i \in R_i(x_{-i})$, there holds $\mu(y_i) > \mu^-(x_N)$.*

Proof. If $\mu(y_i) = k \le \mu^-(x_N) \le m$, then y_i is strongly BR-dominated in Γ^{k-1}; since $x_{-i} \in X_{-i}^{\mu^-(x_N)-1} \subseteq X_{-i}^{k-1}$, this is incompatible with $y_i \in R_i(x_{-i})$. □

Theorem 4.4. *If a finite game Γ is strongly BR-dominance solvable, then it has the FIBRP.*

Proof. Fixing a perfect S-scheme ξ, we consider the functions μ and μ^- defined by (3). Let us show that the preorder represented by μ^-, i.e., $y_N \succeq x_N \rightleftharpoons \mu^-(y_N) \ge \mu^-(x_N)$, is a Cournot quasipotential with $M(x_N) = \operatorname{Argmin}_{i \in N} \mu(x_i)$ when $\mu^-(x_N) \le m$ and $M(x_N) = \emptyset$ otherwise. If $\mu^-(x_N) = m + 1$, then $x_N \in X_N^m$, hence x_N is a Nash equilibrium in Γ by Proposition 3.4.

Let $y_N \rhd^{\mathrm{BR}}{}_i\, x_N$; then $\mu^-(x_N) \le m$, hence Lemma 4.3 is applicable. If $i \notin M(x_N)$, then $\mu^-(y_N) = \mu^-(x_N)$ and $M(y_N) = M(x_N)$; if $i \in M(x_N)$, then $\mu^-(y_N) > \mu^-(x_N)$ because $M(x_N) = \{i\}$. We see that condition (1a) holds. Finally, if $i \in M(x_N)$, then $\mu(x_i) = \mu^-(x_N) \le m$; if $x_i \in R_i(x_{-i})$, then Lemma 4.3 would imply $\mu(x_i) > \mu(x_i)$. Thus, (1b) holds as well. □

Theorem 4.5. *If a finite two person game Γ is strongly BR-dominance solvable, then it has the FBRP.*

Proof. The statement immediately follows from Theorem 4.4 and Corollary to Proposition 2.1. □

The FBRP in the formulation of Theorem 4.5 cannot be replaced with the FIP: if one player has a strongly dominant strategy x_i^+, then any behavior of improvement paths with $x_i^k \ne x_i^+$ is compatible with strong dominance solvability. For the same reason, the FIBRP cannot be replaced with the FBRP in Theorem 4.4.

Theorem 4.6. *If a finite game Γ is strongly BR-dominance solvable, then it has the FSP.*

Proof. Fixing a perfect S-scheme ξ, we consider the functions μ and μ^- defined by (3). Let us show that the strict order represented by μ^-, i.e., $y_N \succ x_N \rightleftharpoons \mu^-(y_N) > \mu^-(x_N)$, is a simultaneous Cournot potential. Let $y_N \rhd^{*\mathrm{BR}} x_N$; then $\mu^-(x_N) \le m$. By Lemma 4.3, $\mu(y_i) > \mu^-(x_N)$ for every $i \in N$, hence $\mu^-(y_N) > \mu^-(x_N)$ as well. □

5 Weak BR-dominance solvability

Lemma 5.1. *Let ξ be a W^{-}-scheme of the length m and $x_N \in X_N$ be such that $\mu^{-}(x_N) \leq m$; then for each $i \in N$ there is $y_i \in R_i(x_{-i})$ such that $\mu(y_i) > \mu^{-}(x_N)$.*

Proof. Let $\mu^{-}(x_N) = k$; for each $i \in N$, we pick y_i maximizing μ over $R_i(x_{-i})$. Lemma 3.3 implies $\mu(y_i) \geq k$ for each $i \in N$ because $x_{-i} \in X_{-i}^{k-1}$. If $\mu(x_i) > k$, then $\mu(y_i) > k$ because μ is injective; let $\mu(x_i) = k$. If $x_i \notin R_i(x_{-i})$, we have $y_i \neq x_i$, hence $\mu(y_i) > \mu(x_i) = k$. Otherwise, $\mu(x_i') \geq k+1$ for every $x_i' \in R_i^{k-1}(x_{-i}) \setminus \{x_i\}$, which set is not empty because x_i is very weakly dominated in Γ^{k-1}; therefore, $\mu(y_i) \geq \mu(x_i') \geq k+1$. $\square$

Theorem 5.2. *If a finite two person game is very weakly BR-dominance solvable, then it has the weak FSP and the weak FBRP.*

Proof. Fixing a perfect W^{-}-scheme ξ, we consider the functions μ and μ^{-} defined by (3), and introduce a binary relation on X_N:

$$\begin{aligned} y_N \succ x_N \rightleftharpoons \big[\, & \mu^{-}(y_N) > \mu^{-}(x_N) \text{ or} \\ & \exists i \in N\, [\mu^{-}(x_N) = \mu(x_i) = \mu^{-}(y_N) \;\&\; x_i \in R_i(x_{-i}) \;\&\; x_{-i} \notin R_{-i}(x_i) \ni y_{-i}]\big]. \end{aligned} \quad (4)$$

The relation is obviously irreflexive; the transitivity is obvious as long as the first disjunctive term in (4) is applicable. Let $y_N \succ x_N$ by the second term. Since $x_{-i} \notin R_{-i}(x_i)$, we have $\mu^{-}(y_N) \leq m$, hence the minimizing $i \in N$ is unique and $x_i = y_i$. Now if $z_N \succ y_N$, then the second disjunctive term in (4) cannot be valid because $y_{-i} \in R_{-i}(y_i)$, hence $\mu^{-}(z_N) > \mu^{-}(y_N) = \mu^{-}(x_N)$, hence $z_N \succ x_N$ by the first term in (4). Similarly, if $x_N \succ z_N$, then the second term in (4) cannot be valid because $x_{-i} \notin R_{-i}(x_i)$, hence $\mu^{-}(y_N) = \mu^{-}(x_N) > \mu^{-}(z_N)$, hence $y_N \succ z_N$.

Let us show that $\succ$ is a weak simultaneous Cournot potential; let $x_N \in X_N$. For each $j \in N$, we define $y_j = x_j$ if $x_j \in R_j(x_{-j})$, and pick y_j maximizing μ over $R_j(x_{-j})$ otherwise. If $y_N = x_N$, then x_N is a Nash equilibrium already; otherwise, $y_N \rhd^{*\mathrm{BR}} x_N$. Let us show $y_N \succ x_N$.

First, we note that $\mu^{-}(x_N) \leq m$, hence $\mu^{-}(x_N) = \mu(x_i)$ for a unique i. By Lemma 5.1, $\mu^{-}(y_N) \geq \mu^{-}(x_N)$. If the inequality is strict, the first disjunctive term in (4) works. Otherwise, we have $y_i = x_i$, hence $x_i \in R_i(x_{-i})$ by the definition of y_i; besides, $y_{-i} \in R_{-i}(x_i)$ by the same definition. Since x_N is not a Nash equilibrium, $x_{-i} \notin R_{-i}(x_i)$. Thus, $y_N \succ x_N$ by the second disjunctive term in (4).

The weak FBRP immediately follows from the weak FSP and Proposition 2.2. $\square$

For more than two players, Theorem 5.2 is wrong as Example 7.2 shows; only a "very weak" FSP, or a "very weak" FBRP, are then ensured. An *individual best response path* is a finite or infinite sequence $\{x_N^k\}_{k=0,1,\ldots}$ such that, whenever x_N^{k+1} is defined, there is $i \in N$ for which $x_{-i}^{k+1} = x_{-i}^k$, $x_i^{k+1} \neq x_i^k$, and $x_i^{k+1} \in R_i(x_{-i}^k)$. A *simultaneous best response path* is a finite or infinite sequence $\{x_N^k\}_{k=0,1,\ldots}$ such that $x_N^{k+1} \neq x_N^k$ and $x_i^{k+1} \in R_i(x_{-i}^k)$ for all $i \in N$ whenever x_N^{k+1} is defined.

Theorem 5.3. *If a finite game is very weakly BR-dominance solvable, then every strategy profile can be connected to a Nash equilibrium with a simultaneous best response path, as well as with an individual best response path.*

Proof. As above, if $\mu^-(x_N) = m+1$, then x_N is already a Nash equilibrium. Otherwise, we pick y_i maximizing μ over $R_i(x_{-i})$ for each $i \in N$; clearly, $\{x_N, y_N\}$ is a simultaneous best response path. By Lemma 5.1, $\mu^-(y_N) > \mu^-(x_N)$. If y_N is not a Nash equilibrium, we make a similar step, and so on. Thus we obtain a simultaneous best response path along which μ^- strictly increases until a Nash equilibrium is reached.

The second statement immediately follows from a straightforward modification of the proof of Proposition 2.2. $\square$

6 On the necessity of BR-dominance solvability

Lemma 6.1. *For every finite two person game Γ, at least one of the following statements holds:*

1. *Every strategy set X_i is a singleton.*
2. *Γ admits a simultaneous Cournot cycle.*
3. *There is a weakly BR-dominated strategy in Γ.*

Proof. Let Statements 1 and 2 not hold. If every strategy profile $x_N \in X_N$ is a Nash equilibrium, then all strategies of the same player are equivalent, hence Statement 3 holds. Otherwise, there is, at least, one pair of strategy profiles such that $y_N \rhd^{*\mathrm{BR}} x_N$. Since there is no simultaneous Cournot cycle, we can pick an $x_N \in X_N$ which is not a Nash equilibrium and for which $x_N \rhd^{*\mathrm{BR}} x'_N$ is impossible for any $x'_N \in X_N$.

For each $i \in N$, we denote $X'_{-i} = R_i^{-1}(x_i) \subseteq X_{-i}$. If $X'_i = \emptyset$ for an $i \in N$, then x_i is even strongly BR-dominated and we are home. Let $X'_N = X'_1 \times X'_2 \neq \emptyset$. Since x_N is not a Nash equilibrium, there must be $i \in N$ and $x_i^0 \in X'_i$ such that $x_i^0 \neq x_i$. If $R_i^{-1}(x_i^0) \supseteq X'_{-i}$, then $x_i^0 \mathrel{\succeq\!\!\!\!\!\succ} x_i$ and we are home again; otherwise, there is $x_{-i}^0 \in X'_{-i}$ such that $x_i^0 \notin R_i(x_{-i}^0)$. Since $x_N \rhd^{*\mathrm{BR}} x_N^0$ is assumed impossible, we must have $x_{-i} \neq x_{-i}^0 \in R_{-i}(x_i^0)$. Again, if $R_{-i}^{-1}(x_{-i}^0) \supseteq X'_i$, then $x_{-i}^0 \mathrel{\succeq\!\!\!\!\!\succ} x_{-i}$. Otherwise, there is $x_i^1 \in X'_i$ such that $x_{-i}^0 \notin R_{-i}(x_i^1)$; we denote $x_N^1 = (x_i^1, x_{-i}^0) \in X'_N$. Since $x_N \rhd^{*\mathrm{BR}} x_N^1$ is assumed impossible, we must have $x_i \neq x_i^1 \in R_i(x_{-i}^0)$; therefore, $x_N^1 \rhd^{*\mathrm{BR}} x_N^0$. Again, if $R_i^{-1}(x_i^1) \supseteq X'_{-i}$, then $x_i^1 \mathrel{\succeq\!\!\!\!\!\succ} x_i$; otherwise, there is $x_{-i}^2 \in X'_{-i}$ such that $x_i^1 \notin R_i(x_{-i}^2)$. We denote $x_N^2 = (x_i^1, x_{-i}^2) \in X'_N$; again, $x_N^2 \rhd^{*\mathrm{BR}} x_N^1 \rhd^{*\mathrm{BR}} x_N^0$, and so on.

Since there is no simultaneous Cournot cycle, the simultaneous Cournot path $x_N^0, x_N^1, \ldots$ cannot be infinite. On the other hand, the next profile x_N^{k+1} cannot be defined only if $x_i^k \mathrel{\succeq\!\!\!\!\!\succ} x_i$ for an $i \in N$. Thus, Statement 3 holds. $\square$

Theorem 6.2. *If a finite two person game Γ has the FSP, then it is weakly BR-dominance solvable.*

Proof. We apply Lemma 6.1. If X_N is a singleton, Γ is even strong BR-dominance solvable. Statement 2 cannot hold by the FSP assumption. Therefore, there is a weakly BR-dominated strategy x_i. The elimination of x_i defines a W-scheme of the length 1 and a fragment Γ^1. By Lemma 3.3, we have $R_i^1(x_{-i}) = R_i(x_{-i}) \cap X_i^1$ for all $i \in N$ and $x_{-i} \in X_{-i}^1$; therefore, the relation $\rhd^{*\mathrm{BR}}$ in Γ^1 is the restriction of $\rhd^{*\mathrm{BR}}$ in Γ to X_N^1, hence Γ^1 also has the FSP, hence Lemma 6.1 applies again. The process only stops when X_N^m is a singleton; then the W-scheme will be perfect (it may become so even before that). □

The Battle of Sexes shows that the FSP in Theorem 6.2 cannot be replaced with the FBRP (or even FIP). This becomes possible under an additional assumption that the set of Nash equilibria is rectangular.

Lemma 6.3. *For every finite two person game Γ, at least one of the following statements holds:*

1. *Every strategy profile $x_N \in X_N$ is a Nash equilibrium.*
2. *Γ admits a Cournot cycle.*
3. *The set of Nash equilibria in Γ is* not *rectangular.*
4. *There is a "not so weakly" BR-dominated strategy in Γ.*

Proof. Let Statements 1, 2, and 3 not hold. We have to show that Statement 4 holds. If there is a strongly BR-dominated strategy in Γ, we are home immediately; suppose there is none.

For each $i \in N$, there is $X_i^0 \subseteq X_i$ such that $X_N^0 = X_N^1 \times X_N^2$ is the set of Nash equilibria of Γ; therefore, $R_i^{-1}(x_i^0) \supseteq X_{-i}^0$ for both $i \in N$ and all $x_i^0 \in X_i^0$. We pick an $x_N \in X_N \setminus X_N^0 \neq \emptyset$ and start a Cournot path from x_N; since Γ has the FBRP, the path must end at an $x_N^0 \in X_N^0$; therefore, $R_i^{-1}(x_i^0) \supset X_{-i}^0$ for an $i \in N$.

We define a binary relation $\rhd$ on X_i:

$$y_i \rhd x_i \rightleftharpoons \exists x_{-i} \in X_{-i}\,[x_i \notin R_i(x_{-i}) \ni y_i \;\&\; x_{-i} \in R_{-i}(x_i) \;\&\; x_{-i} \notin R_{-i}(y_i)]. \tag{5}$$

Let us show that $\rhd$ is acyclic. Supposing to the contrary that $x_i^0, x_i^1, \ldots, x_i^m = x_i^0$ are such that $x_i^{k+1} \rhd x_i^k$ for each $k = 0, \ldots, m-1$, we pick, for each k, an x_{-i}^k from (5). Then we define $x_N^{2k} = (x_i^k, x_{-i}^k)$ and $x_N^{2k+1} = (x_i^{k+1}, x_{-i}^k)$ for each $k = 0, \ldots, m-1$. It follows immediately from (5) that $x_N^0, x_N^1, \ldots, x_N^{2m} = x_N^0$ is a Cournot cycle in Γ, i.e., Statement 2 holds.

Since X_i is finite and $\rhd$ is acyclic, there is $y_i \in X_i$ such that $y_i \rhd x_i$ does not hold for any $x_i \in X_i$. For every $x_{-i} \in R_i^{-1}(y_i)$, we consider two alternatives: If $x_{-i} \in R_{-i}(y_i)$, then (y_i, x_{-i}) is a Nash equilibrium, hence $x_{-i} \in X_{-i}^0$. If $x_{-i} \notin R_{-i}(y_i)$, then we pick $x_i \in R_{-i}^{-1}(x_{-i}) \neq \emptyset$; then $x_i \in R_i(x_{-i})$ because we would have $y_i \rhd x_i$ otherwise; therefore, (x_i, x_{-i}) is a Nash equilibrium, hence $x_{-i} \in X_{-i}^0$ again. Thus, $R_i^{-1}(y_i) \subseteq X_{-i}^0 \subset R_i^{-1}(x_i^0)$, i.e., Statement 4 holds. □

Theorem 6.4. *If a finite two person game Γ has the FBRP and the set of Nash equilibria in Γ is rectangular, then Γ is "not so weakly" BR-dominance solvable.*

Proof. We apply Lemma 6.3 in the same way as Lemma 6.1 was applied in the proof of Theorem 6.2. □

Theorem 6.5. *If a finite two person game Γ has the weak FBRP and the set of Nash equilibria in Γ is rectangular, then Γ is very weakly BR-dominance solvable.*

Proof. We assume that the set of Nash equilibria in Γ is $X_N^0 = X_1^0 \times X_2^0$. For every $x_N \in X_N$, we define $P(x_N)$ as the length of the shortest Cournot path connecting x_N to a Nash equilibrium; then $P(x_N) = 0 \iff x_N \in X_N^0$. Clearly, there is a mapping $\pi \colon X_N \to X_N$ such that: (i) $\pi(x_N) = x_N \iff x_N \in X_N^0$; (ii) $\pi(x_N) \rhd^{\mathrm{BR}} x_N$ and $P(\pi(x_N)) = P(x_N) - 1$ whenever $x_N \notin X_N^0$. The iteration of π defines a "recommended Cournot path" from an arbitrary strategy profile $x_N \in X_N$ to a Nash equilibrium. We denote $M = \max_{x_N \in X_N} P(x_N)$.

If $M \leq 1$, then $X_{-i} = X_{-i}^0$ for at least one $i \in N$ and $R_i^{-1}(x_i) = \emptyset$ for every $x_i \in X_i \setminus X_i^0$, hence Γ is even strongly BR-dominance solvable. Let $P(x_N^*) = M \geq 2$. Without restricting generality, $\pi(x_N^*) = (x_1^*, y_2)$, hence $x_2^* \notin R_2(x_1^*)$ and $P(x_1^*, y_2) = M - 1$.

Claim 6.5.1. *If $x_N \in X_N$ and $\pi_1(x_N) = x_1^*$, then $x_1 = x_1^*$.*

Proof. Suppose the contrary: $\pi(x_N) = (x_1^*, x_2)$ while $x_1 \neq x_1^*$; then $x_1^* \in R_1(x_2)$. Let $\pi(x_1^*, x_2) = (x_1^*, x_2')$, hence $x_2' \in R_2(x_1^*)$, hence $(x_1^*, x_2') \rhd^{\mathrm{BR}} x_N^*$. Now if $P(x_1^*, x_2') < M - 1$, we obtain a contradiction with $P(x_N^*) = M$. If $P(x_1^*, x_2') \geq M - 1$, we obtain $P(x_N) > M$, contradicting the definition of M. □

We see that the elimination of x_1^* would not destroy the weak FBRP. If x_1^* is very weakly BR-dominated, we are home. Suppose the contrary: $R_1(x_2') = \{x_1^*\}$ for an $x_2' \in X_2$; then $x_2' \notin X_2^0$.

Claim 6.5.2. *For any $x_1 \in X_1$, $x_2' \notin R_2(x_1)$.*

Proof. Otherwise, we would have $\pi(x_1, x_2') = (x_1^*, x_2')$, hence $x_1 = x_1^*$ by Claim 6.5.1, hence $(x_1^*, x_2') \in X_N^0$. □

Therefore, x_2' is even strongly BR-dominated and $\pi_2(x_N) \neq x_2'$ for any $x_N \in X_N$, hence the elimination of x_2' will not destroy the weak FBRP. A straightforward inductive argument completes the proof. □

Hypothesis 6.6. *If a finite two person game Γ has the weak FSP, then Γ is very weakly BR-dominance solvable.*

Remark. The existence of very weakly BR-dominated strategies under the conditions is easy to show. However, the elimination of an arbitrary dominated strategy may destroy the weak FSP, even the weak FBRP, making further recursion impossible. In the proof of Theorem 6.5, a suitable candidate for the elimination was pointed out; here, it has not yet been found.

The relationship between BR-dominance solvability and nice best response dynamics becomes especially simple in the case of two person games with unique best responses, as in Moulin

(1984). According to Proposition 2.3 and Lemma 4.1, there is then no need to distinguish between strong and weak versions of the properties. The set of Nash equilibria is rectangular if and only if it is a singleton.

Corollary to Theorem 6.2. *If Γ is a finite two person game such that $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, and Γ has the weak FSP, then Γ is strongly BR-dominance solvable.*

Corollary to Theorem 6.4. *If Γ is a finite two person game such that $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, the set of Nash equilibria in Γ is a singleton, and Γ has the weak FBRP, then Γ is strongly BR-dominance solvable.*

7 "Counterexamples"

Example 7.1 shows that "not so weak" BR-dominance solvability could not be asserted in Proposition 3.1.

Example 7.1. Let us consider the following bimatrix game:

$$\begin{matrix} (3,3) & (2,2) & (0,0) \\ (0,0) & (1,1) & (1,1) \end{matrix}.$$

The middle column weakly dominates the right one; when the latter is deleted, the upper row becomes strongly dominant. Therefore, the game is weakly dominance solvable. On the other hand, none of the strategies is "not so weakly" BR-dominated: each row is the unique best response to a column; the left column is the unique best response to the upper row; both other columns are only best responses to the bottom row.

Example 7.2 shows that Theorems 4.4 and 4.6 become wrong if Γ is only weakly dominance solvable (or "not so weakly" BR-dominance solvable); Example 7.3 shows the same for Theorem 4.5.

Example 7.2. Let us consider a three person $2 \times 3 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} (3,3,3) & (2,1,1) & (1,2,2) \\ (3,3,3) & (1,2,2) & (2,1,1) \end{bmatrix} \quad \begin{bmatrix} (0,0,0) & \underline{(2,1,1)} & \underline{(1,2,2)} \\ (0,0,0) & \underline{(1,2,2)} & \underline{(2,1,1)} \end{bmatrix}.$$

Nash equilibria fill the left column of the left matrix; however, none of the underlined strategy profiles could be connected to any equilibrium with an individual improvement path or with a simultaneous Cournot path. Thus, the game does not have even the weak FIP or the weak FSP. On the other hand, it is weakly dominance solvable: The choice of the left matrix weakly dominates the choice of the right matrix; when the latter is deleted, the left column becomes strongly dominant.

Example 7.3. Let us consider the following bimatrix game:

$$\begin{array}{ccc} (0,1) & \underline{(1,0)} & \underline{(0,1)} \\ (0,1) & \underline{(0,1)} & \underline{(1,0)}. \\ (2,2) & (1,0) & (1,0) \end{array}$$

The bottom row and the left column are weakly dominant; the southwestern corner of the matrix is a unique Nash equilibrium. The underlined fragment is a Cournot cycle (hence a simultaneous Cournot cycle as well).

The Battle of Sexes has the FIP, but is not even very weakly BR-dominance solvable; therefore, the converse to Theorems 4.4 and 4.5 would be wrong. Example 7.4 shows the impossibility to reverse Theorem 4.5 even when the set of Nash equilibria is rectangular. Example 7.5 shows the impossibility to reverse Theorem 4.6, or assert "not so weak" BR-dominance solvability in Theorem 6.4.

Example 7.4. Let us consider a two person 2×2 game:

$$\begin{array}{cc} (0,2) & (2,0) \\ (1,1) & (1,1) \end{array}.$$

The southwestern corner is a unique Nash equilibrium. The game obviously has the FIP. On the other hand, each strategy of each player is a best response to a strategy of the partner; therefore, the game is not strongly BR-dominance solvable.

Example 7.5. Let us consider a two person 2×2 game:

$$\begin{array}{cc} (1,1) & (0,1) \\ (0,1) & (1,1) \end{array}.$$

There are two Nash equilibria: the northwestern and southeastern corners. Simultaneous best response adjustment from any other strategy profile immediately produces a Nash equilibrium, so the game has the FSP. On the other hand, each strategy of player 1 is the unique best response to a strategy of the partner; each strategy of player 2 is a best response to each strategy of the partner. Therefore, the game is not strongly BR-dominance solvable, nor even "not so weak" BR-dominance solvable.

Example 7.6 shows that Theorem 6.2 is wrong for more than two players; Example 7.7 shows the same for Theorem 6.4.

Example 7.6. Let us consider a three person $2 \times 2 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} \underline{(2,1,2)} & (4,4,4) \\ \underline{\underline{(0,0,0)}} & \underline{(1,3,3)} \end{bmatrix} \quad \begin{bmatrix} \underline{\underline{(0,0,0)}} & \underline{(3,2,1)} \\ (4,4,4) & \underline{\underline{(0,0,0)}} \end{bmatrix}.$$

The two Nash equilibria are not underlined. Each of the three strategy profiles underlined once is dominated in the sense of $\rhd^{*\mathrm{BR}}$ only by a Nash equilibrium; each of the three strategy profiles underlined twice is dominated in the same sense only by a strategy profile underlined once. Thus, the game has the FSP. On the other hand, each strategy of each player is a unique best response to a strategy profile of the partners. Therefore, the game is not even very weakly BR-dominance solvable.

Example 7.7. Let us consider a three person $2 \times 2 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} (3,4,3) & (0,0,0) \\ (5,5,5) & (4,3,4) \end{bmatrix} \begin{bmatrix} (2,2,1) & (1,1,2) \\ (0,0,0) & (2,2,1) \end{bmatrix}.$$

The southwestern corner is a unique Nash equilibrium; the FBRP is easy to check. On the other hand, each strategy of each player is the unique best response to a strategy profile of the partners. Therefore, the game is not even very weakly BR-dominance solvable.

Example 7.8 shows that the adverb "very" cannot be dropped in Theorem 6.5 or Hypothesis 6.6.

Example 7.8. Let us consider a two person 6×6 game defined by the left matrix:

$$\begin{matrix} (3,3) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (2,1) & (1,2) & (2,1) & (1,2) & (0,0) \\ (0,0) & (0,0) & (2,1) & (1,2) & (2,1) & (1,2) \\ (0,0) & (1,2) & (0,0) & (2,1) & (1,2) & (2,1) \\ (0,0) & (2,1) & (1,2) & (0,0) & (2,1) & (1,2) \\ (1,2) & (1,2) & (2,1) & (1,2) & (0,0) & (2,1) \end{matrix} \qquad \begin{bmatrix} 0 & 4 & 2 & 4 & 4 & 2 \\ 3 & 4 & 3 & 4 & 5 & 3 \\ 3 & 4 & 4 & 5 & 4 & 3 \\ 5 & 5 & 5 & 6 & 5 & 6 \\ 3 & 4 & 3 & 4 & 4 & 3 \\ 1 & 5 & 2 & 5 & 4 & 2 \end{bmatrix}.$$

The northwestern corner is a unique Nash equilibrium. The weak FSP is easy to check: the right matrix shows the length of the shortest simultaneous Cournot path leading to the equilibrium from every strategy profile. By Proposition 2.2, the game has the weak FBRP as well. On the other hand, none of the sets $R_i^{-1}(x_i)$ include each other for either $i \in N$, even if non-strict inclusion is taken into account. Therefore, there is no weakly BR-dominated strategy.

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