Estimation of Zellner-Revankar Production Function Revisited

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10 December 2006

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Introduction: Arnold Zellner and Nagesh Revankar in their well-known paper "Generalized Production Functions" [Zellner and Revankar, 1969] introduced a new production function, which was illustrated by an example specified as:

$$V \exp(\theta V) = \gamma K^{\alpha(1-\delta)} L^{\alpha\delta} \;:\; 0 < \delta < 1;\; \gamma > 0;\; \alpha > 0. \qquad \ldots (1)$$

where, V, K, L stand for output, capital and labour. The parameters α, δ, $(1-\delta)$ and γ relate to the parameters of returns to scale, output elasticities with respect to labour and capital and efficiency. The parameter θ attribute to other parameters the scale variability character and thus makes the function specified above "general". In particular, for $\theta = 0$ the Zellner-Revankar production function (ZRPF) degenerates into the simple Cobb-Douglas production function. The returns to scale function obtained from the ZRPF is given as $\alpha(V) = \alpha/(1+\theta V)$ that changes with the volume of output.

Estimation of ZRPF: Now we present the Zellner-Revankar method of estimation of the ZRPF parameters. Let us have sample data on output, capital and labour in n observations. Introducing multiplicative random error and log-transforming we have

$$\log(V_i) + \theta V_i = \log(\gamma) + \alpha(1-\delta)\log(K_i) + \delta \log(L_i) + u_i \;: i = 1, 2, ..., \; n \qquad \ldots (2)$$

where u_i 's are random errors, normally and independently distributed, each with mean zero and common variance σ^2. It is also assumed that $\log(K_i)$ and $\log(L_i)$ are distributed independently of the error term, u_i, or they are fixed quantities. Then, the logarithm of the likelihood function, $\log(l)$, is:

$$\log(l) = const. - \tfrac{n}{2}\log(\sigma^2) + \log(J) - \tfrac{1}{2\sigma^2} \sum_{i=1}^{n} \{z_i(\theta) - c_0 - c_1 \log(K_i) - c_2 \log(L_i)\}^2 \qquad \ldots (3)$$

where $z_i(\theta) = \log(V_i) + \theta V_i;\; c_0 = \log(\gamma);\; c_1 = \alpha(1-\delta),\; c_2 = \alpha\delta$ and J is the Jacobian of the transformation from u_i 's to the V_i 's, or

$$J = \prod_{i=1}^{n} \frac{\partial u_i}{\partial V_i} = \prod_{i=1}^{n} \left[\frac{1+\theta V_i}{V_i} \right] \qquad \ldots (4)$$

Now, substituting from (4) in (3) we get

$$\log(l) = const. - \tfrac{n}{2}\log(\sigma^2) + \sum_{i=1}^{n} \log(1+\theta V_i) - \tfrac{1}{2\sigma^2} \sum_{i=1}^{n} \{z_i(\theta) - c_0 - c_1 \log(K_i) - c_2 \log(L_i)\}^2 \qquad \ldots (5)$$

Differentiating (5) partially with respect to σ^2 and setting the derivatives equal to zero we obtain

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \{z_i(\theta) - c_0 - c_1 \log(K_i) - c_2 \log(L_i)\}^2 \qquad \ldots (6)$$

as the conditional maximizing value for σ^2. When $\hat{\sigma}^2$ in (6) is substituted for σ^2 in (5), we obtain

$$\log(l^*) = const. - \frac{n}{2} \log\left[\sum_{i=1}^{n} \{z_i(\theta) - c_0 - c_1 \log(K_i) - c_2 \log(L_i)\}^2 \right] + \sum_{i=1}^{n} \log(1+\theta V_i) \qquad \ldots (7)$$

Now, for any given value of $\theta = \theta_0$, the conditional maximizing values of c_0, c_1 and c_2 may be obtained by regression of $z_i(\theta_0)$ on the explanatory variables, $\log(K_i)$ and $\log(L_i)$ by minimizing

$$\sum_{i=1}^{n} \{z_i(\theta) - c_0 - c_1 \log(K_i) - c_2 \log(L_i)\}^2 \qquad \ldots (8)$$

Minimization of (8) can be done with different trial values of θ, say, $\theta_1, \theta_2, \theta_3, \ldots$ such that we find out the best values of (θ, c_0, c_1, c_2) that obtains the global optimum of the likelihood function in (7). Zellner and Revankar mention that this procedure of maximizing the likelihood function is similar to the procedure described by Box and Cox (1963). *This procedure of estimation will be examined and revisited in this paper.*

Table-A: 1957 U.S. Annual Survey of Manufactures Data for the Transportation Equipment Industry

State	Aggregate value added, V_a	Aggregate capital service flow, K_a	Aggregate man-hours worked, L_a	No. of establishments, N
	(Millions of dollars)		(Millions of Man-hours)	
Alabama	126.148	3.804	31.551	68
California	3201.486	185.446	452.844	1372
Connecticut	690.670	39.712	124.074	154
Florida	56.296	6.547	19.181	292
Georgia	304.531	11.530	45.534	71
Illinois	723.028	58.987	88.391	275
Indiana	992.169	112.884	148.530	260
Iowa	35.796	2.698	8.017	75
Kansas	494.515	10.360	86.189	76
Kentucky	124.948	5.213	12.000	31
Louisiana	73.328	3.763	15.900	115
Maine	29.467	1.967	6.470	81
Maryland	415.262	17.546	69.342	129
Massachusetts	241.530	15.347	39.416	172
Michigan	4079.554	435.105	490.384	568
Missouri	652.085	32.840	84.831	125
New Jersey	667.113	33.292	83.033	247
New York	940.430	72.974	190.094	461
Ohio	1611.899	157.978	259.916	363
Pennsylvania	617.579	34.324	98.152	233
Texas	527.413	22.736	109.728	308
Virginia	174.394	7.173	31.301	85
Washington	636.948	30.807	87.963	179
West Virginia	22.700	1.543	4.063	15
Wisconsin	349.711	22.001	52.818	142

Source: Zellner, A and Revankar, N.S. (1969), p. 249.

Zellner and Revankar apply this procedure for estimating the optimal values of θ, c_0, c_1, c_2, the parameters of the ZRPF, for the U.S. Transportation Equipment Industry. The data used by them have been presented in their paper (reproduced here in Table-A). They measure output (net value added), capital and labour per unit of establishment, that is, $V_i = (V_{a,i} / N_i)$; $K_i = (K_{a,i} / N_i)$; $L_i = (L_{a,i} / N_i)$. They obtain:

$$(\hat{\theta}, \hat{c}_0, \hat{c}_1, \hat{c}_2) = (0.134, 3.0129, 0.3330, 1.1551) \qquad \ldots (9)$$

and, since the estimate of returns to scale parameter, $\hat{\alpha} = \hat{c}_1 + \hat{c}_2$, they also obtain for each state of the U.S. $Est.\ \alpha(V_i) = (\hat{c}_1 + \hat{c}_2)/(1+\hat{\theta}V_i) = 1.49/(1+0.134V_i)$ *approx.* According to their estimates, Indiana, Kentucky, Georgia, Ohio, Connecticut, Missouri, Kansas and Michigan exhibit decreasing returns ($\hat{\alpha}(V)$ decreasing in that order); Illinois, Pennsylvania, New Jersey, Maryland and Washington show $1 < \hat{\alpha} \le 1.1$ while other states have $\hat{\alpha} > 1.1$. Florida has the highest value of $\hat{\alpha} = 1.45$ (see Table-E).

Table-B: 1957 U.S. Annual Survey of Manufactures Data for the Transportation Equipment Industry
[Per establishment]

State	Aggregate value added, V	Aggregate capital service flow, K	Aggregate man-hours worked, L	Figures rounded off at the third place after Decimal		
	a *	b *	c **	a *	b *	c **
Alabama	1.855117647	0.055941176	0.463985294	1.855	0.056	0.464
California	2.333444606	0.135164723	0.330061224	2.333	0.135	0.330
Connecticut	4.484870130	0.257870130	0.805675325	4.485	0.258	0.806
Florida	0.192794521	0.022421233	0.065688356	0.193	0.022	0.066
Georgia	4.289169014	0.162394366	0.641323944	4.289	0.162	0.641
Illinois	2.629192727	0.214498182	0.321421818	2.629	0.214	0.321
Indiana	3.816034615	0.434169231	0.571269231	3.816	0.434	0.571
Iowa	0.477280000	0.035973333	0.106893333	0.477	0.036	0.107
Kansas	6.506776316	0.136315789	1.134065789	6.507	0.136	1.134
Kentucky	4.030580645	0.168161290	0.387096774	4.031	0.168	0.387
Louisiana	0.637634783	0.032721739	0.138260870	0.638	0.033	0.138
Maine	0.363790123	0.024283951	0.079876543	0.364	0.024	0.080
Maryland	3.219085271	0.136015504	0.537534884	3.219	0.136	0.538
Massachusetts	1.404244186	0.089226744	0.229162791	1.404	0.089	0.229
Michigan	7.182313380	0.766029930	0.863352113	7.182	0.766	0.863
Missouri	5.216680000	0.262720000	0.678648000	5.217	0.263	0.679
New Jersey	2.700862348	0.134785425	0.336165992	2.701	0.135	0.336
New York	2.039978308	0.158295011	0.412351410	2.040	0.158	0.412
Ohio	4.440493113	0.435201102	0.716022039	4.440	0.435	0.716
Pennsylvania	2.650553648	0.147313305	0.421253219	2.651	0.147	0.421
Texas	1.712379870	0.073818182	0.356259740	1.712	0.074	0.356
Virginia	2.051694118	0.084388235	0.368247059	2.052	0.084	0.368
Washington	3.558368715	0.172106145	0.491413408	3.558	0.172	0.491
West Virginia	1.513333333	0.102866667	0.270866667	1.513	0.103	0.271
Wisconsin	2.462753521	0.154936620	0.371957746	2.463	0.155	0.372

Computed from Table-A (the last three cols. are the rounded off figures in the 2nd through 4th cols.)
* In millions of dollars per establishment; ** In millions of man-hours per establishment

The Objective of this Paper: We intend to demonstrate here that the estimates of parameters of ZRPF as reported by Zellner and Revankar in their paper are somewhat sub-optimal, that is: $(\hat{\theta},\ \hat{c}_0,\ \hat{c}_1,\ \hat{c}_2) = (0.134,\ 3.0129,\ 0.3330,\ 1.1551)$ do not quite maximize the likelihood function. However, that is so due to the trial and error method used by ZR in which a trial value of θ is chosen, and c_i's are estimated by minimization of (8). This is done repeatedly for different trial values of θ so as to maximize the likelihood function.

In this paper, we use two methods of global optimization, the Particle Swarm [Eberhart and Kennedy, 1995] and Differential Evolution [Storn and Price, 1995] methods, to minimize (8) in which θ, c_0, c_1, c_2 are estimated together. This approach frees us from the risk of obtaining a sub-optimal set of estimated parameters of ZRPF.

Our Estimates by the Methods of Global Optimization: We present here two sets of estimates of the parameters of ZRPF: the one based on highly accurate values of V_i, K_i and L_i (presented in Table-B, 2nd to 4th columns) and the other when these variables are measured with values correct only up to two places after decimal (rounded off at the third place after decimal). We do not know of the accuracy level of the original computations (done by Zellner and Revankar).

Table-C: Estimated Parameters of ZRPF for U.S. Transport Equipment Industry

Accuracy	Method	$\hat{c}_0$	$\hat{c}_1$	$\hat{c}_2$	$\hat{\theta}$	SSQD	(l*)
Low Accuracy (LA)	**Zellner-Revankar**	**3.0129**	**0.3330**	**1.1551**	**0.134**	**1.2016#**	**5.4790**
	Differential Evaln	2.91527	0.352646	1.087540	0.106441	1.0689	5.5769
	R Particle Swarm	2.91476	0.350784	1.090654	0.106506	1.0691	5.5773
High Accuracy (HA)	**Zellner-Revankar**	**3.0129**	**0.3330**	**1.1551**	**0.134**	**1.2118#**	**5.4945**
	Differential Evaln	2.91161	0.350226	1.090161	0.106184	1.0665	5.5917
	R Particle Swarm	2.91587	0.350255	1.092447	0.106811	1.0692	5.5918

SSQD = Sum of Squared Deviations; # = Computed by us; l* = Log Max Likelihood

Table-D: 1957 U.S. Transportation Equipment Industry Value Added [Per Establishment]
Zellner-Revankar Production Function Estimated by Different Methods

	V (Emp)	$V(DE)_{LA}$	$V(RPS)_{LA}$	$V(ZR)_{LA}$	$V(DE)_{HA}$	$V(RPS)_{HA}$	$V(ZR)_{HA}$
Florida	0.193	0.245	0.244	0.245	0.245	0.244	0.241
Maine	0.364	0.306	0.305	0.306	0.306	0.305	0.303
Iowa	0.477	0.478	0.477	0.478	0.477	0.476	0.477
Louisiana	0.638	0.601	0.601	0.601	0.600	0.600	0.608
Massachusetts	1.404	1.363	1.362	1.363	1.363	1.363	1.375
West Virginia	1.513	1.704	1.703	1.704	1.700	1.700	1.723
Texas	1.712	1.997	1.999	1.997	1.998	1.999	2.061
Alabama	1.855	2.378	2.384	2.378	2.381	2.384	2.502
New York	2.040	2.954	2.954	2.954	2.956	2.958	3.012
Virginia	2.052	2.088	2.090	2.088	2.093	2.095	2.140
California	2.333	2.128	2.127	2.128	2.127	2.127	2.124
Wisconsin	2.463	2.510	2.509	2.510	2.507	2.508	2.508
Illinois	2.629	2.354	2.350	2.354	2.354	2.354	2.309
Pennsylvania	2.651	2.762	2.763	2.762	2.765	2.766	2.777
New Jersey	2.701	2.087	2.086	2.087	2.084	2.084	2.063
Maryland	3.219	3.303	3.307	3.303	3.301	3.304	3.321
Washington	3.558	3.134	3.135	3.134	3.136	3.137	3.094
Indiana	3.816	4.979	4.975	4.979	4.972	4.975	4.840
Kentucky	4.031	2.281	2.280	2.281	2.281	2.280	2.188
Georgia	4.289	3.793	3.798	3.793	3.801	3.803	3.742
Ohio	4.440	5.964	5.963	5.964	5.957	5.961	5.783
Connecticut	4.485	5.616	5.622	5.616	5.614	5.619	5.535
Missouri	5.217	4.340	4.342	4.340	4.336	4.336	4.141
Kansas	6.507	5.238	5.254	5.238	5.259	5.261	5.067
Michigan	7.182	6.663	6.657	6.663	6.655	6.652	6.001

Fig-I: 1957 U.S. Transportation Equipment Industry Value Added [Per Establishment]
Zellner-Revankar Production Function Estimated by Different Methods (Low Accuracy)

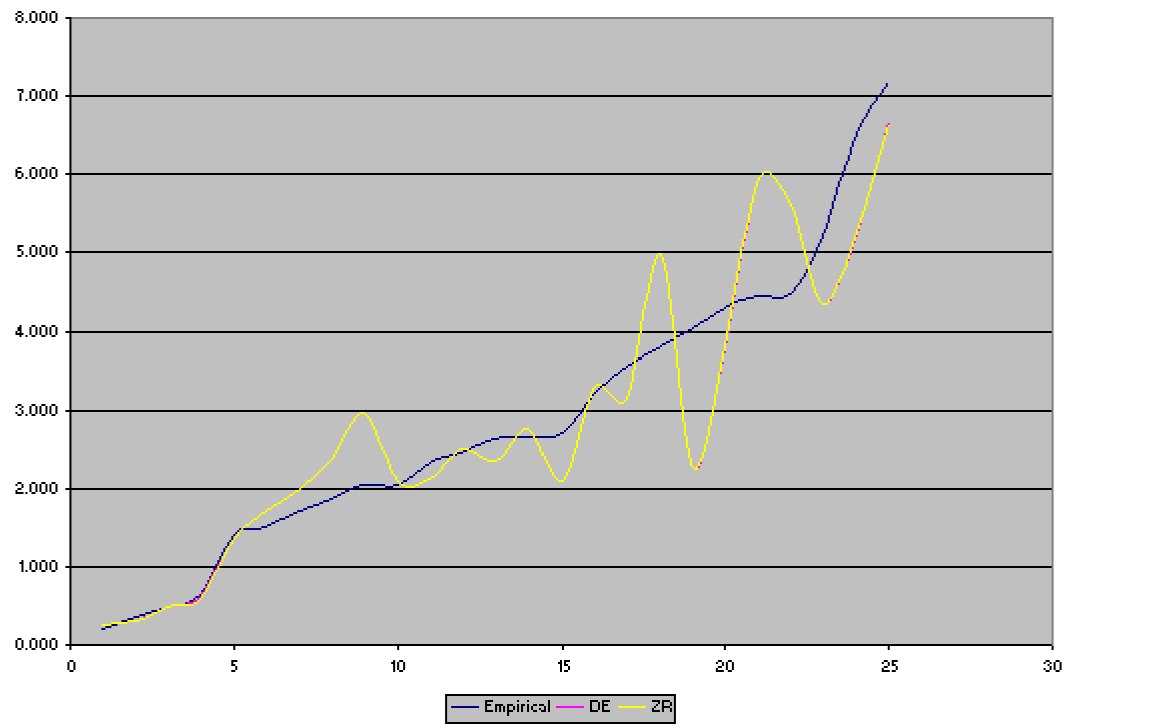
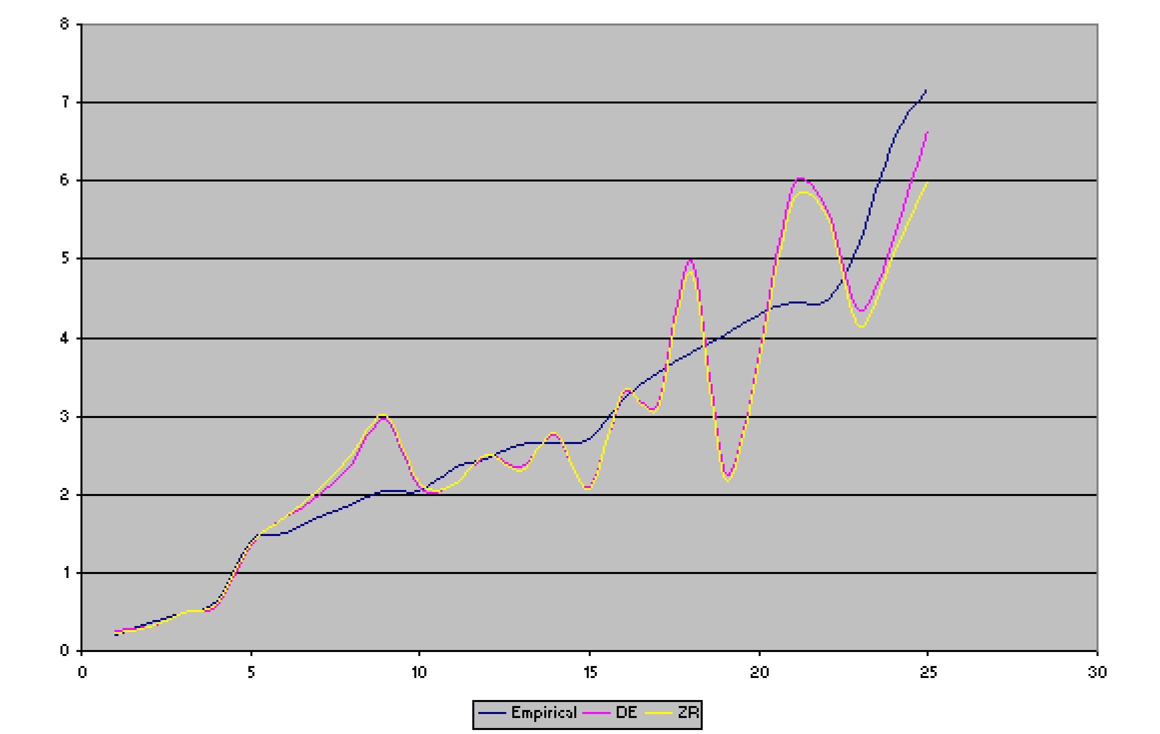


Fig-II: 1957 U.S. Transportation Equipment Industry Value Added [Per Establishment]
Zellner-Revankar Production Function Estimated by Different Methods (High Accuracy)



As it has been shown in Table-C, first, there is no significant difference in the values of estimated parameters (of ZRPF) due to accuracy in computation. HA and LA estimates are more or less same. Secondly, there is no significant difference between the estimated parameters obtained by DE (Differential Evolution) and RPS (Repulsive Particle Swarm). However, the Zellner-Revankar estimates of parameters are quite different from those obtained by the methods of global optimization (DE and RPS). The SSQD (sum of squared deviations) of ZR is larger (and l^* is smaller) than those of DE and RPS. It shows very clearly that the ZR estimates are somewhat sub-optimal. This sub-optimality of ZR estimates may clearly be appreciated by a perusal of Fig-II, although the difference is not observable in Fig.-I. We have arranged the U.S. states in an ascending order of value added per establishment (V) and plotted against each state the observed (empirical) and expected values of V obtained by different methods of estimation. The graphs (that should not ideally have been drawn as curves, since the points on the x axis are discrete) are drawn continuous only to facilitate the visualization of differences between ZR-estimated and DE/RPS-estimated values of V. We observe in Fig-II that DE/RPS estimates are closer to the empirical curve for a majority of points. It appears that ZR computations used rounded off numbers of V, K and L.

Two points deserve a special mention. First, the returns-to-scale parameter, $\hat{\alpha} = \hat{c}_1 + \hat{c}_2$ obtained by DE/RPS method is 1.44 approx, against 1.488 obtained by the ZR estimation. Further, the value of $\hat{\theta}$ obtained by DE/RPS is about 0.106, while it is 0.134 obtained by ZR. A consequence of all these changes is that $\alpha(V_i)$ values for different states are different from those obtained by ZR method. The estimates of $\alpha(V_i)$ are presented in Table-E.

Fig.-III. A Graph of SSQD and Log Max Likelihood With Different Values of Theta

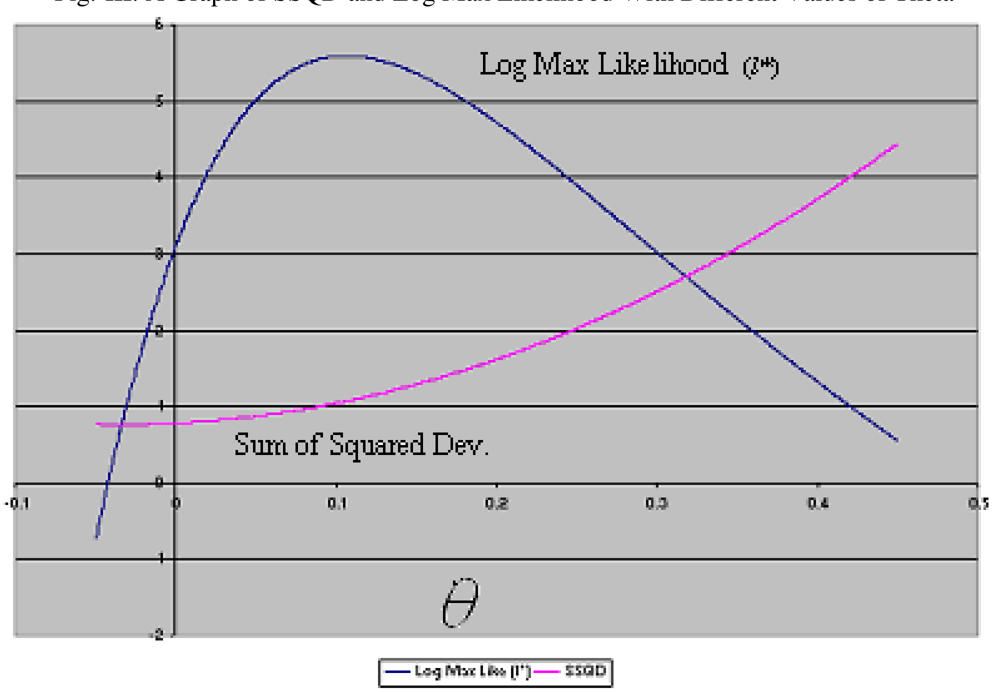


Table-E. Estimated Variable Returns to Scale by Zellner Revankar Method of Estimation							
State	V	Est $\alpha(V)$		State	V	Est . $\alpha(V)$	
		ZR*	DE/RPS			ZR*	DE/RPS
Florida	0.193	1.45	1.41	Pennsylvania	2.651	1.10	1.12
Maine	0.364	1.42	1.39	New Jersey	2.701	1.09	1.12
Iowa	0.477	1.40	1.37	Maryland	3.219	1.04	1.07
Louisiana	0.638	1.37	1.35	Washington	3.558	1.01	1.05
Massachusetts	1.404	1.25	1.25	Indiana	3.816	0.98	1.03
West Virginia	1.513	1.24	1.24	Kentucky	4.031	0.97	1.01
Texas	1.712	1.21	1.22	Georgia	4.289	0.94	0.99
Alabama	1.855	1.19	1.20	Ohio	4.44	0.93	0.98
New York	2.04	1.17	1.18	Connecticut	4.485	0.93	0.98
Virginia	2.052	1.17	1.18	Missouri	5.217	0.88	0.93
California	2.333	1.13	1.15	Kansas	6.507	0.80	0.85
Wisconsin	2.463	1.12	1.14	Michigan	7.182	0.76	0.82
Illinois	2.629	1.10	1.13	* Source: Zellner & Revankar (1969), p. 248			

Concluding Remarks: Zellner-Revankar's paper made two contributions: first, it generalized the production function to allow for the parameters to vary according to the scale of output and secondly it contributed a method to estimate such parameters by the maximum likelihood method. This paper has only an appreciation for the first contribution, but it has shown that the method of estimation (suggested by ZR) is neither convenient nor accurate. It gives us only a local optimum, *not the global optimum*, of the likelihood function. This observation may not sound very impressive when a simple function like Cobb-Douglas's is generalized, but it may be very important if the basic function is intrinsically nonlinear. It is understandable that at the time when the ZR paper was written, there were no effective methods to find global optima of nonlinear functions, especially those with numerous local optima. Now that very effective methods of global optimization have been found, it would be appropriate to estimate the parameters of ZRPF by those advance methods. Our present paper has made a modest attempt to that effect. Using such global optimization methods, we have estimated other nonlinear production functions [Sato's two-level CES and LINEX functions; Mishra, 2006(b)] as well. We have found that the performance of these methods is much better than that of the classical methods of estimation of nonlinear functions [Mishra, 2006(a)].

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