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Abstract

This paper investigates the optimal portfolio rebalancing strategy for assets with cash distributions and proportional transaction costs. A sweep account is an account that is used as the default destination for coupon and dividend proceeds as they arrive. In this study, we incorporate this account and investigate the optimal strategy for the sweep account manager. Our results indicate that the "no-transaction" region is split into two sub-regions, where the cash proceeds are either invested entirely in the riskless asset or in the risky asset, depending on the transaction costs. Additionally, we analyze the impact of the assets' cash distributions and the investors' investment horizon on the demand for the assets. Our findings suggest that changes in the cash distribution of assets, depending on the relative magnitude of transaction costs for risky and riskless assets, can have a varying impact on asset demand. In particular, our results indicate that when the transaction cost for the riskless asset is low, an increase in the cash distributions from the risky asset and an increase in the investor's investment horizon have a positive impact on the liquidity premium of the risky asset.

JEL Codes: G11, G23, D11, D61.

Keywords: Transaction cost, Sweep account, Liquidity premium, Portfolio optimization, Continuous-Time methods.

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Introduction

A sweep account is a type of cash management account that automatically transfers excess cash balances into higher-yielding investment vehicles, such as money market funds or equity investment accounts. The function of a sweep account is to help investors manage the cash received from their investments more effectively and gain higher returns on their cash earnings without having to actively manage their cash balances. It's also worth noting that sweep accounts have become increasingly popular in recent years due to the low interest rate environment and the need for investors to find ways to earn a higher return on their cash. As a result, many banks and financial institutions now offer sweep accounts as a way to help their customers manage their cash more effectively.

This study examines the effect of assets' cash distributions on the optimal portfolio rebalancing strategy. Specifically, this paper studies the problem of optimal portfolio allocation of assets with cash distributions under the assumption of a constant relative risk aversion (CRRA) investor facing proportional transaction costs when purchasing the asset. In this economy, the CRRA investor maximizes their final wealth. We assume that the investor's investment horizon is finite and stochastic. Following the literature, the economy is modeled with two assets - a risky equity and a riskless bond. The investor's portfolio, in turn, consists of two accounts: a bond account that receives coupon payments and an equity account that receives dividend payments.

However, in order to incorporate the impact of assets' cash distribution on the investors' portfolio rebalancing strategy, we add a third account called the sweep account. All cash distributions originating from the bond and equity accounts are automatically deposited into the sweep account, which is then managed to optimize the allocation of cash balances back into the bond and equity accounts. In the absence of liquidity shocks, optimal cash allocation would entail the reinvestment of cash proceeds into either the bond or equity account. This study is designed to provide an analysis of the impact of cash distributions and proportional transaction costs on the optimal portfolio allocation, and how they influence the investment behavior of the investor. Notably, the study's results reveal a binary optimal sweep decision, where the sweep account's entire balance is allocated to either bond or equity purchases.

The results of this analysis indicate that an increase in the dividend payout ratio of the equity asset leads to a rise in demand for equity and a decrease in demand for bonds, assuming that the transaction cost associated with the bond is comparatively lower than those associated with equity. However, beyond a critical ratio of transaction cost of bond to equity, the impact of a higher dividend payout ratio reverses, and the demand for equity starts to wane. Moreover, this paper delves into the impact of a higher coupon rate of bonds on the investment decision of fund managers. We show that when the coupon rate on bonds rises, as the transaction cost of the bond correspondingly increases, the sweep policy does not change regardless of whether coupon rates are high or low. However, this increase in the coupon rate leads to a decrease in the demand for both equity and bonds.

Another critical aspect of the proposed financial model lies in the consideration of the time horizon of the investors. Although these investors are technically maximizing their utility over an indefinite span of time, the assumption that the end of their investment horizon is randomly arriving presents us with an opportunity to gain a deeper understanding of the impact of the expected time horizon on their investment behavior. In this paper, we demonstrate that the relative size of the transaction cost of bonds and equity influences investors' tendency to change their investment behavior as they approach their investment horizon. Specifically, as the investment horizon approaches, in an environment where the transaction cost of bond is comparatively lower than the transaction cost of equity, investors decrease their rate of equity purchases and increase the pace of bond purchases. This result is consistent with the findings of Hopenhayn and Werner (1996), who demonstrated that as investors' investment horizons shorten and they desire to consume earlier, their demand for a more liquid asset with a lower expected payoff would increase.

On the other hand, when the transaction cost of risk-free assets becomes relatively larger compared to the transaction cost of equity, the behavior of investors changes in response to the shortening of their expected time to their investment horizon. In this scenario, investors tend to increase their allocation towards the risky asset and decrease their demand for the risk-free asset, reflecting the impact of cash distribution associated with these assets on the demand for these assets. Furthermore, the study demonstrates that the degree of risk aversion among investors has an adverse effect on the demand for the equity and a positive effect on the demand for the bond.

Finally, this paper builds upon previous studies on the liquidity premium of equities and examines how it is affected by the equity dividend payout and the investment horizon of investors. Constantinides (1986), defines liquidity premium as the extra return that an investor earns as compensation for the lack of liquidity associated with a particular asset. In other words, investors demand a higher rate of return for assets that are less liquid or difficult to sell quickly, compared to assets that are highly liquid and easily tradeable. Drawing from this definition, we demonstrate that higher dividends would have a positive impact on the demand for equities when the transaction cost for bond is low. This increased demand would, in turn, result in higher transaction costs associated with acquiring the asset which leads to an increase in the asset's liquidity premium to compensate investors for their elevated expenditure on transaction costs. In addition to exploring the relationship between cash payments and the liquidity premium of an asset, this study also examines the impact of investors' investment horizons on the liquidity premium for an asset with transaction costs. It is shown that as investors' investment horizons shorten, their demand for the risky asset would decrease and their demand for the riskless asset would increase, depending on the gap in the two transaction costs. This decrease in demand for the risky asset would then reduce the investor's expenditure on transaction costs associated with acquiring this asset. As the demand for the risky asset decreases and the expenditure on transaction costs associated with acquiring the asset decline, the asset would lose some of its liquidity premia.

Constantinides (1986) Showed that transaction costs can increase the liquidity premium of an asset. He asserted that investors in equilibrium should receive a higher mean return for an asset with transaction costs, as compensation for the increased expenditure associated with acquiring the asset. Constantinides further claimed that the liquidity premium is typically lower in magnitude than the transaction cost. However, recent studies have challenged this view. Papers such as Lynch and Tan (2011) have demonstrated that the liquidity premium can be of the same magnitude as the transaction cost, in certain cases. Lynch and Tan (2011) showed that when predictable returns and wealth shocks to labor income are introduced, transaction costs can result in liquidity premia that are on the same order of magnitude as the transaction cost spread. Here, we demonstrate that incorporating two additional factors, namely asset cash payouts and investors' average investment horizon, into the analysis can help explain some of the underestimations of the liquidity premium observed in Constantinides' research.

The findings of this research suggest that the coupon rate of bonds and the dividend payout ratio of equities play a crucial role in determining the optimal portfolio allocation of assets. Furthermore, the results of this study can provide valuable insights for investment decision-making regarding the benefits of sweep accounts. Additionally, it can contribute to a deeper understanding of the impact of cash distributions and transaction costs on the demand for assets within the economy.

Literature Review

The academic literature on optimal portfolio allocation in the presence of transaction costs has been widely explored. Merton (1971) first addressed the issue of optimal portfolio allocation when there are no transaction costs involved. He showed that there exists an optimal ratio of bonds to equity in a portfolio that an investor should maintain by continuously rebalancing the portfolio. However, when transaction costs are considered, the problem becomes more complex. The problem of optimal portfolio rebalancing when investors are facing transaction costs has been studied under different specifications. Constantinides (1979) and Constantinides (1986) analyzed the problem of optimal portfolio choice under the condition of maximizing infinite lifetime consumption and proportional transaction costs. Taksar et al. (1988), Davis and Norman (1990), and Dumas and Luciano (1991) also tackled the issue using the stochastic singular control problem and showed that proportional transaction costs create three distinct regions in the allocation space. The first region, referred to as the no-transaction (NT) region, is a convex cone in the portfolio allocation space where investors do not rebalance their portfolio. The second region is known as the Buy (B) region where the investor rebalances the portfolio by buying the risky asset and selling the riskless asset. The boundary between the Buy region and the No-Transaction region is called the Buy-Boundary. The third region is known as the Sell (S) region, where the investor rebalances the portfolio by selling the risky asset and buying the riskless asset. The boundary between the Sell region and the No-Transaction region is called the Sell-Boundary.

Liu and Wu (2001) further extended the literature and explored the impact of transaction costs on an optimal consumption and investment decision, assuming that security returns have bounded uncertainty. Some other papers studied the impact of fixed transaction cost on optimal portfolio choice. Among them, Liu (2004) studied the optimal portfolio allocation problem with fixed transaction costs and multiple risky assets for investors with constant absolute risk aversion (CARA) utility, while Dybvig (2020) analyzed the problem for mean-variance utility maximizer investors. They found that with fixed transaction costs, investors optimally maintain their portfolio allocation between two constant levels and rebalance their positions as soon as their portfolio allocation reaches either boundary to reach an optimal target. Other research has focused on the optimal consumption and portfolio strategy when taking into account labor income. Bodie, Merton and Samuelson (1992) added to this body of research by examining the influence of the labor-leisure choice on portfolio and consumption decisions throughout an individual's life cycle. Furthermore, Dybvig and Liu (2010) investigated the optimal consumption and portfolio problem in the context of voluntary or mandatory retirement and the presence or absence of a non-negative wealth constraint, which restricts borrowing against future wages. Several academic studies have explored the topic of optimal portfolio selection with transaction costs and a finite investment time horizon. Gennotte and Jung (1994) and Boyle and Lin (2002) are among the seminal works in this field, and they have demonstrated that as the time horizon of investors increases, the no-transaction boundaries grow in a monotonic fashion, ultimately converging to the infinite horizon case in the limit.

The literature in this field has primarily focused on either self-financing portfolios or assets that do not have any cash distribution. To the best of my knowledge, the only study that has briefly mentioned the case of dividend-paying assets is Dumas and Luciano (1991), who state that "the case of a dividend-paying asset would be an interesting case and the extension would require an additional state variable," but they do not provide any in-depth examination. The importance of considering cash payments from assets in investment portfolios was highlighted by Blume (1980), who conducted a survey revealing a strong preference among individual investors for dividend-paying stocks. This trend of dividend-paying stocks being more attractive than non-dividend-paying stocks in the investment world is at odds with the findings of Modigliani and Miller (1958, 1963), who showed that dividends are irrelevant under certain stringent assumptions. To address this apparent contradiction, researchers have attempted to explain why investors prefer dividend-paying stocks. Dybvig and Zender (1991) and Ofer and Thakor (1987) proposed that information asymmetry models could help explain the signals firms send to investors by paying dividends. Frankfurter and Lane (1992) suggested that behavioral biases could also play a role in investors' preference for dividends.

In Section 1, we present a solution to the allocation problem when the transaction cost for bonds is zero. The solution is based on a free boundary ordinary differential equation, which is similar to the solution found in Davis and Norman (1990). In this scenario, the boundary for the sweep falls on the boundary for buying equity, and the no-transaction region is always the region where cash is transferred from the sweep account to the bond account. The main difference between the author's solution in this paper and the solution in Davis and Norman (1990) is the consideration of non-zero dividend payout ratios for equity.

In Section 2, we extend the analysis to the case where the transaction costs for both bonds and equity are non-zero. The author shows that in this scenario, the sweep boundary is strictly inside the no-transaction region. This means that the no-transaction region splits into two sub-regions, and based on the transaction costs for bonds and equity, the investor can transfer cash from the sweep account to either the bond or equity account. By incorporating the impact of dividends in the model, the author analyzes the effect of dividends on the demand for the risky asset and it's liquidity premia.

The Continuous Time Model

In this economy, we assume there is a continuum of investors who live for a finite period, but their investment horizon arrival is not deterministic. The investors can trade two assets, one riskless (bond) and another risky (equity). The shares of the assets are infinitely divisible. Investors take the price of the assets as given and they can only long the assets with zero capital gain tax at the time of sale. Let the equity price follows geometric Brownian motion with mean μ, standard deviation σ, and a constant dividend yield of q. The bond pays a constant rate of return of r with a coupon rate of c. We assume that $\mu - r > 0$.

To formalize the investor's portfolio holdings, we denote $s_0(t)$ as the dollar value of the investor's bond holdings at time t, and $s_1(t)$ as the dollar value of their equity holdings. Assume the investor incurs proportional transaction costs of λ^b only when purchasing bond and λ^s only when purchasing equity. Following the assumption that the sale of assets is free of transaction costs, the investor's wealth at time t is given by $w(t) = s_0(t) + s_1(t)$. Suppose that $s_0(0) = x$ is the initial endowment of bond, and $s_1(0) = y$ is the initial endowment of equity. Thus, the initial wealth of the investor is $w(0) = x + y$. As a result, the law of motion for the investor's holdings, in the absence of any transactions at time t, is as follows:

$$ds_0(t) = s_0(t)(r - c)dt + cs_0(t)dt \tag{1}$$

$$ds_1(t) = (\mu - q)s_1(t)dt + qs_1(t)dt + \sigma s_1(t)dz(t) \tag{2}$$

Equation 1 reflects the investor's bond holding law of motion at time t, and Equation 2 shows the investor's equity holding law of motion at time t. In Equation 2, $z(t)$ follows standard Brownian motion, which represents the stochastic component of the risky asset. In the absence of any transaction costs, changes to the coupon rate or dividend yield will not affect the path of the accounts since these earnings can be reinvested in the same account without incurring any costs. Suppose the investor has a finite time horizon that is exponentially distributed with a parameter value of η. Thus, the probability of the investor meeting their horizon at time $\tau \in dt$ is $\eta e^{-\eta t}$. Consequently, the expected arrival time of the investment horizon is $\frac{1}{\eta}$. This assumption allows us to convert the current finite horizon problem to an infinite time horizon problem, which will later help us to find the stationary solution to the problem.

Assumption 1 *The investor has a finite horizon τ which arrives randomly and it follows an exponential distribution with a parameter value of η. The investor also has a constant relative risk aversion (CRRA) preference over their final wealth, $\frac{w(\tau)^{1-\gamma}}{1-\gamma}$, where $w(\tau)$ is the investor's wealth at time τ, and* $0 < 1-\gamma < 1$.

Following Assumption 1, the investors' objective is to maximize their expected utility over their final wealth. Thus, based on the investor's horizon assumption, the expected value of the investor's utility over their final wealth can be transformed into an expected value over an infinite horizon.

$$E\Big[\frac{w(\tau)^{1-\gamma}}{1-\gamma}\Big] = \eta E\Big[\int_0^\infty e^{-\eta t}\frac{w(t)^{1-\gamma}}{1-\gamma}dt\Big] \tag{3}$$

This transformation of the objective function allows us to study the stationary solution for the optimal portfolio rebalancing strategy. If we maximize Equation 3 in the absence of transaction costs subject to constraints of Equations 1 and 2, the solution would be similar to Merton (1971), where the investor's optimal portfolio allocation is a constant ratio of bonds to equity equal to $\frac{\gamma\sigma^2}{\mu-r} - 1$, and the investor would continuously trade to maintain this optimal ratio of bonds to equity.

Lemma 1 *Let $\eta > (1-\gamma)(r + \frac{(\mu-r)^2}{2\gamma\sigma^2})$, x be the value of the bond account, and y be the value of the equity account of the investor's portfolio. If the investor's objective is to maximize Equation 3 subject to Equations 1, and 2, the investor's optimal portfolio rebalancing policy is $\frac{x}{y} = \frac{\gamma\sigma^2}{\mu-r} - 1$, and the optimal value function is $v(x,y) = \eta\Big(\eta - (1-\gamma)(r + \frac{(\mu-r)^2}{2\gamma\sigma^2})\Big)^{-1}\frac{(x+y)^{1-\gamma}}{1-\gamma}$.*

Proof. Merton (1971) ■

In Lemma 1, it is crucial to note that the assumption of $\eta > (1-\gamma)(r + \frac{(\mu-r)^2}{2\gamma\sigma^2})$ is necessary for the existence of an optimal solution. In the next section, we extend Merton's problem by introducing a positive proportional transaction cost for purchasing equity. We show that this extension would transform the problem into a similar one as discussed in Davis and Norman (1990). However, unlike their study, this paper considers the existence of equity dividends that are transferred to the bond account in the equilibrium.

1 Optimal Portfolio When Only Equity is Subject to Transaction Cost

To incorporate the transaction cost of purchasing the equity, we define non-decreasing, adapted, and right-continuous processes U_t and L_t, where U_t represents the cumulative dollar value of bond purchase and L_t represents the cumulative dollar value of equity purchase from time 0 until time t. To optimize the expected utility of the final wealth, investors must determine the optimal bond, equity, and cash sweep values. Additionally, we define the Sweep account as a repository in which the cash proceeds from bonds and equity accumulate. The fraction $\chi(t)$ denotes the proportion of the sweep account at time t, which the investor directs towards the equity account, while $1-\chi(t)$ denotes the proportion directed towards the bond account.

Problem 1. *Consider an investor facing a proportional transaction cost of $\lambda^s > 0$ when purchasing equity and a zero transaction cost for bonds. Then, the investor seeks to choose $\chi(t), U(t), L(t)$ for $t \in [0,\infty]$ in order to maximize the following problem:*

$$v(x,y) = \max_{L_t, U_t, \chi_t} \mathbb{E}\int_0^\infty \Big[\eta e^{-\eta t}\frac{w(t)^{(1-\gamma)}}{1-\gamma}\Big]dt \tag{4}$$

subject to:

$$ds_0(t) = \Big[(r-c)s_0(t) + (1-\chi_t)\Big(cs_0(t) + qs_1(t)\Big)\Big]dt - (1+\lambda^s)dL_t + dU_t \tag{5}$$

$$ds_1(t) = \Big[(\mu-q)s_1(t) + \frac{1}{1+\lambda^s}\chi_t\Big(cs_0(t) + qs_1(t)\Big)\Big]dt + \sigma s_1(t)dz_t + dL_t - dU_t \tag{6}$$

$$\forall t \quad s_0(t), s_1(t) \geq 0, \qquad s_0(0) = x, \qquad s_1(0) = y$$

In Problem 1, the investor seeks to maximize their expected utility of their final wealth over a random horizon that arrives at a rate of η by choosing the optimal equity strategy L_t, bond strategy U_t, and sweep strategy χ_t. The first constraint, as stated in Equation 5, is the law of motion for the bond account. Bonds held in this account earn a deterministic interest rate of r and receive a coupon rate of c. To purchase dU_t units of bonds, the investor must sell dU_t units of equity, as the sale of equity incurs no transaction costs. The second constraint, as stated in Equation 6, is the law of motion for the equity account. Equity is a risky asset that follows GBM with mean return μ and standard deviation σ. The investor receives dividend payments from the equity at a rate of q at time t. To add dL_t units of equity to the equity account, the investor must sell $(1 + \lambda^s)dL_t$ units of bonds to cover the transaction cost of purchasing the equity. The third condition reflects the assumption that short selling of any asset is not allowed, and that investors can only hold positive balances of each asset. It also ensures that the investment strategy set is compact.

At each time, the investor receives cash distributions from the bond and equity assets held in their portfolio, which are automatically deposited into the sweep account as they arrive. As a result, the cash balance in the sweep account at time t is given by the sum of the cash distributions received from the bond account, which are represented by $cs_0(t)$, and the cash distributions from the equity account, which are represented by $qs_1(t)$. Given the sweep account balance, the investor reallocates a fraction χ_t of this account into the equity account and a fraction $1 - \chi_t$ into the bond account.

Proposition 1 addresses the solution to Problem 1, which pertains to dividend-paying assets and positive transaction costs for the equity asset, while the bond has zero transaction costs (i.e., $\lambda^b = 0$). When the transaction cost of one asset is zero, investors treat that account as a sweep account and deposit the cash proceeds from all accounts into the account with zero transaction cost, which in this case is the bond account. As a result, the boundaries of the sweep account and the zero-transaction cost account overlap in the portfolio allocation space.

The usual assumptions in the relevant literature, such as in Davis and Norman (1990), are that assets do not pay dividends, or if they do, the dividend is assumed to be costlessly reinvested in the same asset. The solution to Problem 1 is similar to that of Davis and Norman (1990), except for the presence of a sweep account to manage the dividend from the risky asset. However, this difference leads to some variations in the no-transaction region.

Lemma 2 *Assume that a solution to Problem 1 exists for all initial values x and y > 0 , and that the value function is twice continuously differentiable with respect to x and y. Then the value function is concave and* *has the following homothetic property:* $v(x,y) = y^{1-\gamma}\psi(\frac{x}{y})$, *and* $\psi(\frac{x}{y})$ *is a* C^2 *function.*

Proof. Davis and Norman (1990), Muzere (2001) ∎

Proposition 1 *In Problem 1, the optimal sweep decision of the investor in problem* 1 *is to always transfer cash into the bond account. Having the optimal rebalancing conditions, setting* $y = 1$, *there exists a unique sell boundary* x_0, *and a unique buy boundary* x_T.

1. *There exists a boundary, called the sell boundary at* x_0, *such that if* $x \leq x_0$, *the investor's optimal strategy is to sell equity and buy bonds until the portfolio reaches the sell boundary. The region in the allocation space to the left of the sell boundary is called the sell region. The investor's value function in the sell region is given by,* $\psi(x) = \frac{1}{1-\gamma}A(x+1)^{1-\gamma}$, *for a constant value of* A.

2. *There exists a boundary, called the buy boundary at* x_T, *such that if* $x \geq x_T$, *the investor's optimal strategy is to buy equity and sell bonds until the portfolio reaches the buy boundary. The region in the allocation space to the right of the buy boundary is called the buy region. The investor's value function in the buy region is given by,* $\psi(x) = \frac{1}{1-\gamma}B(x+1+\lambda^s)^{1-\gamma}$, *for a constant value of* B.

3. *The region in between the sell boundary,* x_0, *and the buy boundary,* x_T, *is called the no transaction region, where the investor would not trade. The value function in this region is solved through the following free boundary differential equation.*

$$\beta_3 x^2 \psi''(x) + (\beta_2 x + q)\psi'(x) + \beta_1 \psi(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0 \tag{7}$$

Where, $\beta_1 = (\mu - q - \frac{1}{2}\sigma^2\gamma)(1-\gamma) - \eta, \quad \beta_2 = \sigma^2\gamma + r - \mu + q, \quad \beta_3 = \frac{1}{2}\sigma^2$

Such that at the boundaries, x_0, *and* x_T, *the following conditions hold:*

$$\frac{\psi'(x_T)}{\psi(x_T)} = \frac{1-\gamma}{x_T+1+\lambda^s} \qquad \frac{\psi'(x_0)}{\psi(x_0)} = \frac{1-\gamma}{x_0+1} \qquad \frac{\psi''(x_T)}{\psi'(x_T)} = \frac{-\gamma}{x_T+1+\lambda^s} \qquad \frac{\psi''(x_0)}{\psi'(x_0)} = \frac{-\gamma}{x_0+1}$$

Proof. *See Appendix* ∎

[Figure 1 here]

Figure 1 displays the results of Proposition 1, which illustrates the asset allocation space, where the horizontal axis represents the value of the bond account, and the vertical axis represents the value of the equity account. The shaded cone-shaped area denotes the no-transaction (NT) region where investors refrain from trading until their portfolio value reaches the boundaries of the cone. The boundaries are straight lines passing through the origin and $(x_0, 1)$ and $(x_T, 1)$, respectively, owing to the homothetic property of the value function and the boundary conditions. The Merton line, located inside the shaded region, shows the portfolio where investors continuously rebalance their portfolios to remain on the line, given zero transaction costs. However, the presence of transaction costs creates this wedge in the asset allocation space. Within the wedge, the cost of rebalancing the portfolio exceeds the benefits, prompting investors to let the portfolio deviate from the Merton line until it reaches either boundary of the NT region. At the boundary, the investor is indifferent, and hence, only aims to maintain the portfolio at the boundaries once it drifts out of the NT region.

The region on the left side of the no-transaction region, marked by *S*, is known as the sell region where investors sell equity and buy bonds to maintain the sell boundary of the no-transaction region. The value function in this region along the trading direction (a line with a slope of -1) is constant. On the right side of the no-transaction region, the buy region, labeled as *B*, represents the area where investors purchase equity and sell bonds to stay on the buy boundary. In this region, the value function is constant along the trading direction, which is a line with a slope of $-\frac{1}{1+\lambda^s}$.

Proposition 1 also demonstrates that the sweep boundary coincides with the buy boundary. In the no-transaction region, the investor uses all the cash proceeds from bonds and equities to buy bonds. In the region to the right of the buy boundary, the investor would use the cash balance in the sweep account to buy equity. As the investor would never let the portfolio drift out of the NT region, the sweep action always moves cash toward buying bonds.

Proposition 2 *Lets assume that* $\eta > (1-\gamma)(r + \frac{(\mu-r)^2}{2\gamma\sigma^2})$. *The exact solution to the ordinary differential equation in proposition 1 is given by,* $\psi(x) = C_1\Psi_1(x) + C_2\Psi_2(x) + \psi_p(x)$, *where,*

$$\Psi_i(x) = x^{-k_i}\Phi(a_i, b_i; \frac{q}{\beta_3}x^{-1})$$

$$\psi_p(x) = \Psi_2(x)\int_0^x \Psi_1(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)} - \Psi_1(x)\int_0^x \Psi_2(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)}$$

$$W(x) = \Psi_1(x)\Psi_2'(x) - \Psi_2(x)\Psi_1'(x)$$

$$k_i = \frac{(\beta_2 - \beta_3) \pm \sqrt{(\beta_2 - \beta_3)^2 - 4\beta_3\beta_1}}{2\beta_3}, \qquad i = 1, 2$$

$$a_i = k_i, \qquad b_i = -\frac{\beta_2}{\beta_3} + 2k_i + 2$$

$$\Phi(a, b; y) = 1 + \Sigma_{j=1}^{\infty} \frac{(a)_j}{(b)_j} \frac{y^j}{j!} \quad \textit{where}, \quad (a)_j = a(a+1)...(a+j-1), \quad (a)_0 = 1.$$

Proof. *See Appendix* ■

Proposition 2 indicates that the exact solution to the ordinary differential equation that governs the optimal value function in the no-transaction region (as introduced in Proposition 1) takes the form of Kummer's confluent hypergeometric function. This proposition implies that the solution to Problem 1 is unique boundaries denoted by x_0 and x_T, such that it is optimal not to trade when the portfolio value lies between these boundaries.

The importance of this extension in our model is that it treats cash distribution optimally, enabling us to examine the impact of the dividend payout ratio on the asset demand in the economy or variations in the no-transaction region. Studying the sensitivity of the no-transaction region is crucial because it determines the optimal trading behavior of investors. For example, if the transaction region widens, investors will trade less frequently to maintain the optimal level of portfolio selection. This, in turn, reduces the demand for assets, which is relevant to the liquidity premium demanded by investors in the economy. In the next subsection, we present the sensitivity of the no-transaction region and the equity's liquidity premium with respect to the dividend payout ratio and investment horizon.

1.1 Comparative Statics

In this part, we show how different parameters such as the dividend payout ratio of equity, and the investor's horizon could affect the no-transaction region boundaries.

[Figure 2 here]

Figure 2 displays the no-transaction region boundaries' shape concerning the transaction cost size for the following parameters: $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $\eta = 0.05$, $r = 0.065$, $q = 0.01$. These parameters were selected to satisfy, $0 < \frac{\gamma\sigma^2}{\mu - r} - 1 < 1$. Dumas (1991) demonstrated the non-dividend paying equity version of this outcome. The result indicates that the no-transaction region widens as the cost of trading equity increases.

In this graph, the increasing line represents the sell boundary's position in relation to different equity transaction cost values, while the decreasing line represents the buy boundary. The Merton line, which is the optimal bond-to-equity ratio when the transaction cost is zero, is the straight line in the middle. The graph demonstrates that as the cost of purchasing equity rises, investors decrease their demand for both equity and bonds. The reduction in equity demand is apparent since investors need a higher return on equity to purchase it when the cost of trading equity rises. Therefore, they wait to buy equity at lower prices to earn a greater return. The decrease in bond demand is less evident. However, as the cost of purchasing equity rises, investors are less inclined to sell their current equity balance because it would be costly for them to repurchase it later if doing so would be optimal. Since investors must sell equity to buy bonds, their demand for bonds also decreases.

[Figure 3 here]

Figure 3 depicts the variation in the no-transaction (NT) region with respect to different equity transaction cost values for two distinct dividend payout ratios. It compares the no-transaction region when the dividend payout ratio increases from 0.01 to 0.1. In this graph, the purple line represents the buy boundary when the dividend payout ratio is 0.1, the red line is the buy boundary when the dividend payout ratio is 0.01, the yellow line corresponds to the sell boundary for a dividend yield of 0.1, and the blue line corresponds to the sell boundary for a dividend payout ratio of 0.01. The graph demonstrates that increasing the equity's dividend payout ratio results in a downward shift in both the sell and buy boundaries.

Proposition 1 showed that in a scenario where the transaction cost on the bond is zero, the best strategy is to convert all cash received from dividends into bonds in each period. When the equity payout ratio increases, more cash flows from the equity account into the bond account through the equity's dividend payout channel. In other words, as the dividend payout ratio rises, equity gets converted to bonds faster. This means that the investor would want to buy the stock sooner, at any transaction cost value, so the buy boundary moves down. On the other hand, if the dividend yield goes up, the sell boundary should become more relaxed. Because the equity payout ratio is converting equity to bonds faster, investors have less reason to sell bonds and buy equity. So the sell boundary also moves down.

Another important aspect of the model is its consideration of the transformed expected utility that assumes investors seek to maximize their utility over an infinite period of time. However, the model also accounts for the fact that investors have a randomly arriving horizon. This means that investors may have a certain expectation about when they will need to use their investments. Specifically, the model defines the probability of the horizon's arrival as η. Since the investment horizon is exponentially distributed, an increase in η leads to a decrease in the expected time until the horizon's arrival, which is represented as $\frac{1}{\eta}$. When the investor expects their horizon to arrive sooner, they treat bonds and equity differently due to the fact that trading equity incurs costs while trading bonds does not. As the investor approaches their investment horizon, they convert bonds to equity at a slower pace and equity to bonds at a faster pace. This anticipated behavior stems from the investor's desire to spend less on rebalancing their portfolio when they are closer to maturity. It's noteworthy to mention that the benefit for investing in the equity is forward-looking, and the time to maturity is directly related to the investor's marginal benefit from purchasing the equity. As the investor's investment horizon approaches, the marginal cost of trading equity remains fixed at any given equity price, but the marginal benefit of purchasing the equity decreases. Therefore, the model suggests that the expected time until the investor's horizon arrival affects their trading behavior, which has important implications for portfolio management.

[Figure 4 here]

In Figure 4, the sensitivity of the no-transaction boundaries is depicted with respect to the investors' horizon. The graph highlights that as the η value increases from 0.05 to 0.5, or in other words, as the expected time to maturity or investment horizon decreases, the buy boundary of the no-trading region increases. This implies that the investor's demand for bond increases as they approach their investment horizon, which is in line with their goal of preserving their capital and reducing risk as they get closer to needing their investments. Conversely, the sell boundary of the no-transaction region also increases as η increases. This means that the investor's demand for equity decreases as they approach their investment horizon. Therefore, as the investor approaches their investment horizon, they would prefer to hold a more conservative portfolio with a higher allocation of bond, which has zero transaction costs, and a lower allocation of equity, which has positive transaction costs.

Finally, the paper examines the impact of transaction costs on the liquidity premium, following the work of Constantinides (1986). Constantinides suggests that transaction costs have a second-order effect on the liquidity premium, which is defined as the excess in the mean return of an asset subject to transaction costs compared to an asset that is exempt from transaction costs. To measure the liquidity premium, Constantinides considers the case of two assets with perfectly correlated rates of return and equal standard deviations of their rates of return. In this scenario, the expected rate of return of the asset with transaction costs must exceed that of the exempted asset in equilibrium. The liquidity premium, denoted by $\delta(\lambda)$, represents the additional return that investors require to be compensated for the transaction costs, λ, and is given by the difference in the mean returns of the two assets.

Constantinides defines the liquidity premium, $\delta(\lambda)$, as the excess in the mean return of the asset with transaction cost compared with the asset without transaction cost which makes the investor indifferent between holding either of the assets at the optimal portfolio allocation under no transaction cost, $x^* = \frac{\gamma\sigma^2}{\mu - r} - 1$. In other words, in equilibrium the liquidity premium, $\delta(\lambda)$ that the investors require to be compensated for the transaction cost λ, must satisfy the following equation.

$$\psi(x^*) = \eta\Big(\eta - (1-\gamma)(r + \frac{(\mu - \delta(\lambda) - r)^2}{2\gamma\sigma^2})\Big)^{-1} \frac{(x^*+1)^{1-\gamma}}{1-\gamma} \tag{8}$$

The left-hand side of Equation 8 represents the expected utility of the investor from holding portfolio x^* under transaction cost, which we derived explicitly from proposition 2. On the other hand, the right-hand side of the equation represents the expected utility under no transaction cost, which is derived in lemma 1. This equation provides a measure of liquidity premium that estimates the excess return required to maintain the optimal portfolio inside the no-transaction region.

[Figure 5 here]

Figure 5 illustrates the liquidity premium, $\delta(\lambda)$, under two distinct dividend payout policies. The dashed line pertains to $q = 0.1$, whereas the solid line corresponds to $q = 0.01$. Subsequent to the rise in the transaction cost, the liquidity premium almost doubles. As Constantinides (1986) argued, the liquidity premium is typically an order of magnitude below the transaction cost. Nevertheless, current literature suggests that the liquidity premium may be of the same order of magnitude as the transaction cost spread. This study aims to establish that dividend payout from the asset is one of the contributing factors that could escalate the liquidity premium. The rationale behind the increase in premium lies in the fact that higher dividend payout leads to a surge in the demand for the asset, which, in turn, lowers the buy boundary. Consequently, the investor ends up procuring the asset more frequently, leading to higher transaction costs. Hence, the investor has to be compensated for the amplified cost of buying the asset.

[Figure 6 here]

In Figure 6, we observe a positive association between the liquidity premium and the investors' horizon. Specifically, as the parameter η increases from 0.05 to 0.5, the expected time to maturity or investment horizon of the investors decreases. Consequently, the liquidity premium decreases as the investor's expected horizon shorten. The reason for this positive correlation is rooted in the behavior of the investor as they approach their investment horizon. Investor tends to reduce their demand for the asset and trade equity less frequently. As a result of the decreased transaction cost, the investor would be required to be compensated less for trading the asset.

We can conclude that in an environment where the transaction cost of bonds is zero, high investment horizon and high dividend payout ratio are two potential contributing factors to the higher liquidity premium observed in empirical findings as compared to Constantinides' (1986) findings. The interplay between these factors leads to a shift in the demand for the asset with transaction costs, ultimately influencing the liquidity premium.

However, these relationships may change when transaction cost of bond is positive. In the next section, we will explore a comprehensive model that accounts for positive transaction costs for both riskless and risky assets.

2 Optimal Portfolio When Bond and Equity are Subject to Transaction Cost

In this section, we delve into the full problem that includes positive values of proportional transaction costs for purchasing bonds and equity. As in the previous case, the transaction cost is proportional to the size of the trade and is only incurred when the investor purchases the asset. However, this time, the investor does not have a trivial sweep decision like in the case of zero transaction cost of bond, where the investor always converts cash to the bond. The presence of transaction cost of bond alters the investor's sweep decision process and influences their choice of asset.

In this problem, the investor at time t chooses how much to buy/sell bond and equity and also chooses how to reallocate the sweep account cash balance in the bond or the equity account. As before, let U_t be the cumulative dollar value of bond purchase at time t, L_t be the cumulative dollar value of equity purchase at time t, and χ_t be the fraction of the sweep account at time t that the investor transfers from the sweep account into the equity account.

Problem 2 *Consider an investor facing a proportional transaction cost of $\lambda^s > 0$ when purchasing equity, and $\lambda^b > 0$ when purchasing bond. Then, the investor seeks to choose $\chi(t), U(t), L(t)$ for $t \in [0, \infty]$ in* *order to maximize the following problem:*

$$v(x,y) = \max_{L_t,U_t,\chi_t} \mathbb{E} \int_0^\infty \left[\eta e^{-\eta t} \frac{w(t)^{(1-\gamma)}}{1-\gamma} \right] dt \tag{9}$$

Subject to:

$$ds_0(t) = \Big[(r-c)s_0(t) + \frac{1}{1+\lambda^b}(1-\chi_t)\Big(cs_0(t) + qs_1(t)\Big)\Big]dt - (1+\lambda^s)dL_t + dU_t$$

$$ds_1(t) = \Big[(\mu-q)s_1(t) + \frac{1}{1+\lambda^s}\chi_t\Big(cs_0(t) + qs_1(t)\Big)\Big]dt + \sigma s_1(t)dz_t + dL_t - (1+\lambda^b)dU_t$$

$$\forall t \quad s_0(t), s_1(t) \geq 0, \qquad x = s_0(0), \qquad y = s_1(0)$$

Similar to Problem 1, the first constraint is the law of motion for the bond account. To purchase dU_t units of bonds, the investor must sell $(1+\lambda^b)dU_t$ units of equity to cover the transaction cost of purchasing the bond. The second constraint is the law of motion for the equity account. To add dL_t units of equity to the equity account, the investor must sell $(1+\lambda^s)dL_t$ units of bonds to cover the transaction cost of purchasing the equity.

In problem 2 the cash proceeds from the equity account are given by $qs_1(t)$, while the cash proceeds from the bond account are given by $cs_0(t)$. As a result, the cash balance that is held in the sweep account at time t is given by $cS_0(t) + qs_1(t)$. The investor must then decide how to allocate this cash balance between the equity account and the bond account. Specifically, the investor can allocate a fraction $0 \leq \chi_t \leq 1$ of the sweep account balance to the equity account, incurring a transaction cost of λ^s in the process. The remaining balance can then be allocated to the bond account, incurring a transaction cost of λ^b proportional to the size of the transaction.

Proposition 3 *Let λ^s, $\lambda^b > 0$, and assume there exists a solution to the problem 2. Under this specification,*

1. *There exists a boundary, called the sell boundary at x_0, such that if $x \leq x_0$, the investor's optimal strategy is to sell equity and buy bonds until the portfolio reaches the sell boundary. The region in the allocation space to the left of the sell boundary is called the sell region. The investor's value function in the sell region is given by, $\psi(x) = \frac{1}{1-\gamma}A(x + \frac{1}{1+\lambda^b})^{1-\gamma}$, for a constant value of A.*

2. *There exists a boundary, called the buy boundary at x_T, such that if $x \geq x_T$, the investor's optimal strategy is to buy equity and sell bonds until the portfolio reaches the buy boundary. The region in the allocation space to the right of the buy boundary is called the buy region. The investor's value function in the buy region is given by, $\psi(x) = \frac{1}{1-\gamma}B(x + 1 + \lambda^s)^{1-\gamma}$, for a constant value of B.*

3. *The region in between the sell boundary, x_0, and the buy boundary, x_T, is called the no transaction region, where the investor would not trade. In this region there exists a sweep boundary, x_e, which $x_0 < x_e < x_T$. In the region where $x_0 \leq x \leq x_e$ (NT_0) the investor's optimal strategy is to sweep the cash into the bond account. The value function in this region is solved through the following free boundary differential equation,*

$$\beta_3 x^2 \psi_2''(x) + (\beta_2 x + \frac{q}{1+\lambda^b})\psi_2'(x) + \beta_1 \psi_2(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0$$

Where, $\beta_1 = (\mu - q - \frac{1}{2}\sigma^2\gamma)(1-\gamma) - \eta, \quad \beta_2 = \sigma^2\gamma + r - \frac{\lambda^b}{1+\lambda^b}c - \mu + q, \quad \beta_3 = \frac{1}{2}\sigma^2$

4. *In the region where $x_e \leq x \leq x_T$ (NT_1) the investor's optimal strategy is to sweep the cash into the equity account. The value function in this region follows,*

$$\beta_3 x^2 \psi_1''(x) + (\beta_2 x - \frac{1}{1+\lambda^s}cx^2)\psi_1'(x) + \Big(\beta_1 + \frac{1-\gamma}{1+\lambda^s}cx\Big)\psi_1(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0$$

$$\beta_1 = \Big(-\frac{1}{2}\sigma^2\gamma + \mu - \frac{\lambda^s}{1+\lambda^s}q\Big)(1-\gamma) - \eta, \qquad \beta_2 = \sigma^2\gamma + r - c - \mu + \frac{\lambda^s}{1+\lambda^s}q, \qquad \beta_3 = \frac{1}{2}\sigma^2$$

5. *The following conditions hold at the boundaries:*

$$\frac{\psi_1'(x_T)}{\psi_1(x_T)} = \frac{1-\gamma}{x_T + 1 + \lambda^s} \qquad \frac{\psi_2'(x_0)}{\psi_2(x_0)} = \frac{(1-\gamma)(1+\lambda^b)}{x_0(1+\lambda^b)+1}$$

$$\frac{\psi_1''(x_T)}{\psi_1'(x_T)} = \frac{-\gamma}{x_T + 1 + \lambda^s} \qquad \frac{\psi_2''(x_0)}{\psi_2'(x_0)} = \frac{-\gamma(1+\lambda^b)}{x_0(1+\lambda^b)+1}$$

$$\psi_1(x_e) = \psi_2(x_e) \qquad \psi_1'(x_e) = \psi_2'(x_e) \qquad \psi_1''(x_e) = \psi_2''(x_e)$$

Proof. *See Appendix* ■

As shown in Figure 7, there are two sub-regions in the no-transaction region. One of the sub-regions is denoted by NT_0, and the other line is denoted by NT_1. In NT_0, the investor fully reinvests the cash balance in the sweep account into the bond account, and in NT_1, the investor fully reinvests the cash balance in the sweep account into the equity account.

[Figure 7 here]

Proposition 3 demonstrates that the optimal solution to problem 2 is a unique buying boundary of the equity, a unique selling boundary of the equity, and a unique boundary for the sweep account. Similar to the previous case, the no-transaction region is a convex cone, where the investor would not rebalance the portfolio in that area. However, the sweep decision boundary for the investor, in this case, lies strictly inside the no-transaction region. The sweep boundary indicates that, in the area to the left of this boundary, the investor would only purchase bonds with the cash balance in the sweep account, and in the region to the right of the sweep boundary, the investor would reallocate the cash balance in the sweep account into the equity account.

Proposition 4 *Lets assume that $\eta > (1-\gamma)(r + \frac{(\mu-r)^2}{2\gamma\sigma^2})$. The exact solution to the ordinary differential equation in proposition 3 for the NT_1 region is as followed, $\psi_1(x) = C_{11}\Psi_{11} + C_{12}\Psi_{12}(x) + \psi_{1p}(x)$ where,*

$$\Psi_{1i}(x) = x^{k_i}\Phi(a_i, b_i; \frac{c}{\beta_3(1+\lambda^s)}x)$$

$$\psi_{1p}(x) = \Psi_{12}(x)\int_0^x \Psi_{11}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)} - \Psi_{11}(x)\int_0^x \Psi_{12}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)}$$

$$W(x) = \Psi_{11}(x)\Psi'_{12}(x) - \Psi_{12}(x)\Psi'_{11}(x)$$

$$k_i = \frac{(\beta_3-\beta_2) + \sqrt{(\beta_3-\beta_2)^2 - 4\beta_3\beta_1}}{2\beta_3} \qquad i = 1, 2$$

$$a_i = k_i - 1 + \gamma, \qquad b_i = \frac{\beta_2}{\beta_3} + 2k_i$$

$$\Phi(a, b; y) = 1 + \Sigma_{j=1}^{\infty}\frac{(a)_j}{(b)_j}\frac{y^j}{j!} \quad \textit{where,} \quad (a)_j = a(a+1)...(a+j-1), \quad (a)_0 = 1.$$

And the solution to the ordinary differential equation in proposition 3 for the NT_0 region is as followed, $\psi_2(x) = C_{21}\Psi_{21}(x) + C_{22}\Psi_{22}(x) + \psi_{2p}(x)$, where,

$$\Psi_{2i}(x) = x^{-k_i}\Phi(a_i, b_i; \frac{q}{(1+\lambda^b)\beta_3}x^{-1})$$

$$\psi_{2p}(x) = \Psi_{22}(x)\int_0^x \Psi_{21}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)} - \Psi_{21}(x)\int_0^x \Psi_{22}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)}$$

$$W(x) = \Psi_{21}(x)\Psi'_{22}(x) - \Psi_{22}(x)\Psi'_{21}(x)$$

$$a_i = k_i, \qquad b_i = -\frac{\beta_2}{\beta_3} + 2k_i + 2$$

Proof. *See Appendix* ■

Proposition 4 demonstrates that the exact solution to the ODE introduced in proposition 3 takes the form of Kummer's Confluent Hypergeometric function. This proposition indicates that the solution to the problem would be a unique set of boundaries denoted by x_0, x_T, and x_e, such that it is optimal not to trade when the portfolio value lies in the region between x_0 and x_T. Additionally, it shows that it is optimal to reinvest the cash balance from the sweep account into the bond account when the portfolio is in the region between x_0 and x_e, and it is optimal to reinvest the cash balance from the sweep account into the equity account when the portfolio is in the region between x_e and x_T.

2.1 Comparative Statics

In this section, we will be examining the results derived from Proposition 3. The first step is to replicate the changes in the no-transaction region as the transaction costs of bonds and equity are varied.

[Figure 8 here]

Figure 8 depicts the changes in the boundaries with respect to variations in the transaction cost of bond while fixing transaction cost equity. In this section, the parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, q = 0.01, \eta = 0.05, c = 0.02, \lambda^s = 0.01$ unless it is specified otherwise.

The buy boundary is increasing, which indicates that the investor's demand for equity is decreasing. This is because the investor is less inclined to sell their bond to purchase equity since it would be more expensive to buy back their bond. The sell boundary is decreasing as well, as the cost of bond purchases is increasing. That is because the investor would want the price of equity to increase enough to cover the higher cost of bond purchase. The straight line in the center is the Merton line.

The sweep boundary overlaps with the buy boundary when the transaction cost of bonds is zero, as we learned in the first section. However, as the transaction cost of bonds increases, the sweep boundary declines and creates two sub-regions within the NT region. The first sub-region, labeled NT_1, lies between the buy boundary and the sweep boundary, where the investor invests the entire cash balance of the sweep account in the equity account. The second sub-region, labeled NT_0, lies between the sell boundary and the sweep boundary, where the investor invests the entire cash balance of the sweep account in the bond account. The area of NT_1 increases as the transaction cost for bonds increases, while the area of NT_0 decreases.

[Figure 9 here]

Figure 9 displays the changes in the boundaries as the transaction cost of bonds is held constant at

$\lambda^b = 0.01$, while the transaction cost of equity varies. Notably, the sweep boundary starts at the sell boundary when the transaction cost of equity is zero and subsequently rises as the cost of purchasing equity escalates. This trend is attributed to investors being more likely to allocate the cash balances of the sweep account to the bond account when the cost of purchasing equity becomes high. Knowing the boundary variations with respect to changes in the transaction costs of assets, we can compare these graphs under different asset characteristics.

[Figure 10 here]

Figure 10 depicts the boundaries under two different dividend payout policies as the transaction cost of bonds increases. The solid lines indicate the buy, sell, and sweep boundaries for a dividend payout policy of 0.01, while the dashed lines represent the corresponding boundaries for a dividend payout policy of 0.1.

From the graph, we can observe that at lower levels of bond transaction costs, the demand for equity increases as the equity pays more dividends. A similar case was previously demonstrated in section 1 when the transaction cost of bonds was zero. However, as the transaction cost of bonds increases, the buy boundary for higher dividend payout cases intersects with the buy boundary under the low dividend payout case at some level of the bond transaction cost. This implies that if the bond transaction cost is relatively high compared to the equity transaction cost, the demand for equity would decrease after an increase in the equity payout.

At lower levels of bond transaction cost, investors sweep cash into bonds at low cost and prefer to reinvest the cash in the equity account as the equity balance is declining due to the high dividend level. This results in higher demand for equity. Once the transaction cost of bonds surpasses a certain threshold, investors reduce their demand for both equity and bonds as the dividend payout increases. At higher levels of bond transaction cost, investors face high costs for sweeping cash into the bond account, which increases the likelihood of transferring cash distribution into the equity account. In this case, the demand for equity would decrease for two reasons. First, dividends are more likely to be reinvested in the equity account, resulting in a lower need for purchasing more equity. Second, since the bond is more expensive, investors hesitate to sell bonds and buy equity in return because they will have to pay high transaction costs on that trade when they want to sell that equity in the future to buy bonds. To sum up, the effect of the dividend payout ratio on demand for equity is ambiguous and depends on the relative magnitude of the transaction cost of equity and bonds.

[Figure 11 here]

Figure 11 shows how a coupon bond affects asset demand under two scenarios: a low coupon rate of 0.02 and a high rate of 0.06, with $\lambda^b = 0.01$. When equity transaction costs are very low, investors demand more bonds to replace depleted bond balances due to high coupon rates. But as equity transaction costs rise, investors demand fewer bonds due to decreased likelihood of transferring cash to equity.

Figure 12 shows the impact of investor horizon on asset demand. This figure shows the position of boundaries under a high investment horizon, $\eta = 0.05$ and a low investment horizon, $\eta = 0.5$, when transaction cost for equity, λ^s, is 0.01.

[Figure 12 here]

The figure shows that when the transaction cost of bond is low relative to equity investor demand the riskless asset more and demand the risky asset. This is consistent with the literature. However, the more controversial effect of investment horizon happens when the transaction cost of equity gets elevated. Demand for bond decreases and the demand for equity actually increases under very high transaction cost for bond.

Conclusion

This paper explores the optimal allocation of cash precedes received from a portfolio of dividend-paying equity and coupon bond in presence of proportional transaction cost. Merton (1971) showed that the optimal allocation of non-dividend paying bonds and equities in the absence of transaction cost is a constant allocation in which investors continuously trade assets to maintain the optimal allocation. Davis and Norman (1990) showed that when investors face proportional transaction cost and non-dividend-paying assets, the investors would not rebalance their portfolio in a convex cone region in the allocation space around the Merton line. In this paper, we argue that when investors are investing in dividend-paying stocks the optimal decision of reinvesting the cash proceeds is determined by a boundary which is called *Sweep* boundary. This paper illustrates that the No Transaction region when the assets pay higher dividends shifts downward such that the buy boundary and the sell boundary decrease. Additionally, this paper displays the impact of the investment horizon and illustrates that when the investors' investment horizon becomes shorter, they delay the portfolio rebalancing by lowering their demand for the illiquid asset and they increase the pace at which they purchase the more liquid asset. Furthermore, we show the impact of asset cash payout and investors' horizon on the liquidity premium. We argue that when the investor's investment horizon shortens they would decrease their demand to decrease their expenditure on transaction costs. The investors' lower expenditure on transaction costs translates to lower liquidity premia in equilibrium. We also show that when assets increase their cash distributions in terms of dividends or coupons, investors would increase their demand for these assets and the increased demand for the asset means that the investors' expenses on transaction cost would increases therefore the liquidity premia would rise as a result.

This paper extends the literature on optimal portfolios with transaction cost by including cash payments from the assets. This extension sheds light on the impact of dividends and coupons on the portfolio optimization decision and subsequently demand and liquidity premia for the assets with transaction cost.

Appendix

Proof of Proposition 1

Define the following Martingale,

$$M_t \equiv \frac{\eta}{1-\gamma}\int_0^t e^{-\eta s} w_s^{1-\gamma} ds + e^{-\eta t} v(x,y)$$

By the Ito formula,

$$M_t - M_0 = \frac{\eta}{1-\gamma}\int_0^t e^{-\eta s} w_s^{1-\gamma} ds + \int_0^t e^{-\eta s} dv(x,y) - \eta\int_0^t e^{-\eta s} v(x,y)dt$$

$$dM = e^{-\eta t}\Big\{\Big[\frac{\eta}{1-\gamma}w_t^{1-\gamma} - \eta v + \frac{1}{2}\sigma^2 y^2 v_{yy} + (\mu - q)y v_y + \Big(rx + qy\Big)v_x\Big]dt$$

$$+\Big(\frac{1}{1+\lambda^s}v_y - v_x\Big)\chi_t\Big(cx+qy\Big)dt + \Big(v_y - (1+\lambda^s)v_x\Big)dL_t + \Big(v_x - v_y\Big)dU_t + \sigma y v_y dz_t\Big\}$$

The HJB to be solved for v in 4 is:

$$\max_{l,u,\chi}\Big\{\frac{1}{2}\sigma^2 y^2 v_{yy} + \Big(rx+qy\Big)v_x + \Big(\mu - q\Big)y v_y + \frac{\eta}{1-\gamma}w^{1-\gamma} - \eta v$$

$$+\Big(\frac{1}{1+\lambda^s}v_y - v_x\Big)\chi\Big(cx+qy\Big) + \Big(v_y - (1+\lambda^s)v_x\Big)E[\frac{dL}{dt}] + \Big(v_x - v_y\Big)E[\frac{dU}{dt}]\Big\} = 0$$

Where $\quad v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y}, \quad v_{xx} = \frac{\partial^2 v}{\partial x^2}, \quad \text{and} \quad v_{yy} = \frac{\partial^2 v}{\partial y^2}.$

To maximize the HJB, the choices of χ, L, and U are:

$$\chi\begin{cases} = 1 & \textit{if}\ \ v_y > (1+\lambda^s)v_x \\ \in [0,1] & \textit{if}\ \ v_y = (1+\lambda^s)v_x \\ = 0 & \textit{if}\ \ v_y < (1+\lambda^s)v_x \end{cases} \quad (10) \qquad dL\begin{cases} > 0 & \textit{if}\ \ v_y \geq (1+\lambda^s)v_x \\ = 0 & \textit{if}\ \ v_y \leq (1+\lambda^s)v_x \end{cases} \quad (11) \qquad dU\begin{cases} > 0 & \textit{if}\ \ v_x \geq v_y \\ = 0 & \textit{if}\ \ v_x \leq v_y \end{cases} \quad (12)$$

Thus, χ has a bang-bang solution. The No-Trade region, in this case, is exactly the same as Davis and Norman's NT, i.e., $v_x \leq v_y \leq (1+\lambda^s)v_x$. Thus in the NT region $\chi = 0$. In the NT region, $dL = dU = 0$ and $\chi = 0$. The partial differential equation that defines the value function in the NT region is as followed:

$$\frac{1}{2}\sigma^2 y^2 v_{yy} + \Big(rx+qy\Big)v_x + \Big(\mu - q\Big)y v_y + \frac{\eta}{1-\gamma}(x+y)^{1-\gamma} - \eta v = 0 \qquad (13)$$

By the homothetic property of the utility function, I conjecture the value function that satisfies the above is,

$v(x, y) = y^{1-\gamma}\psi(\frac{x}{y})$.

As v is constant along lines of slope -1 in region S and along $-(1+\lambda^s)^{-1}$ in region B,

At $x \le x_0$, there is constant A such that:

$$\psi(x) = \frac{1}{1-\gamma}A(x+1)^{1-\gamma} \tag{14}$$

And at $x \ge x_T$, there is a constant B such that:

$$\psi(x) = \frac{1}{1-\gamma}B(x+1+\lambda^s)^{1-\gamma} \tag{15}$$

So that we can write the differentials at 13 in terms of ψ as:

$$v_y(x, y) = (1-\gamma)y^{-\gamma}\psi(\frac{x}{y}) - y^{-\gamma}\frac{x}{y}\psi'(\frac{x}{y}), \qquad v_x(x, y) = y^{-\gamma}\psi'(\frac{x}{y})$$

$$v_{yy}(x, y) = -\gamma y^{-\gamma-1}\Big((1-\gamma)\psi(\frac{x}{y}) - \frac{x}{y}\psi'(\frac{x}{y})\Big) - y^{-\gamma-1}\Big((1-\gamma)\frac{x}{y}\psi'(\frac{x}{y}) - \frac{x}{y}\psi'(\frac{x}{y}) - (\frac{x}{y})^2\psi''(\frac{x}{y})\Big)$$

$$v_{xx} = y^{-\gamma-1}\psi''(\frac{x}{y}) \qquad v_{xy} = -\gamma y^{-\gamma-1}\psi'(\frac{x}{y}) - y^{-\gamma-1}\frac{x}{y}\psi''(\frac{x}{y})$$

Setting $y = 1$, equation 13 reduces to below for $x \in [x_0, x_T]$,

$$\beta_3 x^2\psi''(x) + (\beta_2 x + q)\psi'(x) + \beta_1\psi(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0 \tag{16}$$

$$\beta_1 = (\mu - q - \frac{1}{2}\sigma^2\gamma)(1-\gamma) - \eta, \qquad \beta_2 = \sigma^2\gamma + r - \mu + q, \qquad \beta_3 = \frac{1}{2}\sigma^2$$

Using 11 at $x = x_T$, or by the value matching property, at this boundary, the value of the utility function in the direction of trade must be unchanged. The direction of trade at this boundary is selling $(1+\lambda^s)dL$ bonds and buying dL stocks.

$$v(x, y) = v(x - (1+\lambda^s)dL, y + dL)$$

Expanding above we get

$$v_y = (1+\lambda^s)v_x \quad \Rightarrow \quad (1-\gamma)\psi(x_T) = (x_T + 1 + \lambda^s)\psi'(x_T) \tag{17}$$

Similarly at $x = x_0$ direction of trade is to sell dU stocks and buy dU bonds. By 12 or by the value-matching property at this boundary.

$$v(x,y) = v(x-dU, y+dU)$$

$$v_y = v_x \quad \Rightarrow \quad (1-\gamma)\psi(x_0) = (x_0+1)\psi'(x_0) \tag{18}$$

For optimality[1], at the boundaries, the derivative of the value function must stay unchanged in the direction of the trade. At $x = x_T$ the smooth pasting property dictates

$$v_y(x-(1+\lambda^s)dL, y+dL) = (1+\lambda^s)v_x(x-(1+\lambda^s)dL, y+dL)$$

Expanding above

$$(1+\lambda^s)v_{yx} - v_{yy} = (1+\lambda^s)^2 v_{xx} - (1+\lambda^s)v_{xy} \quad \Rightarrow \quad \frac{\psi''(x_T)}{\psi'(x_T)} = \frac{-\gamma}{x_T+1+\lambda^s} \tag{19}$$

And, based on the smooth pasting condition at x_0;

$$v_{yx} - v_{yy} = v_{xx} - vxy \quad \Rightarrow \quad \frac{\psi''(x_0)}{\psi'(x_0)} = \frac{-\gamma}{x_0+1} \tag{20}$$

Having the differential equation 16, for $\psi(x)$ as in 15, we can find B at $x = x_T$, and for $\psi(x)$ as in 14, we can find A at $x = x_0$.

Proof of Proposition 2

To solve for the homogeneous solution of the above differential equation, let $\psi(x) = x^{-k}w(x^{-1})$ where k is the root for the following quadratic equation, $\beta_3 k^2 + (\beta_3 - \beta_2)k + \beta_1 = 0$ and w satisfied the following equation, ([24]):

$$\beta_3 z w''(z) - [qz + \beta_2 - 2\beta_3(k+1)]w'(z) - kqw(z) = 0$$

$$k_i = \frac{(\beta_2 - \beta_3) \pm \sqrt{(\beta_2-\beta_3)^2 - 4\beta_3\beta_1}}{2\beta_3}, \qquad i = 1,2$$

Where k_1 is the positive root and k_2 is the negative root of the above equation. Note, this quadratic function has two roots, one is negative and one is positive. To solve for the solution of the transformed differential equation we use transformation $y = -\frac{q}{\beta_3}x^{-1}$, and for every k_i, $w(x^{-1}) = \Phi(a_i, b_i; y)$, where Φ is the Kummer's confluent hypergeometric function. Defining the parameters a_i, b_i for Kummer's confluent

[1] Harrison and Taksar (1983) named this condition 'smooth pasting' or 'high contact' when the optimality requires the marginal utility before and after the regulation to be equal. Dumas (1991) names this condition 'super contact' when the optimality requires a higher degree of tangency which in this case the second derivative is required for optimality.

hypergeometric function:

$$a_i = k_i, \qquad b_i = -\frac{\beta_2}{\beta_3} + 2k_i + 2 \qquad \Phi(a,b;y) = 1 + \Sigma_{j=1}^{\infty} \frac{(a)_j}{(b)_j}\frac{y^j}{j!}$$

where, $\quad (a)_j = a(a+1)...(a+j-1), \quad (a)_0 = 1.$

If b is not a non-positive integer, then the general solution has the following form:

$$\psi(x) = C_1\Psi_1(x) + C_2\Psi_2(x) + \psi_p(x) \tag{21}$$

$$\Psi_i(x) = x^{-k_i}\Phi(a_i, b_i; \frac{q}{\beta_3}x^{-1})$$

$$\psi_p(x) = \Psi_2(x)\int_0^x \Psi_1(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)} - \Psi_1(x)\int_0^x \Psi_2(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)}$$

$$W(x) = \Psi_1(x)\Psi_2'(x) - \Psi_2(x)\Psi_1'(x)$$

Based on the optimal value function, 21, and from the boundary conditions of 17, 18, 19, and 20 we can find C_1, C_2, x_0, and x_T,

Proof of Proposition 3

In this case the assumption is, $\quad \lambda^b \neq 0 \quad \lambda^s \neq 0.$

Define the following Martingale, $\quad M_t \equiv \frac{\eta}{1-\gamma}\int_0^t e^{-\eta s} w_s^{1-\gamma} ds + e^{-\eta t} v(x,y)$

By the Ito formula,

$$M_t - M_0 = \frac{\eta}{1-\gamma}\int_0^t e^{-\eta s} w_s^{1-\gamma} ds + \int_0^t e^{-\eta s} dv(x,y) - \eta\int_0^t e^{-\eta s} v(x,y) dt$$

$$dM = e^{-\eta t}\Big\{\Big[\frac{\eta}{1-\gamma}w_t^{1-\gamma} - \eta v + \frac{1}{2}\sigma^2 y^2 v_{yy} + (\mu - q) y v_y + \Big((r - \frac{\lambda^b c}{1+\lambda^b})x + \frac{q}{1+\lambda^b}y\Big)v_x\Big]dt$$

$$+\Big(\frac{1}{1+\lambda^s}v_y - \frac{1}{1+\lambda^b}v_x\Big)\chi_t\Big(cx + qy\Big)dt + \Big(v_y - (1+\lambda^s)v_x\Big)dL_t + \Big(v_x - (1+\lambda^b)v_y\Big)dU_t + \sigma y v_y dz_t\Big\}$$

The HJB becomes:

$$\max_{L,U,\chi}\Big\{\frac{1}{2}\sigma^2 y^2 v_{yy} + \Big((r - \frac{\lambda^b c}{1+\lambda^b})x + \frac{q}{1+\lambda^b}y\Big)v_x + (\mu - q) y v_y + \frac{\eta}{1-\gamma}w^{1-\gamma} - \eta v$$

$$+\Big(\frac{1}{1+\lambda^s}v_y - \frac{1}{1+\lambda^b}v_x\Big)\chi\Big(cx + qy\Big) + \Big(v_y - (1+\lambda^s)v_x\Big)E[\frac{dL}{dt}] + \Big(v_x - (1+\lambda^b)v_y\Big)E[\frac{dU}{dt}]\Big\} = 0$$

Solving for the optimal χ :

$$\chi \begin{cases} = 0 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_x > v_y \\ \in [0,1] & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_x = v_y \\ = 1 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_x < v_y \end{cases} \tag{22}$$

As $\frac{1}{1+\lambda^b} \leq \frac{1+\lambda^s}{1+\lambda^b} \leq 1+\lambda^s$, Thus the optimal choice of χ in 22 splits the NT region into two sub-regions; NT_0 where $\chi = 0$, and NT_1 where $\chi = 1$.

Solving for the HJB at NT_1:

$$\frac{1}{2}\sigma^2 y^2 v_{1yy} + (\mu - \frac{\lambda^s}{1+\lambda^s} q) y v_{1y} + \frac{c}{1+\lambda^s} x v_{1y} + (r-c) x v_{1x} + \frac{\eta}{1-\gamma}(x+y)^{1-\gamma} - \eta v_1 = 0 \tag{23}$$

As before, based on the homothetic property of the value function, We conjecture that the value function has the following form: $\quad v_1(x,y) = y^{1-\gamma}\psi_1(x/y)$.

Based on this transformation of the value function, equation 23 reduces to below for $x \in [x_e, x_T]$:

$$\beta_3 x^2 \psi_1''(x) + (\beta_2 x - \frac{1}{1+\lambda^s} c x^2)\psi_1'(x) + (\beta_1 + \frac{1-\gamma}{1+\lambda^s} c x)\psi_1(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0 \tag{24}$$

$$\beta_1 = \Big(-\frac{1}{2}\sigma^2\gamma + \mu - \frac{\lambda^s}{1+\lambda^s} q\Big)(1-\gamma) - \eta \qquad \beta_2 = \sigma^2\gamma + r - c - \mu + \frac{\lambda^s}{1+\lambda^s} q, \qquad \beta_3 = \frac{1}{2}\sigma^2$$

The free boundaries in this region are:

$$\chi \begin{cases} = 0 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_{1x} > v_{1y} \\ \in [0,1] & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_{1x} = v_{1y} \\ = 1 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_{1x} < v_{1y} \end{cases} \tag{25}$$

$$dL \begin{cases} > 0 & \text{if } v_{1y} \geq (1+\lambda^s) v_{1x} \\ = 0 & \text{if } v_{1y} \leq (1+\lambda^s) v_{1x} \end{cases} \tag{26}$$

26 reflects value matching property at $x = x_T$. At this boundary, the trade occurs in the direction of $(1+\lambda^s)dL$ bond sale and purchase of dL stocks. By the value matching property, the value function should be unchanged when this trade occurs at this boundary.

$$v_1(x,y) = v_1(x - (1+\lambda^s)dL, y + dL)$$

$$v_{1y} = (1+\lambda^s) v_{1x} \quad \Rightarrow \quad (1-\gamma)\psi_1(x_T) = (x_T + 1 + \lambda^s)\psi_1'(x_T) \tag{27}$$

Equation 25 reflects the value matching property at $x = x_e$, where the Sweep regulator must be indifferent between buying bonds or buying stock with the cash at the center.

$$v_1(x + \frac{1}{1+\lambda^b}(cx+qy)dt, y) = v_1(x, y + \frac{1}{1+\lambda^s}(cx+qy)dt)$$

Expanding above gives us the value-matching property at the Sweep boundary.

$$v_{1y} = \frac{1+\lambda^s}{1+\lambda^b} v_{1x} \quad \Rightarrow \quad (1-\gamma)\psi_1(x_e) = (x_e + \frac{1+\lambda^s}{1+\lambda^b})\psi_1'(x_e) \tag{28}$$

Based on the smooth pasting condition at $x = x_T$, the derivative of the value function at the boundary of $x = x_T$ must be fixed at the direction of trade by the regulator.

$$v_{1y}(x - (1+\lambda^s)dL, y + dL) = (1+\lambda^s)v_{1x}(x - (1+\lambda^s)dL, y + dL)$$

$$2(1+\lambda^s)v_{1yx} = v_{1yy} + (1+\lambda^s)^2 v_{1xx} \quad \Rightarrow \quad \frac{\psi_1''(x_T)}{\psi_1'(x_T)} = \frac{-\gamma}{x_T + 1 + \lambda^s} \tag{29}$$

Based on the optimality of smooth pasting condition at the Sweep boundary, $x = x_e$, derivative of the value function must stay unchanged at the direction of the trade at this boundary.

$$v_{1y}(x + \frac{1}{1+\lambda^b}(cx+qy)dt, y) = \frac{1+\lambda^s}{1+\lambda^b} v_{1x}(x + \frac{1}{1+\lambda^b}(cx+qy)dt, y) \quad \Rightarrow \quad v_{1yx} = (\frac{1+\lambda^s}{1+\lambda^b})^2 v_{1xx}$$

$$v_{1y}(x, y + \frac{1}{1+\lambda^s}(cx+qy)dt) = \frac{1+\lambda^s}{1+\lambda^b} v_{1x}(x, y + \frac{1}{1+\lambda^s}(cx+qy)dt) \quad \Rightarrow \quad v_{1yy} = (\frac{1+\lambda^s}{1+\lambda^b})^2 v_{1xy}$$

Thus

$$v_{1yy} = (\frac{1+\lambda^s}{1+\lambda^b})^2 v_{1xx} \quad \Rightarrow \quad \frac{\psi_1''(x_e)}{\psi_1'(x_e)} = \frac{-\gamma(1+\lambda^b)}{x_e(1+\lambda^b) + (1+\lambda^s)} \tag{30}$$

In the NT_0 region the HJB is as followed:

$$\frac{1}{2}\sigma^2 y^2 v_{2yy} + \Big((r - \frac{\lambda^b c}{1+\lambda^b})x + \frac{q}{1+\lambda^b} y\Big) v_{2x} + (\mu - q) y v_{2y} + \frac{\eta}{1-\gamma}(x+y)^{1-\gamma} - \eta v_2 = 0 \tag{31}$$

Based on the homothetic property of the value function, We conjecture that the value function has the following form, $\quad v_2(x, y) = y^\gamma \psi_2(x/y)$.

Equation 31 reduces to below for $x \in [x_0, x_e]$,

$$\beta_3 x^2 \psi_2''(x) + (\beta_2 x + \frac{q}{1+\lambda^b})\psi_2'(x) + \beta_1 \psi_2(x) + \frac{\eta}{1-\gamma}(x+1)^{1-\gamma} = 0 \tag{32}$$

$$\beta_1 = (\mu - q - \frac{1}{2}\sigma^2\gamma)(1-\gamma) - \eta, \quad \beta_2 = \sigma^2\gamma + r - \frac{\lambda^b}{1+\lambda^b}c - \mu + q, \quad \beta_3 = \frac{1}{2}\sigma^2$$

The free boundaries in NT_0 are:

$$\chi \begin{cases} = 0 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_{2x} \geq v_{2y} \\ = 1 & \text{if } \frac{1+\lambda^s}{1+\lambda^b} v_{2x} \leq v_{2y} \end{cases} \quad (33) \qquad dU \begin{cases} > 0 & \text{if } \frac{1}{1+\lambda^b} v_{2x} \geq v_{2y} \\ = 0 & \text{if } \frac{1}{1+\lambda^b} v_{2x} \leq v_{2y} \end{cases} \quad (34)$$

At the Buy boundary,$x = x_0$, equation 34 reflects the value matching property of v_2 in the direction of trade at this point.

$$v_2(x, y) = v_2(x + dU, y - (1 + \lambda^b)dU)$$

$$v_{2x} = (1 + \lambda^b)v_{2y} \;\; \Rightarrow \;\; (1-\gamma)(1+\lambda^b)\psi_2(x_0) = (x_0(1+\lambda^b)+1)\psi_2'(x_0) \quad (35)$$

Similar to 28, the value matching of v_2 at the sweep boundary, $x = x_e$, is reflected by 33.

$$v_{2y} = \frac{1+\lambda^s}{1+\lambda^b}v_{2x} \quad \Rightarrow \quad (1-\gamma)\psi_2(x_e) = (x_e + \frac{1+\lambda^s}{1+\lambda^b})\psi_2'(x_e) \quad (36)$$

Based on the smooth pasting condition at $x = x_0$, the derivative of the value function in the NT_1 region must be unchanged at the direction of trade at the buy boundary.

$$v_{2x}(x + dU, y - (1+\lambda^b)dU) = (1+\lambda^b)v_{2y}(x + dU, y - (1+\lambda^b)dU)$$

$$-v_{2xx} + (1+\lambda^b)v_{2xy} = -(1+\lambda^b)v_{2yx} + (1+\lambda^b)^2 v_{2yy} \quad \Rightarrow \quad \frac{\psi_2''(x_0)}{\psi_2'(x_0)} = -\frac{\gamma(1+\lambda^b)}{x_0(1+\lambda^b)+1} \quad (37)$$

And similar to 30, based on the smooth pasting condition at $x = x_e$.

$$v_{1yy} = (\frac{1+\lambda^s}{1+\lambda^b})^2 v_{1xx} \quad \Rightarrow \quad \frac{\psi_2''(x_e)}{\psi_2'(x_e)} = -\frac{\gamma(1+\lambda^b)}{x_e(1+\lambda^b)+(1+\lambda^s)} \quad (38)$$

Combining 25 and 33 at $x = x_e$:

$$\psi_1'(x_e) = \psi_2'(x_e) \quad (39)$$

Combining 30 and 38 at $x = x_e$:

$$\psi_1''(x_e) = \psi_2''(x_e) \quad (40)$$

Also, at $x = x_e$:

$$\psi_1(x_e) = \psi_2(x_e) \quad (41)$$

Conditions of 39, 40, 41 show that at the sweep boundary on top of the value matching and the first derivative the second derivatives must be similar in the regions on both sides of the sweep boundary.
We can find A at $x = x_0$ for $\psi(x) = \frac{1}{1-\gamma}A(x(1+\lambda^b)+1)^{1-\gamma}$ in 32.
We can find B at $x = x_T$ for $\psi(x) = \frac{1}{1-\gamma}B(x+1+\lambda^s)^{1-\gamma}$ in 24.

Proof of Proposition 4

To solve for the homogeneous solution of the above differential equation, using transformations $\psi_1(x) = x^k w(x)$ where k is the root for the quadratic equation $\beta_3 k^2 + (\beta_2 - \beta_3)k + \beta_1 = 0$. This leads to the following equation:

$$\beta_3 x w''(x) + [(\frac{-c}{1+\lambda^s})x + 2\beta_3 k + \beta_2]w'(x) + \Big[-(\frac{1}{1+\lambda^s})ck + \frac{1-\gamma}{1+\lambda^s}c\Big]w(x) = 0$$

$$k_i = \frac{(\beta_3 - \beta_2) + \sqrt{(\beta_3-\beta_2)^2 - 4\beta_3\beta_1}}{2\beta_3} \qquad i = 1, 2$$

Note, this quadratic function has two roots, one is negative and one is positive. To solve for the solution of the transformed differential equation we use transformation, $y = -\frac{c}{\beta_3(1+\lambda^s)}x$, and $w(x) = e^{-y}\Phi(y)$, where Φ is the Kummer's confluent hypergeometric function.

Defining the parameters a, b for the Kummer's confluent hypergeometric function, $\Phi(a, b; y)$ as a particular solution;

$$a_i = k_i - 1 + \gamma, \qquad b_i = \frac{\beta_2}{\beta_3} + 2k_i \quad \Phi(a, b; y) = 1 + \Sigma_{j=1}^{\infty}\frac{(a)_j}{(b)_j}\frac{y^j}{j!}$$

where, $\quad (a)_j = a(a+1)...(a+j-1), \quad (a)_0 = 1.$

If b is not a non-negative integer, k_1 being the positive root and k_2 the negative root then the general solution has the following form:

$$\psi_1(x) = C_{11}\Psi_{11} + C_{12}\Psi_{12}(x) + \psi_{1p}(x) \tag{42}$$

$$\Psi_{1i}(x) = x^{k_i}\Phi(a_i, b_i; \frac{c}{\beta_3(1+\lambda^s)}x)$$

$$\psi_{1p}(x) = \Psi_{12}(x)\int_0^x \Psi_{11}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)} - \Psi_{11}(x)\int_0^x \Psi_{12}(z)\frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3}\frac{dz}{W(z)}$$

$$W(x) = \Psi_{11}(x)\Psi'_{12}(x) - \Psi_{12}(x)\Psi'_{11}(x)$$

To solve for the homogeneous solution of the differential equation in NT_0, using transformations $x = z^{-1}$, $\psi_2(x) = x^k w(z)$ where k is the root for the quadratic equation $\beta_3 k^2 + (\beta_3 - \beta_2)k + \beta_1 = 0$. This leads to the following equation:

$$\beta_3 z w''(z) - [\frac{q}{1+\lambda^b} z + \beta_2 - 2\beta_3(k+1)]w'(z) - k\frac{q}{1+\lambda^b} w(z) = 0$$

$$k_i = \frac{(\beta_2 - \beta_3) \pm \sqrt{(\beta_2 - \beta_3)^2 - 4\beta_3\beta_1}}{2\beta_3} \qquad i = 1, 2$$

Note, this quadratic function has two roots, one is negative and one is positive. To solve for the solution of the transformed differential equation we use transformation $y = -\frac{q}{(1+\lambda^b)\beta_3} z$, and $w(z) = e^{-y}\Phi(y)$, where Φ is the Kummer's confluent hypergeometric function. Defining the parameters a, b for the Kummer's confluent hypergeometric function, $\Phi(a, b; y)$ as a particular solution:

$$a_i = k_i, \qquad b_i = -\frac{\beta_2}{\beta_3} + 2k_i + 2 \quad \Phi(a, b; y) = 1 + \Sigma_{j=1}^{\infty} \frac{(a)_j}{(b)_j} \frac{y^j}{j!}$$

where, $\quad (a)_j = a(a+1)...(a+j-1), \quad (a)_0 = 1.$

If b is not a non-negative integer, k_1 being the positive root and k_2 the negative root then the general solution has the following form:

$$\psi_2(x) = C_{21}\Psi_{21}(x) + C_{22}\Psi_{22}(x) + \psi_{2p}(x) \tag{43}$$

$$\Psi_{2i}(x) = x^{-k_i}\Phi(a_i, b_i; \frac{q}{(1+\lambda^b)\beta_3} x^{-1})$$

$$\psi_{2p}(x) = \Psi_{22}(x) \int_0^x \Psi_{21}(z) \frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3} \frac{dz}{W(z)} - \Psi_{21}(x) \int_0^x \Psi_{22}(z) \frac{\eta(z+1)^{1-\gamma}}{(1-\gamma)\beta_3} \frac{dz}{W(z)}$$

$$W(x) = \Psi_{21}(x)\Psi'_{22}(x) - \Psi_{22}(x)\Psi'_{21}(x)$$

From 27, 29, 35, 37, 39, 40,and 41 we can find C_{11}, C_{12}, C_{21}, C_{22}, x_0, x_T, and x_e.

Figures

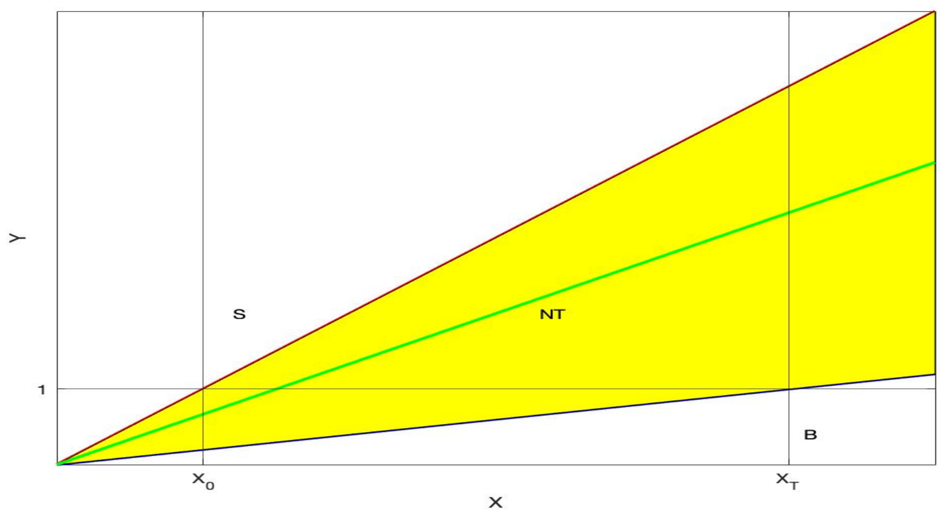

Figure 1

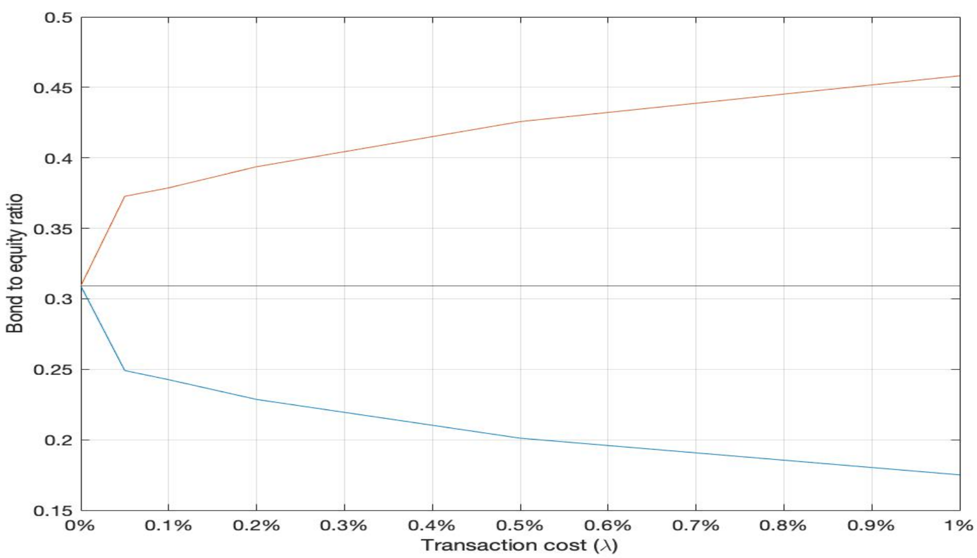

Figure 2: parameters are $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $\eta = 0.05$, $r = 0.065$, $q = 0.01$

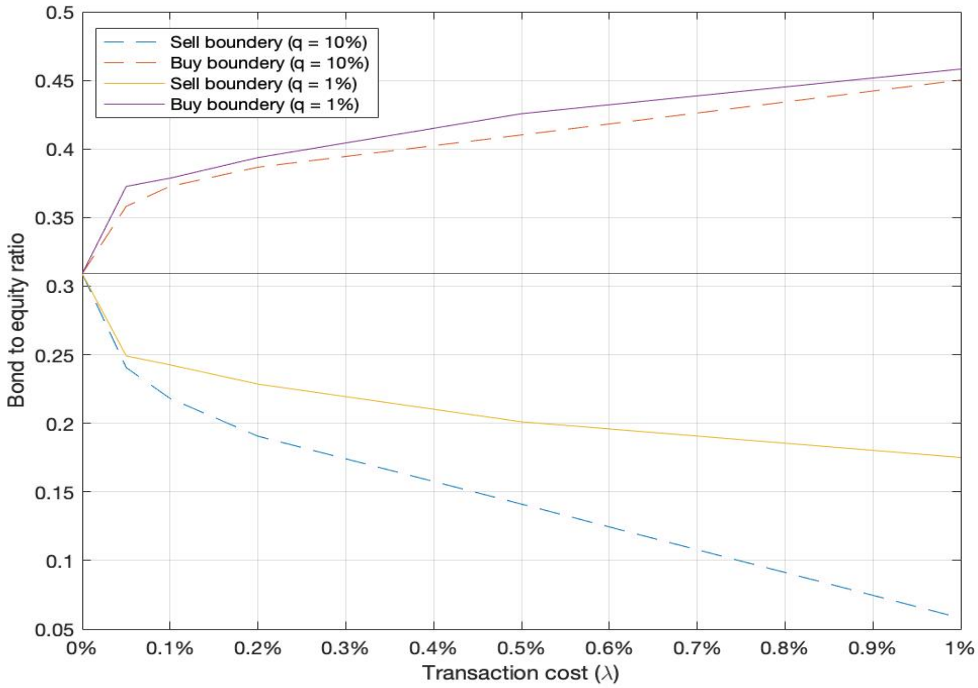

Figure 3: parameters are $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $\eta = 0.05$, $r = 0.065$

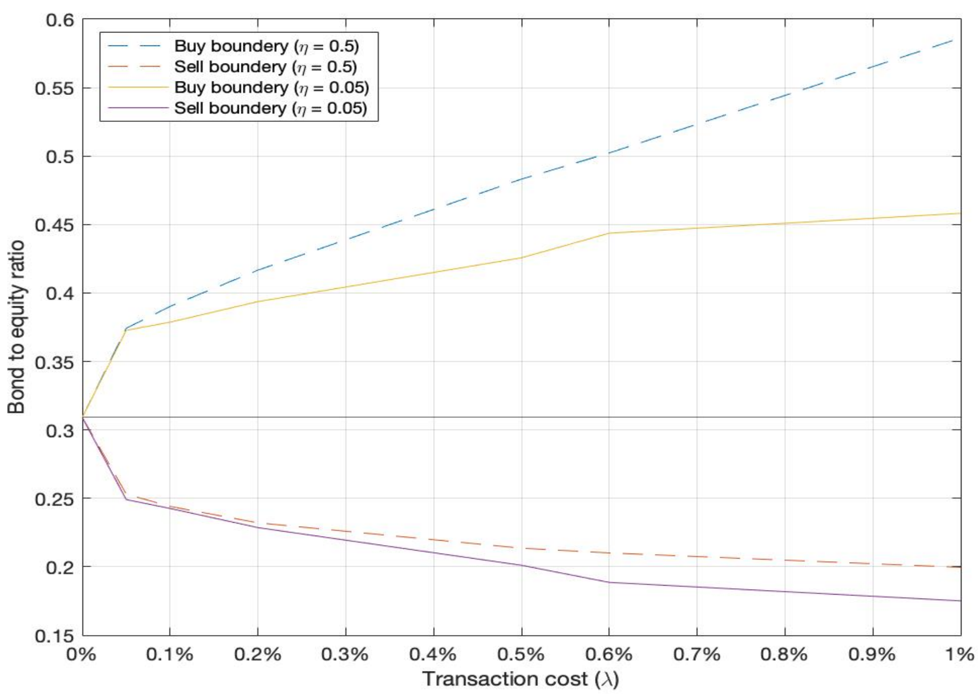

Figure 4: parameters are $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $r = 0.065$, $q = 0.01$

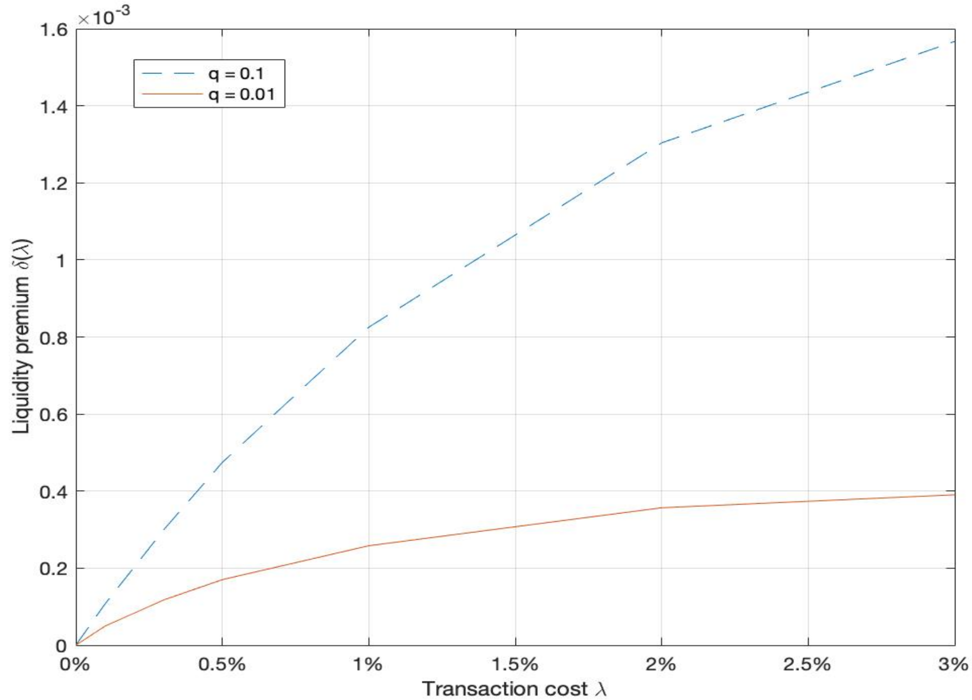

Figure 5: parameters are $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $\eta = 0.05$, $r = 0.065$

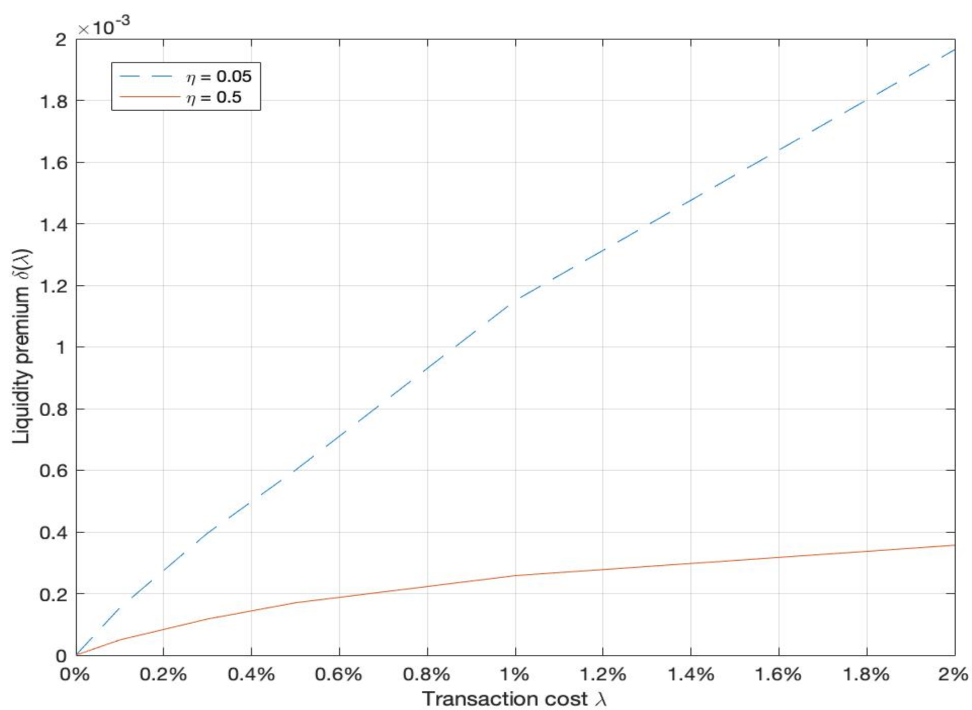

Figure 6: parameters are $\mu = 0.12$, $\sigma = 0.4$, $\gamma = 0.45$, $r = 0.065$, $q = 0.01$

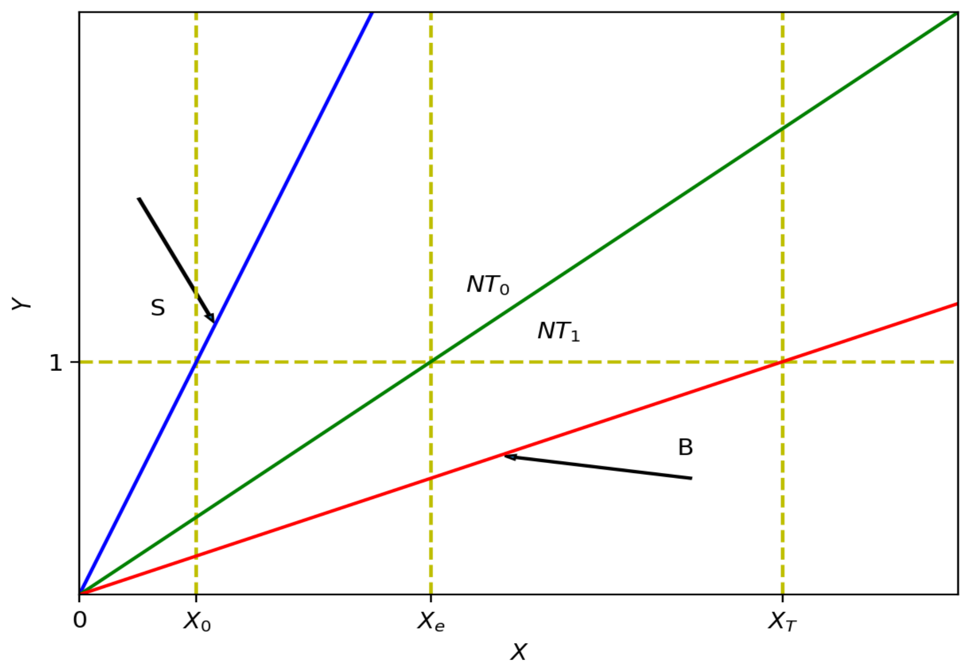

Figure 7

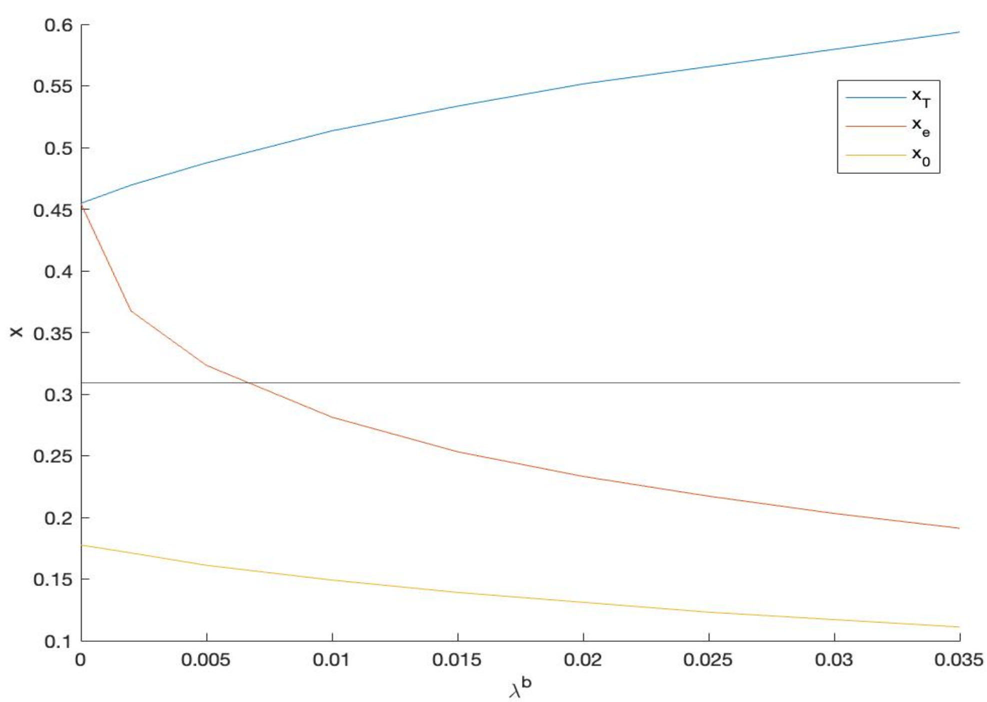

Figure 8: Parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, q = 0.01, \eta = 0.05, c = 0.02, \lambda^s = 0.01$

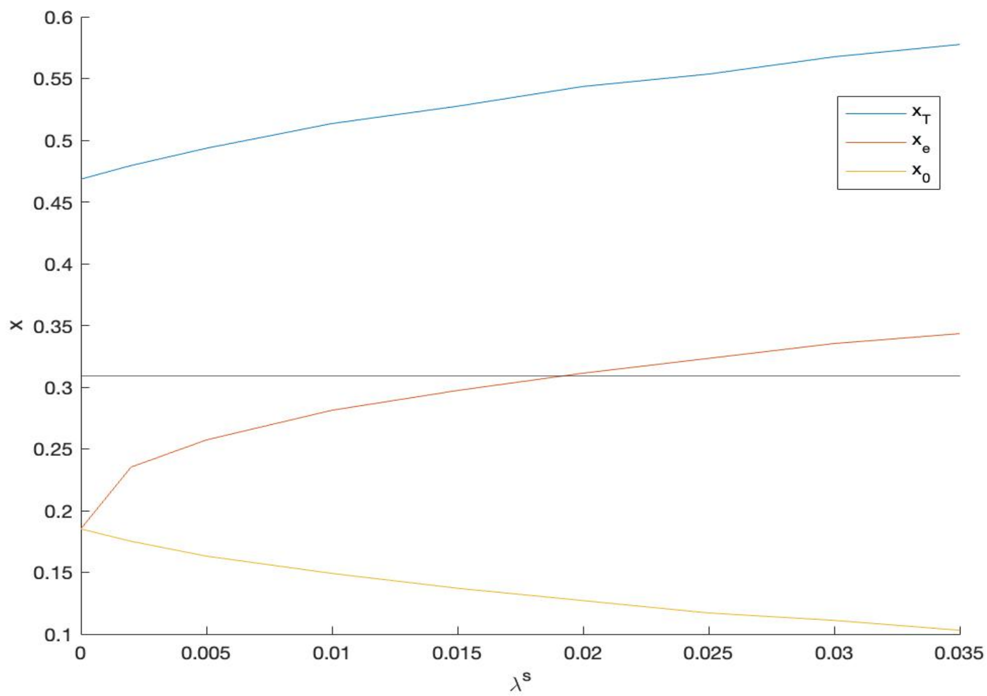

Figure 9: Parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, q = 0.01, \eta = 0.05, c = 0.02, \lambda^b = 0.01$

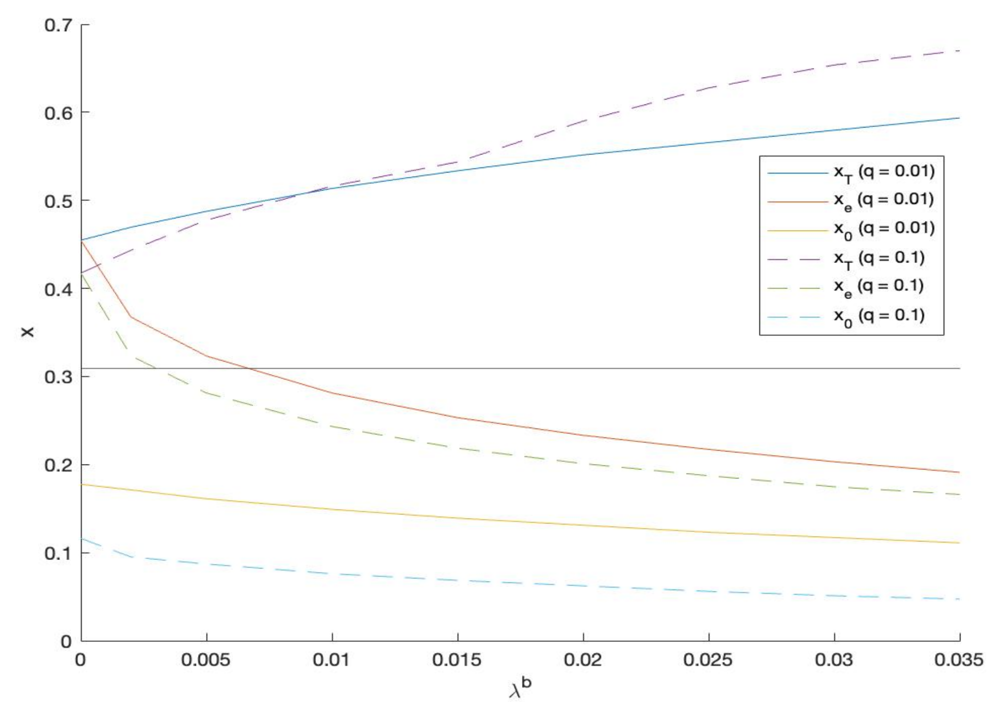

Figure 10: Parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, \eta = 0.05, c = .02, \lambda^s = 0.01$

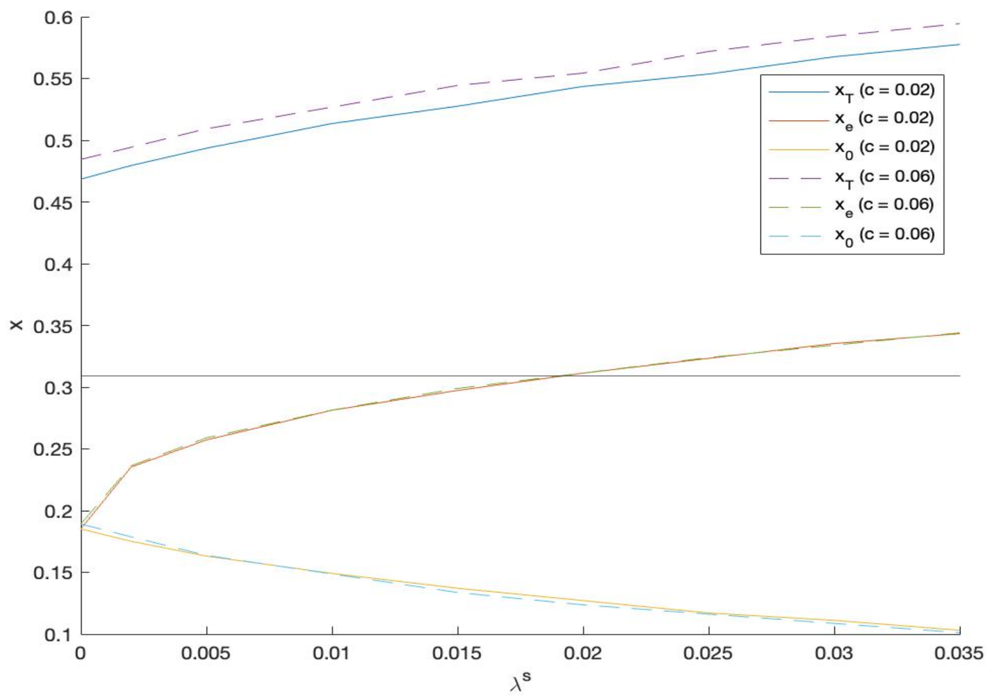

Figure 11: Parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, q = 0.01, \eta = 0.05, \lambda^b = 0.01$

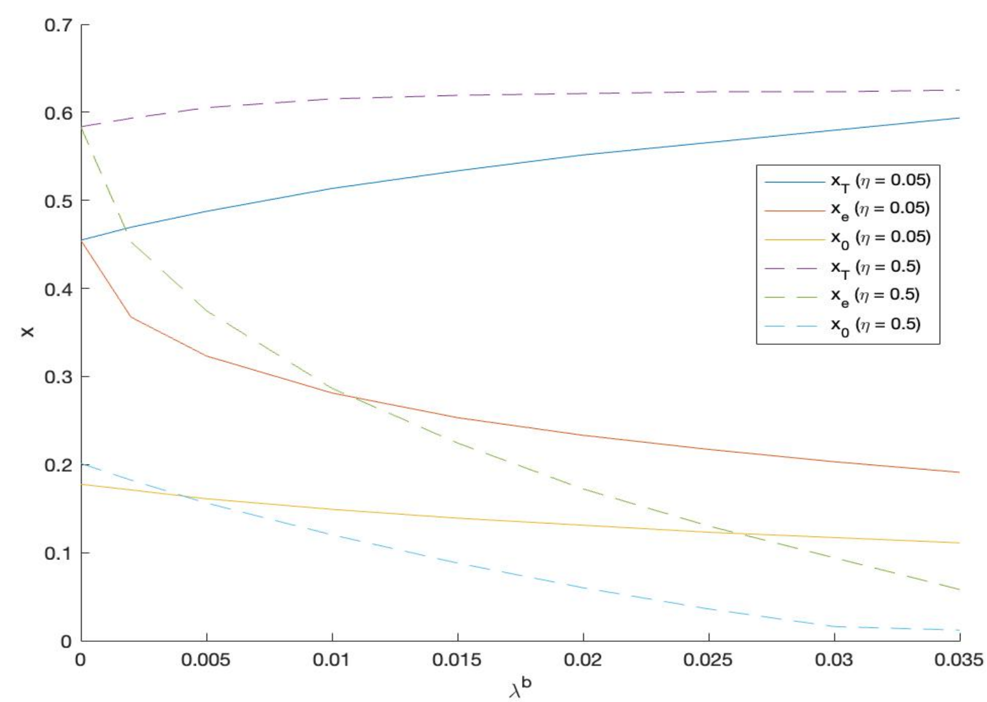

Figure 12: Parameters are $\mu = 0.12, \sigma = 0.4, \gamma = 0.45, r = 0.065, q = 0.01, c = 0.02, \lambda^s = 0.01$

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