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29 April 2023

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Abstract

The paper tries to answer one of the more nascent questions in the literature on general equilibrium theory by investigating the equivalence between the set of club equilibrium states and the bargaining set for a club economy. Clubs in this framework are treated in a parallel fashion to private goods as articles of choice. Each club membership is composed of three components: (i) the individual's characteristics; (ii) the profile of the club. and (iii) the club project. Thus clubs are identified through their profile and the particular project they undertake. We introduce the bargaining set for such an economy in lieu of Mas-Colell [22] and define a two-step veto mechanism. In this paper we establish that non-equilibrium states are those against which there exist a set of agents who agree upon a mutually beneficial trade agreement amongst themselves or in other words there exists a Walrasian objection to such states. In what follows from the literature is that Walrasian objections are justified as well which thereby helps us establish our equivalence.

JEL Code: D50, D51, D60,D71, D11, D00

Keywords: Club goods, Bargaining set, Walrasian objection, Justified objection, Bargaining-Equilibrium equivalence.

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1 Introduction

The notion of the core has been often criticized in the literature by highlighting the myopic nature of the agents. To deal with such short-sighted behavior Aumann and Maschler [4] introduced the concept of bargaining set to inject further credibility into the veto mechanism. They allowed for counter-objections to objections leading to ruling out frivolous objections. To further highlight the economic scenarios where the core falls short Aumann in his paper [3] provides an example of how forming atoms in continuum economies allows for non-preferable allocations for the atom in the core. Maschler [20] in his paper showed that the bargaining set is robust to such a setting and thus advantageous over the core. This formed a major motivation for studying the bargaining set in economy games and investigating its relation to the core and the set of competitive equilibrium allocations. Later Mas-Colell [22] introduced the notion of bargaining set to exchange economies and established the equivalence between competitive equilibrium and the bargaining set. It was later that Anderson et al. [1] pointed out that if one constructs a sequence of replicated economies as in Debreu and Scarf [9], the sequence of bargaining sets in those replicated economies fails to converge to the set of Walrasian equilibrium allocation. The main departure between the two definitions of the bargaining set that leads to the departure between the notions concerns the counter-objection mechanism. Anderson et al. [1] assumed contrary to Mas-colell that agents who are not members of the objecting coalition but have similar type agents belonging to the same need not necessarily improve upon the objecting allocation and it is sufficient for them to improve upon the initial allocation while counter-objecting. Later Hervés-Estévez, Javier, and Emma Moreno-García [16] adopted the definition of Mas-Colell's bargaining set in a sequence of replicated economies and showed that the bargaining sets of the sequence of replicated economies converge to the set of Walrasian allocations, thereby establishing a parity to Mas-Colell [22]. It is important to observe that such equivalence is similar to that of Debreu and Scarf [9] extended to the two-step veto mechanism of the bargaining set. Aubin in his work [5] introduced a new class of coalitions which were termed "Aubin coalitions". These are the class of coalitions where agents participate in multiple objecting coalitions with a fraction of their endowments. Aubin showed that in a finite economy, the Aubin core is equivalent to the set of Walrasian equilibrium allocations. Hervés-Estévez and Moreno-García [17] adopted the notion of Aubin coalitions to the two-step veto mechanism and defined Aubin bargaining set for a finite economy. They further characterized the Walrasian equilibrium allocations in terms of the Aubin bargaining set for such an economy. They also cite that coalition formation is costly and thereby restrict coalition formation to obtain further characterizations of the Aubin bargaining set. in the specific case where agents participate in blocking coalitions with rational participation rates, they show that one can obtain the veto power when the finite economy is enlarged via replicas. Later, Hervés-Beloso et al. [15] and Liu [18] extend the results of Hervés-Estévez and Moreno-García [17] to the case of a finite production economy and to the case of a finite coalitional production economy respectively. Liu and Zhang [19] provided a characterization of the competitive equilibrium allocations of a continuum economy with production in the spirit of Mas-Colell [22] and established that the bargaining sets for such economies are equivalent to the set of competitive allocation allocations. Recently Bhowmik and Saha [7] considered a mixed club economy and extended the core-equivalence theorem of Ellickson et al. [10] by means of interpretation through atomless economies. They further consider restricted coalition formations in terms of Schmeidler [24] and Vind [28] and extend both the Schmeidler and Vind's theorem not only to a club economy but also to an economy with atoms with ordinary coalitions. They also introduce a notion of "ε"-robust efficiency which primarily follows from the characterization of club equilibrium through "approximate" robustly efficient states proposed by Bhowmik and Kaur [6]. In their paper Bhowmik and Kaur [6] adopted such approximations as the standard notion of robust efficiency introduced by Hervés-Beloso and Moreno-García [14], a fact that can be noted from the failure of the second welfare theorem in the case of an economy with club goods. In regards to studies concerning bargaining sets in club economies, Saha [23] in his work has studied the equivalence between Walrasian and justified objection. However, there is no result concerning the equivalence between the bargaining set and the set of club equilibrium states.

Another developing strand of literature is that of club goods. The theory of clubs has remained fairly confined in determining the optimal sizes of clubs. Club goods can be characterized as a convex combination of public and private goods. Thus, they embody a notion of excludability in themselves. Alongside, members of a club have unlimited access to its resources and projects which highlights the nonrival nature of such goods. Buchanan [8] in his work further emphasized that any public project entailed membership sizes that were infinite, although in everyday life one comes across public goods for which membership sizes are finite or in other words the extent of "publicness" is limited. So Buchanan's work focused on determining the membership margin, so to speak, the size of the most desirable cost and consumption-sharing arrangement. Later Scotchmer and Wooders [27] developed a model for club economies with anonymous crowding. Anonymous crowding refers to the fact that individuals with different attributes or characteristics can share the same facilities. They show that in an economy with a finite number of types intended to approximate an economy where all agents may differ, consumers with different tastes can be grouped together where relevant similarity in tastes among efficient groups results in the similarity of demands for facility size and crowding. However, with anonymous crowding, one cannot guarantee the existence of an optimal solution if the optimal club size is not an integer multiple of the size of the population. Scotchmer [25] introduced a notion of non-anonymous crowding in her work. However, with non-anonymous crowding, there is an added dimension of difficulty in forming groups containing the right composition of different types. Scotchmer [25] found a solution to the price-equilibrium by relaxing the assumption of perfect competition by granting firms market power when firm size is non-trivial relative to the economy. Later Scotchmer [26] in her work introduced a notion of approximate equilibrium as a solution to the same problem. She said that in equilibrium firms are optimizing whereas all but a small set of agents are not optimizing. Engl and Scotchmer [12] expressed the pricing mechanism in the presence of non-anonymous crowding as a linear function of externalities producing attributes of consumers. They achieve this result by using a super additivity assumption on attributes. Giles and Scotchmer[13] made significant contributions to studying the decentralization of the core in the club economy framework. Their work was the first to incorporate multiple private goods in the analysis. However, all of the above studies remained restricted to the case of a finite number of agents, and this in tandem with the core indivisible nature of club goods resulted in non-competitive scenarios. Ellickson et al. [10] posited a model that treated club goods in a perfectly competitive setting. The main aspect of their work is the parallel treatment of social interaction and consumption governed by a few important subtleties. To begin with, club sizes must be finite to retain perfect competition in the model. Also, finite club sizes allow externalities arising from clubs to be confined within. Secondly, the feasibility of an allocation is quite different from the traditional literature. In addition, it requires that inputs to club projects are part of material balance and that club memberships must be consistent. For, example if a third of the population are women married to men, then a third of the population must be men married to be women. And lastly, the equilibrium condition in a club economy additionally requires budget balance for club types i.e. the sum of memberships for a particular club is just enough to pay for its inputs to club projects. Ellickson et al. [10] showed that equilibrium exists for such an economy and further the equilibrium states can be decentralized through the core of the economy [1]. In this paper, we refine the veto mechanism and introduce the notion of bargaining set in lines of Mas-Colell [22] and further aim to establish the equivalence between the set of equilibrium states

[1] One can refer to Ellickson et al. [11] for similar results in the case of large finite economies.

and the bargaining set of a club economy.

The rest of the paper is organized as follows. Section 2 talks about the economic models and the crucial assumptions for our framework. Section 3 introduces the notions of different equilibriums for our economy. Alongside we talk about the cooperative or veto mechanism for such a framework and introduce the concepts of core and bargaining set in this section. Section 4 presents the main result which establishes the equivalence between the set of club equilibrium states and the bargaining set of a club economy. Section 5 concludes.

2 Economic Model

We assume that the space of agents for our economy is a complete, finite, positive measure space. We denote it by (A, Σ, λ) with A being the set of agents and Σ as the corresponding σ-$algebra$ whose economic weights on the market are given by the measure λ. The measure space is an atomless one or in words comprises agents whose economic weight in the market is negligible. Now let N denote the set of private commodities. We assume that the private commodity space is denoted by the N-dimensional Euclidean space $\mathbb{R}^N$. Thus private goods in our setup are perfectly divisible[2]. The consumption set of these commodities is restricted to the non-negative orthant $\mathbb{R}^N_+$ for each agent. Furthermore, let $\mathbb{R}^N_{++}$ denote the strictly positive elements of $\mathbb{R}^N$. Under standard notations for any two bundles $x, y \in \mathbb{R}^N_+$, $x \geq y$ implies $x_i \geq y_i$ for all $i \in N$; $x > y$ implies that $x \geq y$, however $x \neq y$; and $x \gg y$ implies that $x_i > y_i$ for each $i \in N$. We denote $\|x\|_1 := \sum_{n=1}^{N} x_n$.

2.1 Clubs

Each potential member of a club, as in Ellickson et al [10] is bestowed with some characteristics that are external in nature. This implies that not only characteristics are observable to other members and also create externality within clubs. Examples of such characteristics can be sex, appearance, religion, etc. To capture such externalities, we define a broad set of **finite characteristics** Ω, from which an agent may accrue. An element $\omega \in \Omega$ denotes the characteristic of an individual agent relevant to other members. Each club can be characterized by the composition of its members along the domain of external characteristics. For that we define, a map $\pi : \Omega \to \mathbb{Z}_+$, $\mathbb{Z}_+$ being the set of non-negative integers. We identify the composition of a club with such a

[2]Without loss of generality we assume that N also denotes the cardinality for the set commodities.

map and term it as **profile** of a club. Thus, for any $\omega \in \Omega$ the number $\pi(\omega)$ denotes the number of individuals having characteristic ω. Therefore, for a profile π of a club, the total number of members is $\|\pi\|_1 := \sum_{\omega\in\Omega} \pi(\omega)$.

Each club endorses a public project (local to the club) which is termed as **activities**. Such activities are members of a finite abstract set of club activities available to the profile of agents. The abstract nature of the activities is an adaptation from Mas-Colell [21]. The set is abstract in the sense that there does not exist a pre-defined linear order over this set of activities and ranking is entirely subjective to individual members. We denote the set of such activities by Γ. Activities are not traded and ranking amongst them may be influenced by private goods consumption. We define a **club type** as a pair (π, γ), where π denotes the profile of the club and $\gamma \in \Gamma$ denotes the activity the club partakes in. In our economy, there exist only a finite set of possible club types, denoted by $Clubs := \{(\pi, \gamma)\}$. Now club projects are to be funded and operated by members of the clubs only. In absence of the notion of money in our model, inputs to club projects are made via a contribution through private goods. Thus, the requirement of inputs for a club type, denoted by $inp\,(\pi, \gamma)$, is a vector of $\mathbb{R}^N_+$.

The next concept we define is pertaining to club memberships. Memberships, reserve rights of admission to individuals for clubs. An agent of external characteristic $\omega \in \Omega$ can become a member of club type (π, γ) if and only if the particular club type allows individuals with characteristic ω to be a non-zero fraction in the club composition, i.e., $\pi(\omega) \geq 1$ in absolute terms. A **club membership** is thus a triplet $m = (\omega, \pi, \gamma)$, where $(\pi, \gamma) \in \mathcal{C}lubs$ and $\pi(\omega) \geq 1$. The set of all club memberships is denoted by $\mathscr{M}$. An agent may purchase memberships of multiple clubs or none and also can purchase more than one subscription of one particular club type. We define a map $\mathscr{L} : \mathscr{M} \to \{0, 1, 2, \dots\}$, where $\mathscr{L}\,(\omega, \pi, \gamma)$ of a membership $m = (\omega, \pi, \gamma)$ denotes the number of that membership being bought. We term the above-defined map as *list*. The set of all such possible *list* is denoted by the following notation:

$$Lists = \{\mathscr{L} : \mathscr{L} \text{ is a } list\}.$$

Letting $\mathbb{R}^{\mathscr{M}}$ be the set of all mappings from the set $\mathscr{M}$ to the real line, we can frequently view $Lists$ as a subset of $\mathbb{R}^{\mathscr{M}}$. Throughout the rest of the paper, we also assume that there is an exogenously given upper bound M on the number of memberships an individual may choose.

2.2 Club Economy

In a club economy, each individual agent is identified by his or her external characteristic $\omega \in \Omega$. The total amount of private goods in the economy is characterized by the distribution e_a for all $a \in A$. We refer to e_a as the endowment of agent a of private goods. Endowments are said to be **desirable** in our framework if we have $u_a(e_a, 0) > u_a(0, l_a)$ for every agent $a \in A$ and $l_a \in Lists_a$. In simple words, an agent will prefer to stay put with his or her initial endowment compared to consuming only club memberships. Unlike standard private goods economies, in our framework consumption is a combination of both private and club goods. We capture the set of all such feasible consumption bundles by the set X_a. Thus $X_a \subseteq \mathbb{R}^N \times Lists$. Since club goods possess a nature of excludability in them, it is natural to assume that each agent may be barred from access to certain club types (for example a swimming pool may be restricted to use only by females). We denote the set of feasible club memberships for agent a as $Lists_a \subseteq Lists$. Thus, $X_a = \mathbb{R}^N_+ \times Lists_a$. The utility function of agent a is $u_a : X_a \to \mathbb{R}$.

Definition 2.1. A **club economy** $\mathscr{E}$ is a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$, satisfying the following conditions:

(i) The mapping $a \mapsto \omega_a$ is measurable for all $a \in A$;

(ii) The correspondence $a \mapsto X_a$ is a measurable for all $a \in A$;

(iii) The endowment mapping $a \mapsto e_a$ is integrable for all $a \in A$ and each endowment is strictly positive i.e. $e_a \in \mathbb{R}^N_{++}$ for all $a \in A$; and

(iv) The mapping $(a, f, l) \mapsto u_a(f, l)$ is jointly measurable with $u_a(\cdot, l)$ is continuous and strongly monotonic for all $a \in A$.

2.3 States, Allocations, and Consistency

It is worthwhile to reiterate that clubs in equilibrium are composed of only a finite number of members. This in conjunction with the fact that club types are finite in our framework leads to the fact the number of feasible club memberships that can be purchased by each agent is upper-bounded. All these make clubs infinitesimal compared to the entire economy. Also, as pointed out earlier that externalities emerging from member characteristics are confined within clubs, and hence our economy remains devoid of any hindrance to perfect competition.

Consumption in our economy consists of feasible choices composed of both private and club goods. This feature is in sharp distinction with traditional economies consisting of only private goods. Therefore we define an allocation or what we refer to as state in our economy next.

Definition 2.2. A **state** of $\mathscr{E}$ is basically a measurable mapping $(f, l) : A \to \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$, which specifies for any agent $a \in A$ the amount of private good consumption f_a and the club membership vector l_a.

A state is said to be **individually feasible** if $(f_a, l_a) \in X_a$ λ-a.e. Club membership choices made by agents are intrinsically indivisible in nature. Equilibrium for club goods, therefore, requires the existence of the desired number of club types to match the demand of agents. To this end, we define a consistency condition in our economy.

Definition 2.3. Given a membership vector $\overline{\mu} \in \mathscr{R}^{\mathscr{M}}$, if for each club type $(\pi, \gamma) \in$ *Clubs*, there exists a number $\psi(\pi, \gamma) \in \mathbb{R}_+ \setminus \{0\}$ such that

$$\overline{\mu}(\omega, \pi, \gamma) = \psi(\pi, \gamma)\, \pi(\omega)$$

for all $\omega \in \Omega$, then we call such a membership vector $\overline{\mu}$ **consistent**.

In the above definition, the number $\psi(\pi, \gamma)$ may be interpreted as the *relative number* of clubs of type (π, γ) demanded for in $\overline{\mu}$. Define

$$\mathscr{C}ons := \left\{\overline{\mu} \in \mathscr{R}^{\mathscr{M}} : \overline{\mu} \text{ is consistent}\right\}.$$

Note that $\mathscr{C}ons \subseteq \mathscr{R}^{\mathscr{M}}$ and satisfies the properties of a vector subspace. Next, we shall try and investigate when a choice of club membership is consistent for any positive measurable subset of agents.

Definition 2.4. A **coalition** is a measurable subset B of A whose measure is positive. Furthermore, a **sub-coalition** of a coalition B is a coalition B' such that $B' \subseteq B$. For any coalition B, a choice function $\mu : B \to Lists$ is **consistent for *B*** if the corresponding aggregate membership vector $\overline{\mu}_B = \int_B \mu_a d\lambda \in \mathscr{C}ons$.

Notice that the co-ordinate $\overline{\mu}(\omega, \pi, \gamma)$ of $\overline{\mu}$ specifies the total number of memberships chosen by the members in B with external characteristics ω for club type (π, γ). Consistency is the requirement that these numbers are in the same proportion as in the club type.

We next define feasibility for our club economy. We have already defined the feasibility condition for club memberships through the "consistency" condition. Private goods in our framework also need clearance. Juxtaposed to the traditional general equilibrium model, private goods in our setup are also used as inputs to club projects in parallel to consumption. We capture *social feasibility* for any state of the economy through the **material balance** condition. For the club type (π, γ), we divide the inputs to the club in equal proportions amongst the members of the club. As in Ellickson et al. [10] for an agent $a \in A$ with membership choice l_a, let $\tau(l_a)$ denote his share for club project where

$$\tau(l_a) := \sum_{(\omega,\pi,\gamma)} \frac{1}{\|\pi\|_1} inp\,(\pi, \gamma)\, l_a\,(\omega, \pi, \gamma)\,.$$

Definition 2.5. A state (f, l) is **feasible for a coalition *B*** if it abides by the following conditions:

- **Individual Feasibility**: $(f_a, l_a) \in X_a$ λ-a.e. on B;
- **Material Balance**: $\int_B f_a d\lambda + \int_B \tau(l_a) d\lambda = \int_B e_a d\lambda$; and
- **Consistency**: $\int_B l_a \in \mathscr{C}ons$.

For $B = A$ then we simply call it **feasible**.

3 Equilibrium, Transfers and Bargaining set

In this subsection, we shall lay out some definitions. We first outline the definitions pertaining to the solution concepts for the market mechanism and then follow it up with the solution concepts from the cooperative mechanism. For the cooperative mechanism, we resort to both strong and weak notions of some concepts.

Definition 3.1. A **club equilibrium** of $\mathscr{E}$ consists of a feasible state (f, l) and a price vector $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, such that:

- **Budget Feasibility** : For λ-a.e. on A, we have $(p, q) \cdot (f_a, l_a) = p \cdot f_a + q \cdot l_a \leq p \cdot e_a$;
- **Optimization**: For λ-a.e. on A, we have $(g_a, \mu_a) \in X_a$ and $u_a\,(g_a, \mu_a) > u_a\,(f_a, l_a)$ together imply $p \cdot g_a + q \cdot \mu_a > p.e_a$.
- **Budget Balance for Club types** : For each $(\pi, \gamma) \in \mathscr{C}lubs$,

$$\sum_{\omega \in \Omega} \pi(\omega) q\,(\omega, \pi, \gamma) = p.inp\,(\pi, \gamma)\,.$$

Let $\mathscr{W}(\mathscr{E})$ denote the set of club equilibrium states of the economy $\mathscr{E}$. A **club quasi-equilibrium** of $\mathscr{E}$ also consists of a feasible state (f, l) and a price vector $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, satisfies the first and third conditions in the definition of a club equilibrium, but instead of the second condition it satisfies:

- **Quasi-optimization**: For λ-a.e. on A, we have $(g_a, \mu_a) \in X_a$ and $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ together imply $p \cdot g_a + q \cdot \mu_a \geq p.e_a$.

Definition 3.2. A pure-transfer club equilibrium consists of a feasible state (f, l) and prices $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, such that:

- **Budget Feasibility for Individuals** : For almost all $a \in A$,

$$p \cdot f_a + q \cdot l_a + p \cdot \tau(l_a) \leq p.e_a;$$

- **Optimisation**: For almost all $a \in A$, $(g_a, \mu_a) \in X_a$ and

$$u_a(g_a, \mu_a) > u_a(f_a, l_a) \Rightarrow p \cdot g_a + +q \cdot \mu_a + p \cdot \tau(\mu_a) > p.e_a; \text{ and}$$

- **Pure Transfers** : $q \in \mathcal{T}rans$.

In the next definition, we introduce the notion of club irreducibility which further aids us in establishing that quasi-demand is demand for each agent $a \in A$. The importance of this result is captured in the proof of Proposition 4.2, where we show that any state which is not a club equilibrium must be objected to by a set of agents who are willing to trade amongst themselves at the given prices.

Definition 3.3. Let (f, l) be a feasible state of the club economy $\mathscr{E}$. We say that (f, l) is **club linked** whenever $\{I, J\}$ is a partition of the set of private goods, $\{1, 2, \cdots, N\}$ and $f_{ai} = 0$ for all $i \in I$ and almost all $a \in A$, then for almost all $a \in A$ there exist $r \in \mathbb{R}_+$, $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(f_a, l_a),$$

where δ_j is a vector containing one unit of good j and no other goods. An economy $\mathscr{E}$ is said to be **club irreducible** if every feasible state is club linked.

Lemma 3.4. *Let $\mathscr{E}$ be a club economy satisfying Definition 2.1 and for which endowments are desirable too. Then every for each agent $a \in A$, if $((f_a, l_a), (p, q))$ is a quasi-demand and club-linked then $p \gg 0$ and $((f_a, l_a), (p, q))$ can be sustained as a demand.*

Definition 3.5. : A feasible state (f, l) of the economy $\mathscr{E}$ is said to be:

(i) **strongly objected** if there exists some coalition B and a state (g,μ) such that (g,μ) is feasible for B and $u_a(g_a,\mu_a) > u_a(f_a,l_a)$ for almost all $a \in B$.

(ii) **weakly objected** if there exists some coalition B and a state (g,μ) such that (g,μ) is feasible for B, $u_a(g_a,\mu_a) \geq u_a(f_a,l_a)$ for almost all $a \in B$ and $u_a(g_a,\mu_a) > u_a(f_a,l_a)$ for all agents $a \in B'$ for some sub-coalition B' of B.

A feasible state (f,l) is said to be in the **weak core** of the economy $\mathscr{E}$ if it is not strongly objected. A feasible state (f,l) is similarly said to be in the **strong core** of the economy $\mathscr{E}$ if it is not weakly objected.

To introduce our next concepts, we say that a pair $(S,(g,\mu))$ constitutes an **objection** to a state (f,l) if the state (f,l) is strongly objected by the coalition S via the state (g,μ).

Definition 3.6. We say that an objection $(S,(g,\mu))$ to a feasible state (f,l) is **counter-objected** if there exists some coalition T and a state (h,ν) such that (h,ν) is feasible for T:

(i) $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ for all $a \in T \cap S$; and

(ii) $u_a(h_a,\nu_a) > u_a(f_a,l_a)$ for all $a \in T \setminus S$.

We call an objection to be **justified** if there exists no counter-objection to it. The **bargaining set** is the set of all feasible states for which there does not exist any justified objection. We denote the bargaining set of the economy $\mathscr{E}$ by $\mathscr{B}(\mathscr{E})$.

Now we introduce a veto mechanism through the market system, known as Walrasian Objection. Roughly speaking, it simply means that given a competitive price pair $(p,q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$ there exists a set of agents who are willing to trade private goods amongst themselves rather than accept the proposed state.

Definition 3.7. We say that an objection $(S,(g,\mu))$ to a feasible state (f,l) is **Walrasian** if there exists a price vector $(p,q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$ such that

(i) $p \cdot h_a + q \cdot \nu_a > p \cdot e_a$ whenever $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ and $a \in S$;

(ii) $p \cdot h_a + q \cdot \nu_a > p \cdot e_a$ whenever $u_a(h_a,\nu_a) > u_a(f_a,l_a)$ and $a \in A \setminus S$; and

(iii) **Budget Balance for Club types** : For each $(\pi,\gamma) \in \mathscr{C}lubs$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega,\pi,\gamma) = p.inp(\pi,\gamma).$$

However, again for the convenience of the proof, we resort to the pure-transfers case.

Definition 3.8. We say that an objection $(S,(g,\mu)$ to a feasible state (f,l) is a **pure-transfer Walrasian** if there exists a price vector $(p,q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$ such that

(i) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a$ whenever $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ and $a \in S$;

(ii) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a$ whenever $u_a(h_a,\nu_a) > u_a(f_a,l_a)$ and $a \in A \setminus S$; and

(iii) $q \in \mathcal{T}rans$.

Lemma 3.9. *Let $\mathscr{E}$ be a club economy and $q, q^* \in \mathbb{R}^{\mathscr{M}}$ be such that*

$$q^*(\omega,\pi,\gamma) = q(\omega,\pi,\gamma) + \frac{1}{\|\pi\|_1} p \cdot inp(\pi,\gamma)$$

Then $(S,(g,\mu))$ is a Walrasian objection under price (p,q) if and only if $(S,(g,\mu))$ is a pure-transfer Walrasian objection under price (p,q^*).

Proof. The proof is immediate from Lemma 3.4 of Ellickson et al. [10].

□

4 Main result

In this section, we illustrate the equivalence between the set of club equilibria states and the bargaining set of a club economy. Prior to that, we state a lemma that appears in Ellickson et al. [10] which shall aid in our proof. We present it for completeness, and one shall refer to Ellickson et al. [10] for the complete proof. We call a subset $L \subset Lists_M$ to be **strictly balanced** if there are strictly positive real numbers $\{\beta_L(\ell) : \ell \in L\}$ (called balancing weights) such that $\sum_{\ell \in L} \beta_L(\ell)\ell \in \mathscr{C}ons$.

Lemma 4.1. *Let $R^* > 0$ be a constant such that if $L \subset Lists_M$ is a strictly balanced collection and $q \in \mathcal{T}rans$ is a pure transfer then*

$$\max_{\ell \in L} q \cdot \ell \geq -R^* \min_{\ell \in L} q \cdot \ell.$$

Now we present one of our main results, which states that any state that fails to qualify as an equilibrium of the club economy $\mathscr{E}$, must be objected to by a coalition of agents who are willing to trade among themselves at the given prices to achieve a mutually beneficial outcome.

Proposition 4.2. *Let $\mathscr{E}$ be a club economy for which endowments are desirable. Further, let $\mathscr{E}$ be club linked. Then any feasible state that is not a club equilibrium, must have a Walrasian objection against it.*

Proof. Let (f, l) be a feasible state which is not a club equilibrium. Without loss of generality, we can assume that (f, l) is not a pure-transfer equilibrium state.

Step 1: Constructing modified economies

Without loss of generality, we assume that $\lambda(A) = 1$. We assume that the individual endowments are **uniformly bounded** above by $W_0\mathbf{1}$, where $\mathbf{1} = (1, 1, \cdots, 1)$. Let $W = \max\{W_0, 1\}$.

Apart from the original set of agents, we introduce an auxiliary set of agents in our economy. For each integer $k > 0$, choose a family $\{A^k_\omega : \omega \in \Omega\}$ of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$, each of measure $\frac{1}{k}$ with $A \cap A^k_\omega = \emptyset$, for all $\omega \in \Omega$. Define

$$B^k := \bigcup \{A^k_\omega : \omega \in \Omega\}$$

and consider a measure space of agents $(B^k, \mathscr{B}^k, m)$, where $\mathscr{B}^k$ is the σ-algebra of Lebesgue measurable subsets of B^k and m is the Lebesgue measure on $\mathscr{B}^k$. For each $k \geq 1$, we thus, introduce an economy $\mathscr{E}^k$ whose space of agents is defined by the measure space $(A^k, \Sigma^k, \lambda^k)$, where

(1) $A^k := A \cup B^k$;

(2) $\Sigma^k := \Sigma \otimes \mathscr{B}^k = \{C \cup D^k : C \in \Sigma,\ D^k \in \mathscr{B}^k\}$; and

(3) $\lambda^k : \Sigma^k \to \mathbb{R}_+,$ where $\lambda^k(C) := \lambda(C \cap A) + m(C \cap B^K)$.

We assume that for each agent $a \in A$, his or her external characteristics, choice set, initial endowment, and utility function are the same as that in the original economy $\mathscr{E}$. For each agent $a \in A^k_\omega$, we define

$$\begin{aligned} \omega_a &:= \omega; \\ X_a &:= \mathbb{R}^N_+ \times \{l(\omega', \pi, \gamma) : l(\omega', \pi, \gamma) = 0 \text{ if } \omega' \neq \omega\}; \\ e_a &:= W\mathbf{1};\ \text{and} \\ u_a(x_a, \mu_a) &:= \|x_a\|_1 \text{ for all } (x_a, \mu_a) \in X_a. \end{aligned}$$

Before proceeding, we state two important conditions that aid our proof. Choose a small enough $\varepsilon \in \left(0, \frac{1}{N}\right)$ such that

$$[1 - (N-1)\varepsilon]\left[\frac{W}{kN\varepsilon} - W\left(1 + \frac{|\,\Omega\,|}{k}\right)\right] - \varepsilon(N-1)W\left(1 + \frac{|\,\Omega\,|}{k}\right) > 0 \qquad (4.1)$$

Having chosen ε, choose a sufficiently large $R \in \mathbb{R}_+$ such that $R > 2\|\tau(\ell)\|_1 M^*$ for all $\ell \in Lists_M$ and

$$[1-(N-1)\varepsilon]\left[\frac{R}{2kNM^*} - W\left(1+\frac{|\,\Omega\,|}{k}\right)\right] - \varepsilon(N-1)W\left(1+\frac{|\,\Omega\,|}{k}\right) > 0 \quad (4.2)$$

where $M^* = max\left\{\|\pi\|_1 : (\pi,\gamma) \in \mathcal{C}lubs\right\}$. It is worthwhile to point out that both ε, R depend on k.

<u>Step 2: Constructing aggregate excess demand</u>

Since the utilities for added agents are strongly monotone in private goods, equilibrium prices cannot be arbitrarily low. Thus, for each $0 < \varepsilon \leq \frac{1}{N}$, we define the price space for private goods to be

$$\Delta_\varepsilon := \left\{p \in \mathbb{R}^N_+ : \sum_{n\in N} p_n = 1 \text{ and } p_n \geq \varepsilon \text{ for each } n \in N\right\}.$$

For any $q \in \mathcal{T}rans$, if a particular membership price is positive and large enough, there must be some other membership price that is large enough and negative. Moreover, from the utilities of the added agents, it follows that such agents derive no utilities from consuming club goods. Consequently, such agents shall purchase memberships that provide them with large subsidies. This, however, leads to excess demand for private goods in the economy. Thus, membership prices must belong to a compact domain, which is defined as

$$\mathcal{Q}_R := \{q \in \mathcal{T}rans : \|q_m\|_1 \leq R \text{ for all } m \in \mathscr{M}\}.$$

Fix an $\varepsilon > 0$, for a given $(p,q) \in \Delta_\varepsilon \times \mathcal{Q}_R$, the **budget set** for agent $a \in A$ is defined as

$$\mathbb{B}(a,p,q) := \{(x_a,\mu_a) \in X_a : p\cdot x_a + q\cdot \mu_a + p\cdot\tau(\mu_a) \leq p\cdot e_a\}.$$

The corresponding **demand** and **excess demand** sets are defined as

$$\mathbb{D}(a,p,q) := \arg\max\{u_a(x_a,\mu_a) : (x_a,\mu_a) \in \mathbb{B}(a,p,q)\}$$

and

$$\zeta(a,p,q) := \{(x_a+\tau(\mu_a),\, \mu_a) - (e_a,0) : (x_a,\mu_a) \in \mathbb{D}(a,p,q)\}.$$

Since memberships are bounded above by M and $q \in \mathcal{Q}_R$, the subsidies from memberships are bounded above by RM. It follows that $\mathbb{B}(a,p,q)$ is compact. The continuity of u_a further implies that $\mathbb{D}(a,p,q) \neq \emptyset$, and hence $\zeta(a,p,q) \neq \emptyset$. Since individual endowments are bounded above by W, for all $a \in A^k$, the excess demand for each such agent satisfies the following inequalities:

$$-W \leq \zeta(a,p,q) \leq \frac{1}{\varepsilon}(W+RM)$$

Thus, aggregate excess demands for private goods for economy $\mathscr{E}^k$ belong to the following compact subset of $\mathbb{R}^N$.

$$\mathcal{X} := \left\{ x \in \mathbb{R}^N : -\lambda^k \left(A^k\right) W \leq x_n \leq \lambda^k \left(A^k\right) \frac{1}{\varepsilon}(W+RM), \text{ for each } n \in N \right\}.$$

Aggregate demand for club memberships belongs to the set

$$\mathcal{C} := \left\{ \overline{\mu} \in \mathbb{R}_+^{\mathscr{M}} : \sum_{m \in \mathscr{M}} \overline{\mu}(m) \leq \lambda^k \left(A^k\right) M \right\}.$$

We define the aggregate excess demand correspondence $Z : \Delta_\varepsilon \times Q_R \rightrightarrows \mathbb{R}^N \times \mathbb{R}_+^{\mathscr{M}}$ as

$$Z(p,q) := \int_{A^k} \zeta(a,p,q) d\lambda^k.$$

Step 3: Constructing modified aggregate excess demand

We define the modified aggregate excess demand correspondence $Z^* : \Delta_\varepsilon \times Q_R \rightrightarrows \mathbb{R}^N \times \mathbb{R}_+^{\mathscr{M}}$ as

$$Z^*(p,q) := \int_{A^k} \zeta^*(a,p,q) d\lambda^k,$$

where for $a \in B^k$, $\zeta^*(a,p,q) := \zeta(a,p,q)$ and for $a \in A$[3]

$$\zeta^*(a,p,q) := \begin{cases} \zeta(a,p,q), \text{ if } u_a(\mathbb{D}(a,p,q)) > u_a(f_a,l_a); \\ \\ \zeta(a,p,q) \cup \{(0,0)\}, \text{ if } u_a(\mathbb{D}(a,p,q)) = u_a(f_a,l_a); \text{ and} \\ \\ \{(0,0)\}, \text{ if } u_a(\mathbb{D}(a,p,q)) < u_a(f_a,l_a). \end{cases}$$

For a given $(p,q) \in \Delta_\varepsilon \times \mathcal{Q}_R$, define

$$S_{p,q} := \{a \in A : (f_a,l_a) \notin \mathbb{D}(a,p,q)\}.$$

[3] Here, $u_a(\mathbb{D}(a,p,q)) = u_a(x_a,\mu_a)$ for $(x_a,\mu_a) \in \mathbb{D}(a,p,q)$.

Since (f, l) is not a pure transfer club equilibrium state, we must have $\lambda(S_{p,q}) > 0$.

Step 4: Constructing a pure-transfer club equilibrium of $\mathscr{E}^k$

We define a correspondence $\Phi^* : \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C} \rightrightarrows \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C}$ by

$$\Phi^*(p, q, x, \overline{\mu}) = [\arg\max \{(p^*, q^*).(x, \overline{\mu}) : (p^*, q^*) \in \Delta_\varepsilon \times \mathcal{Q}_R\}] \times Z^*(p, q)$$

By a standard argument, the correspondence Φ^* satisfies upper hemi-continuity and is non-empty, compact, and convex valued. Hence from Kakutani's fixed point theorem, we can guarantee the existence of a fixed point $((p^k, q^k), (\overline{z}^k, \overline{\mu}^k)) \in \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C}$ of Φ^* that maximizes the value of $(p^*, q^*).(x, \overline{\mu})$, where

$$(\overline{z}^k, \overline{\mu}^k) = \int_{A^k} (z_a^k, \mu_a^k) d\lambda^k$$

for some element $(z_a^k, \mu_a^k) \in \zeta^*(a, p^k, q^k)$ λ^k-a.e. Now using Equation (4.3) and Equation (4.4), from *Step 5* and *Step 6* of the proof of Theorem 6.1 in Elllickson et al. we obtain: (i) $q^k \in \mathcal{T}rans$ and $\overline{\mu}^k \in \mathscr{C}ons$; (ii) $p^k \in \Delta_\varepsilon$; (iii) $\overline{z}^k = 0$; and (iv) the state (x_a^k, μ_a^k), defined by $x_a^k := z_a^k + e_a - \tau(\mu_a^k)$, satisfies $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, q^k)$ for all $a \in C_k$, where

$$C_k := \left\{a \in A : (z_a^k, \mu_a^k) \in \zeta(a, p^k, q^k)\right\}$$

has a positive measure due to the fact that $S_{p^k,q^k} \subseteq C_k$.

Step 5: Constructing a bounded sequence $\{\overline{q}^k : k \geq 1\}$ of membership prices

Since membership prices q^k are bounded above by R, which in turn varies with k, we need to guarantee that a bounded sequence of membership prices exists. So passing to a subsequence if necessary, we may assume without loss of generality that for each $\ell \in Lists_M$ the sequence $\{q^k \cdot \ell : k \geq 1\}$ converges to a limit G_ℓ which may be finite or infinite. We define the following sets:

$$\mathbb{L} = \left\{\ell \in Lists_M : q^k \cdot \ell \to G_\ell \in \mathbb{R}\right\};$$

$$\mathbb{L}_+ = \left\{\ell \in Lists_M : q^k \cdot \ell \to +\infty\right\};$$

and

$$\mathbb{L}_- = \left\{\ell \in Lists_M : q^k \cdot \ell \to -\infty\right\}.$$

Consequently, for each $\ell \in \mathbb{L}$, the sequence $\{q^k \cdot \ell : k \geq 1\}$ is bounded. This implies the existence of a large enough $\overline{G} \in \mathbb{R}$ such that $|q^k \cdot \ell| \leq \frac{\overline{G}}{\#\mathbb{L}}$ [4] for each $k \geq 1$ and all $\ell \in \mathbb{L}$. Define $T : \mathcal{T}rans \to \mathbb{R}^{\mathbb{L}}$ as $T(q)_\ell = q \cdot \ell$. Recognized that T is a linear

[4] Note that $\#\mathbb{L}$ denotes the number of elements belonging to the set $\mathbb{L}$.

transformation, and we denote by ran$T := T(\mathcal{T}rans) \subseteq \mathbb{R}^{\mathbb{L}}$ the range of T. Now, let ker$T := T^{-1}(0) \subseteq \mathcal{T}rans$, denote the null space of T or the kernel of T. It follows from the fundamental theorem of linear algebra that there exists a subspace H of $\mathcal{T}$rans such that $H \cap \text{ker}T = \{0\}$ and $H + kerT = \mathcal{T}rans$. Denote by $T|_H$, the linear transformation T when the domain is restricted to H. It follows that the map $T|_H : H \to ranT$ is one-to-one and onto, and thus admits an inverse. Let $S : \text{ran}T \to H$ denote the inverse. Since S is a linear functional, it must be continuous. Thus, there exists a constant K such that $\|S(x)\|_1 \leq K\|x\|_1$ for each $x \in ranT$. Let R^* be a constant constructed as in Lemma 4.1. Choose a large enough k_0 such that for all $k \geq k_0$, we have the following

$$q^k \cdot \ell > 2K\overline{G}M + W \text{ if } \ell \in \mathbb{L}_+;$$

and

$$q^k \cdot \ell < -2K\overline{G}M - \frac{W}{R^*} \text{ if } \ell \in \mathbb{L}_-.$$

We define ST as a composition of S with T. For each $k \geq k_0$, define

$$\overline{q}^k := ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \mathcal{T}rans. \tag{4.3}$$

Recognize that

$$\|ST(q^k)\|_1 \leq K\|T(q^k)\|_1 \leq K\overline{G}.$$

Similarly, $\|ST(q^{k_0})\|_1 \leq K\overline{G}$. Thus, from Equation (4.3), we have

$$|\overline{q}^k \cdot \ell| \leq 2K\overline{G}M + |q^{k_0} \cdot \ell|$$

for all $k > k_0$ and all $\ell \in Lists_M$. Thus, the prices of lists are bounded. In view of the fact that singleton memberships are themselves lists, we conclude that $\{\overline{q}^k : k \geq 1\} \subseteq \mathcal{T}rans$ is also a bounded sequence.

Step 6: $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, \overline{q}^k)$ for all $a \in C_k$ for all sufficiently large k

Since $(x^k, \mu^k), (p^k, q^k)$ is a pure transfer equilibrium for $\mathscr{E}^k$, it suffices to show that the choice (x_a^k, μ_a^k) is budget feasible and optimal at $(p^k, \overline{q}^k)$ for every $a \in C_k$. First, we assert that for any $k > k_0$, $\mu_a^k \in \mathbb{L}$ for any $a \in C_k$. If $a \in C_k$, since endowments are bounded above by W it follows quite trivially that $q^k \cdot \mu_a^k \leq W$. Thus, $\mu_a^k \notin \mathbb{L}_+$. Again, since the collection $\{\mu_a^k : a \in A^k\}$ is strictly balanced and $q^k \in \mathcal{T}rans$, it follows from Lemma 4.1 that

$$\min\left\{q^k \cdot \mu_a^k : a \in A^k\right\} \geq -\frac{1}{R^*} \max\left\{q^k \cdot \mu_a^k : a \in A^k\right\} \geq -\frac{W}{R^*}.$$

Consequently, $\overline{\mu}_a^k \notin \mathbb{L}_-$ for each $a \in C_k$. Hence, $\mu_a^k \in \mathbb{L}$ for each $a \in C_k$. On the other hand, it follows from Equation (4.3) that

$$T(\overline{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k). \tag{4.4}$$

The last equality in the above equation follows from the fact that since $S, T|_H$ are inverses, the composition TS is an identity. Since $\mu_a^k \in \mathbb{L}$ for each $a \in C_k$, in view of Equation (4.4), we have $\overline{q}^k \cdot \mu_a^k = q^k \cdot \mu_a^k$ for each $a \in C_k$. Thus, $(x_a^k, \mu_a^k) \in \mathbb{B}(a, p^k, \overline{q}^k)$ for all $a \in C_k$. To show that (x_a^k, μ_a^k) is optimal at $(p^k, \overline{q}^k)$ for each $a \in C_k$, we first choose $k_1 \geq k_0$ such that

$$q^k \cdot \ell < q^{k_0} \cdot \ell - 2K\overline{G}M$$

for all $\ell \in \mathbb{L}_-$ and $k \geq k_1$. Choose and fix $a \in C_k$. Suppose that there exists some $(y, \nu) \in X_a$ such that $(y, \nu) \in \mathbb{B}(a, p^k, \overline{q}^k)$ and $u_a(y, \nu) > u_a(x_a^k, \mu_a^k)$. Budget feasibility of (y, ν) at prices $(p^k, \overline{q}^k)$ implies that $\overline{q}^k \cdot \nu \leq W$ and hence, by Equation (4.3), we have $q^{k_0} \cdot \nu \leq W + 2K\overline{G}M$ as $\|ST(q^k)\|_1 \leq K\overline{G}$ and $\|ST(q^{k_0})\|_1 \leq K\overline{G}$. Thus $\nu \notin \mathbb{L}_+$. For $\ell \in \mathbb{L}_-$ and $k > k_1$, by Equation (4.3), we simply obtain

$$\overline{q}^k \cdot \ell > q^{k_0} \cdot \ell - 2K\overline{G}M > q^k \cdot \ell.$$

Thus, $\overline{q}^k \cdot \ell > q^k \cdot \ell$ for $\ell \in \mathbb{L}_-$. Thus, we have $\overline{q}^k \cdot \ell \geq q^k \cdot \ell$ for all $\ell \in \mathbb{L} \cup \mathbb{L}_-$. This along with the fact that $(y, \nu) \in \mathbb{B}(a, p^k, \overline{q}^k)$ further imply $(y, \nu) \in \mathbb{B}(a, p^k, q^k)$. Coupled with our assumption that $u_a(y, \nu) > u_a(x_a^k, \mu_a^k)$, it implies that $(x_a^k, \mu_a^k) \notin \mathbb{D}(a, p^k, q^k)$, contradicting **Step 4**. Therefore, we conclude that $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, \overline{q}^k)$ for all $a \in C_k$ and $k \geq k_1$.

Step 7: Constructing Walrasian objection

Since $\left\{(p^k, \overline{q}^k, \overline{\mu}^k) : k \geq 1\right\}$ is a bounded sequence, passing to a subsequence, if necessary we may assume that $p^k \to p^* \in \Delta$, $\overline{q}^k \to q^* \in \mathcal{T}rans$, $\overline{\mu}^k \to \overline{\mu}^* \in \mathscr{C}ons$. Since the sequence $\left\{\mu^k : k \geq 1\right\}$ is uniformly bounded, it is uniformly integrable. By Schmeidler's version of Fatou's Lemma, there exists some $(z^*, \mu^*) : A \to \mathbb{R}^N \times \mathbb{R}_+^{\mathscr{M}}$ such that

(i) $(z_a^*, \mu_a^*) \in Ls\left\{\left(z_a^k, \mu_a^k\right) : k \geq 1\right\}$ for λ-a.e. on A;

(ii) $\int_A (z_a^*, \mu_a^*)\, d\lambda \leq \lim_{k \to \infty} \int_A \left(z_a^k, \mu_a^k\right) d\lambda$; and

(iii) $\overline{\mu}^* = \int_A \mu_a^* d\lambda$.

Define

$$S_0 := \left\{a \in A : (z_a^*, \mu_a^*) \in \zeta\left(a, p^*, q^*\right)\right\}.$$

Claim 1. $\lambda(S_0) > 0$**.** To see this, define

$$\widetilde{X_a} = \{(y_a, \nu_a) \in X_a : u_a(y_a, \nu_a) > u_a(f_a, l_a)\}$$

and

$$X_a^k = \left\{(y_a, \nu_a) \in \widetilde{X_a} : p^k \cdot y_a + \overline{q}^k \cdot \nu_a + p^k \cdot \tau(\nu_a) < p^k \cdot e_a\right\}.$$

Let

$$R := \left\{a \in A : p^* \cdot y_a + q^* \cdot \nu_a + p^* \cdot \tau(\nu_a) < p^*.e_a \text{ for some } (y_a, \nu_a) \in \widetilde{X_a}\right\}.$$

and

$$R_m := \left\{a \in A : \bigcap_{k=m}^{\infty} X_a^k \neq \emptyset\right\}$$

for all $m \geq 1$. From the definition of R_m, it follows that $\{R_m : m \geq 1\}$ is an ascending sequence of coalitions such that

$$R \subseteq \bigcup \{R_m : m \geq 1\}.$$

We know that (f, l) is not a pure transfer equilibrium state under prices (p^*, q^*). Therefore, by Lemma 3.4, we conclude that (f, l) is not a pure transfer quasi-equilibrium state under prices (p^*, q^*). Therefore, we must have $\lambda(R) > 0$. We show that $R \subseteq S_0$. To this end, pick an element $a \in R$. Then there is an $m_0 \geq 1$ such that $a \in R_{m_0}$, which further implies $u_a(\mathbb{D}(a, p^k, \overline{q}^k)) > u_a(f_a, l_a)$ for all $k \geq m_0$. It follows from the definition of ζ^* that $\zeta^*\left(a, p^k, \overline{q}^k\right) = \zeta\left(a, p^k, \overline{q}^k\right)$, which further implies $a \in C_k$ and thus, $(x_a^k, \mu_a^k) \in \mathbb{D}\left(a, p^k, \overline{q}^k\right)$ for all $k \geq m_0$, where $x_a^k := z_a^k + e_a - \tau(\mu_a^k)$. Letting $x_a^* := z_a^* + e_a - \tau(\mu_a^*)$, from (i), we note that

$$(x_a^*, \mu_a^*) \in Ls\left\{\left(x_a^k, \mu_a^k\right) : k \geq 1\right\}.$$

Then there is a subsequence $\{k_r : r \geq 1\}$ of positive integers such that

$$(x_a^*, \mu_a^*) = \lim_{r \to \infty} \left(x_a^{k_r}, \mu_a^{k_r}\right).$$

It can be readily verified that $(x_a^*, \mu_a^*) \in \mathbb{B}\left(a, p^*, q^*\right)$ and it quasi-optimizes the utility over the budget set $\mathbb{B}\left(a, p^*, q^*\right)$. By Lemma 3.4, we conclude that $p \gg 0$ and $(x_a^*, \mu_a^*) \in \mathbb{D}\left(a, p^*, q^*\right)$. Thus, $(z_a^*, \mu_a^*) \in \zeta\left(a, p^*, q^*\right)$ and this is true for all $a \in R$. Consequently, $R \subseteq S_0$.

Claim 2. $(S_0, (x^*, \mu^*))$ **forms a Walrasian objection to** (f, l)**.** To see this, note that

$$\int_{S_0} \mu_a^* d\lambda = \int_A \mu_a^* d\lambda = \overline{\mu}^* \in \mathscr{C}ons$$

and

$$\int_{S_0} z_a^* d\lambda = \int_A z_a^* d\lambda \leq \lim_{k\to\infty} \int_A z_a^k d\lambda = \lim_{k\to\infty} \overline{z}^k = 0.$$

Since $q^* \in \mathcal{T}rans$, we derive that

$$q^* \cdot \int_{S_0} \mu_a^* d\lambda = 0 \text{ and } \int_{S_0} (x_a^* + \tau(\mu_a^*))\, d\lambda \leq \int_{S_0} e_a d\lambda.$$

On the other hand, $p^* \cdot x_a^* + q^* \cdot \mu_a^* + p^* \cdot \tau(\mu_a^*) = p^* \cdot e_a$ for all $a \in S_0$, which immediately yields

$$p^* \cdot \int_{S_0} (x_a^* + \tau(\mu_a^*))\, d\lambda = p^* \cdot \int_{S_0} e_a d\lambda.$$

Since $p^* \gg 0$, we have

$$\int_{S_0} (x_a^* + \tau(\mu_a^*))\, d\lambda = \int_{S_0} e_a d\lambda.$$

Let $a \in S_0$ and choose $(y_a, \nu_a) \in X_a$ such that $u_a(y_a, \nu_a) > u_a(x_a^*, \mu_a^*)$. As a consequence of the definition of demand, we have $p^*.y_a + q^*.\nu_a + p^* \cdot \tau(\nu_a) > p^*.e_a$. We now assume $a \in A \setminus S_0$ and choose any $(y_a, \nu_a) \in X_a$ such that $u_a(y_a, \nu_a) > u_a(f_a^*, l_a^*)$. From the fact that $a \notin S_0$, it follows that $u_a(f_a, l_a) \geq u_a(\mathbb{D}(a, p^*, q^*))$ Consequently, $u_a(y_a, \nu_a) > u_a(\mathbb{D}(a, p^*, q^*))$. Thus, $p^*.y_a + q^*.\nu_a + p^* \cdot \tau(\nu_a) > p^*.e_a$. Thus, we have established that $(S_0, (x^*, \mu^*))$ is a pure transfer Walrasian objection to (f, l). It can be further inferred from Lemma 3.9 that there exists a price $\widetilde{q}^* \in \mathbb{R}^{\mathscr{M}}$ such that $(S_0, (x^*, \mu^*))$ constitutes a Walrasian objection to (f, l) under price $(p^*, \widetilde{q}^*)$. □

Theorem 4.3. *Let $\mathscr{E}$ be a club economy for which endowments are desirable. Further, let $\mathscr{E}$ be club linked. Then the bargaining set of the economy $\mathscr{E}$ coincides with the set of club equilibrium states.*

Proof. We first intend to show that $\mathscr{W}(\mathscr{E}) \subseteq \mathscr{B}(\mathscr{E})$. This is quite obvious. From the definition of bargaining set, it follows that the core of the economy $\mathscr{C}(\mathscr{E}) \subseteq B(\mathscr{E})$. It can be easily claimed that $\mathscr{W}(\mathscr{E}) \subseteq \mathscr{C}(\mathscr{E})$. Hence, it automatically follows that $\mathscr{W}(\mathscr{E}) \subseteq B(\mathscr{E})$. For the reverse direction, we use the contrapositive argument. So let $(f, l) \notin \mathscr{W}(\mathscr{E})$. Then from Proposition 4.2 we conclude that there exists a Walrasian objection $(S, (g, \mu))$ against it. We claim that $(S, (g, \mu))$ is a justified objection. If not, then there exists a counter-objection $(T, (h, \nu))$ to $(S, (g, \mu))$ such that

(i) $\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$;

(ii) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ for all $a \in T \cap S$;

(iii) $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ for all $a \in T \setminus S$; and

(iv) $\int_T \nu_a d\lambda \in \mathscr{C}ons$.

From (i), one can immediately obtain that

$$\int_T p \cdot h_a d\lambda + \int_T p \cdot \tau(\nu_a) d\lambda = \int_T p \cdot e_a d\lambda. \tag{4.5}$$

From our definition of $\tau(\cdot)$, we have the following expression

$$p \cdot \tau(\nu_a) = \frac{1}{\|\pi\|_1} \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} p \cdot inp(\pi,\gamma)\nu_a(\omega,\pi,\gamma).$$

From Definition 2.3 and **budget balance for club types** condition in Definition 3.1, it follows that

$$p \cdot \int_T \tau(\nu_a) d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{\omega\in\Omega} \psi(\pi,\gamma)\pi(\omega)q(\omega,\pi,\gamma)$$

We can simplify further to obtain

$$p \cdot \int_T \tau(\nu_a) d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \psi(\pi,\gamma)\pi(\omega)q(\omega,\pi,\gamma)$$

Again from Definition 2.3 we can claim that

$$p \cdot \int_T \tau(\nu_a) d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \int_T \nu_a(\omega,\pi,\gamma) \cdot q(\omega,\pi,\gamma) d\lambda.$$

It can be further noted that the above equation simplifies to

$$p \cdot \int_T \tau(\nu_a) d\lambda = \int_T \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \nu_a(\omega,\pi,\gamma) \cdot q(\omega,\pi,\gamma) d\lambda.$$

Thus, we get that

$$p \cdot \int_T \tau(\nu_a) d\lambda = \int_T q \cdot \nu_a d\lambda.$$

Thus, Equation (4.5) can be re-written as

$$\int_T p \cdot h_a d\lambda + \int_T q \cdot \nu_a d\lambda = \int_T p \cdot e_a d\lambda$$

This is a contradiction as $(S,(g,\mu))$ is a Walrasian objection. Hence, it follows that $(S,(g,\mu))$ is a justified objection to (f,l), which further implies that $(f,l) \notin \mathscr{B}(\mathscr{E})$. This completes the proof.

□

5 Conclusion

The bargaining set was introduced by Maschler [20] as a cooperative solution concept advantageous over the core. Mas-Colell [22]extended the definition of bargaining set to the case of market economies. He adopted the large economy framework of Aumann [2] and showed that for such economies the bargaining set coincides with the set of equilibrium allocations. In other words, objections to equilibrium allocations that are frivolous are not formed in the economy. We extend Mas-Colell's seminal result in a club economy framework where club goods are treated as articles of choice and club formation is endogenous. We adapt the model proposed by Ellickson et al. [10] and introduce the two-step veto mechanism of objection and counter-objection. We extend the concepts of justified and Walrasian objection to our framework. Our main result shows that if a feasible state in the economy $\mathscr{E}$ is not a club equilibrium state, then there must exist some Walrasian objection against it. In other words, we show that there exists a coalition of agents who will be willing to trade among themselves at the given price to achieve a mutually beneficial outcome for themselves. Finally, by virtue of Walrasian objection being justified, we can immediately claim the equivalence between club equilibrium states and the bargaining set for a club economy.

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