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Uncovering seeds*

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Abstract

We provide the theoretical foundations for a new estimation algorithm that non-parametrically infers higher-order beliefs in generalized guessing games with heterogeneous interactions within an extended level-k framework. The algorithm takes the strategic dependencies of the game and subjects' choices as input and returns a detailed histogram (a "pseudo-spectrogram") of *seeds* that represent population beliefs about the behavior of level-0 players. As a by-product, the algorithm also returns the estimated population composition of reasoning levels. Estimating individual seeds in a highly parametrized model without requiring strong distributional assumptions is made possible by incorporating the game-theoretical structure into a mixture model.

The main contributions are as follows. First, we study the equilibrium properties of generalized guessing games and provide an ordinal (visual) characterization for uniqueness. Second, within the level-k model, our key theoretical results establish conditions on the subjective beliefs or the game structure so that the population distributions of level-k choices and the population distribution of seeds are alike. These results are obtained without any distributional assumptions on the seeds. We also present a central limit result that supports the use of parametric gaussian approaches often used in the literature. Third, on the basis of the theoretical results, we construct a new non-parametric maximum likelihood estimation algorithm that fully identifies the seed pattern. Fourth, we apply the algorithm to existing experimental data. It is found that seeds cluster around a few focal points and that a few seeds are able to explain a high percentage of observed behavior. Finally, our theoretical results can also be useful in the design of laboratory guessing games with good estimation properties.

Keywords: beliefs, expectation-maximization algorithm, level-k, maximum-likelihood estimation, mixture model, networks, centrality.

1 Introduction

Motivating example

Using an extended level-k approach, this paper analyzes the microfoundations of a non-parametric method for estimating beliefs about level-0 behavior in generalized guessing games from laboratory choices.[1,2] As a simple example, consider a large population of experimental subjects anonymously arranged into different groups of n players who face the following simultaneous move game. Each player $i \in \{1, 2, \ldots, n\}$ in a group has to choose a real number $x_i \in [0, 100]$. The utility of player i for a given profile of choices $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ within her group is $u_i(\mathbf{x}) = -(x_i - t_i)^2$, where

$$t_i = p \cdot \frac{\sum_{j=1}^{n} x_j}{n}$$

is said to be the target of player i. We set $p = 2/3$ in the example. This value is well-known to result in a unique Nash equilibrium: $\mathbf{x}^* = (0, 0, \ldots, 0)$.

Our analysis allows for more general games in the sense that targets may be asymmetric (*i.e.*, defined by a network) and may include exogenously given anchors (*i.e.*, constants like 50 or 80). We call this class of games *anchored guessing games*. Proposition 1 shows that an anchored guessing game has a unique Nash equilibrium whenever the dominant eigenvalue of the $n \times n$ matrix that

[1]Prior research on level-k reasoning has typically employed models with initial belief distributions that are fixed or parametric. In contrast, this paper develops a more flexible approach by allowing for free subjective beliefs as targets for estimation, thus enabling a more nuanced analysis that is focused on beliefs rather than reasoning levels. Nagel (1995) and Stahl and Wilson (1994, 1995) are the seminal contributions to the level-k model. See, Crawford, Costa-Gomes, and Iriberri (2013) for a more recent literature overview and Camerer, Ho and Chong (2004) for cognitive hierarchies.

[2]The reader can use the algorithm with their own dataset on any computer or mobile device via the Javascript notebook titled "Uncovering seeds," which is publicly available at `https://observablehq.com/@coballester`.

represents the strategic dependencies is less than 1.[3] In the example, the entries of this dependency matrix are all equal to p/n, and the condition of Proposition 1 is satisfied since this eigenvalue is precisely $p = 2/3$.

Next, we derive the optimal choices in the extended level-k framework when players have subjective beliefs about how all players in the group would behave if they were level-0 players (see, Ho, Camerer and Weigelt 1998 and Burchardi and Penczynski 2014). In the motivating example:

- A level-0 player chooses a number from $[0, 100]$ (not necessarily uniformly at random).
- A level-1 player maximizes utility given her subjective beliefs about level-0 behavior. Consider the level-1 subject Alice (a). Her optimal choice is

  $$x^{1(a)} = \frac{2}{3} \cdot e^{(a)},$$

  where $e^{(a)} \in [0, 100]$ is Alice's subjective belief (expectation) about the average choice of the n players in her group when they are all level-0. We refer to $e^{(a)}$ as Alice's *seed*.
- A level-2 player views all other players as level-1. For instance, suppose that Bob (b) is a level-2 subject. His optimal choice is

  $$x^{2(b)} = \left(\frac{2}{3}\right)^2 \cdot e^{(b)},$$

  where $e^{(b)}$ is Bob's seed over which he iterates twice. Observe that the seeds are allowed to be heterogeneous between-subjects; *i.e*, $e^{(a)}$ and $e^{(b)}$ may differ.

[3] This type of result is standard in the literature on network economics. See, for instance, Ballester, Calvó-Armengol and Zenou (2006), Bramoullé and Kranton (2007), Acemoğlu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012), Elliot and Golub (2020), and Galeotti, Golub and Goyal (2020).

- And so forth for all level-k players. In general, the optimal choice of a level-k subject s in this example is $x^{k(s)} = (2/3)^k \cdot e^{(s)}$ and it falls into the interval $[0, (2/3)^k 100]$.

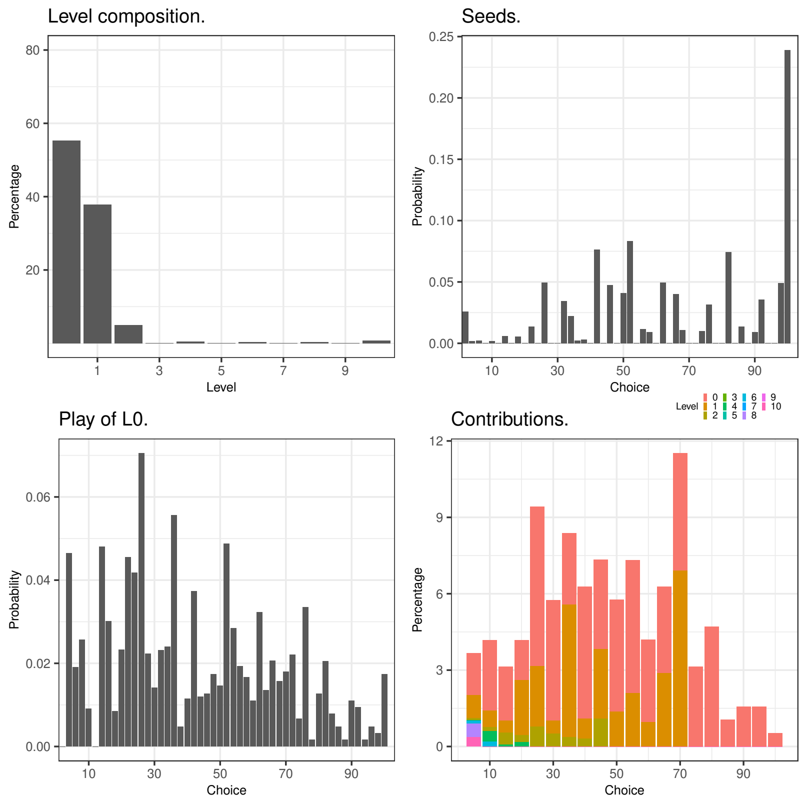


Figure 1: Estimation outcome for Brañas-Garza *et al.* (2012). Top-left: estimated distribution of reasoning levels. Top-right: estimated distribution of seeds. Bottom-left: estimated destribution of level-0 play. Bottom-right: stacked histogram of choices by estimated reasoning level ($p = 2/3$). Observations: 1146.

Our objective is to estimate the unknown distribution of seeds from observed laboratory choices without making any distributional assumptions. As an illustration, we show in Figure 1 our estimation outcome for the pooled data (over six treatments of p) of Brañas-Garza, García-Muñoz and Hernán González (2012).[4] The main output of the estimation is the top-right panel which

[4]Brañas-Garza, García-Muñoz and Hernán González (2012) analyze how personal characteristics such as attention, proxied by performance in the cognitive reflection test, and visual reasoning, measured by Raven's progressive

displays the estimated seed distribution. It consists of various peaks (notably: 100, 50, 40, and 80), each of which can be interpreted as a focal point for the starting seed of the level-k reasoning process.[5] For example, 24% of the choices associated with a strictly positive reasoning level have a seed at 100. If $p = 2/3$, the level-2 play for this seed is $(2/3)^2 \cdot 100 = 44.44$. The top-left panel presents the estimated distribution of reasoning levels. It turns out that 55% of the guesses are level-0 and 38% level-1 play, but a few choices have a rather high reasoning level as well. Since the four above-mentioned seeds of 100, 50, 40, and 80 alone account for 47% of all seeds of subjects with a non-zero reasoning level and since the estimated fraction of level-0 subjects is 55%, these seeds explain $47\% \cdot 45\% = 21\%$ of all choices. The bottom-left panel shows the estimated level-0 choices. These are the choices that are not generated from the estimated seeds, that is, they are not generated by a level-k iterative process. Finally, the bottom-right panel analyzes the degree to which each reasoning level explains the data. Because of the structure imposed by the level-k model on the estimation, the algorithm produces a non-trivial output in the sense that it does not necessarily assign each choice to its maximum compatible reasoning level. For instance, in the first bar of this histogram –the interval [0,5)– the two lowest reasoning levels explain about 70% of these choices, while levels of 8 and beyond explain about 30%.

With this brief illustration the reader should have a clear idea about the objectives and the output of the estimation procedure. In Section 5, we apply our approach to the data of the

matrices, affect behavior in beauty contest games. In a slight variation to our setting, in beauty contest games only the player whose guess is closest to the target receives a strictly positive payoff.

[5]There are at least three different sources for seeds. First, a seed might be natural or intrinsic to the subject or group (like "popular" numbers). Second, it might be part of the description of the game. In our anchored guessing games, constants may naturally arise as focal points. Third, focal points may also emerge through framing effects.

newspaper experiments of Bosch-Domènech, Montalvo, Nagel and Satorra (2002) and to one of the anchored guessing games of Ballester, Rodriguez-Moral and Vorsatz (2022). We stress the fact that Section 5 is only an illustration of our estimation technique which is applied to datasets that probably come from low-power experimental designs, *i.e.*, designs that cannot discriminate among the possibly many behavioral rules subjects may be using.

Main results

To detail our estimation technique assume that choices from a large experimental population have been collected in our example. The random variable $\tilde{e}$, whose distribution has to be estimated, represents the population seeds. This random variable produces, through the reduced-form equations of the example game, the random variable $\tilde{x}^k$ of choices made by the level-k population: $\tilde{x}^k = (2/3)^k \cdot \tilde{e}$, for $k \geq 1$, The central econometric building block of our estimation is a *mixture model* that consists of

1. **Level composition**: $p_0, p_1, \ldots, p_K$ are the unknown fractions of each reasoning level. The maximum reasoning level is set to K.

2. **Density functions**: $f^0, f^1, \ldots, f^K$ are the unknown density functions of $\tilde{x}^0, \tilde{x}^1, \ldots, \tilde{x}^K$, and they are discretized by means of an arbitrary number of rectangular buckets.

The main idea of the mixture model is that the density function of the observed data is a convex combination of the densities $f^0, f^1, \ldots, f^K$ with weights $p_0, p_1, \ldots, p_K$ corresponding to the $K+1$ mixture components (*i.e.*, reasoning levels). The mixture model can then be solved by using maximum likelihood estimation techniques. The key challenge is to introduce structural restrictions into the mixture model. One possible approach would be to consider the full set of reduced-form equations. But since the mixture model then becomes hard to implement and computationally intractable for general anchored guessing games, we instead impose well-founded structural conditions that approximate the reduced-form equations. This leads to an easy implementation and a quick and consistent estimation. More formally, Theorems 1 and 2 allow us to assume in the estimation that the random variables $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^K$ have the *same shape* as the seeds $\tilde{e}$, that is, $\tilde{x}^k$ is an affine transformation of $\tilde{e}$ for $k \geq 1$. As a consequence, the mixture model yields estimates for f^0 and the density function of $\tilde{e}$. This means that we are reducing the effective number of mixture components to be estimated from $K + 1$ to 2.

The key assumption in Theorems 1 and 2 is that for all reasoning levels $k \geq 1$, $\tilde{e}$ is independent of k, *i.e.*, Alice and Bob may have different beliefs about level-0 behavior for reasons that are not related with their respective reasoning levels. To the best of our knowledge, the vast majority of level-k models implicitly operate under this level-independence assumption. Theorem 1 establishes that if beliefs are homogeneous *within* subjects (*i.e.*, each subject holds identical beliefs about all level-0 players in her group), then each random variable of optimal choices $\tilde{x}^k$, $k \geq 1$, has exactly the same shape as $\tilde{e}$. Closest to Theorem 1 is Burchardi and Penczynski (2014) who estimate beliefs in the beauty contest game under the assumption that beliefs are homogenous within subjects and follow a normal distribution. Our result is distribution-free and applies to the more general class of anchored guessing games. Second, Theorem 2 obtains a weaker implication –the shape of $\tilde{x}^k$ approximates the shape of $\tilde{e}$– in exchange for dropping the assumption of homogenous beliefs within subjects. This approximation also turns out to be better in anchored guessing games with a lower *eigenvalue ratio*.[6] Consequently, if one cannot rely on belief homogeneity within subjects required

[6]The eigenvalue ratio is the absolute value of the ratio between the second and the first eigenvalue of the matrix that represents the strategic dependencies of the game and it can be taken as a measure of the balancedness of the

by Theorem 1, the mixture model estimation (which assumes equal shapes) is more consistent for games with a lower eigenvalue ratio due to Theorem 2. The p-beauty contest of this introduction is an extreme case where the eigenvalue ratio is 0 and the approximation is thus exact.

Finally, we derive a new central limit result that provides microfoundations for gaussian mixture models that have been used in the level-k literature. In particular, Proposition 2 complements Theorem 2 by showing that if the number of players n in each group grows and no player receives too much weight in the guessing targets of all players in her group, then, under certain independence assumptions, the random variables $\tilde{x}^k$, $k \geq 1$, become normally distributed. The introductory example satisfies the condition on the strategic dependencies because targets assign the same weight to all players in each group.

Related literature

The level-k model has received considerable attention. The main insight from the early experimental studies of guessing games in Nagel (1995), Ho, Camerer and Weigelt (1998), Bosch-Domènech, Montalvo, Nagel and Satorra (2002), and Costa-Gomes and Crawford (2006) as well as from the normal-form game experiments in Stahl and Wilson (1994,1995) and Costa-Gomes, Crawford and Broseta (2001) is that a substantial fraction of the non-equilibrium play is consistent with k-rationalizability for low levels of k ($k \leq 3$) if level-0 subjects are assumed to choose uniformly at random over the entire action space.[7] However, according to Crawford, Costa-Gomes and Iriberri

link distribution in the dependency network. For example, Golub and Jackson (2012) use the second eigenvalue to measure homophily in a network, which affects the speed of convergence to consensus.

[7] Agranov, Potamites, Schotter and Tegiman (2012), Georganas Healy and Weber (2015), and Aloui and Penta (2016) introduce the notion of a strategic bound. A player with bound $\bar{k}$ strategically chooses the reasoning level $k \leq \bar{k}$ at which she operates. We abstain from these game-theoretical considerations.

(2013) “More work is needed to evaluate the credibility of the models’ explanations and to assess their domains of applicability, their portability, and the stability of their parameter estimates across types of games.”[8] Not only is the class of anchored guessing games a generalized framework that is ideally suited to compare parameter estimates across different game specifications; more importantly, we develop microfoundations for a new non-parametric estimation technique that extracts beliefs about the level-0 players directly from the choice data. Our approach therefore complements others in which information about beliefs could be obtained with the help of alternative experimental designs which add power to the estimation. For instance, Bhatt and Camerer (2005) extract reasoning levels via fMRI; Costa-Gomes and Weizäcker (2008) ask subjects to state their beliefs; Wang, Spezio and Camerer (2010) use eyetracking devices to monitor subjects’ decisions; Burchardi and Penczynski (2014) use group chat protocols from team decisions and Fragiadakis, Kovaliukaite and Rojo-Arjona (2019) use an incentive compatible mechanism to elicit beliefs.

We proceed as follows. Section 2 introduces the class of all anchored guessing games and studies their equilibrium properties. The main theoretical results regarding the level-k model are derived in Section 3. Section 4 describes the estimation algorithm. Section 5 illustrates the algorithm with the help of data from various experiments. The conclusion shows how our results can be helpful for designing laboratory experiments. All proofs are relegated to the Appendix.

[8] Among others, the level-k model has been applied to strategic information transmission by Cai and Wang (2006), auctions and hide and seek games by Crawford and Iriberri (2007a, 2007b), the 11-20 game by Arad and Rubinstein (2012), and the centipede game by García-Pola, Iriberri and Kovářík (2020).

2 Anchored guessing games

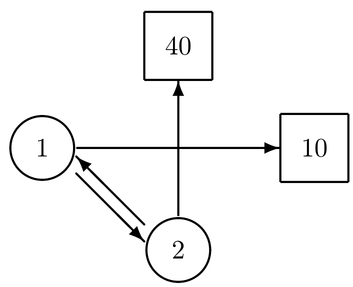
Example 1.

Consider the following simultaneous move game. There are two players $i \in \{1, 2\}$. Each player i has to choose a real number $x_i \in [0, 100]$. Each player has a target number:

The target t_1 of player 1 is the average of 10 and the choice x_2 of player 2.

The target t_2 of player 2 is the average of 40 and the choice x_1 of player 1.

Players have the incentive to minimize the distance between their choices and targets. This game has a very simple graphical representation. In the diagram, nodes are players and cells are numbers that we call anchors. Arrows departing from a player point to the information that determines her target. We omit the weights of the arrows, which are 1/2 in all instances.



The unique equilibrium is $\mathbf{x}^* = (20, 30)$, which is the unique solution of the linear system

$$x_1 = \frac{x_2 + 10}{2}$$
$$x_2 = \frac{x_1 + 40}{2}.$$

We are ready to introduce the formal model. Let $N = \{1, \ldots, n\}$ be a finite set of n players. Players have to simultaneously and independently choose a number from the interval $X \equiv [\underline{x}, \overline{x}] \subseteq$

$\mathbb{R}$, where $\overline{x} > \underline{x}$.[9] Let $x_i \in X$ be a particular strategy for player i. The column vector $\mathbf{x} = (x_1, \dots, x_n)$ is said to be a strategy vector. The $n \times n$ non-negative matrix $\mathbf{W}$ with generic element w_{ij} describes the network of strategic dependencies between players. For example, if the target of player 1 is the average of x_2 and x_3, then $w_{12} = w_{13} = 0.5$. The element w_{ii} may or may not be equal to zero. If $\mathbf{W}$ has a zero main diagonal, as in Example 1, then we say that the game is represented in its best-reply form. Targets also depend on constants. Formally, there is a non-empty set of anchors $M = \{1, \dots, m\}$ with an associated $m \times 1$ vector of real anchor values $\mathbf{v} = (v_1, \dots, v_m)^\top \in \mathbb{R}^m$. Anchor values enter into the players' objectives through the $n \times m$ non-negative matrix $\mathbf{A}$, which describes how much weight each player assigns to each of the m anchor values. By convention, if there are no anchors, we set $m = 1$ and take $\mathbf{A}$ to be an $n \times 1$ vector of zeroes $\mathbf{0}$ and $\mathbf{v}$ the 1×1 vector (0). Given a strategy vector $\mathbf{x}$, the target of player i is equal to

$$t_i(\mathbf{x}) = \sum_{l=1}^{m} a_{il}\, v_l + \sum_{j=1}^{n} w_{ij}\, x_j.$$

Expressed in matrix form, targets are thus

$$\mathbf{t}(\mathbf{x}) = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{x}.$$

The utility function of player $i \in N$ is $u_i(\mathbf{x}) = -\left(x_i - t_i(\mathbf{x})\right)^2$. Let $\mathbf{1}$ be the vector of ones. Targets are said to always be within the strategic bounds if for all $i \in N$ and all $\mathbf{x} \in X^n$, $t_i(\mathbf{x}) \in [\underline{x}, \overline{x}]$. We concentrate throughout on games that satisfy this interiority requirement, which can be succinctly written as follows.

Assumption 1 (Interiority)**.** $\underline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1} \leq \mathbf{A}\mathbf{v} \leq \overline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1}$.

[9] The strategy space is assumed to be common exclusively for notational simplicity. The model and results can be adapted in a straightforward way to allow for heterogeneous strategy spaces.

While Assumption 1 is restrictive, it still leaves a lot of room for designing meaningful games for the laboratory.[10] The class of all simultaneous move games $(N, X^n, (u_i)_{i\in N})$, parametrized by $\underline{x}$, $\overline{x}$, $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{v}$, that satisfy interiority are called *anchored guessing games.*

Order the (possibly complex) eigenvalues of $\mathbf{W}$ by their absolute value, *i.e.*, $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Since $\mathbf{W}$ is non-negative, it is well-known by the Perron-Frobenius Theorem that λ_1 is real and $\lambda_1 \geq 0$. Also, by Assumption 1, $\lambda_1 \leq 1$ and each row sum of $\mathbf{W}$ is less than or equal to one (see, Lemma 1 and Lemma 2 in the Appendix). Proposition 1 shows that there always is a Nash equilibrium in pure strategies. The equilibrium is unique if and only if $\lambda_1 < 1$. Proposition 1 provides for all anchored guessing games an ordinal characterization of this uniqueness condition. For this purpose, we say that player $i \in N$ *has a path to an anchor* if there is a sequence of players $(i = j_0, j_1, j_2, \ldots, j_k = j)$ such that $\sum_{l\in N} w_{jl} < 1$ and for all $s = 1, 2, \ldots, k$, $w_{j_{s-1}j_s} > 0$. That is, player i has a path leading to a player $j \in N$ (not necessarily distinct from i) whose row sum $\sum_l w_{jl}$ is strictly smaller than 1. This condition is essentially graph-theoretical (*i.e.*, ordinal) in the sense that only the existence of links in the graph, but not the particular weights, matter. Moreover, it can be easily verified by inspecting the network.

Proposition 1.

(a) Every anchored guessing game has Nash equilibrium in pure strategies.

(b) The equilibrium is unique if and only if every player $i \in N$ has a path to an anchor.

(c) If the equilibrium is unique, then $\mathbf{x}^* = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{A}\mathbf{v}$.

[10] In particular, given a non-negative matrix $\mathbf{W}$ whose row sums are less than or equal than 1 and a strategy space X, one can always find a set of anchors $\mathbf{A}$ and values $\mathbf{v}$ so that the interiority assumption holds.

(d) If the equilibrium is unique, then the game is dominance solvable, the equilibrium is globally stable, and it can be obtained by applying to any initial $\mathbf{x}^0 \in X^n$ the iterative vector function

$$\mathbf{x}^{\tau+1} = \mathbf{t}(\mathbf{x}^{\tau}), \ \textit{where } \tau = 0, 1, \ldots .$$

Proposition 1 follows from standard results. Since the game is smooth supermodular, existence follows from Milgrom and Roberts (1990). The ordinal characterization is useful because uniqueness can be checked by merely inspecting the network of strategic dependencies. The asymptotic convergence factor of the iterative process $\mathbf{x}^{\tau+1} = \mathbf{t}(\mathbf{x}^{\tau})$ towards $\mathbf{x}^*$ is λ_1 and, therefore, a smaller dominant eigenvalue λ_1 is associated with a higher speed of convergence of this process to the unique equilibrium. Finally, all results in Proposition 1 except for speed of convergence are robust to changes in the targets $\mathbf{t}$ whenever the underlying best-reply mapping remains unaffected. We conclude with another example.

Example 2 (p-beauty contest).

We now expand on our introductory example. Let there be n players. The strategy space is $X = [0, \overline{x}]^n$, where $\overline{x} > 0$. Targets are equal to a fraction $p \geq 0$ of the mean choice, that is, for all $i \in N$, $t_i = (p/n) \sum_{j \in N} x_j$. Thus, $\mathbf{v} = (0)$, $\mathbf{A} = \mathbf{0}$, and $\mathbf{W} = (p/n)\,\mathbf{U}$, where $\mathbf{U} = \mathbf{1}\mathbf{1}^{\top}$ is an $n \times n$ matrix of ones. It is easy to see that the interiority assumption is satisfied if and only if $p \leq 1$. Since the dominant eigenvalue of $\mathbf{W}$ is $\lambda_1 = p$, the equilibrium is unique if and only if $p < 1$. Observe that in this particular game, the asymptotic speed of convergence towards the equilibrium is decreasing in p and that it does not depend on the number of players. Section 3 further analyzes this game in its best-reply form. $\square$

3 A level-k framework

3.1 Decision-making process

In this section, we extend the classical level-k framework by introducing the possibility of free subjective beliefs about level 0. Each player is endowed with a reasoning level $k \in \{0, 1, 2, \ldots\}$. We do not make any assumption on the behavior of level-0 players. A player with reasoning level $k \in \{1, 2, \ldots\}$ is defined recursively as one who best replies to the subjective belief that all other players employ reasoning level $k-1$. This creates a belief hierarchy that collapses at level 0. We also note that the actual level-0 behavior may differ from the beliefs that players hold about level-0. Both are free of distributional assumptions in this paper.

In order to introduce our results, suppose that a large number of experimental subjects are randomly assigned into groups of n to play a given anchored guessing game. From an econometric point of view, each experimental subject s is endowed with an observable role $i \in N$, an unobservable reasoning level $k \in \{0, 1, 2, \ldots\}$, and unobservable beliefs $\mathbf{e}^{(s)} = (e_1^{(s)}, \ldots, e_n^{(s)}) \in [\underline{x}, \overline{x}]^n$. Here, $e_j^{(s)} \in [\underline{x}, \overline{x}]$ denotes, from the perspective of subject s, her subjective belief about the play of a level-0 subject in role j.[11] The joint distribution of subjective beliefs in the population is a random vector $\tilde{\mathbf{e}} = (\tilde{e}_1, \ldots, \tilde{e}_n)$. Denote the restriction of $\tilde{\mathbf{e}}$ to any given role $i \in N$ by $\tilde{\mathbf{e}}^{(i)}$. Also, define the random variable of *seeds* of the subjects in role i as $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j \, \tilde{e}_j^{(i)} \in [\underline{x}, \overline{x}]$, where $\boldsymbol{\gamma}$ is a vector of non-negative weights that add up to 1. It is assumed throughout that for all non-zero reasoning levels, $\tilde{\mathbf{e}}^{(i)}$ is independent k. This assumption, which is implicit in much of the level-k

[11] More concretely, the subjective beliefs over level-0 play are a probability distribution over X. By the linearity of the targets and interiority, an expected utility maximizing subject only considers the mathematical expectation of this probability distribution. Hence, formally $e_j^{(s)}$ is the mathematical expectation of the subjective beliefs.

literature, turns out to be crucial for our estimation strategy in Section 4.

Assumption 2 (Level-independent beliefs)**.** *For all roles, beliefs are independent of non-zero reasoning levels. That is, for all* $i \in N$*,* $\mathbf{e} \in [\underline{x}, \overline{x}]^n$ *and all* $k, k' \geq 1$*,*

$$\Pr(\tilde{\mathbf{e}}^{(i)} \leq \mathbf{e} \,|\, k) = \Pr(\tilde{\mathbf{e}}^{(i)} \leq \mathbf{e} \,|\, k').$$

Assumption 2 underlies the level-k literature, including the classical beauty contest game, where a level-0 individual is believed to play uniformly in the interval $[0, 100]$, independently of her reasoning level or her label in the game. In this case, the vector $\tilde{\mathbf{e}}^{(i)}$ takes the degenerate form of $\tilde{\mathbf{e}}^{(i)} = 50 \cdot \mathbf{1}$ for each role i, independently of the reasoning level. Even in more complex setups of the literature where $\tilde{\mathbf{e}}^{(i)}$ is non-degenerate, it is still assumed to be independent of the reasoning level.

We next derive the optimal choices under the level-k model. Subject s in role i with reasoning level $k \geq 1$ and beliefs $\mathbf{e}^{(s)}$ iterates k times the target function $\mathbf{t}$ over $\mathbf{e}^{(s)}$. That is, she solves the following vector iterative process:

$$\begin{aligned} &\mathbf{x}^{1(s)} = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{e}^{(s)} \text{ and} \qquad (1) \\ &\mathbf{x}^{\tau(s)} = \mathbf{t}(\mathbf{x}^{(\tau-1)(s)}) = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{x}^{(\tau-1)(s)} \text{ for all } \tau \in \{2, 3, \ldots, k\}. \end{aligned}$$

The optimal choice $x_i^{k(s)}$ of subject s corresponds to the i-th entry of the k-th iteration vector $\mathbf{x}^{k(s)}$. Let $\tilde{x}_i^k$ be the random variable induced by the optimal choices of all subjects with reasoning level k in role i. Since $\tilde{x}_i^k$ depends (linearly) on the beliefs $\tilde{\mathbf{e}}^{(i)}$, we write $\tilde{x}_i^k = x_i^k(\tilde{\mathbf{e}}^{(i)})$ to make this dependence explicit. We refer to $\tilde{x}_i^k = x_i^k(\tilde{\mathbf{e}}^{(i)})$, for $k \geq 1$ and $i \in N$, as the set of *reduced-form equations.*

By Proposition 1, if there is a unique equilibrium $\mathbf{x}^*$, the random variable $\tilde{x}_i^k$ degenerates towards the Nash equilibrium action x_i^* as k grows. The speed of convergence depends inversely on the dominant eigenvalue λ_1 of $\mathbf{W}$. We thus highlight here the importance of λ_1 as a crucial parameter in anchored guessing games under the level-k model because it helps delimiting scenarios where the level-k theory can explain out-of-equilibrium behavior. For instance, the level-k model is more likely to explain out-of-equilibrium behavior of players with moderate levels of reasoning in games with a high λ_1 (because these games converge slower). Also, note that the domain of $\tilde{x}_i^k$ is the set of k-iterated undominated strategies $U_i^k \equiv [\underline{x}_i^k, \overline{x}_i^k]$. Due to Assumption 1 the bounds $\underline{x}_i^k$ and $\overline{x}_i^k$ can be computed from the iterative formula in (1) by substituting the initial vector $\mathbf{e}^{(s)}$ by $\underline{x}\,\mathbf{1}$ and $\overline{x}\,\mathbf{1}$, respectively. The bounds of U_i^k then correspond to the i-th entries of the k-th iteration vectors.

3.2 Main results

Our main purpose is to learn the features of the random vector of beliefs $\tilde{\mathbf{e}}^{(i)}$, given the choices of the subjects in a given role $i \in N$. From a theoretical point of view, we are interested in analyzing conditions on the game characteristics and on the beliefs so that the "shapes" of the random variables $\tilde{x}_i^k$, $k \geq 1$, are alike. Formally, consider two real random variables $\tilde{y}$ and $\tilde{z}$, and two domain intervals $[\underline{y}, \overline{y}]$ and $[\underline{z}, \overline{z}]$ that contain the supports of $\tilde{y}$ and $\tilde{z}$, respectively. We say that $\tilde{y}$ and $\tilde{z}$ are *equally shaped*, written $\tilde{y} \overset{\mathcal{S}}{=} \tilde{z}$, if their normalizations within these domain intervals are equally distributed:

$$\frac{\tilde{y} - \underline{y}}{\overline{y} - \underline{y}} \overset{\mathcal{D}}{=} \frac{\tilde{z} - \underline{z}}{\overline{z} - \underline{z}}.$$

The geometric interpretation of the condition is that the probability distribution functions of $\tilde{y}$ and $\tilde{z}$ have exactly the same visual form within their respective domains (*i.e.*, the random variables $\tilde{y}$ and $\tilde{z}$ are affine transformations of each other).[12] Let $\{\tilde{y}^k\}_{k=1}^{\infty}$ be a sequence of random variables defined on the domains $\{[\underline{y}^k, \overline{y}^k]\}_{k=1}^{\infty}$ and let $\tilde{y}$ be a random variable with given domain $[\underline{y}, \overline{y}]$. We say that the random variables $\{\tilde{y}^k\}_{k=1}^{\infty}$ *converge in shape to* $\tilde{y}$ (written $\tilde{y}^k \xrightarrow{\mathcal{S}} \tilde{y}$) if

$$\frac{\tilde{y}^k - \underline{y}^k}{\overline{y}^k - \underline{y}^k} \xrightarrow{\mathcal{D}} \frac{\tilde{y} - \underline{y}}{\overline{y} - \underline{y}} \quad \text{as } k \to \infty.$$

We then say that the shape of $\tilde{y}^k$ approaches the shape of $\tilde{y}$ as k grows.[13]

Theorems 1 and 2 establish that under suitable conditions, the random variables of choices $\tilde{x}_i^k$, $k \geq 1$, have the same or similar shapes as the random variable of seeds $\tilde{\mu}^{(i)}$. In Theorem 1, equal shapes are the consequence of imposing restrictions on beliefs. Formally, the beliefs $\tilde{\mathbf{e}}^{(i)}$ are said to be *individually homogeneous* if for each subject s in role $i \in N$, $e_1^{(s)} = e_2^{(s)} = \ldots = e_n^{(s)} \equiv e^{(s)}$. In other words, if beliefs are individually homogeneous, then the random belief vector $\tilde{\mathbf{e}}^{(i)}$ is equal

[12] In statistics, two random variables $\tilde{y}$ and $\tilde{z}$ are said to be of the same type whenever

$$\frac{\tilde{y} - \mathbb{E}[\tilde{y}]}{\sigma_y} \stackrel{\mathcal{D}}{=} \frac{\tilde{z} - \mathbb{E}[\tilde{z}]}{\sigma_z}.$$

Our definition is stronger than this notion because it is parametrized by the given domain intervals.

[13] Convergence in distribution ($\xrightarrow{\mathcal{D}}$) to a non-degenerate random variable is a special case of convergence in shape with suitably defined domains. For instance, $\mathcal{U}(0, 1 + 1/k) \xrightarrow{\mathcal{S}} \mathcal{U}(0, 1)$ when the domains are the corresponding supports of the uniform distributions (in fact, the shapes are equal). However, we stress the importance of imposing a non-degenerate limit random variable $\tilde{y}$. For instance, $\mathcal{U}(0, 1/k)$ converges in distribution to the degenerate random variable 0. However, $\mathcal{U}(0, 1/k)$ does not converge in shape to any degenerate random variable. In fact, for each k, the shape of $\mathcal{U}(0, 1/k)$ is equal to the shape of $\mathcal{U}(0, 1)$ with corresponding domains $[0, 1/k]$ and $[0, 1]$. In this sense, convergence in shape to $\tilde{y}$ means convergence in shape to the whole class of random variables that have the same shape as $\tilde{y}$.

to $\tilde{\mu}^{(i)}\mathbf{1}$ for any weight vector $\boldsymbol{\gamma}$. Note that beliefs remain nevertheless heterogeneous between subjects because two different subjects s and s' may think differently about how level-0 subjects behave. Observe that while $\tilde{x}_i^k$ has domain U_i^k, $\tilde{\mu}^{(i)}$ has domain $[\underline{x}, \overline{x}]$.

Theorem 1. *Suppose that all subjects in role $i \in N$ have individually homogenous beliefs. Then, the random variables $\{\tilde{x}_i^k\}_{k=1}^{\infty}$ are equal in shape. In particular, for all $k \in \{1, 2, \ldots\}$, $\tilde{x}_i^k \overset{\mathcal{S}}{=} \tilde{\mu}^{(i)}$.*

Next, we drop the assumption of individually homogeneous beliefs. For this purpose, we suppose for simplicity that $\mathbf{W}$ is diagonalizable, which holds generically. To proceed, the following graph-theoretical definitions are needed. An anchored guessing game is said to be connected if the dependency matrix $\mathbf{W}$ is irreducible, that is, if any two players can be connected by some path in the network represented by $\mathbf{W}$. The matrix $\mathbf{W}$ is primitive[14] if there is some integer $k > 0$ such that all entries of $\mathbf{W}^k$ are strictly positive. It is well-known that if $\mathbf{W}$ is primitive, then the corresponding anchored guessing game is connected. Theorem 2 guarantees shape convergence whenever the underlying dependency matrix $\mathbf{W}$ is primitive (without imposing any condition on beliefs). The theorem states that the shape of $\tilde{x}_i^k$ converges to the shape of the seeds $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j \, \tilde{e}_j^{(i)}$, where the weight γ_j assigned to each role $j \in N$ is precisely the centrality of player j in the network induced by $\mathbf{W}$. The centrality vector $\boldsymbol{\gamma}$ is computed as the left eigenvector (with $\sum_j \gamma_j = 1$) associated with the dominant eigenvalue λ_1 of $\mathbf{W}$. That is, $\boldsymbol{\gamma}^\top \mathbf{W} = \lambda_1 \boldsymbol{\gamma}^\top$. The centrality γ_j summarizes the influence of role j on the choices of all players in the same group.

Theorem 2. *Suppose that $\mathbf{W}$ is primitive. Then, for each role $i \in N$, the random variables $\{\tilde{x}_i^k\}_{k=1}^{\infty}$ converge in shape. In particular, $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$ as $k \to \infty$.*

[14]Primitive matrices are generic. Even if $\mathbf{W}$ has a zero diagonal, primitivity remains generic for $n \geq 3$.

The asymptotic convergence factor of this process is given by the eigenvalue ratio $|\lambda_2|/\lambda_1 < 1$. An extreme case of Theorem 2 occurs when $\mathbf{W}$ has rank one and can thus be written as $\mathbf{W} = \mathbf{r}\,\boldsymbol{\gamma}^\top$. The shape of $\tilde{x}_i^k$ for $k \geq 1$ then turns out to be equal to the shape of $\tilde{\mu}^{(i)}$ because the eigenvalue ratio reaches its minimum value of 0. The p-beauty contest of Example 2 has an associated rank-one matrix $\mathbf{W}$ with $\mathbf{r} = p\,\mathbf{1}$ and $\boldsymbol{\gamma} = (1/n)\,\mathbf{1}$. The centrality of player in role i is $1/n$ and the random variable of seeds is $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \tilde{e}_j^{(i)}/n$.

We also highlight that Theorem 2 provides insights about the proper design of anchored guessing games for the laboratory because games with a low eigenvalue ratio $|\lambda_2|/\lambda_1$ are those for which shape convergence may be observed at the earliest non-zero reasoning levels.[15] Our simulations show that an eigenvalue ratio of 0.65 seems generally low enough to obtain similar shapes starting at levels 1 or 2.[16] Under these circumstances it is therefore more than justified to impose in the econometric analysis developed in Section 4 that the $\tilde{x}_i^k$ for $k \geq 1$ are equally shaped. As an illustration, consider the p-beauty contest with at least three players in its best-reply form (zero main diagonal) with targets $t_i(\mathbf{x}) = p/(n-p)\sum_{j\neq i} x_j$. Also, suppose that beliefs are not assumed individually homogenous. Then, $\mathbf{W}$ is primitive and Theorem 2 applies (but not Theorem 1). It follows from the anonymity of the interactions in this beauty contest that the centrality of each player i is $\gamma_i = 1/n$, which implies that $\tilde{x}_i^k$ converges in shape to $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \tilde{e}_j^{(i)}/n$. Moreover, the eigenvalue ratio $|\lambda_2|/\lambda_1 = (n-1)^{-1}$ is bounded above by 0.5 when there are at least three players, which facilitates similar shapes at low reasoning levels.

[15] In the discussion section, we address this design problem by comparing a variety of games with $n \leq 4$.

[16] Convergence in shape is not only determined by the eigenvalue ratio, which should be interpreted as an optimistic measure of convergence. It is also affected by the random vector of beliefs $\tilde{\mathbf{e}}^{(i)}$ as well as by other graph-theoretical features like the diameter of the network associated with the dependency matrix $\mathbf{W}$.

3.3 Large games

Next, we study games with a growing number of players n. Proposition 2 below identifies conditions under which $\tilde{x}_i^k$ is approximated by a (truncated) normal distribution. The proposition thus rationalizes the application of parametric gaussian estimation techniques in certain types of anchored guessing games. For example, Ho, Camerer and Weigelt (1998) and Burchardi and Penczynski (2014) assume that a normal distribution for level-0 choices or beliefs is transmitted to all non-zero reasoning levels. In our setting, normality arises naturally without making this kind of assumption. Two types of convergences take place under our result. First, the random variables $\tilde{x}_i^k$ and $\tilde{\mu}^{(i)}$ have a similar shape by Theorem 2. Second, $\tilde{\mu}^{(i)}$ is distributed normally as n grows by a central limit theorem.

We introduce the necessary notation and definitions. Given role $i \in N$, beliefs are said to be *role-independent* if the random variables $\tilde{e}_1^{(i)}, \tilde{e}_2^{(i)}, \ldots, \tilde{e}_n^{(i)}$ are independently (but not necessarily identically) distributed. Since role-independence implies that beliefs are not individually homogenous, this condition is incompatible with Theorem 1. We thus have to build the central limit result on top of Theorem 2. Remember that in Theorem 2, $\tilde{x}_i^k$ converges in shape towards $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j \, \tilde{e}_j^{(i)}$. Let $\overline{\gamma} = \max_j\{\gamma_j\}$ be the maximum network centrality, $\mathrm{Var}(\tilde{e}_j^{(i)})$ be the variance of the beliefs subjects in role i have about the play of level-0 subjects in role j, and $\bar{\gamma}_j = \gamma_j/\overline{\gamma} \in [0, 1]$ be the centrality of role j relative to the maximum centrality. In the context of a growing network all parameters and variables associated with the game, except for the strategy space, like $\mathbf{W}$, $\tilde{\mathbf{e}}^{(i)}$, $\overline{\gamma}$, $\bar{\gamma}_j$, and $\tilde{x}_i^k$, may vary with the size of the network n. However, in order to keep the notation as simple as possible, we abstain from making theses dependencies explicit.

Proposition 2. *Consider any role $i \in N$ and suppose that $\mathbf{W}$ is primitive for all $n \geq 1$. If beliefs* *are role-independent and if* $\sum_{j=1}^{n} \overline{\gamma}_j^2 \operatorname{Var}(\tilde{e}_j^{(i)}) \to \infty$ *as* $n \to \infty$, *then* $\tilde{x}_i^k$ *is asymptotically normal. In particular,*

$$\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}_{[\underline{x}_i^k, \overline{x}_i^k]} \left(\underline{x}_i^k + \frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}} \left(\mathbb{E}\left[\tilde{\mu}^{(i)}\right] - \underline{x} \right), \left(\frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}} \right)^2 \operatorname{Var}\left(\tilde{\mu}^{(i)}\right) \right) \ \text{as } k, n \to \infty,$$

where $\mathcal{N}_{[\underline{x}_i^k, \overline{x}_i^k]}$ *is the truncation of a normal distribution to the interval* $[\underline{x}_i^k, \overline{x}_i^k]$.

To see why role independence is a necessary condition imagine an extreme case where beliefs are individually homogeneous with $\tilde{e}^{(i)} \sim \mathcal{U}(\underline{x}, \overline{x})$. Then, by Theorem 1, choices are uniformly (*i.e.*, not normally) distributed. Next, we discuss the condition on the variance of the beliefs. First, it is required that $\operatorname{Var}(\tilde{e}_j^{(i)})$ does not vanish too fast (if it vanishes) as n grows. In fact, without enough belief variability, it would be difficult to obtain normally distributed choices. Second, no player should have an excessive impact on the targets of other players, that is, as n grows, there is no role j whose centrality γ_j becomes large compared to that of the other roles. Corollary 1 below illustrates this point when the belief variance is bounded away from zero: normality arises when the maximum centrality $\overline{\gamma}$ vanishes faster than $n^{-1/2}$.

Corollary 1. *Consider any role* $i \in N$ *and suppose that* $\mathbf{W}$ *is primitive for all* $n \geq 1$. *If beliefs are role-independent, if there is* $\varepsilon > 0$ *independent of* n *such that for all* $j \in N$, $\operatorname{Var}(\tilde{e}_j^{(i)}) \geq \varepsilon$ *as* $n \to \infty$, *and if* $\lim_{n\to\infty} \overline{\gamma}\sqrt{n} = 0$, *then* $\tilde{x}_i^k$ *is asymptotically normal as* $k, n \to \infty$.

A simple scenario of a moderate maximum influence, which includes the p-beauty contest with $n \geq 3$ players, occurs when $\mathbf{W}$ approaches a column-regular matrix, *i.e.*, there is $c > 0$ such that $\mathbf{1}^\top \mathbf{W} \to c\,\mathbf{1}^\top$ as $n \to \infty$. Then, for all $j \in N$, we have that $\gamma_j \simeq \overline{\gamma} \simeq 1/n$ for large n, which vanishes sufficiently fast. More generally, anonymous games (*i.e.*, games where roles cannot be distinguished from each other in the network $\mathbf{W}$) satisfy the conditions for $\overline{\gamma}$ of Corollary 1.

4 Maximum likelihood estimation

We propose a mixture model, whose components are the different reasoning levels, in order to non-parametrically infer the beliefs about the level-0 behavior. By Theorem 1 or by Theorem 2, the random variables $\tilde{x}_i^k$, for all $k \geq 1$, are assumed to have the same shape as the random variable of seeds $\tilde{\mu}^{(i)}$. This reduces considerably the dimensionality of the statistical model. Since, by our results, these random variables only differ in known scale and shift parameters from the random variable of seeds, solving the mixture model therefore accomplishes our goal of estimating the seed distribution.

Formally, let x_l, $l = 1, ..., L$, be a typical observation or choice. For notational simplicity, it is assumed that all choices belong to the same role $i \in N$ in a given anchored guessing game. Since the role is fixed, we suppress the index i throughout this section. In the log-likelihood function

$$\mathcal{L} = \sum_{l=1}^{L} \log \left(\sum_{k=0}^{K} p_k \, f^k(x_l) \right),$$

K denotes the maximum reasoning level in the estimation, $p_k \in [0, 1]$ corresponds to the (unknown and to be estimated) fraction of observations that are generated by the population with reasoning level k, and f^k is the (unknown and to be estimated) conditional density function that describes the choices of the level-k population. The domain of the density function f^k is the k-iterated undominated set $U^k = [\underline{x}^k, \overline{x}^k]$.

The interest of Theorems 1 and 2 lies in that they produce shape restrictions that reduce the number of unknown density functions from $K + 1$ to 2 (f^0 and f).[17] In particular, by applying

[17]Instead of considering an unrestricted behavior for level-0 subjects, in the appendix we impose restrictions on f^0 that further reduce the dimensionality of the mixture model.

either of these results, we obtain that for each $k \geq 1$ and each $x \in \mathbb{R}$,

$$f^k(x) = \begin{cases} \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}\, f\left(\underline{x} + \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}(x-\underline{x}^k)\right) & \text{if } x \in U^k \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Here, f is the unknown density of seeds $\tilde{\mu}$ with domain $[\underline{x}, \overline{x}]$. Given an observation x, let $k(x)$ be the highest level k such that $x \in U^k$. That is, $k(x)$ is the maximum reasoning level that is compatible with choice x. Substituting (2) into the original log-likelihood function yields

$$\mathcal{L} = \sum_{l=1}^{L} \log \left(p_0\, f^0(x_l) + \sum_{k=1}^{k(x_l)} p_k\, \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}\, f\left(\underline{x} + \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}(x_l-\underline{x}^k)\right)\right).$$

We discretize each density f^k by partitioning its domain in a precise way that takes advantage of our theoretical results. In particular, each set $U^k = [\underline{x}^k, \overline{x}^k]$ is divided into a total of B buckets of equal width $(\overline{x}^k - \underline{x}^k)/B$. These buckets form a histogram that approximates the density function f^k. The non-negative bucket areas (to be estimated) for reasoning level k are denoted by $a_1^k, a_2^k, \ldots, a_B^k$. The total bucket area for reasoning level k is $\sum_{b=1}^{B} a_b^k = 1$. For each observation x and each reasoning level k such that $x \in U^k$, let $b^k(x)$ be the index $1, 2, \ldots, B$ of the bucket that contains x in the histogram of f^k. The discrete version of the original log-likelihood function (without shape restrictions) is then

$$\mathcal{L}_D = \sum_{l=1}^{L} \log \left(\sum_{k=0}^{k(x_l)} p_k\, \frac{a^k_{b^k(x_l)}\, B}{\overline{x}^k - \underline{x}^k} \right),$$

where $a^k_{b^k(x_l)}\, B/(\overline{x}^k - \underline{x}^k)$ is the height of the $b^k(x_l)$-th bucket that indicates the approximated value of $f^k(x_l)$. Theorems 1 and 2 allow us to assume equal bucket areas across non-zero levels. That is, for each bucket $b = 1, 2 \ldots, B$, we have that $a_b^1 = a_b^2 = \cdots = a_b^K \equiv a_b$. The log-likelihood maximization is then given by:

$$\max_{(p_k)_{k=0}^K \in \Delta^K, (a_b^0)_{b=1}^B, (a_b)_{b=1}^B \in \Delta^{B-1}} \sum_{l=1}^{L} \log \left(p_0 \, \frac{a^0_{b^0(x_l)} \, B}{\overline{x} - \underline{x}} + \sum_{k=1}^{k(x_l)} p_k \, \frac{a_{b^k(x_l)} \, B}{\overline{x}^k - \underline{x}^k} \right), \tag{3}$$

where Δ^r is the r-dimensional simplex. The total number of parameters to be estimated is $K + 2(B-1)$, which is possibly a large number.

We adapt the expectation maximization (EM) algorithm to solve the maximization problem (3) iteratively. Maximizing the likelihood function in mixture models is a computationally challenging task. However, due to the utilization of a closed-form formula that we derive for the M-step of the algorithm, instead of solving an interim maximization program, the algorithm is fast and efficient in practice. It typically converges to a local maximum, regardless of the starting point, and can be executed multiple times to generate a set of local maxima that can be analyzed further. In the following, superscript $[t]$ denotes the current parameter value at time t during a single execution of the algorithm. Start at time 0 with arbitrary parameter values $(p_k^{[0]})_{k=0}^K \in \Delta^K$ and $(a_b^{0[0]})_{b=1}^B, (a_b^{[0]})_{b=1}^B \in \Delta^{B-1}$. Then, at time $t = 0, 1, \ldots$, repeat the following two steps until the distance between two consecutive solutions reaches a desired tolerance level:

- **E-step (expectation).**

  Given the iteration t density functions $f^{0[t]}(x_l) = a^{0[t]}_{b^0(x_l)} \, B/(\overline{x} - \underline{x})$ and $f^{k[t]}(x_l) = a^{[t]}_{b^k(x_l)} \, B/(\overline{x}^k - \underline{x}^k)$ for all $k \geq 1$, compute for each reasoning level k and each observation x_l,

  $$q_k^{[t]}(x_l) = \frac{p_k^{[t]} \, f^{k[t]}(x_l)}{\sum_{k'=0}^{K} p_{k'}^{[t]} \, f^{k'[t]}(x_l)}.$$

- **M-step (maximization).**

The parameters are updated.[18,19]

1. Update the level composition. For each $k \geq 0$,

$$p_k^{[t+1]} = \frac{1}{L} \sum_{l=1}^{L} q_k^{[t]}(x_l).$$

2. Update bucket areas. For each bucket $b = 1, \ldots, B$,

$$a_b^{0[t+1]} = \frac{1}{L\, p_0^{[t+1]}} \sum_{\substack{l=1,\ldots,L \\ b^0(x_l)=b}} q_0^{[t]}(x_l) \text{ and } a_b^{[t+1]} = \frac{1}{L\left(1-p_0^{[t+1]}\right)} \sum_{k=1}^{K} \sum_{\substack{l=1,\ldots,L \\ b^k(x_l)=b}} q_k^{[t]}(x_l).$$

5 Illustration

This section estimates the mixture model for the data from the newspaper experiments of Bosch-Domènech, Montalvo, Nagel and Satorra (2002), who implement the p-beauty contest with $p = 2/3$ in the Financial Times (England), Spektrum (Germany), and Expansión (Spain), and for the first of the three anchored guessing games detailed in Ballester, Rodriguez-Moral and Vorsatz (2022).[20] The latter study is tailored as a more direct test of equilibrium behavior in Proposition 1 and of whether the estimated seeds depend on exogenous characteristics such as attention levels. In

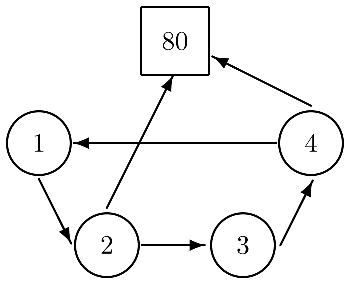
[18] It can be shown that the the closed-form formulae for level composition and bucket areas given in the M-step are the solution to the interim maximization program

$$\max_{(p_k^{[t+1]})_{k=0}^{K} \in \Delta^{\bar{K}},\, (a_b^{0[t+1]})_{b=1}^{B},\, (a_b^{[t+1]})_{b=1}^{B} \in \Delta^{B-1}} \sum_{l=1}^{L} \sum_{k=0}^{K} q_k^{[t]}(x_l) \log\left(p_k^{[t+1]} f^{k[t+1]}(x_l)\right).$$

[19] When a solution is reached after a high number of iterations T, the probability that the choice $x \in [\underline{x}, \overline{x}]$ is generated by the reasoning level k is approximated by $q_k^{[T]}(x)$ from the E-step.

[20] We also performed estimations for the beauty contest dataset of Matthew O. Jackson from his Game Theory course in Coursera. The results are similar to those obtained in the newspaper experiments, but they are noisier and the estimation yields, maybe due to the absence of incentives, lower reasoning levels.

Ballester, Rodriguez-Moral and Vorsatz (2022) subjects are asked, using an incentive compatible mechanism, to register during 8 periods as many choices as they wish to without receiving feedback regarding the actions or the payoffs of any player.



$$[\mathbf{W}|\mathbf{A}] = \left(\begin{array}{cccc|c} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0 & 0 & 0.25 \end{array}\right)$$

$\mathbf{x}^* = (11, 23, 11, 23)$

$\lambda_1 = 0.35$

Figure 2: Anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2022).

The targets, induced by means of the matrix in Figure 2, have simple verbal equivalents that are used in the instructions in order to keep the experiment as simple as possible. Subjects are also provided with the graphical representation of the strategic dependencies (network structure) to further simplify the experiment. Players 1 and 3 as well as players 2 and 4 are isomorphic in the sense that their respective positions in the game are identical. We pool all registered choices because not much additional insight is gained from separating player roles or periods.

Theorem 1 reasonably applies to Bosch-Domènech, Montalvo, Nagel and Satorra (2002) because their game is anonymous, which supports Assumption 2. But one could also make use of Theorem 2 since the eigenvalue ratio $(n-1)^{-1}$ is negligible when subjects use best-replies and n is large. For Ballester, Rodriguez-Moral and Vorsatz (2022) we have to impose Assumption 2 in order to apply Theorem 1. Theorem 2 cannot be used because $\mathbf{W}$ is not primitive. We set the maximal reasoning level to $K = 10$, approximate the density functions via $B = 50$ buckets, and fix an error tolerance of 10^{-8}. The results are quite stable with respect to these choices. Finally, since the algorithm identifies local maxima for each starting point, we run for each data set a total of 1,000 estimations with random starting points. In most cases, we get one or a few local maximum with small differences in the value of the log-likelihood function. The global maximum is always the most common outcome. The reported outcome is the average of these 1,000 estimations.

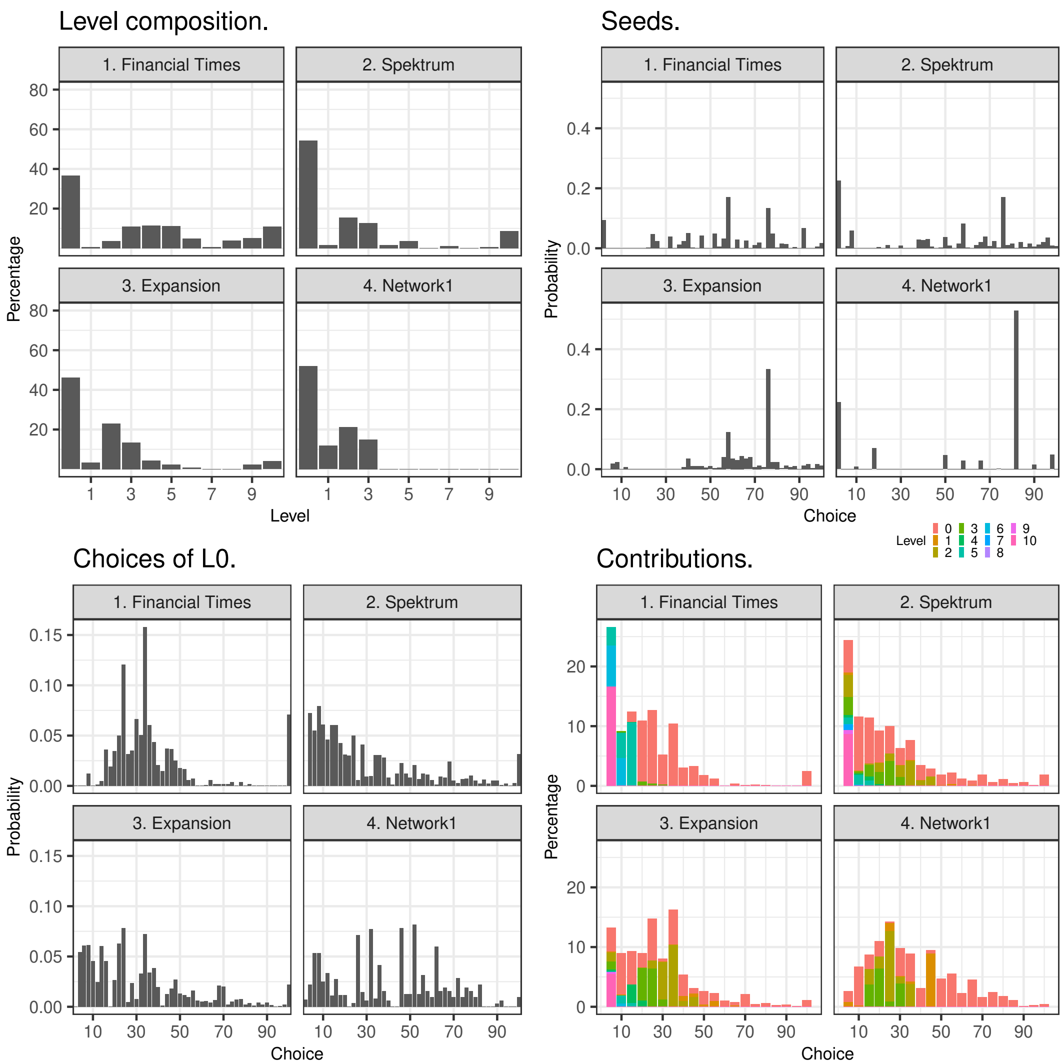


Figure 3: Estimation results.

The results of the estimations are presented in Figure 3. The top-left panel shows the percentages of the subjects that are assigned to the different reasoning levels. We find that the percentage of level-0 subjects is between 35% and 55%. Some subjects are assigned to the highest reasoning level in the newspaper experiments, which is different for the anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2022), where the highest reasoning level with positive estimated mass is $k = 3$. Also, the percentage of level-1 subjects is rather low and smaller than the percentage of level-2 subjects. However, this has a straightforward explanation that directly brings us to the top-right panel.

The top-right panel, the main output of the algorithm, shows the estimated seed distribution. This estimated distribution is much more concentrated than the original choice histograms in the bottom-right panel, which suggests that subjects usually start their reasoning process from a few common focal points. To understand the lack of level-1 subjects in the level compositions of the newspapers' beauty contests, note that many subjects choose the number 33 in this game. If subjects assume that level-0 play is uniformly distributed on [0,100], then the average belief is 50 and the choice of 33 is best associated with a level-1 behavior. Since our estimation does not make specific assumptions (other than shape similarity) about the distribution of seeds, a guess of 33 is now not only consistent with a level-1 play for a seed of 50, but also with a level-2 play for a seed of 75. In fact, under shape similarity, a single observation of 33 is more likely to be generated by level 2 than by level 1, which illustrates the algorithm's tendency to assign subjects to higher levels. To see this, suppose that the seeds 50 and 75 are equally likely, that is, $f(50) = f(75)$. Then, by shape similarity,

$$f^2(33) = \left(\frac{3}{2}\right)^2 f(75) > \left(\frac{3}{2}\right) f(50) = f^1(33).$$

The tendency to assign a guess of 33 in the newspaper experiments to level 2 can be observed in the top-right panel: instead of a spike at 50, a lot of mass is assigned to the 70-80 range.

With respect to Ballester, Rodriguez-Moral and Vorsatz (2022), about 50% of the subjects with a non-zero reasoning level have a seed of 80. This suggests that the unique anchor of the game description is a strong focal point in the belief formation process. Moreover, about 20% of these subjects have a seed of 0, which may be caused by the fact that this number involves simpler calculations. Since 50% of all choices are level-0 play in this game (see, the top-left panel), the seeds of 80 and 0 explain about 35% of all choices.

The estimated behavior of the level-0 players in the bottom-left panel is interpreted as the residual of observations that the algorithm cannot assign to higher reasoning levels. Notably, the distributions are not uniformly distributed.

The stacked histograms in the bottom-right panel show for every range of guesses, the percentage of choices in that range that correspond to each level. These percentages are obtained from the E-step after the execution of the algorithm. For example, in case of the newspaper experiment in the Financial Times, it turns out that choices below 15 come predominantly from subjects with a reasoning level $k \geq 6$, whereas for the Spektrum and Expansión treatments, choices in this range are partly attributed to level-0 subjects. For Ballester, Rodriguez-Moral and Vorsatz (2022), the interior equilibrium causes that the choices between 10 and 30 are mainly assigned to subjects with a relatively higher reasoning level and that choices close to the extremes of the action space correspond almost exclusively to level-0 subjects.

Concluding discussion

This paper contributes to the literature on strategic models of bounded rationality and, in particular, on the level-k model, by proposing for a general class of guessing games a new maximum likelihood estimation procedure that uncovers the belief structure that underlies actual choices and by deriving the microeconomic/statistical foundations of this econometric approach. Section 2 generalizes p-beauty contests, which have been widely applied in laboratory experiments, by allowing for asymmetries and constants in the players' best reply functions. These anchored guessing games have a natural visualization in terms of networks. It is worth mentioning in this respect that the characterization of equilibrium uniqueness in Proposition 1 is ordinal (*i.e.* purely graph-theoretical). Section 3 studies the optimal choices under the level-k framework and derives the foundations of our estimation technique of Section 4, which allow us to reduce the effective number of mixture components to two. Throughout, the central assumption is that beliefs are independent of reasoning levels. We then find that if each subject assigns the same belief to all level-0 players in all roles, then the distributions of level-k choices, for $k \geq 1$, must have exactly the same shape as the unknown seed distribution (Theorem 1). If beliefs are not individually homogenous, the distributions of level-k choices, for $k \geq 1$, approach the shape of the seed distribution (Theorem 2). We also provide extra conditions on Theorem 2 so that the seed distribution is normal as the number of players n grows (Proposition 2). Using Theorems 1 and 2, Section 4 presents the maximum likelihood estimation procedure of a mixture model with a free-form seed distribution, with the underlying assumption that all level-k choice distributions, for $k \geq 1$, have the same shape as the seed distribution. We solve the maximization problem iteratively with the help of the expectation maximization algorithm, which turns out to be fast since we can rely in the maximization step on closed-form formulae. Finally, we illustrate the estimation technique on the data of Bosch-Domènech, Montalvo, Nagel, and Satorra (2002) and Ballester, Rodriguez-Moral and Vorsatz (2022). We identify clustered seeds in both data sets. Two main focal points appear for the anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2022). The first focal point is 80, which is precisely the unique anchor value of the game. The second focal point is 0, which is the lower bound of the strategy space.

In the remainder of this section, we adapt a mechanism design perspective. Apart from personal research interests and other factors like simplicity in terms of the number of player and the number of connections in the network, the following factors affect the quality of the estimation:

1. **Connectedness**. If a game is connected (in the associated network, there is a path between any two players), the choice of a player is potentially determined by her level-0 beliefs about all players. Theorem 2 requires that the dependency matrix $\mathbf{W}$ is primitive, which implies connectedness.

2. **Eigenvalue ratio** $|\lambda_2|/\lambda_1$. The estimation algorithm assumes common shapes for all reasoning levels $k \geq 1$, but Theorem 2 applies only in the limit. As we have seen in Section 3, games with a lower eigenvalue ratio theoretically provide better estimations because convergence emerges at lower reasoning levels.

3. **Heterogeneity**. As the game becomes more heterogeneous in terms of dependencies or anchors, it is richer in terms of variability across different roles. However, the assumption of individually homogeneous beliefs necessary for Theorem 1 is probably harder to sustain as heterogeneity increases. In those cases, one must rely on Theorem 2 and make sure that the eigenvalue ratio is moderate. In a similar fashion, if Theorem 2 does not apply (because the matrix is not primitive or, more generally, the eigenvalue ratio is close to one), one has to rely entirely on Theorem 1. Anchor value variability should then probably be avoided.

4. **Dominant eigenvalue** λ_1. Recall that lower values of λ_1 imply faster convergence towards the equilibrium as the reasoning level increases. Next, we establish an additional result, which shows that intermediate values of the dominant eigenvalue λ_1 may improve the quality of the estimation. The intuition is simple in the p-beauty contest when it is taken into account that $\lambda_1 = p$. Suppose that $p = 0.1$. This has two effects. First, it allows us to classify all observations in the interval $[10, 90]$ as level 0. Second, it reduces the choice range for all $k \geq 1$ to $[0, 10]$, which can make it difficult to correctly discriminate between these levels when players make mistakes. The same idea applies when p is too high. If $p = 0.9$, it not so easy to classify some choices as low-level. As we show below, $p = 2/3$ is a compromise eigenvalue when it is the objective to estimate 4 reasoning levels ($K = 3$).

To formalize the last point, let the *discriminating sets* of player i be the disjoint sets

$$D_i^k = \begin{cases} U_i^k \setminus U_i^{k+1} & \text{if} \quad k < K \\ U_i^K & \text{if} \quad k = K. \end{cases}$$

For $k < K$, the k-th discrimination set D_i^k for a player in role i contains a particular choice $x \in X$ if and only if the maximum reasoning level x can be assigned to is exactly k. And D_i^K is the set of choices with a maximum reasoning level of at least K. Intuitively, we would like to jointly maximize the size of all discriminating sets of all players subject to the interiority condition. For that we say that the dependency matrix $\mathbf{W}$ *admits interiority* if $\underline{x}\ (\mathbf{I} - \mathbf{W})\,\mathbf{1} \leq \overline{x}\ (\mathbf{I} - \mathbf{W})\,\mathbf{1}$, that is, if $\mathbf{W}$ is substochastic. We fix a baseline non-negative matrix $\mathbf{H}$ in the sense that the maximum of its row sums is one. Then, we solve the maximization problem for games with dependency matrix $\mathbf{W} = q\,\mathbf{H}$, where q is a scaling factor such that $\mathbf{W}$ admits interiority, that is, $q \in [0, 1]$. The dominant eigenvalue of the dependency matrix $\mathbf{W}$ is simply $\lambda_1(\mathbf{W}) = q\,\lambda_1(\mathbf{H})$. The following result assumes that $\mathbf{H}$ (and hence $\mathbf{W}$) is regular[21], which includes all anonymous games such as the p-beauty contest.

Proposition 3. *Fix a regular baseline matrix $\mathbf{H} \geq \mathbf{0}$. The optimal scale $\hat{q}$ that jointly maximizes the size of discriminating sets of the anchored guessing game with matrix $\mathbf{W} = q\mathbf{H}$ is*

$$\hat{q} \equiv \underset{q\in[0,1]}{\operatorname{argmax}} \prod_{i=1}^{n} \prod_{k=0}^{K} |D_i^k| = \frac{K+1}{K+3}.$$

In order to illustrate how the before-mentioned factors may affect the quality of the estimation, we consider a family of anchored guessing games with two types of players. Players of type A have one out-link in the network represented by $\mathbf{W}$ and are also directly connected to an anchor. Players of type B have two neighbors in the network $\mathbf{W}$. We assume that $\underline{x} \leq 0 \leq \overline{x}$ and that the dependency matrix $\mathbf{W}$ has a zero main diagonal. The targets are

$$t_i = \begin{cases} q\,\frac{x_{i_1}+v_i}{2} & \text{if } i \text{ is of type A} \\ q\,\frac{x_{i_1}+x_{i_2}}{2} & \text{if } i \text{ is of type B,} \end{cases}$$

where $q > 0$ is a parameter, v_i is the anchor value of player i of type A, and i_j denotes a neighbor of player i in the dependency network $\mathbf{W}$. It is not difficult to see that the interiority assumption translates into $q \leq 1$ and for any type-A player i, $v_i \in [(2-q)\underline{x}/q, (2-q)\overline{x}/q]$.

Figure 4 shows all connected three-player and four-player games in this class. Type-A players are indicated by black dots and type-B players by white dots. The games are ordered by the network structure and the eigenvalue ratio $r = |\lambda_2|/\lambda_1$. The number of links in the network is e and the dominant eigenvalue λ_1 is shown as a function of the parameter q. For each game in the figure, $\hat{q}$ is the value of q that maximizes the objective function of Proposition 3 with $K = 3$.[22]

[21] A non-negative matrix $\mathbf{S}$ is regular if $\mathbf{S1} = s\mathbf{1}$ for some constant $s > 0$.

[22] We have numerically solved the maximization problem for each game.

By Proposition 1(b), all games with at least one type-A player must have a unique equilibrium (every player has a path to an anchor). And all games with only type-B players have a unique equilibrium ($\mathbf{x}^* = \mathbf{0}$) if and only if $q < 1$. Finally, Theorem 2 only applies if $r < 1$ (*i.e.*, $\mathbf{W}$ is primitive).[23] Interestingly, Theorem 2 cannot be applied to any of the "square" games.

Consider the first game in the second row of all "diamond" games, which consists of two players of each type. In this game, the eigenvalue ratio is $r = 0.62$, which is substantially bounded away from 1 so that shape similarity should ceterus paribus be observed at relatively early non-zero reasoning levels by Theorem 2. And it should be rather easy for subjects to understand the game. The number of links is relatively small ($e = 6$) and $\hat{q} \approx 3/4$. Finally, setting $v_1 = 0$ and $v_2 = 100$ does not impose too complex calculations on experimental subjects. This example shows how Figure 4 can be used to design anchored guessing games for laboratory experiments with reasonable estimation properties.

[23] The second game in the first row of all "diamond" games is an exception because $\mathbf{W}$ is not diagonalizable.

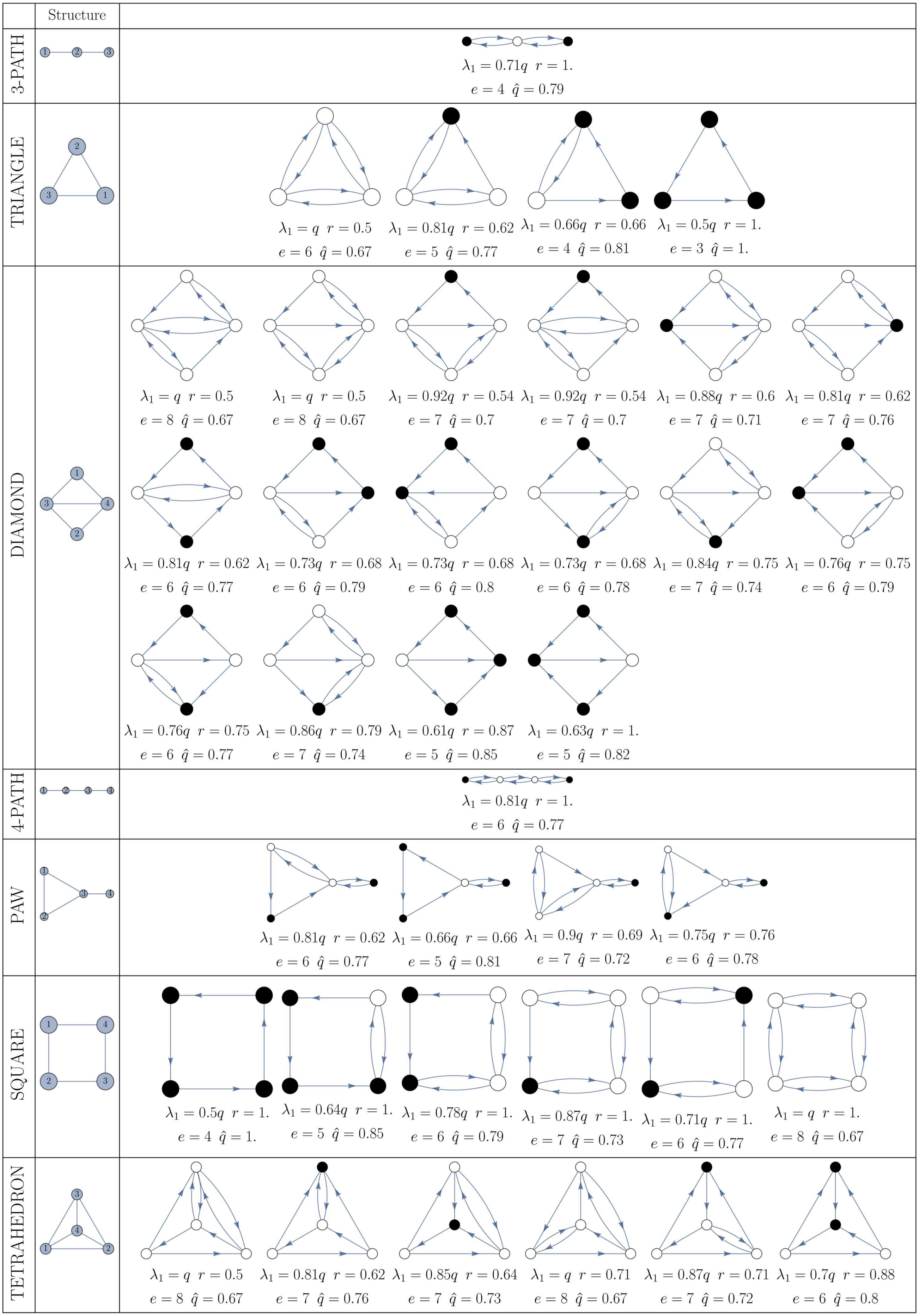


Figure 4: Three-player and four-player anchored guessing games.

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Appendix: Proofs

We establish a couple of auxiliary results before proving Proposition 1. A matrix is (row)-substochastic if all its entries are non-negative and all row sums are smaller than or equal to 1. If all row sums are exactly equal to 1, the matrix is called (row)-stochastic.

Lemma 1. *For all anchored guessing games,* $\mathbf{W}$ *is row-substochastic.*

Proof of Lemma 1. Suppose by contradiction that $\mathbf{W}$ is not row-substochastic. Then, there is a player $i \in N$ for whom $\sum_j w_{ij} \equiv r_i > 1$. By Assumption 1, $\underline{x}\,(\mathbf{I}-\mathbf{W})\,\mathbf{1} \leq \mathbf{A}\mathbf{v} \leq \overline{x}\,(\mathbf{I}-\mathbf{W})\,\mathbf{1}$. Considering the i-th row of this system of equations, we can see that $\underline{x}(1-r_i) \leq \overline{x}(1-r_i)$. Since $r_i > 1$, this contradicts the assumption that $\underline{x} < \overline{x}$. □

Lemma 2. *For all anchored guessing games, the dominant eigenvalue* λ_1 *of* $\mathbf{W}$ *satisfies* $\lambda_1 \in [0,1]$.

Proof of Lemma 2. First, $\lambda_1 \geq 0$ because $\mathbf{W} \geq \mathbf{0}$. Also, $\lambda_1 \leq 1$ because (a) λ_1 is monotone in $\mathbf{W}$ due to Perron-Frobenius theory, (b) $\lambda_1 = 1$ whenever $\mathbf{W}$ is row-stochastic, and (c) since $\mathbf{W}$ is row-substochastic by Lemma 1, it can be obtained by decreasing a stochastic matrix. □

Proof of Proposition 1.

(a) An anchored guessing game is smooth supermodular whenever for all $i \in N$, the strategy space X_i is a closed interval, the utility function u_i is twice continuously differentiable on X_i, and for all $j \in N \setminus \{i\}$, $\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \geq 0$. Since $X_i = [\underline{x}, \overline{x}]$, and $\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} = 2w_{ij} \geq 0$, the game is smooth supermodular. Equilibrium existence in pure strategies follows from Theorem 5 in Milgrom and Roberts (1990).

(b) Suppose that every player $i \in N$ has a path to an anchor. This, combined with the fact that $\mathbf{W}$ is row-substochastic by Lemma 1, implies that there is an integer k such that all row sums of $\mathbf{W}^k$ are strictly less than 1. The dominant eigenvalue of $\mathbf{W}^k$ is λ_1^k and it is bounded by the maximum of its row sums. Hence, $\lambda_1^k < 1$. This implies that $\lambda_1 < 1$. Consequently, $\mathbf{I} - \mathbf{W}$ has an inverse. By interiority, every equilibrium $\mathbf{x}^*$ must satisfy $\mathbf{x}^* = \mathbf{Av} + \mathbf{Wx}^*$. Then, $\mathbf{x}^* = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{Av}$ must be the unique equilibrium.

Suppose next that some player $i \in N$ does not have a path to an anchor. Consider the set $S \subseteq N$ of all such players. By construction, the square matrix $\mathbf{W}_S$, the restriction of $\mathbf{W}$ to the set S, is stochastic. Multiplicity of equilibria is attained as follows. For each $c \in [\underline{x}, \overline{x}]$, we construct an equilibrium $\mathbf{x}^*(c)$ such that $x_i^*(c) = c$ for all $i \in S$. To see this, note that the interiority assumption restricted to the set S becomes $\mathbf{0} \leq (\mathbf{Av})_S \leq \mathbf{0}$, which implies that $(\mathbf{Av})_S = \mathbf{0}$, that is, the target of each member of S is a weighted average of choices made within S. This means that all members of S are best-replying by choosing the number c. Now, taking this action c as a (possibly) new anchor for players in $N \setminus S$, these players choose their unique equilibrium strategies (since they all have a path to some anchor). Hence, each choice c by all the members of S determines an equilibrium of the entire game.

(c) Since there is a unique equilibrium by assumption, each player has a path to anchor by part (b). We have already seen in the proof of part (b) that a player has a path to an anchor if and only if the dominant eigenvalue λ_1 of $\mathbf{W}$ is such that $\lambda_1 < 1$. This implies that the matrix $\mathbf{I}-\mathbf{W}$ is invertible. Since best replies are interior by assumption, we have that the unique solution must be $\mathbf{x}^{*}=(\mathbf{I}-\mathbf{W})^{-1}\mathbf{A}\mathbf{v}$.

(d) Iterative convergence and dominance solvability follows from smooth supermodularity and equilibrium uniqueness. Global stability follows from iterative convergence and equilibrium uniqueness.

This concludes the proof of the proposition. $\square$

Proof of Theorem 1. Let s be a level-k subject playing role i. By the interiority assumption, for all $t \geq 1$,

$$\overline{\mathbf{x}}^{t}=\mathbf{A}\mathbf{v}+\mathbf{W}\overline{\mathbf{x}}^{t-1}$$

and

$$\underline{\mathbf{x}}^{t}=\mathbf{A}\mathbf{v}+\mathbf{W}\underline{\mathbf{x}}^{t-1}.$$

By Equation (1),

$$\mathbf{x}^{k(s)}-\underline{\mathbf{x}}^{k}=\mathbf{W}^{k}\left(\mathbf{e}^{(s)}-\underline{\mathbf{x}}^{0}\right)=(e^{(s)}-\underline{x})\mathbf{W}^{k}\mathbf{1}$$

and

$$\overline{\mathbf{x}}^{k}-\underline{\mathbf{x}}^{k}=\mathbf{W}^{k}\left(\overline{\mathbf{x}}^{0}-\underline{\mathbf{x}}^{0}\right)=(\overline{x}-\underline{x})\mathbf{W}^{k}\mathbf{1},$$

where $\underline{\mathbf{x}}^{0}=\underline{x}\,\mathbf{1}$ and $\overline{\mathbf{x}}^{\mathbf{0}}=\overline{x}\,\mathbf{1}$ due to the fact that the strategy space is common, and $\mathbf{e}^{(s)}=e^{(s)}\,\mathbf{1}$

by individually homogeneous beliefs. Then,

$$\begin{aligned}
\mathbf{x}^{k(s)} - \underline{\mathbf{x}}^k &= \mathbf{W}^{k-1}\left(\mathbf{x}^{1(s)} - \underline{\mathbf{x}}^1\right) \\
&= \mathbf{W}^{k-1}\left(\mathbf{A}\mathbf{v} + e^{(s)}\mathbf{W}\mathbf{1} - \mathbf{A}\mathbf{v} - \mathbf{W}\underline{\mathbf{x}}^0\right) \\
&= \mathbf{W}^{k-1}\left(e^{(s)}\mathbf{W}\mathbf{1} - \underline{x}\mathbf{W}\mathbf{1}\right) \\
&= \left(e^{(s)} - \underline{x}\right)\mathbf{W}^k\mathbf{1} \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\left(\overline{x} - \underline{x}\right)\mathbf{W}^k\mathbf{1} \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\mathbf{W}^k\left(\overline{\mathbf{x}}^0 - \underline{\mathbf{x}}^0\right) \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\left(\overline{\mathbf{x}}^k - \underline{\mathbf{x}}^k\right).
\end{aligned}$$

The optimal choice $x_i^{k(s)}$ of subject s is the i-th entry of $\mathbf{x}^{k(s)}$. It follows then from the former equation that

$$\frac{x_i^{k(s)} - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} = \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}.$$

Since $\tilde{e}^{(i)}$ is independent of the reasoning level k by Assumption 2,

$$\frac{\tilde{x}_i^k - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} \overset{\mathcal{D}}{=} \frac{\tilde{e}^{(i)} - \underline{x}}{\overline{x} - \underline{x}}.$$

Thus, $\tilde{x}_i^k \overset{\mathcal{S}}{=} \tilde{e}^{(i)}$ for each role $i \in N$. □

Proof of Theorem 2. We have for each level-k subject s playing role i that

$$\begin{aligned}
\frac{x_i^{k(s)} - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} &= \frac{\left(\mathbf{W}^k\right)_i \left(\mathbf{e}^{(s)} - \underline{\mathbf{x}}^0\right)}{\left(\mathbf{W}^k\right)_i \left(\overline{\mathbf{x}}^0 - \underline{\mathbf{x}}^0\right)} \\
&= \frac{\sum_{j=1}^n (\mathbf{W}^k)_{ij} \left(e_j^{(s)} - \underline{x}\right)}{\sum_{j=1}^n (\mathbf{W}^k)_{ij} \left(\overline{x} - \underline{x}\right)} \\
&\underset{k\to\infty}{\rightarrow} \frac{\sum_{j=1}^n \gamma_j \left(e_j^{(s)} - \underline{x}\right)}{\sum_{j=1}^n \gamma_j \left(\overline{x} - \underline{x}\right)} \\
&= \frac{\sum_{j=1}^n \gamma_j e_j^{(s)} - \underline{x}}{\overline{x} - \underline{x}},
\end{aligned}$$

where $\gamma_j > 0$ for all $j \in N$ and $\sum_{j=1}^n \gamma_j = 1$. The limit follows because $\mathbf{W}$ is primitive and, by the Perron-Frobenius theorem, it has a unique dominant eigenvalue $\lambda_1 > 0$ ($\lambda_1 > |\lambda_j|$ for $j = 2, 3, \ldots, n$) whose associated right and left eigenvectors, $\mathbf{r}$ and $\boldsymbol{\gamma}$, with $\sum_{j=1}^n \gamma_j = 1$, are strictly positive. Thus, for high k, $(\mathbf{W}^k)_{ij} \sim C\, \lambda_1^k r_i \gamma_j$ for some constant $C > 0$ independent of k. By definition, $\tilde{\mu}^{(i)} = \sum_{j=1}^n \gamma_j\, \tilde{e}_j^{(i)}$ is the random variable of seeds with domain $[\underline{x}, \overline{x}]$. Since $\tilde{\mu}^{(i)}$ is independent of the reasoning level k by Assumption 2, it follows that

$$\frac{\tilde{x}_i^k - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} \overset{\mathcal{D}}{\to} \frac{\tilde{\mu}^{(i)} - \underline{x}}{\overline{x} - \underline{x}}.$$

This means that $\tilde{x}_i^k \overset{\mathcal{S}}{\to} \tilde{\mu}^{(i)}$ for each role $i \in N$. Indeed, converge is point-wise. □

Proof of Proposition 2. First, we apply the result of Theorem 2 to see that for all $i \in N$, $\tilde{x}_i^k \overset{\mathcal{S}}{\to} \tilde{\mu}^{(i)} = \sum_{j=1}^n \gamma_j\, \tilde{e}_j^{(i)}$, where $\gamma_j \geq 0$ for all $j \in N$ and $\sum_j \gamma_j = 1$. Let $m_j^{(i)} = \mathbb{E}[\tilde{e}_j^{(i)}]$, ${\sigma_j^{(i)}}^2 = \mathrm{Var}(\tilde{e}_j^{(i)})$ and $\mathbf{m}^{(i)} = (m_1^{(i)}, \ldots, m_n^{(i)})$. Also, define

$$m^{(i)} \equiv \mathbb{E}[\tilde{\mu}^{(i)}] = \sum_{j=1}^n \gamma_j\, m_j^{(i)}$$

and

$$s^{(i)^2} \equiv \mathrm{Var}(\tilde{\mu}^{(i)}) = \sum_j \gamma_j^2 \, \sigma_j^{(i)^2},$$

where the last equality follows because the random variables $\tilde{e}_1^{(i)}, \ldots, \tilde{e}_n^{(i)}$ are independent by assumption. We first show that $(\tilde{\mu}^{(i)} - m^{(i)})/s^{(i)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$ by applying Lyapunov's Central Limit Theorem to the independent (but not identically distributed) random variables $\gamma_1 \tilde{e}_1^{(i)}, \ldots, \gamma_n \tilde{e}_n^{(i)}$. To shorten notation, let $\tilde{\mathbf{z}}^{(i)} = \tilde{\mathbf{e}}^{(i)} - \mathbf{m}^{(i)}$. Lyapunov's CLT requires the existence of $\delta > 0$ (independent of n) such that

$$\lim_{n\to\infty} \frac{1}{s^{(i)^{2+\delta}}} \sum_{j=1}^{n} \mathbb{E}\left[\left| \gamma_j \, \tilde{e}_j^{(i)} - \gamma_j \, m_j^{(i)} \right|^{2+\delta} \right] = 0.$$

We find an upper bound that tends to zero. Note that $\tilde{\mathbf{z}}^{(i)}$ is bounded because both $\tilde{\mathbf{e}}^{(i)}$ and $\mathbf{m}^{(i)}$ are. So let $D \geq |\tilde{z}_j^{(i)}|$ for all $i, j \in N$, where D is fixed, independent of n. Then,

$$\begin{aligned}
\frac{1}{s^{(i)^{2+\delta}}} \sum_{j=1}^{n} \mathbb{E}\left[\left| \gamma_j \, \tilde{e}_j^{(i)} - \gamma_j \, m_j^{(i)} \right|^{2+\delta} \right] &= \frac{1}{s^{(i)^{2+\delta}}} \sum_{j=1}^{n} \gamma_j^{2+\delta} \, \mathbb{E}\left[\left| \tilde{z}_j^{(i)} \right|^{2+\delta} \right] \\
\leq \frac{1}{s^{(i)^{2+\delta}}} \sum_{j=1}^{n} (D \, \gamma_j)^{\delta} \, \gamma_j^2 \, \mathbb{E}\left[\left(\tilde{z}_j^{(i)} \right)^2 \right] &\leq \frac{1}{s^{(i)^{2+\delta}}} \, (D \, \bar{\gamma})^{\delta} \sum_{j=1}^{n} \gamma_j^2 \, \mathbb{E}\left[\left(\tilde{z}_j^{(i)} \right)^2 \right] \\
= \frac{1}{s^{(i)^{2+\delta}}} \, (D \, \bar{\gamma})^{\delta} \, s^{(i)^2} &= \left(\frac{D \, \bar{\gamma}}{s^{(i)}} \right)^{\delta}
\end{aligned}$$

Since D and δ are independent of n, Lyapunov's condition is satisfied if $\bar{\gamma}/s^{(i)} \to 0$ as $n \to \infty$. As $n \to \infty$, we have that

$$\frac{\bar{\gamma}}{s^{(i)}} = \frac{\bar{\gamma}}{\sqrt{\sum_j \gamma_j^2 \sigma_j^{(i)^2}}} = \frac{1}{\sqrt{\sum_j \overline{\gamma}_j^2 \sigma_j^{(i)^2}}} \to 0,$$

where the limit follows from the condition $\sum_j \overline{\gamma}_j^2 \sigma_j^{(i)^2} \to \infty$. Thus, $(\tilde{\mu}^{(i)} - m^{(i)})/s^{(i)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, which is equivalent to $\tilde{\mu}^{(i)} \xrightarrow{\mathcal{S}} \mathcal{N}(0,1)$. Since $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$ by Theorem 2, we obtain that $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \mathcal{N}(0,1)$. Then, since the normal distribution is a stable distribution (*i.e.*, a normal random variable remains normal after affine transformations), we conclude that $\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}$. In particular,

$$\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}\left(\underline{x}_i^k + \frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}}(m^{(i)} - \underline{x}), \left(\frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}}\right)^2 s^{(i)2}\right).$$

Finally, given that the support of $\tilde{x}_i^k$ is $U_i^k = [\underline{x}_i^k, \overline{x}_i^k]$, the limiting distribution of $\tilde{x}_i^k$ is the corresponding truncated normal in U_i^k. □

Proof of Corollary 1. We have that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \overline{\gamma}_j(n)^2 \, {\sigma_j^{(i)}}^2(n) \geq \varepsilon \lim_{n\to\infty} \sum_{j=1}^{n} \overline{\gamma}_j(n)^2 = \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2} \sum_{j=1}^{n} \gamma_j(n)^2 \geq \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2} \sum_{j=1}^{n} \frac{1}{n^2} =$$
$$= \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2 \, n} = \infty.$$

This concludes the proof. □

Proof of Proposition 3. The matrix $\mathbf{W}$ is regular and, therefore, $\mathbf{W1} = \lambda_1 \mathbf{1}$ by the Perron-Frobenius theorem, where λ_1 is the dominant eigenvalue of $\mathbf{W}$. Also $\lambda_1 = q$ because $\lambda_1(\mathbf{H}) = 1$ by the regularity of the baseline matrix $\mathbf{H}$. Since for all $i \in N$ and all $k \in \mathbb{N}_+$, U_i^{k+1} is a subinterval of U_i^k , the vector $\mathbf{D}^k \equiv (D_i^k)_{i=1}^n$ that contains the k-th discrimination sets of all players is equal to

$$\mathbf{D}^k = \begin{cases} [\underline{\mathbf{x}}^k, \underline{\mathbf{x}}^{k+1}] \cup [\overline{\mathbf{x}}^{k+1}, \overline{\mathbf{x}}^k] & \text{if} \quad k < K \\ \\ [\underline{\mathbf{x}}^K, \overline{\mathbf{x}}^K] & \text{if} \quad k = K. \end{cases}$$

Given a reasoning level $k < K$, the sizes of the k'th discrimination sets $|\mathbf{D}^k| \equiv (|D_i^k|)_{i=1}^n$ are given by the expression

$$\begin{aligned}|\mathbf{D}^k| &= (\underline{\mathbf{x}}^{k+1} - \underline{\mathbf{x}}^k) + (\overline{\mathbf{x}}^k - \overline{\mathbf{x}}^{k+1}) \\ &= (\overline{\mathbf{x}}^k - \underline{\mathbf{x}}^k) - (\overline{\mathbf{x}}^{k+1} - \underline{\mathbf{x}}^{k+1}) \\ &= (\overline{x} - \underline{x}) \left(\mathbf{W}^k \mathbf{1} - \mathbf{W}^{k+1}\mathbf{1}\right) \\ &= (\overline{x} - \underline{x})\lambda_1^k(1-\lambda_1)\mathbf{1}.\end{aligned}$$

Similarly, $|\mathbf{D}^K| = (\overline{x} - \underline{x})\lambda_1^K\mathbf{1}$. Hence,$|D_i^k|$ is independent of the player role i, which allows us to write the objective function to be maximized as follows:

$$L(\lambda_1; K) \equiv \prod_{i=1}^{n}\prod_{k=0}^{K}|D_i^k| = \left((\overline{x} - \underline{x})^{K+1}\lambda_1^{\frac{K(K+1)}{2}}(1-\lambda_1)^K\right)^n.$$

We finally show that the global maximum is attained at $\hat{\lambda}_1 = (K+1)/(K+3)$. Since $\lambda_1 \in [0,1]$, the base of the exponentiation in $L(\lambda_1; K)$ is non-negative on the whole domain of λ_1, which implies that the we can maximize instead the function $\tilde{L}(\lambda_1; K) = \lambda_1^{\frac{K(K+1)}{2}}(1-\lambda_1)^K$ in λ_1. It follows from straightforward calculus that the unique critical point of $\tilde{L}(\lambda_1; K)$ is $\hat{\lambda}_1 = (K+1)/(K+3)$. Finally, since $L(0;K) = L_2(1;K) = 0$ and since $\tilde{L}$ is continuous and non-negative on $\lambda_1 \in [0,1]$, it must be the case that $\hat{\lambda}_1 = (K+1)/(K+3)$ is a global maximum. □

Appendix: Restricted estimations

We consider two possilble restrictions on $L0$ behavior. In an $L0$-*fixed* estimation, a concrete f^0 is imposed. For example, if level-0 choices are assumed to be uniformly distributed, then we set for all $x \in [\underline{x}, \overline{x}]$ and all iterations t, $f^{0[t]}(x) = 1/(\overline{x} - \underline{x})$. Alternatively, in an $L0$-*consistent* estimation, the density f^0 of level-0 play is assumed to be equal to the density f of average beliefs. In this case, the average beliefs are correct and we substitute a_b^0 by a_b for all buckets $b = 1, \ldots, B$ and update the bucket areas in the M-step as

$$a_b^{[t+1]} = \frac{1}{L} \sum_{k=0}^{K} \sum_{\substack{l=1,\ldots,L \\ b^k(x_l)=b}} q_k^{[t]}(x_l).$$

Finally, if Proposition 2 applies, we may adopt a gaussian approach in which the algorithm has to estimate the parameters μ and σ of the normal density f of seeds.[24] [25]

For the sake of simplicity in the estimation, we do not truncate the normal distributions. We can still consider $L0$-consistent and $L0$-fixed estimations for the gaussian case. In the equations below, the $L0$-consistent estimation corresponds to the model parameter $\mathcal{C} = 0$, while an $L0$-fixed estimation is obtained if $\mathcal{C} = 1$. Let

$$z^k(x) = \underline{x} + \frac{\overline{x} - \underline{x}}{\overline{x}^k - \underline{x}^k}(x - \underline{x}^k)$$

be the normalization of $x \in U^k$ from $[\underline{x}^k, \overline{x}^k]$ to $[\underline{x}, \overline{x}]$. Start at time $t = 0$ with arbitrary parameter values $(p_k^{[0]})_{k=0}^K \in \Delta^K$, $\mu^{[0]} \in [\underline{x}, \overline{x}]$, and $\sigma^{[0]} > 0$. The E-step and M-step for the gaussian estimation are as follows:

- **E-step (expectation)**.

  From current parameter values available at time t, compute for each reasoning level k and

[24]In typical gaussian mixture models, gaussian level-0 beliefs imply gaussian choices of higher reasoning levels. We allow for any distribution of seeds and gaussian choices arise by the central limit result of Proposition 2.

[25]It is important to observe that a gaussian estimation with no restrictions on L0-behavior would generally lead to trivial solutions with $p_0 = 1$ because B buckets of f^0 tend to explain the data better alone than when mixed with gaussian choices derived from f for levels $k \geq 1$. Therefore, we only consider an L0-restricted framework for the gaussian estimation.

each observation x_l

$$q_k^{[t]}(x_l) = \frac{p_k^{[t]}\, f^{k[t]}(x_l)}{\sum_{k'=0}^{K} p_{k'}^{[t]}\, f^{k'[t]}(x_l)},$$

where for all $k \geq \mathcal{C}$,

$$f^k(x) = \begin{cases} \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k} \frac{1}{\sqrt{2\pi\sigma^{[t]2}}} \exp\left\{-\frac{\left(z^k(x)-\mu^{[t]}\right)^2}{2\sigma^{[t]2}}\right\} & \text{if } x \in U^k \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{C} = 1$, we have $f^{0[t]} = f^0$ (fixed ex-ante).

- **M-step (maximization)**.

  The parameters are updated.

  1. Update the level composition. For each $k \geq 0$,

  $$p_k^{[t+1]} = \frac{1}{L} \sum_{l=1}^{L} q_k^{[t]}(x_l).$$

  2. Update the gaussian parameters:

  $$\mu^{[t+1]} = \frac{1}{L\left(1-\mathcal{C}\, p_0^{[t+1]}\right)} \sum_{l=1}^{L} \sum_{k=\mathcal{C}}^{k(x_l)} q_k^{[t]}(x_l)\, z^k(x_l)$$

  and

  $$\sigma^{[t+1]2} = \frac{1}{L\left(1-\mathcal{C}\, p_0^{[t+1]}\right)} \sum_{l=1}^{L} \sum_{k=\mathcal{C}}^{k(x_l)} q_k^{[t]}(x_l)\ \left(z^k(x_l) - \mu^{[t+1]}\right)^2.$$

Figure 5 presents the results of the $L0$-fixed estimations for the data of section 5 when level-0 play is assumed to be distributed uniformly on the interval $[0, 100]$ (see, bottom-left panel). If we compare Figures 3 and 5, then it is evident that the level composition in the top-left panel changes substantially. Most strikingly, the percentage of level-0 subjects in the population is now much lower. The level composition has consequently a tendency to shift upwards for the $L0$-fixed estimation. An important question to be answered is whether the change in the level composition between the estimation procedures is mainly due to the different behavior of the level-0 subjects or whether this model assumption causes the estimated behavior of subjects with a higher reasoning level to be different. A comparison of the corresponding top-right panel is helpful in providing an answer to this question. One observes that the estimated seeds are rather robust between. The estimated histograms display peaks at similar locations under both methodologies, with a slightly less clustered distribution in the $L0$-fixed estimation. The explanation for the reduced clustering of the $L0$-fixed estimation is that this procedure is generally forced to classify more observations above level 0, which implies that the algorithm needs to figure out new beliefs that can generate the new observations that are not level 0. All in all, the shape of the choice distributions of non-zero levels does not vary too much between the two approaches and the differences in the level composition seems to be mainly driven by the level-0 subjects themselves.

Finally, Bosch-Domènech, Montalvo, Nagel and Satorra (2010) estimate a mixture model for the data from the newspaper experiments using generalized beta distributions when level-0 play is uniformly distributed on the interval [0,100], the estimated distributions are anchored at the theoretical play, the width of the support of the distributions is exogenous and equal to 20, and beliefs about level-0 play are common and set to 50 or 100, respectively. They find for the pooled data that about 30% of the subjects are level-0, about 10% are level-1, about 20% are level-2, and about 40% are level-∞. Our $L0$-fixed estimations assign between 15% and 20% of the choices to level-0 play, but finds similar percentages of level-1 and level-2 guesses as in Bosch-

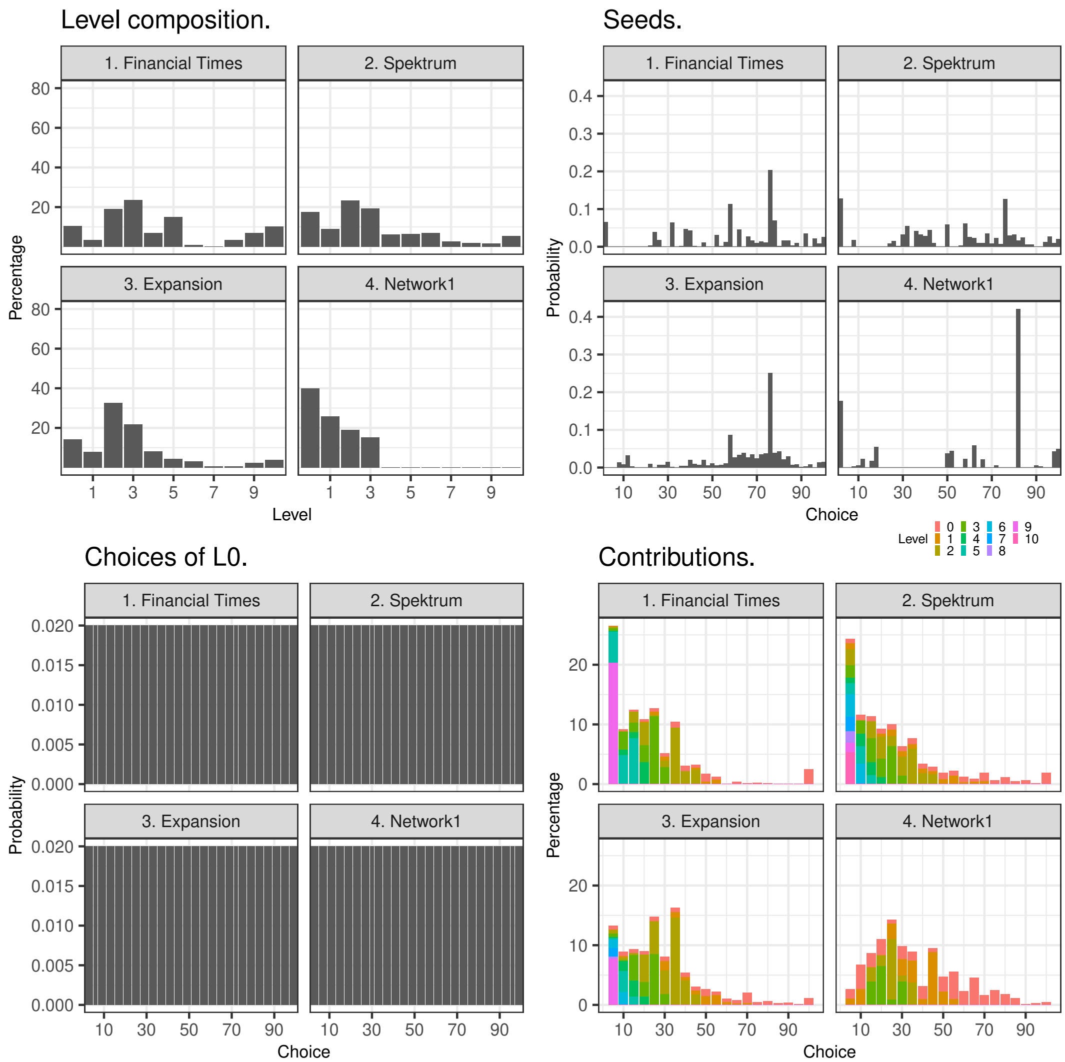

Figure 5: Estimation results for uniformly distributed $L0$ choices.

Domènech, Montalvo, Nagel and Satorra (2010). A direct comparison of higher levels play is not possible because Bosch-Domènech, Montalvo, Nagel and Satorra (2010) pool all remaining levels at infinity. Overall, and due to the different underlying model assumptions, the results of the two studies are difficult to compare.