Bertrand-Cournot Ranking Reversal for Optimal Privatization Level

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Abstract

We consider a differentiated product mixed duopoly market where a public and private firm compete in the downstream market. The public firm is partially privatized and a welfare maximizing regulator chooses the privatization level. The production of the final commodity requires a key input that is supplied by a foreign monopolist, in the upstream market, who may practice either uniform or discriminatory pricing. We show that with uniform pricing regime the privatization is always larger under Cournot competition while in case of discriminatory pricing regime, the privatization level under Bertrand competition is always larger. We also find that under discriminatory pricing regime, the Cournot-Bertrand rankings of other relevant variables are sensitive to the degree of substitutability.

Keywords: partially private firm, price (Bertrand) competition, quantity (Cournot) competition, optimal privatization, vertical market

JEL classification code: D4, D6, H4, L1,L2

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1 Introduction

Privatization is often a debatable issue in developing countries like India and China. Disputes related to the privatization policy of the government are very common in newspapers, news channels and social media. In recently years (January 2021) the Tata Group took full ownership of Air India that was earlier a public enterprise burdened with massive debt. Supporters of privatization generally have the following arguments.

(F-1) Public firms are generally inefficient than the private firm. Therefore, privatization may reduce inefficiency in terms of cost reduction (Choi [9]).

(F-2) Government can raise the fund to finance its deficits through the privatization policy (Emmanuaelle & Picard [12]).

On the other hand, opposition of the privatization have the following views against privatization:

(A-1) The privatization policy shift the objective of public firms from more welfare oriented to less. Consequently, we have the detrimental effect on the welfare of society (Matsumura [22]).

(A-2) Further, people generally have larger faith on the government than private organization therefore privatization policies may lead to trust issues thereby further destabilize the economy.

Generally, macroeconomists are more interested in verifying the validity of the statement (A-2) whereas the validity of the statement (F-2) is mainly the research area of the public finance. The industrial economists are generally interested in the trade off involved in statements (F-1) and (A-1). This paper is primarily a study of industrial organization hence our focus is also on the related trade off as indicated.

The comparison of Cournot and Bertrand competition with differentiated products, produced by two competing firms, comprise a large part of this literature. In this study we are going to consider one of these two competing firms is a public firm. If the level of privatization of the public firm is captured as a continuous variable then the concept of optimum privatization is of utmost relevance. When the regulator behaves optimally the level of privatization under Cournot competition may differ from that under Bertrand competition. Fujiwara [13] shows if firms compete under Cournot competition and there is no inefficiency gap between the public and private firms then the public firm becomes partially privatized. On the other hand keeping the framework fixed, Ohnishi [26] shows that the public firm becomes fully public under Bertrand competition.[1] Therefore, by comparing these two existing analysis we may conclude that with no inefficiency gap, the optimum privatization level, when the firms compete in quantity, is larger than that when the firms compete in price.

The structure of the model considered by both Fujiwara [13] and Ohnishi [26] are similar in terms of stages through which the sequence of events occurred. For both the model, in the first stage, the optimum privatization was determined and then the market competition had taken place. Whereas, we allow for a vertical structure by introducing a new stage between the privatization level determining stage and the market competition. In this new stage a foreign monopolist charges the optimum input price. Assuming the vertical structure of the model we show that the ranking of optimum privatization between Cournot and Bertrand competition depends on the regime of input pricing. The optimum privatization under Cournot competition is larger than Bertrand competition when we have the uniform input pricing. However, the optimum privatization under Bertrand competition is larger than Cournot competition when we have the discriminatory input pricing.

We also consider the Cournot-Bertrand rankings for other relevant market variables (for example prices, quantities, profits and welfare) separately for uniform and discriminatory pricing regime. Our results indicate that assuming uniform input pricing, the Cournot-Bertrand rankings for all most all the relevant equilibrium market outcomes are identical to the rankings that we have in Ghosh and Mitra [14]. However, assuming discriminatory pricing regime, the Cournot-

[1] A recent study by Mitra.et.al [21] establishes this conclusion for a wide class of demand functions. Moreover, Mitra.et.al [21] shows that if firms are allowed to endogenize the strategic variable then in equilibrium the price competition emerges and in equilibrium the partially private firm becomes fully public one.

Bertrand rankings for most of the market variables are sensitive to the degree of product substitution. Specifically, the Cournot-Bertrand rankings corresponding to market outcomes relating to the private firm and the welfare of society gets reversed with respect to the rankings that we have in Ghosh and Mitra [14] for low degree of product substitution.

These results are very important in the context of privatization literature. Our results identify a scenario in which the privatization is desirable for the society. Specifically it is due to the existence of foreign monopolist who supplies the input. If we have the vertical structure then the monopoly power of the input supplier would raise the degree of distortion faced by the society. Further, if the monopolist can discriminate, then it would exploit the private firm more than the public firm. Therefore to protect the private firm, the government can use privatization as the infant industry policy instrument. Our result is also important in terms of the Cournot-Bertrand comparison. Not only the existing Cournot-Bertrand ranking of optimum privatization (in absence of the vertical structure) gets reversed under vertical structure with price discrimination but if products are sufficiently differentiated then the Cournot-Bertrand rankings of all relevant market outcomes relating to the private firm and welfare of the society differ between uniform and discriminating pricing regime.

We also consider different extensions of our results to consider the robustness check. Firstly, we consider the extension through the introduction of operational inefficiency of the public firm. Secondly, we consider the extension through the mixed oligopoly structure. However, we get the same Cournot-Bertrand ranking in terms of optimal privatization along both the lines of extension. These are some relatable realistic scenarios that resemble the structure that we study here.

Our study is applicable to various mixed oligopoly industries where the foreign input is used as key input of that industry; specifically the Banking and Insurance industries where foreign capital is one very important key input. It is evident from the study of Chen.et.al [8] that privatization level indeed increased in case of the Chinese banking industry after allowing for the foreign equity. We can apply our model to the health sector as well. In most of the developing countries like India and China the health sector follows the mixed oligopoly structure and the specialized machineries are imported and often the public health sector follows P-P-P kind of structure for its functioning. We started with the example of Air India being recently privatized. Note that the Air India still operates a mixed fleet of Boeing and Airbus planes and so is true for most of the private airlines that operate in India.

The paper is organized as follows. We conclude this section with a brief discussion on the related literature. In section 2, the readers are introduced to the basic framework and the Game structure. In section 3, we present all our results. section 4, introduces two possible extensions of our main result. Finally in section 5, we conclude.

1.1 Related literature

Industrial economists are often interested in comparing different market structures based on respective market outcomes and then try to determine the best market structure considering either the society's welfare or the firm's profit and sometimes considering both. In this context, the Cournot-Bertrand comparison is one such important criterion that has often been analyzed in the literature of industrial economics. The first study with differentiated products was by Singh and Vives [30]. They conclude that under Cournot duopoly each firm in the industry produces less, charges more and earns higher profit than under Bertrand duopoly. Further, they argued that the latter is efficient than the former in terms of welfare ranking. We refer to these rankings as the standard rankings. Subsequent studies are mainly classified in two categories. One branch of literature focuses on verifying the robustness of the standard ranking (see Amir and Jin [2], Vives [33], Okuguchi [28], Hsu and Wang [17]). Other branch of the literature is interested in analyzing the circumstances where these standard rankings are either partially reversed or fully reversed. Moreover, we can classify the second branch in two different sub-categories. First sub-category focuses on private oligopoly market. In this direction, studies by Hackner [16] with quality differences; Mukherjee [24] and Cellini.et.al [7] with free entry; Symeonidis [31] and Lin and Saggi [19] with endogenous Research & Development expenditure; López and Naylor [20] in presence of the wage bargaining, provided evidence on partial reversal of the standard rankings. Arya.et.al [4] and Alipranti.et.al [1] have shown the complete reversal of the standard rankings with a vertically related producers. The relatively newer second sub-category focuses on mixed oligopoly market. The literature of mixed oligopoly became popular after Bȯs [6]. Then the study by Matsumura [22] opened up a new direction under mixed oligopoly with privatization. However, the Bertrand-Cournot comparison under mixed oligopoly was rather unexplored. Ghosh and Mitra [15] made the first attempt to introduce Cournot-Bertrand comparison in this context and found the complete reversal of the standard ranking.

Further, we can classify the literature of mixed oligopoly in two broad categories. The first category includes studies where issues other than the privatization get importance (see Choi [9], Choi [10], Dong and Wang [11], Mitra.et.al [21], Matsumura and Sunada [23], Scrimitore [29], Nakamura and Takami [25] and many more). The second category includes studies where privatization is further classified into two parts: privatization as a discrete variable (See Anderson et al. [3], Barcena-Ruiz and Garzon [5].) and privatization as a continuous variable (see Matsumura [22], Fujiwara [13], Ohnishi [26], Ohnishi [27], Wang and Chen [34], Wang and Chiou [35], Wang and Chiou [36], Wen and Yuan [37].).

2 The Framework

Consider a simple economy consisting of two sectors, namely: a competitive sector that produces a nueméraire commodity (money) and an imperfectly competitive sector that produces commodities that are not perfect substitutes. Further, the imperfectly competitive sector consists of two firms: one publicly regulated firm (Firm 0) and one private firm (Firm 1).

2.1 Demand Side

In this subsection we describe the demand side of the economy. The utility of the representative consumer is quasi-linear in the competitive sector's output and is given by $\mathcal{U}(q_0, q_1, y) = U(q_0, q_1) + y$ where for all $i = \{0, 1\}$, q_i be the consumption of quantity of output of Firm i and y be the consumption of quantity of output of the competitive sector. The sub-utility that depends on the commodity bundle purchased from imperfectly competitive sector is assumed to be quadratic and summarized by the following equation,

$$U(q_0, q_1) = a(q_0 + q_1) - \frac{1}{2}\left[q_0^2 + q_1^2 + 2sq_0q_1\right],\ a > 0,\ s \in (0, 1),$$

where a and s respectively represent the taste parameter and the parameter of degree of product substitution.[2] Therefore the representative consumer's problem is to maximize $\mathcal{U}(q_0, q_1, y) = U(q_0, q_1) + y$ by choosing (q_0, q_1, y) subject to $p_0q_0 + p_1q_1 + y \leq I$ where for all $i = \{0, 1\}$, p_i be the price of good i charged by Firm i and I be the income of the consumer. Given the quasi-linear utility function, the consumers' problem can be reduced to maximizing $U(q_0, q_1) - p_0q_0 - p_1q_1$ by choosing (q_0, q_1). Therefore from the first order condition of the consumer's optimization we have the inverse demand function that Firm i faces

$$P_i(q_0, q_1) = \frac{\partial U(q_0, q_1)}{\partial q_i} = a - q_i - sq_j \quad \forall i, j = 0, 1 \ \&\ i \neq j. \tag{1}$$

Given $s \in (0, 1)$ the inverse demand function is invertible and we can solve for q_i to obtained the direct demand function that Firm i faces

$$D_i(p_0, p_1) = \frac{a}{1+s} - \frac{p_i}{1-s^2} + \frac{sp_j}{1-s^2} \quad \forall i, j = 0, 1 \ \&\ i \neq j. \tag{2}$$

[2] Note that $s = 0$ and $s = 1$ respectively represent goods that are independent to each other and that are perfect substitutes. Therefore $s \in (0, 1)$ represents the case when goods are imperfect substitute to each other.

Therefore the consumer surplus in term of prices is

$$\overline{CS}(p_0, p_1) = U(D_0(p_0, p_1), D_1(p_0, p_1)) - p_0 D_0(p_0, p_1) - p_1 D_1(p_0, p_1). \tag{3}$$

Using the price-quantity duality, the consumer surplus in terms of quantities is

$$CS(q_0, q_1) = U(q_0, q_1) - P_0(q_0, q_1)q_0 - P_1(q_0, q_1)q_1. \tag{4}$$

2.2 Supply side

We assume the production of final commodity requires a key input on one-to-one basis. The key input is supplied by a foreign monopolist (Firm M) who applies the liner pricing rule. Suppose w_i denotes the price that Firm M charges to Firm i. Further, there is no other cost of production. Therefore, the cost of production of Firm i to produce q_i unit of quantity is $C_i(q_i; w_i) = w_i q_i$ for all $i = 0, 1$.[3] Therefore, the profit of Firm i in terms of quantities is

$$\pi_i(q_0, q_1; w_i) = P_i(q_0, q_1)q_i - C_i(q_i; w_i). \tag{5}$$

and using the price-quantity duality the profit of firm i in terms of prices is

$$\overline{\pi}_i(p_0, p_1; w_i) = p_i D_i(p_0, p_1) - C_i\left(D_i(p_0, p_1); w_i\right). \tag{6}$$

Assuming no cost of input production, the profit of the input monopolist is

$$\pi_M(q_0, q_1, w_0, w_1) = w_0 q_0 + w_1 q_1 \tag{7}$$

[3] One may consider slightly general type of technology where producing one unit of final commodity involves α unit of the key input along with some others input that cost β per unit. The cost function of firm $i = \{0, 1\}$ is $C_i(q_i) = (\alpha w_i + \beta)q_i$. However such modification will not essentially change our results. Hence we consider the simple one to one technology.

2.3 Welfare of the Society

The welfare of society is the sum of consumer surplus and total profit. Therefore the welfare in terms of quantities is

$$\begin{aligned} W(q_0, q_1; w_0, w_1) &= CS(q_0, q_1) + \pi_0(q_0, q_1; w_0) + \pi_1(q_0, q_1; w_1) \\ &= U(q_0, q_1) - w_0 q_0 - w_1 q_1. \end{aligned} \tag{8}$$

Similarly the welfare in terms of prices is

$$\begin{aligned} \overline{W}(p_0, p_1; w_0, w_1) &= \overline{CS}(p_0, p_1) + \overline{\pi}_0(p_0, p_1, w_0) + \overline{\pi}_1(p_0, p_1, w_1) \\ &= U(D_0(p_0, p_1), D_1(p_0, p_1)) - w_0 D_0(p_0, p_1) - w_1 D_1(p_0, p_1). \end{aligned} \tag{9}$$

2.4 Game Structure

The sequence of events are given by the following three stage game.

- **Stage-I** Regulator or planner chooses the optimal privatization ratio $\theta \in [0, 1]$ to maximize social welfare.
- **Stage-II** Firm M chooses input price to maximize own profit. We consider two regimes of input choice: uniform pricing and discriminatory pricing.
- **Stage-III** Firm 0 and Firm 1 compete in the market. We allow for the two usual modes of competition, namely, Cournot competition and Bertrand competition. In case of former, firms compete in quantity and for the latter firms compete in price.

We use backward induction method to solve this three-stage game separately for different regimes (uniform pricing and discriminatory pricing) of input pricing and different modes of competition (Cournot and Bertrand). Note that the Firm 0 is a publicly regulated firm where the instrument of regulation is the level of privatization (denoted by $\theta \in [0, 1]$). Given the choice of θ by the social planer at Stage-I, the payoff of Firm 0 is the weighted average of his own profit and the society's welfare where the weight attached to its profit is the privatization ratio (or level). Our objective is to compare between the Cournot and Bertrand competition for different regimes of input pricing and check how the privatization level and relevant market outcomes vary between the different input pricing regimes.

3 Results

In this section we list all our results with explanations.[4]

3.1 Uniform Pricing

Observation 1 Suppose the input monopolist practices uniform pricing in upstream market (that is, $w_1 = w_2 = w$) then we have the following key outcomes.

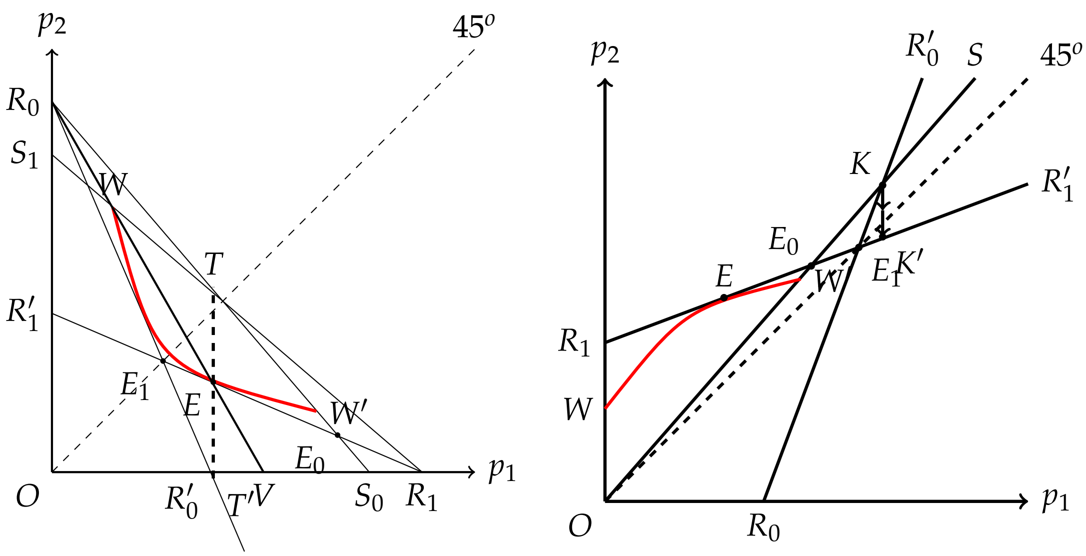
(i) In both Cournot and Bertrand the input monopolist charges $a/2$.

(ii) The social planner always partially privatizes the publicly regulated firm under Cournot competition. However, the publicly regulating firm becomes completely nationalized under Bertrand competition.

(iii) The Bertrand Cournot ranking of all the other market outcomes (such as price, quantity and profit of both the firms as well as the social welfare and Consumer surplus) remains unaltered as in Ghosh and Mitra [14]. However, as the products become highly substuitutable ($s > 0.9063$), the (Cournot-Bertrand) ranking corresponding to the profit of Firm 0 as well as the ranking Consumer surplus gets reversed.

We know that under vertical structure the optimum input price is determined by double marginalization and the source of double marginalization is the ratio of total input demand to marginal input demand (see Tirole[32], page 169). For Cournot and Bertrand competition with linear demand, this ratio is independent of the privatization level. Therefore the monopolist optimally

[4]Proofs are available in the Appendix.



(a) Cournot (b) Bertrand

Figure 1: Optimum privatization determination with uniform pricing.

charges $a/2$ which is independent of the level of privatization. The optimum privatization level is same as Fujiwara [13] for Cournot competition and Ohnishi [26] for Bertrand competition. The optimal privatization levels for Cournot and Bertrand competition are illustrated by figure (1a) and figure (1b) respectively (a detailed explanation of the same is available in Mitra et al.[21]). Finally, rankings other relevant market outcomes (except the consumer surplus and the profit of Firm 0) are just as found by Ghosh and Mitra [14], since the privatization level (of Firm 0) under Cournot competition, though positive, is not sufficient enough to alter the Bertrand and Cournot rankings.

3.2 Discriminatory pricing regime:

Suppose the foreign monopolist is able to discriminate between firms on the basis of input pricing. Unlike the uniform pricing regime, here the privatization level has a significant role in determining the input prices (only for Bertrand competition) as described below in Lemma 1.

Lemma 1 (i) Under Cournot competition, the foreign monopolist input supplier does not discriminate in terms of input prices and optimally charges $a/2$.

(ii) Under Bertrand competition, given any $\theta \in [0,1)$ set by the regulator, the foreign monopolist discriminates between firms through input prices. If $w_i^{BD}(\theta)$ denotes the input price of Firm i (where $i \in \{0,1\}$), then we have $w_0^{BD}(\theta) < w^{BI} < w_1^{BD}(\theta)$ where $w^{BI} = a/2$, the optimum input price charged by the monopolist under uniform pricing.

(iii) Finally, the ratio $\omega(\theta) = (w_1^{BD}(\theta) - w^{BI})/(w^{BI} - w_0^{BD}(\theta))$ is decreasing in $\theta \in (0,1)$.

Given the presence of one-to-one relation between inputs and outputs, the final stage quantities are nothing but the demands that the monopolist faces from two different downstream firms. The foreign monopolist with input-price discriminating capability acts like a multi-product monopolist (as the input demanded by downstream firms are imperfect substitutes). This is particularly so as the final commodities are imperfect substitutes.

Under Cournot competition in Stage-III, the quantities of Firm 0 and Firm 1 are respectively

$$q_0^{CD}(w_0, w_1, \theta) = \frac{2(a - w_0) - s(a - w_1)}{2(1+\theta) - s^2} \tag{10}$$

and

$$q_1^{CD}(w_0, w_1, \theta) = \frac{(1+\theta)(a - w_1) - s(a - w_0)}{2(1+\theta) - s^2}. \tag{11}$$

These Stage-III optimal quantities are nothing but the demand functions faced by the monopolist in Stage-II. Observe that the cross effects of input prices are equal, that is, $(\partial q_0^{CD}/\partial w_1) = (\partial q_1^{CD}/\partial w_0) = s/[2(1+\theta) - s^2]$.[5] In case of discriminatory input pricing the final input price primarily depends on two factors a) difference in the technology used by downstream firms and b) cross effects of input prices. Yoshida [38] shows in absence of cross effect differences, input prices would differ due to difference in the technology used by downstream firms. Since there is no difference in technology for both the downstream firms and the cross effects are identical in our context, we find no price discrimination under Cournot competition.

On the other hand, if we have the Bertrand competition in Stage-III, then the demand functions

[5]The negative externality generated via reduction of the input price of private firm on the demand of the public firm is identical to the negative externality generated via reduction the input price of public firm on the demand of the private firm.

faced by the monopolist in Stage-II are given by the following conditions:

$$q_0^{BD}(w_0, w_1, \theta) = \frac{(2-s)(a-w_0) - s[1+(1-\theta)(1-s^2)](a-w_1)}{(1-s^2)[2(1+\theta)-s^2]} \tag{12}$$

and

$$q_1^{BD}(w_0, w_1, \theta) = \frac{[1+\theta(1-s^2)](a-w_1) - s(a-w_0)}{(1-s^2)[2(1+\theta)-s^2]}. \tag{13}$$

Note that, in case of Bertrand the cross effects are not equal, specifically, $(\partial q_0^{BD}/\partial w_1) > (\partial q_1^{BD}/\partial w_0) > 0$[6].

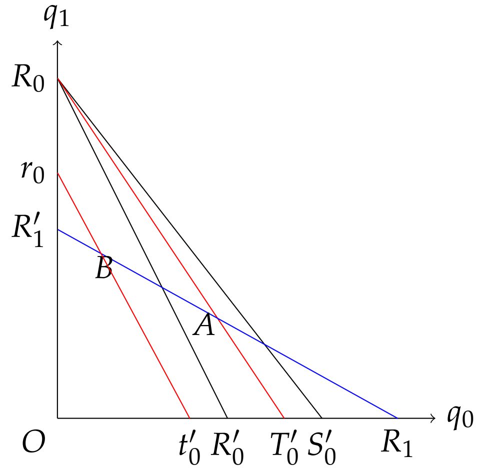
To explain the cross effect differences of the input prices on the Stage-III quantities specifically under Bertrand competition, we need to consider the reaction equation downstream firms separately for Cournot and Bertrand competition. Under Cournot competition in the Stage-III, the reaction equations of the public and private firms are respectively

$$P_0(\boldsymbol{q}) - w_0 + \theta q_0 \frac{\partial P_0}{\partial q_0}(\boldsymbol{q}) = 0 \text{ and } P_1(\boldsymbol{q}) - w_1 + q_1 \frac{\partial P_1}{\partial q_1}(\boldsymbol{q}) = 0.$$

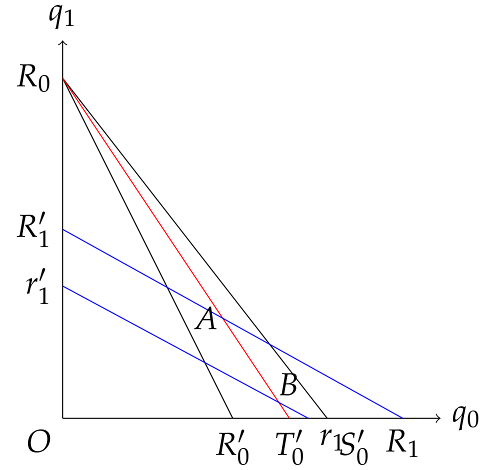
Note that the input price of Firm i only affects Firm i's reaction equation. To identify the cross effect consider the Figure 2. Left and right panel of Figure 2 are respectively explaining the effect of change in equilibrium due to w_0 and w_1. As in this case the quantities are strategic substitutes, in both the panels of Figure 2 the reaction curves are downward sloping.Given any w_0, R_0R_0' and R_0S_0' are respectively the reaction curves of Firm 0 corresponding to $\theta = 1$ and $\theta = 0$, therefore, for any $\theta \in [0, 1]$, R_0T_0' represents the reaction curve of Firm 0. Further, given any w_1, R_1R_1' represents the reaction curve of Firm 1. Finally, there interaction gives us the equilibrium point A for initial input price vector (w_0, w_1). Now as w_0 increases R_0T_0' shifts downward to r_0t_0' (Figure 2 (a)) and consequently the equilibrium point shift from A to B. Hence, q_1 increases due to w_0 increase via shift of the Firm 0's reaction curve only. Similarly, Figure 2 (b) explains how q_0 increases due to increase in w_1 via downward shift in Firm 1's reaction curve only. Therefore, under Cournot competition the cross effect of input price (w_j) on equilibrium quantity (q_i) is obtained due to

[6] In case of the Bertrand competition, the negative externality generated by reduction of the input price of private firm on the demand of the public firm is greater than the negative externality generated by reduction of the input price of public firm on the demand of the private firm.

the shift in Firm $i \in \{0,1\}$'s ($i \neq j$) reaction curve only. On the other hand, under Bertrand



(a) Change of w_0



(b) Change of w_1.

Figure 2: Explanation of the cross effect under Cournot competition

competition in Stage-III, the reaction equations of Firm 0 and Firm 1 are respectively

$$(p_0 - w_0)\frac{\partial D_0}{\partial p_0}(\boldsymbol{p}) + \theta D_0(\boldsymbol{p}) + (1-\theta)(p_1 - w_1)\frac{\partial D_1}{\partial p_0}(\boldsymbol{p}) = 0 \tag{14}$$

and

$$(p_1 - w_1)\frac{\partial D_1}{\partial p_1}(\boldsymbol{p}) + D_1(\boldsymbol{p}) = 0.$$

Using the equilibrium price vector $(p_0^{BD}(\boldsymbol{w},\theta), p_1^{BD}(\boldsymbol{w},\theta))$ and the demand function we get the quantities $(q_0^{BD}(\boldsymbol{w},\theta), q_1^{BD}(\boldsymbol{w},\theta))$. Note that w_0 affects only the reaction equation of Firm 0 therefore like Cournot competition the cross effect of w_0 on $q_1^{BD}(\boldsymbol{w},\theta)$ is obtained via the shift of the reaction curve of Firm 0 only. However, w_1 affects both the reaction equations. Therefore, the cross effect of w_1 on $q_0^{BD}(\boldsymbol{w},\theta)$ is obtained through two sources (i) like Cournot competition due to the shift of the reaction curve of Firm 1 and (ii) unlike Cournot competition due to the shift of the reaction curve of Firm 0. Therefore, the latter effect of w_1 variation on $q_0^{BD}(\boldsymbol{w},\theta)$ creates the cross effect differences and that is specifically captured by the term $T = (1-\theta)(p_1 - w_1)(\partial D_1(\boldsymbol{p}^{BD}(\boldsymbol{w},\theta))/\partial p_0)$. If w_1 increases then due to the law demand, $q_1^{BD}(\boldsymbol{w},\theta)$

decreases. Given the Firm 1's reaction equation one can write

$$p_1^{BD}(\boldsymbol{w},\theta) - w_1 = -\frac{1}{\partial D_1(\boldsymbol{p}^{BD}(\boldsymbol{w},\theta))/\partial p_1} q_1^{BD}(\boldsymbol{w},\theta)$$

hence, $p_1^{BD}(\boldsymbol{w},\theta) - w_1$ decreases due to w_1 increases[7]. Consequently, we have decrease in T. Hence, given w_0 fixed, decrease in T leads to either decrease in the negative first term of the condition (14) or increase in positive second term of the condition (14) to maintain the equality. Thus, under Bertrand competition, the impact w_1 increase affects $q_0^{BD}(\boldsymbol{w},\theta)$ positively through more than one channels explained above. Hence, $\partial q_0^{BD}/\partial w_1 > \partial q_1^{BD}/\partial w_0$.

The mechanism of input price discrimination lies on the comparison between monopolist optimization condition under uniform and discriminatory pricing. The uniform pricing problem is $\max_{(w_0,w_1)} \pi_M^{BD}(w_0,w_1,\theta) = w_0 q_0^{BD}(\boldsymbol{w},\theta) + w_1 q_1^{BD}(\boldsymbol{w},\theta)$ subject to $w_1 = w_0$. Therefore, if w^{BI} be the optimum input price for Bertrand competition then

$$\frac{\partial \pi_M^{BD}}{\partial w_0}(w^{BI},w^{BI},\theta) + \frac{\partial \pi_M^{BD}}{\partial w_1}(w^{BI},w^{BI},\theta) = 0$$

which implies the sum of the marginal profit with respect to w_0 and w_1 at optimum input price is zero. Therefore, either both the terms are individually zero or they are of different in sign[8]. These terms are respectively

$$\frac{\partial \pi_M^{BD}}{\partial w_0}(w^{BI},w^{BI},\theta) = q_0^{BD}(w^{BI},w^{BI},\theta) + w^{BI}\left(\frac{\partial q_0^{BD}}{\partial w_0} + \frac{\partial q_1^{BD}}{\partial w_0}\right) \tag{15}$$

and

$$\frac{\partial \pi_M^{BD}}{\partial w_1}(w^{BI},w^{BI},\theta) = q_1^{BD}(w^{BI},w^{BI},\theta) + w^{BI}\left(\frac{\partial q_1^{BD}}{\partial w_1} + \frac{\partial q_0^{BD}}{\partial w_1}\right). \tag{16}$$

Here in both the conditions (15) and (16) the first term on the right hand side is marginal term due to w_i change on $\pi_M^{BD}(w_0,w_1,\theta)$ at optimum uniform input price $w_0 = w_1 = w^{BI}$ that is

[7] This means p_1 increases lesser proportion to which w_1 increases.
[8] This is true under unconstrained optimization

$q_i^{BD}(w^{BI}, w^{BI}, \theta)$. Under uniform pricing public firm always produces more than private firm that is $q_0^{BD}(w^{BI}, w^{BI}, \theta) > q_1^{BD}(w^{BI}, w^{BI}, \theta) > 0$. Further, in both the conditions (15) and (16) the second term on the right hand side is infra-marginal term due to w_i change on $\pi_M^{BD}(w_0, w_1, \theta)$ at optimum uniform input price $w_0 = w_1 = w^{BI}$; specifically that is $w^{BI}[(\partial q_i^{BD}/\partial w_i) + (\partial q_j^{BD}/\partial w_i)]$. Since the own-price effect dominates the cross-price effect, both the infra-marginal terms are negative. Adding up the established fact $(\partial q_0^{BD}/\partial w_1) > (\partial q_1^{BD}/\partial w_0) > 0$ (evaluated at $w_0 = w_1 = w^{BI}$) we may say the infra-marginal term in condition (15) dominates that of the condition (16). Finally, if the cross effect difference is sufficiently high then the negative infra-marginal term of condition (16) may be dominated by the positive marginal term. Hence, we have $(\partial \pi_M^{BD}(w^{BI}, w^{BI}, \theta)/\partial w_1) > 0$ and at the same time $(\partial \pi_M^{BD}(w^{BI}, w^{BI}, \theta)/\partial w_0) < 0$. Consequently, we have $w_0^{BD} < w^{BI} < w_1^{BD}$.

Note that $\omega(\theta)$ as defined in Lemma 1(iii), is a measure of the degree of price discrimination. As the privatization level chosen by the regulator increases, the ideological difference between the two firms in competition reduces and as a result the term T weakens and finally as a consequence, the cross effect difference between two sources of input demands decreases. Thus the degree of price discrimination decreases.

Proposition 1 We have partial privatization under Cournot and Bertrand competition. Further the following results hold true:

(i) Given any $s \in [0, 1]$ and $a > 0$, the optimum privatization under Cournot is $\theta^{CD} = \theta^{CI}$.

(ii) Given any $s \in (0, 1)$ and $a > 0$, there exist an unique optimum privatization $\theta^{BD} \in (0, 1)$ under Bertrand competition.

(iii) Under Bertrand competition, the public firm is privatized more compared to Cournot competition.

We need to consider the expression of the social welfare to understand the determination of the optimal privatization for both the competition. The social welfare in terms of quantities can be reduced to $W(q_1, q_2) = U(q_1, q_2) - w_1 q_1 - w_2 q_2 = U(q_1, q_2) - \pi_M(q_1, q_2, w_0, w_1)$. The first term

$(U(q_1, q_2))$ represents the gross benefit that society received due to consumption of the bundle (q_1, q_2) and the second term $(\pi_M(q_1, q_2, w_0, w_1))$ represent the distortion that foreign monopolist brings in while providing the bundle (q_1, q_2) to the society. Given the fact that privatization shifts the objective of the public firm from benefiting society as a whole to its own private surplus, if the regulator increases the privatization (θ) then the gross benefit reduces via the increase in the oligopolistic distortion (overall out put reduction) that occurs in Stage-III. However, privatization also leads to reduction of the distortion by reducing the foreign monopolist's profit. Therefore, privatization policy in our context faces trade off between reduction of gross benefit and distortion. When θ is very low then the public firm produces very high amount and given the diminishing marginal utility, the loss of utility through due to output reduction gets more than compensated by the gain in distortion reduction, and consequently, full nationalization is not desirable for the society. Similarly, if θ is very high, the public firm produces very low amount and given the diminishing marginal utility, the benefit of utility increase due to output increase outruns the loss through distortion and therefore full privatization is not also desirable for the society. Hence, we have partial privatization for both the competition. Now for Bertrand competition the distortion $(\pi_M(\boldsymbol{q}^{BD}(\boldsymbol{w}^{BD}(\theta), \theta), \boldsymbol{w}^{BD}(\theta)))$ can be decomposed into two parts; (i) distortion due to double marginalization (i.e. $\pi_M(\boldsymbol{q}^{BD}(\boldsymbol{w}^{BI}, \theta), \boldsymbol{w}^{BI})$) and (ii) distortion due to price discrimination $(\pi_M(\boldsymbol{q}^{BD}(\boldsymbol{w}^{BD}(\theta), \theta), \boldsymbol{w}^{BD}(\theta)) - \pi_M(\boldsymbol{q}^{BD}(\boldsymbol{w}^{BI}, \theta), \boldsymbol{w}^{BI}))$. Note that the second type of distortion is not present under Cournot competition and following Lemma 1(iii), this distortion reduces with privatization. Therefore, to get these advantage the social planer privatizes more under Bertrand than under Cournot.

Proposition 2 While we compare the uniform input-pricing regime with the discriminating pricing regime, the following results hold ture:

(i) The Bertrand and Cournot rankings of all the market outcomes for the publicly regulated firm remain same.

(ii) If goods are sufficiently differentiated then the Bertrand and Cournot rankings of all the market outcomes related to private firm get altered.

(iii) If goods are sufficiently differentiated then the Bertrand and Cournot ranking of social welfare gets altered.

The explanation of the Proposition (2) comes from the comparison of the optimum privatization between the Cournot and Bertrand competition and how this comparison of optimum privatization varies with the degree of substitutablity. Recall that, the term responsible for input price discrimination is $T = (1-\theta)(p_1 - w_1)(\partial D_1/\partial p_0)$. Due to the quasi-linear utility structure the income effect on the demand of Firm 1 is absent and therefore the partial derivative $(\partial D_1/\partial p_0)$ represents the Hicksian substitution effect. Hence, $(\partial D_1/\partial p_0)$ must be positively related to the degree of substitution (s). Therefore, for lower degree of substitution, T is low. Consequently we have lesser price discrimination ($\omega(\theta)$ is low). Therefore, monopolistic exploitation via price discrimination is low and so is the distortion. Hence, a low θ would bring higher benefits to the society by increasing utility to consumers. These comparison is shown in Figure 3. It is quite evident from Figure 3 that for low value of the degree of substitutability, the difference between the optimal privatization under Bertrand and that under Cournot is not very large. However, the difference is quite significant when the degree of substitutability is high enough. Therefore, at a lower degree of production substitution, the privatization as an *infant industry support* to the private firm, is not very effective under Bertrand competition (as compared to Cournot competition). In price discrimination under Bertrand competition, the behavior of the private firm changes quite drastically as the degree of substitutability varies drastically (low to high). Hence, we have the ranking reversal for private firm variables. Observe that the privatization has direct bearing over the public firm and the input-price discrimination takes the public firm in relatively advantageous position than the private firm and the behavior of the public firm remains unchanged under Bertrand competition leading to same ranking for the public firm's variables. The ranking reversal of the private firm affects the society to a significant extent and as a result we observe the reversal of the social welfare ranking.

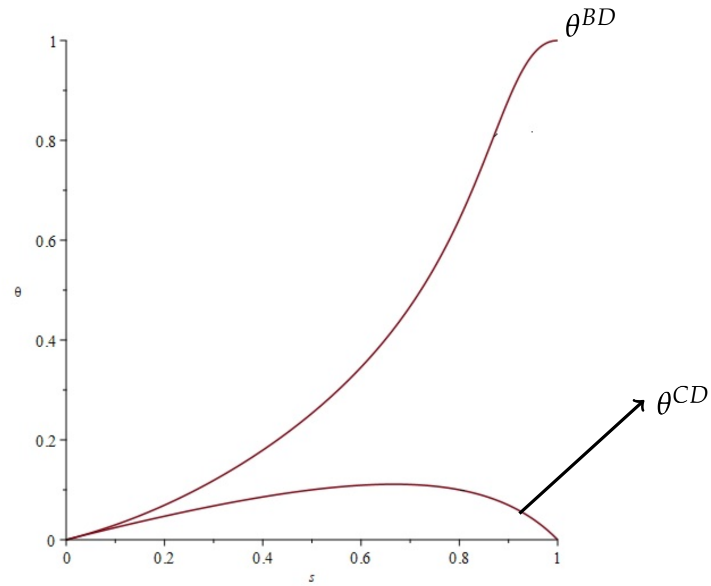


Figure 3: Optimum Privatization level: Cournot vs Bertrand.

4 Extension and Discussion

The key difference of the our analysis and the existing literature is under discriminating pricing regime the optimum privatization is more under Bertrand competition than under Cournot competition. Here in this section we check the robustness of this fact under different plausible extension of our analysis.

4.1 Inefficient Public Firm

Suppose the demand structure remains same but consider a general type of asymmetric technology such that produce one unit of final commodity Firm $i \in \{0, 1\}$ requires α_i unit of key input supplied by foreign monopolist along with some other inputs which cost β_i per unit of output to Firm i. Therefore, the cost function of Firm i is $C_i(q_i) = (\alpha_i w_i + \beta_i)q_i$. We assume Firm 0 is inefficient firm. Here we can classify three types of inefficiency of Firm 0 (for detail see Yoshida [38]).

(i) **Only α-inefficiency:** In this case we have $\alpha_0 \geq \alpha_1$ and $\beta_0 = \beta_1$.

(ii) **Only β-inefficiency:** In this case we have $\alpha_0 = \alpha_1$ and $\beta_0 > \beta_1$.

(iii) **Only $\alpha\beta$-inefficiency:** In this case we have $\alpha_0 \geq \alpha_1$ and $\beta_0 \geq \beta_1$.

To keep the analysis simple we us the following normalization, $\alpha_1 = 1$ and $\beta_1 = 0$. Further to keep the notation simple we denote $\alpha_0 = \alpha \geq 1$ and $\beta_0 = \beta \geq 0$. Therefore we assume the cost function of Firm 0 is $C_0(q_0) = (\alpha w_0 + \beta)q_0$ and that Firm 1 is $C_1(q_1) = w_1 q_1$. Let us introduce $w_i^{CD}(\theta)$ and $w_i^{BD}(\theta)$ for all $i \in \{0,1\}$ that respectively denote the Stage-II input price under discriminating pricing that the monopolist will charged to Firm i under Cournot and Bertrand competition. Now under discriminating pricing the Stage-III choices depends on the effective input prices αw_0 and w_1 respectively for Firm 0 and 1. Further, the monopolist's profit in the Stage-II also depends on the effective input prices. Therefore in the Stage-II the monopolist determine the effective input prices. Hence in the determination of the Stage-I optimum privatization the effective input prices only matters and not the individual input prices. Hence one can ignore the α-inefficiency and focus on the β-inefficiency only to check the ranking between the Cournot and Bertrand ranking in terms of optimum privatization. Using only β-inefficiency we do the simulation to check the ranking. The simulation results is summarized in the Table 1 and 2 (see the Appendix). The comparison of Table 1 and 2 reveals that the privatization is always higher under Bertrand that the Cournot.

4.2 Mixed Oligopoly

Consider the following extension of our model in case of mixed oligopoly structure. Assume that in the imperfectly competitive sector there are total $N+1$ number of firms with $N \geq 1$ and S denotes set of all firms. Further, assume that the Firm 0 is the only publicly regulated firm and any Firm $i \in S \setminus \{0\}$ is the private firm. Therefore, the utility function of the representative consumer changes to $V(\boldsymbol{q}, y) = U(\boldsymbol{q}) + y$ where $\boldsymbol{q} = (q_0, q_1, \ldots, q_N)$ be the vector of imperfectly competitive sector. The function $U(\boldsymbol{q})$ is a quadratic and given by

$$U(\boldsymbol{q}) = a\sum_{j\in S} q_j - \frac{1}{2}\left[\sum_{j\in S} q_j^2 + s\sum_{j\in S}\sum_{j'\in S\setminus\{j\}} q_j q_{j'}\right].$$

The inverse demand function that Firm $i \in S$ faces is $P_i(\boldsymbol{q}) = a - q_i - s\sum_{j\in S\setminus\{i\}} q_j$. We assume that the own effect dominates the sum of the cross effects in absolute terms that is $1 - Ns > 0$. Given $Ns < 1$ and $s \in (0,1)$ one can show that under discriminating regime the privatization ratio under Bertrand competition is always larger than that under Cournot competition. This happens due to the fact that when we have N private firms then the part of the infra-marginal term of the reaction equation of the public firm, that is originated due to presence of the welfare maximizing character of the public firm, increases with the number of private firm.[9] These leads to the larger price discrimination. Therefore, to protect the private firms from the negative effect of price discrimination, we have the larger privatization under Bertrand competition.

5 Conclusion

The findings of this analysis are relevant to the literature of Cournot-Bertrand comparison and privatization policy. In terms of privatization policy the study suggests that privatization is a desirable policy to the society when we have vertical structure and the practice of price discrimination in the input market. Further, the optimal privatization policy differs in terms of the mode of competition prevailing in the market. Whether a partially private firm in a duopolistic or oligopolistic market structure should be further privatized or not is much linked to several factors like the source of input of the production is a foreign monopolist firm, what kind of pricing strategy is used by the foreign firm and what is the mode of competition between the oligopoly firms. The crucial finding of our analysis is the difference of the privatization policy between different of downstream competition when regime of input pricing changes. These fact leads to

[9]That is, the counterpart of the term $(1-\theta)(p_1 - w_1)\frac{\partial D_1(p_0,p_1)}{\partial p_0} / \frac{\partial D_0(p_0,p_1)}{\partial p_0}$ becomes $(1-\theta)\sum_{i\in S\setminus\{0\}}(p_i - w_i)\frac{\partial D_i(p_0,p_1)}{\partial p_0} / \frac{\partial D_0(p_0,p_1)}{\partial p_0}$ which is increasing in N since in Stage-II choices $w_i = w$ and $p_i = p$ for all $i \in S \setminus \{0\}$.

the changes in the Cournot-Bertrand ranking of some of the important market outcomes for high degree of product differentiation. Apparently the linear demand structure assumed through out our analysis might highlight the limitation of our findings. However, when the demand system varies, these findings can only be violated under the condition that the aggregate effect of the second order terms works in opposite to the aggregate effect of the first order terms and the former will dominate the latter. We also have applied robustness check of our findings in this paper under different plausible extension setup by introducing inefficient partially privatized public firm or by introducing mixed oligopoly where we have multiple private firms along with the partially privatized public firm along with the input supplier who is still a foreign monopolist. However the findings of these extensions are very much aligned with the findings under the simplistic structure introduced in the paper.

A Appendix

Proof of the Observation 1:
Proof of Observation 1 (i):
Derivation of the input price under Cournot: If in Stage-III firms competing with Cournot competition, then given the quantity of Firm 1, q_1, Firm 0 will maximize

$$V^I(q_0, q_1; w, \theta) = V(q_0, q_1; w, w, \theta) = \theta\pi_0(q_0, q_1; w) + (1-\theta)W(q_0, q_1; w, w),$$

by choosing its own output q_0; given quantity of Firm 0, q_0, Firm 1 will maximize $\pi_1(q_0, q_1; w)$ by choosing its output q_1. Given any $\theta \in [0, 1]$ of Stage-I and w of Stage-II, if $(q_0^{CI}(w, \theta), q_1^{CI}(w, \theta))$ be the Stage-III choice vector then $(q_0^{CI}(w, \theta), q_1^{CI}(w, \theta))$ simultaneously satisfy the following reaction equations

$$\frac{\partial V^I}{\partial q_0}(q_0, q_1, w, \theta) = P_0(q_0, q_1) - w + \theta \frac{\partial P_0}{\partial q_0}(q_0, q_1) = 0 \tag{17}$$

and

$$\frac{\partial \pi_1}{\partial q_1}(q_0, q_1, w) = P_1(q_0, q_1) - w + q_1 \frac{\partial P_1}{\partial q_1}(q_0, q_1) = 0. \tag{18}$$

Evaluating condition (17) and (18) at $(q_0^{CI}(w, \theta), q_1^{CI}(w, \theta))$ we get the system of equations involving the Stage-III choices are

$$\begin{aligned}(1+\theta)q_0^{CI}(w,\theta)+sq_1^{CI}(w,\theta)&=a-w\\ sq_0^{CI}(w,\theta)+2q_1^{CI}(w,\theta)&=a-w\end{aligned}$$

Solving for quantities we have

$$q_0^{CI}(w,\theta)=\frac{(2-s)(a-w)}{2(1+\theta)-s^2} \quad (19)$$

and

$$q_1^{CI}(w,\theta)=\frac{(1+\theta-s)(a-w)}{2(1+\theta)-s^2}. \quad (20)$$

Therefore, substituting in the equation (7) we get

$$\pi_M^{CI}(w,\theta)=\pi_M(q_0^{CI}(w;\theta),q_1^{CI}(w;\theta);w,w)=\frac{w(a-w)(3+\theta-2s)}{2(1+\theta)-s^2} \quad (21)$$

Differentiating the condition (21) with respect to w and evaluating at $w=w^{CI}$ then setting equal to zero we get $(\partial\pi_M^{CI}\left(w=w^{CI},\theta\right)/\partial w)=\left(a-2w^{CI}\right)(3+\theta-2s)/[2(1+\theta)-s^2]=0$. Solving for w^{CI} we have $w^{CI}=a/2$. Note that $\partial^2\pi_M^{CI}/\partial w^2=-2(3+\theta-2s)/[2(1+\theta)-s^2]<0$. Hence the second order condition also satisfied for Stage-II.

Derivation of the input price under Bertrand: If in Stage-III firms competing with Bertrand competition, then given the price of Firm 1, p_1, Firm 0 will maximize

$$\overline{V}^I(p_0,p_1,w,\theta)=\overline{V}(p_0,p_1;w,w,\theta)=\theta\overline{\pi}_0(p_0,p_1;w)+(1-\theta)\overline{W}(p_0,p_1;w,w),$$

by choosing its price p_0 and given price of Firm 0, p_0, Firm 1 will maximize $\overline{\pi}_1(p_0,p_1;w)$, by choosing its own price p_1. Given any $\theta\in[0,1]$ of Stage-I and w of Stage-II, if $(p_0^{BI}(w,\theta),p_1^{BI}(w,\theta,))$ be the Stage-III choice vector then $(p_0^{BI}(w,\theta),p_1^{BI}(w,\theta))$ simultaneously satisfy the following reaction equations

$$\frac{\partial\overline{V}^I}{\partial p_0}(p_0,p_1,w,\theta)=\theta D_0(p_0,p_1)+(p_0-w)\frac{\partial D_0}{\partial p_0}(p_0,p_1)+(1-\theta)(p_1-w)\frac{\partial D_1}{\partial p_0}(p_0,p_1)=0 \quad (22)$$

and

$$\frac{\partial\overline{\pi}_1}{\partial p_1}(p_0,p_1,w)=D_1(p_0,p_1)+(p_1-w)\frac{\partial D_1}{\partial p_0}(p_0,p_1)=0 \quad (23)$$

Evaluating condition (22) and (23) at $(p_0^{BI}(w,\theta),p_1^{BI}(w,\theta))$ we get the system of equations involving the Stage-III choices are

$$\begin{aligned}(1+\theta)p_0^{BI}(w,\theta) - sp_1^{BI}(w,\theta) &= \theta(1-s)a + (1-s+\theta s)w \\ -sp_0^{BI}(w,\theta) + 2p_1^{BI}(w,\theta) &= (1-s)a + w\end{aligned}$$

Solving for prices we have

$$p_0^{BI}(w,\theta) = w + \frac{(1-s)(2\theta+s)(a-w)}{2(1+\theta)-s^2} \tag{24}$$

and

$$p_1^{BI}(w,\theta) = w + \frac{(1-s)(1+\theta+\theta s)(a-w)}{2(1+\theta)-s^2}. \tag{25}$$

Therefore the resulting quantities are

$$q_0^{BI}(w,\theta) = D_0\left(p_0^{BI}(w,\theta), p_1^{BI}(w,\theta)\right) = \frac{\left[2-(1-\theta)s^2+\theta s\right](a-w)}{(1+s)\left[2(1+\theta)-s^2\right]} \tag{26}$$

and

$$q_1^{BI}(w,\theta) = D_1\left(p_0^{BI}(w,\theta), p_1^{BI}(w,\theta)\right) = \frac{(1+\theta+s\theta)(a-w)}{(1+s)\left[2(1+\theta)-s^2\right]}. \tag{27}$$

Therefore, substituting in the equation (7) we get,

$$\pi_M^{BI}(w,\theta) = \pi_M(q_0^{BI}(w;\theta), q_1^{BI}(w;\theta); w, w) = \frac{w(a-w)\left[3+\theta+2s\theta-(1-\theta)s^2\right]}{(1+s)\left[2(1+\theta)-s^2\right]}. \tag{28}$$

Differentiating the condition (28), with respect to w and evaluating at $w = w^{BI}$ then setting equal to zero we get $(\partial\pi_M^{BI}(w = w^{BI},\theta)/\partial w) = (a - 2w^{BI})\left[3+\theta+2s\theta-(1-\theta)s^2\right]/(1+s)\left[2(1+\theta)-s^2\right] = 0$. Solving for w^{BI}, we have $w^{BI} = a/2$. Note that $(\partial^2\pi_M^{BI}/\partial w^2) = -2\left[3+\theta+2s\theta-(1-\theta)s^2\right]/(1+s)\left[2(1+\theta)-s^2\right] < 0$. Hence, the second order condition also satisfied for Stage-II.

Hence, Observation 1 (i)

Proof of Observation 1 (ii):

Optimum privatization under Cournot competition: Substituting the optimum input price in the condition (19) and (20), we will respectively have the resulting quantities $q_0^{CI}(w^{CI},\theta) = (2-s)a/2\left[2(1+\theta)-s^2\right]$ and $q_1^{CI}(w^{CI},\theta) = (1+\theta-s)a/2\left[2(1+\theta)-s^2\right]$. Substituting in the equation (8) we will get the resulting welfare

$$\hat{W}^{CI}(\theta) = \frac{\left[3\theta^2+2(7-5s)\theta+(7-6s-2s^2+2s^3)\right]a^2}{8\left[2(1+\theta)-s^2\right]^2}. \tag{29}$$

Differentiation with respect to θ and evaluating at $\theta = \theta^{CI}$, then setting equals to zero, we get $\left(\partial \hat{W}^{CI}\left(\theta = \theta^{CI}\right)/\partial\theta\right) = (2-s)\left[s(1-s) - (4-3s)\theta\right]/4\left[2(1+\theta) - s^2\right]^3 = 0$. Solving for θ^{CI}, we get $\theta^{CI} = s(1-s)/(4-3s)$.

Optimum privatization under Bertrand competition: Substituting the optimum input price w^{BI} in the condition (26) and (27) we will have the resulting quantities respectively $q_0^{BI}(w^{BI}, \theta) = \frac{\left[2-(1-\theta)s^2+\theta s\right]a}{2(1+s)[2(1+\theta)-s^2]}$ and $q_1^{BI}(w^{BI}, \theta) = \frac{(1+\theta+s\theta)a}{2(1+s)[2(1+\theta)-s^2]}$. Substituting in the equation (8), we will get the resulting welfare

$$\hat{W}^{BI}(\theta) = \frac{\left[(1+s)(3+4s-3s^2)\theta^2 + (14-4s-3s^2)\theta + (7+s-7s^2-s^3+2s^4)\right]a^2}{8(1+s)\left[2(1+\theta)-s^2\right]^2}. \tag{30}$$

Differentiation with respect to θ we get $\left(\partial \hat{W}^{BI}(\theta)/\partial\theta\right) = -(2+s)(1-s)^3\left[s(1+s) + (4+3s)\theta\right]/4(1+s)\left[2(1+\theta) - s^2\right]^3 < 0$. Therefore, $\theta^{BI} = 0$.

Hence, Observation 1 (ii)

Proof of the Observation 1 (iii):

The equilibrium quantities produced by the publicly regulated firm are $\tilde{q}_0^{CI} = q_0^{CI}\left(w^{CI}, \theta^{CI}\right) = (2-s)(4-3s)a/2\left[8-4s-6s^2+3s^3\right]$ and $\tilde{q}_0^{BI} = q_0^{BI}\left(w^{BI}, \theta^{BI}\right) = a/2(1+s)$ respectively for Cournot and Bertrand competition. Therefore, for all $s \in (0,1)$, we have $\tilde{q}_0^{BI} - \tilde{q}_0^{CI} = -sa/2(4+4s-3s^2-3s^3) < 0$. Hence, proved.

The equilibrium prices charged by the publicly regulated firm under Cournot competition is $\tilde{p}_0^{CI} = P_0\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}\right) = \left(8-2s-9s^2+4\right)a/2\left[8-4s-6s^2+3s^3\right]$ and that under Bertrand competition is $\left(\tilde{p}_0^{BI}\right) = P_0^{BI}\left(w^{BI}, \theta^{BI}\right) = \left(2+s-2s^2\right)a/2(2-s^2)$. Therefore, for all $s \in (0,1)$, we have $\tilde{p}_0^{BI} - \tilde{p}_0^{CI} = s(2-3s-s^2+3s^3-s^4)a/(2-s^2)(8-4s-6s^2+3s^3) > 0$. Hence, proved.

The equilibrium profits earned by the publicly regulated firm are $\tilde{\pi}_0^{CI} = \pi_0\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}; w^{CI}\right) = s(1-s)(4-3s)a^2/4\left[4-3s^2\right]^2$ and $\tilde{\pi}_0^{BI} = \pi_0\left(\tilde{q}_0^{BI}, \tilde{q}_1^{BI}; w^{BI}\right) = s(1-s)a^2/4(1+s)(2-s^2)$ respectively for Cournot and Bertrand competition. Therefore, we have $\tilde{\pi}_0^{BI} - \tilde{\pi}_0^{CI} = s(1-s)(8-2s-14s^2+s^3+6s^4)a^2/4(1+s)(2-s^2)(4-3s^2)^2 \gtreqless 0$ for all $s \lesseqgtr \hat{s}_1$.[10] Hence proved.

The equilibrium quantities produced by the private firm are $\tilde{q}_1^{CI} = q_1^{CI}\left(w^{CI}, \theta^{CI}\right) = (2-3s+s^2)a/2\left[8-4s-6s^2+3s^3\right]$ and $\tilde{q}_1^{BI} = q_1^{BI}\left(w^{BI}, \theta^{BI}\right) = a/2(1+s)(2-s^2)$ respectively for Cournot and Bertrand competition. Therefore, we have for all $s \in (0,1)$, $\tilde{q}_1^{BI} - \tilde{q}_1^{CI} = s^2(3-2s^2)a/2(1+s)(2-s^2)(4-3s^2) > 0$. Hence, proved.

The equilibrium prices charged by the private firm under Cournot competition is $\tilde{p}_1^{CI} = P_1\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}\right) = \left(12-10s-4s^2+3\right)a/2\left[8-4s-6s^2+3s^3\right]$ and that under Bertrand competition is $\tilde{p}_1^{BI} = p_1^{BI}\left(w^{BI}, \theta^{BI}\right) = \left(3-s-s^2\right)a/2(2-s^2)$. Therefore, we have for all $s \in (0,1)$, $\tilde{p}_1^{BI} - \tilde{p}_1^{CI} = -s^2(2-3s+s^2)a/2(2-s)(2-s^2)(4-3s^2) < 0$. Hence, proved.

[10] We could not fully identify the value of $\hat{s}_1$ although it is a unqie positive real number and $\hat{s}_1 \in (0.8359, 0.8438)$.

The equilibrium profits earned by the private firm are $\tilde{\pi}_1^{CI} = \pi_1\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}; w^{CI}\right) = (2-3s+s^2)^2a^2/\left[8-4s-6s^2+3s^3\right]^2$ and $\tilde{\pi}_1^{BI} = \pi_1\left(\tilde{q}_0^{BI}, \tilde{q}_1^{BI}; w^{BI}\right) = (1-s)a^2/4(1+s)(2-s^2)^2$ respectively for Cournot and Bertrand competition. Therefore, we have for all $s \in (0,1)$, $\tilde{\pi}_1^{BI} - \tilde{\pi}_1^{CI} = s^2(1-s)(8-11s^2+4s^4)a^2/4(1+s)(2-s^2)^2(4-3s^2)^2 > 0$. Hence proved.
The equilibrium values of the consumer surplus are $\left(\tilde{CS}^{CI}\right) = CS\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}\right) = (5-4s)a^2/8(4-3s^2)$ and $\left(\tilde{CS}^{BI}\right) = CS\left(\tilde{q}_0^{BI}, \tilde{q}_1^{BI}\right) = (5-s-3s^2+s^3)a^2/8(1+s)(2-s^2)^2$ respectively for Cournot and Bertrand competition. Therefore, we have $\tilde{CS}^{BI} - \tilde{CS}^{CI} = s(1-s)(8-s-12s^2+4s^3)a^2/8(1+s)(2-s^2)^2(4-3s^2) \lesseqgtr 0$ for all $s \lesseqgtr \hat{s}_2$.[11] Hence proved.
The equilibrium values of the welfare are $\left(\tilde{W}^{CI}\right) = W\left(\tilde{q}_0^{CI}, \tilde{q}_1^{CI}, w^{CI}, w^{CI}\right) = (7-6s)a^2/8(4-3s^2)$ and $\tilde{W}^{BI} = W\left(\tilde{q}_0^{BI}, \tilde{q}_1^{BI}, w^{BI}, w^{BI}\right) = (7+s-7s^2-s^3+2s^4)a^2/8(1+s)(2-s^2)^2$ respectively for Cournot and Bertrand competition. Therefore, we have for all $s \in (0,1)$,

$$\tilde{W}^{BI} - \tilde{W}^{CI} = \frac{s^2(1-s)(3-2s^2)a^2}{8(1+s)(2-s^2)^2(4-3s^2)} > 0.$$

Hence proved.

Proof of the Lemma 1:
Proof of Lemma 1 (i) If in Stage-III firms competing with Cournot competition, then assuming the quantity of Firm 1 (q_1) is fixed, Firm 0 will maximize

$$V(q_0, q_1; w_0, w_1, \theta) = \theta\pi_0(q_0, q_1; w_0) + (1-\theta)W(q_0, q_1; w_0, w_1)$$

by choosing its own output q_0 and similarly assuming quantity of Firm 0 (q_0) is fixed, Firm 1 will maximize $\pi_1(q_0, q_1; w_1)$ by choosing its output q_1. Given any $\theta \in [0,1]$ of Stage-I and (w_0, w_1) of Stage-II, if the Stage-III choice vector is $(q_0^{CD}(w_0, w_1, \theta), q_1^{CD}(w_0, w_1, \theta))$ then $(q_0^{CD}(w_0, w_1, \theta), q_1^{CD}(w_0.w_1, \theta))$ simultaneously satisfy the following reaction equations

$$\frac{\partial V}{\partial q_0}(q_0, q_1, w_0, w_1, \theta) = P_0(q_0, q_1) - w_0 + \theta q_0 \frac{\partial P_0}{\partial q_0}(q_0, q_1) = 0 \tag{31}$$

and

$$\frac{\partial \pi_1}{\partial q_1}(q_0, q_1, w_1) = P_1(q_0, q_1) - w_1 + \theta q_1 \frac{\partial P_1}{\partial q_1}(q_0, q_1) = 0. \tag{32}$$

Evaluating condition (31) and (32) at $(q_0^{CD}(w_0, w_1, \theta), q_1^{CD}(w_0, w_1, \theta))$ we get the system of equa-

[11]We could not fully identify the value of $\hat{s}_2$ although it is a unqie positive real number and $\hat{s}_2 \in (0.8984, 0.9063)$.

tions involving the Stage-III choices are

$$
\begin{aligned}
(1+\theta)q_0^{CD}(w_0,w_1,\theta) + sq_1^{CD}(w_0,w_1,\theta) &= a - w_0\\
sq_0^{CD}(w_0,w_1,\theta) + 2q_1^{CD}(w_0,w_1,\theta) &= a - w_1.
\end{aligned}
$$

Solving for quantities we have conditions (10) and (11).

Therefore, substituting $q_0 = q_0^{CD}(w_0,w_1,\theta)$ and $q_1 = q_1^{CD}(w_0,w_1,\theta)$ in the equation (7) we get

$$
\pi_M^{CD}(w_0,w_1,\theta) = \frac{w_0\left[2(a-w_0) - s(a-w_1)\right] + w_1\left[(1+\theta)(a-w_1) - s(a-w_0)\right]}{2(1+\theta) - s^2} \tag{33}
$$

If w_0^{CD} and w_1^{CD} are respectively the optimum input price for publicly regulated and private firm then w_0^{CD} and w_1^{CD} satisfy the first order conditions $\partial\pi_M^{CD}\left(w_0 = w_0^{CD}, w_1 = w_1^{CD}, \theta\right)/\partial w_0 = 0$ and $\partial\pi_M^{CD}\left(w_0 = w_0^{CD}, w_1 = w_1^{CD}, \theta\right)/\partial w_1 = 0$. Therefore, the system of equation involving the Stage-II choices are $2(a - 2w_0^{CD}) - s(a - 2w_1^{CD}) = 0$ and $(1+\theta)(a - 2w_1^{CD}) - s(a - 2w_0^{CD}) = 0$. Solving for w_0^{CD} and w_1^{CD} we have $w_0^{CD} = w_1^{CD} = a/2$. Note that $\partial^2\pi_M^{CD}/\partial w_0^2 = -4/[2(1+\theta) - s^2] < 0$, $\partial^2\pi_M^{CD}/\partial w_1^2 = -2(1+\theta)/[2(1+\theta) - s^2] < 0$, and $H^{CD} = \left(\partial^2\pi_M^{CD}/\partial w_0^2\right)\left(\partial^2\pi_M^{CD}/\partial w_1^2\right) - \left(\partial^2\pi_M^{CD}/\partial w_0\partial w_1\right) = 4/[2(1+\theta) - s^2] > 0$ therefore, the second order condition also satisfied for Stage-II. Hence proved.

Proof of Lemma 1 (ii): If in Stage-III firms competing with Bertrand competition, then assuming the price of Firm 1 (p_1) is fixed Firm 0 will maximize

$$
\overline{V}(p_0,p_1;w_0,w_1,\theta) = \theta\overline{\pi}_0(p_0,p_1;w_0) + (1-\theta)\overline{W}(p_0,p_1;w_0,w_1)
$$

by choosing its own price, p_0 and assuming the price of Firm 0 (p_0) is fixed Firm 1 will maximize $\overline{\pi}_1(p_0,p_1;w_1)$ by choosing its own price, p_1. Given any $\theta \in [0,1]$ of Stage-I and (w_0,w_1) of Stage-II, if the Stage-III choice vector is $(p_0^{BD}(w_0,w_1,\theta), p_1^{BD}(w_0,w_1,\theta))$ then $(p_0^{BD}(w_0,w_1,\theta), p_1^{BD}(w_0.w_1,\theta))$ simultaneously satisfy the following reaction equations

$$
\frac{\partial\overline{V}}{\partial p_0}(p_0,p_1,w_0,w_1,\theta) = (p_0 - w_0)\frac{\partial D_0}{\partial p_0}(p_0,p_1) + \theta D_0(p_0,p_1) + (1-\theta)(p_1 - w_1)\frac{\partial D_1}{\partial p_0}(p_0,p_1) = 0 \tag{34}
$$

and

$$
\frac{\partial\overline{\pi}}{\partial p_1}(p_0,p_1,w_1) = (p_1 - w_1)\frac{\partial D_1}{\partial p_1}(p_0,p_1) + D_1(p_0,p_1) = 0. \tag{35}
$$

Evaluating condition (34) and (35) at $(p_0^{BD}(w_0,w_1,\theta), p_1^{BD}(w_0,w_1,\theta))$ we get the system of equations involving the Stage-III choices are

$$(1+\theta)p_0^{BD}(w_0,w_1,\theta) - sp_1^{BD}(w_0,w_1,\theta) = \theta(1-s)a + w_0 - s(1-\theta)w_1$$
$$-sp_0^{BD}(w_0,w_1,\theta) + 2p_1^{BD}(w_0,w_1,\theta) = (1-s)a + w_1$$

Solving for prices we have

$$p_0^{BD}(w_0,w_1,\theta) = w_0 + \frac{(2\theta - s^2)(a-w_0) - s(2\theta-1)(a-w_1)}{2(1+\theta)-s^2}$$

and

$$p_1^{BD}(w_0,w_1,\theta) = w_1 + \frac{[1+\theta(1-s^2)](a-w_1) - s(a-w_0)}{2(1+\theta)-s^2}.$$

Hence we have the conditions (12) and (13).

Therefore, substituting $q_0 = q_0^{BD}(w_0,w_1,\theta)$ and $q_1 = q_1^{BD}(w_0,w_1,\theta)$ in the equation (7) we get

$$\pi_M^{BD}(w_0,w_1,\theta) = \frac{\left[\begin{array}{c} w_0\left[(2-s)(a-w_0) - s(1+(1-\theta)(1-s^2))(a-w_1)\right] \\ +w_1\left[[1+\theta(1-s^2)](a-w_1) - s(a-w_0)\right]\end{array}\right]}{(1-s^2)\left[2(1+\theta)-s^2\right]} \tag{36}$$

If w_0^{BD} and w_1^{BD} are respectively the optimum input price for publicly regulated and private firm then w_0^{BD} and w_1^{BD} satisfy the first order conditions $\partial\pi_M^{BD}\left(w_0 = w_0^{BD}, w_1 = w_1^{BD}, \theta\right)/\partial w_0 = 0$ and $\partial\pi_M^{BD}\left(w_0 = w_0^{BD}, w_1 = w_1^{BD}, \theta\right)/\partial w_1 = 0$. Therefore, the system of equation involving the Stage-II choices are $2(2-s^2)w_0^{BD} - s(2+(1-\theta)(1-s^2))w_1^{BD} = (1-s)(2+s\theta-s^2+\theta s^2)a$ and $-s[2+(1-\theta)(1-s^2)]w_0^{BD} + 2[1+\theta-\theta s^2]w_1^{BD} = (1-s)(1+\theta+s\theta)a$. Solving for w_0^{BD} and w_1^{BD} we have $w_0^{BD}(\theta) = [s(1-s^2)\theta^2 + (4-s^2-s^3)\theta + (4-s-s^2)]a/[-s^2(1-s^2)\theta^2 + 2(4+s^2-s^4)\theta + (8-5s^2+s^4)]$ and $w_1^{BD}(\theta) = [-s^2(1-s^2)\theta^2 + (4-2s+3s^2+s^3-2s^4)\theta + (2-s)(2-s^2)(1+s)]a/[-s^2(1-s^2)\theta^2 + 2(4+s^2-s^4)\theta + (8-5s^2+s^4)]$. Note that $\partial^2\pi_M^{BD}/\partial w_0^2 = -2(2-s^2)/(1-s^2)[2(1+\theta)-s^2] < 0$, $\partial^2\pi_M^{BD}/\partial w_1^2 = -2(1+\theta-\theta s^2)/(1-s^2)[2(1+\theta)-s^2] < 0$, and $H^{BD} = \left(\partial^2\pi_M^{BD}/\partial w_0^2\right)\left(\partial^2\pi_M^{BD}/\partial w_1^2\right) - \left(\partial^2\pi_M^{BD}/\partial w_0\partial w_1\right) = [(8-5s^2+s^4) + 2(4+s^2-s^4)\theta - s^2(1-s^2)\theta^2]/(1-s^2)[2(1+\theta)-s^2]^2 > 0$, therefore, the second order condition for Stage-II also satisfied.

Moreover, we have

$$w^{BI} - w_0^{BD}(\theta) = \frac{s(1-s)(1-\theta)(2+s)[1+\theta-s(1-\theta)]a}{2\left[8(1+\theta) - \{4+(1-\theta)^2\}s^2 + (1-\theta)^2 s^4\right]} > 0 \tag{37}$$

and

$$w_1^{BD}(\theta) - w^{BI} = \frac{s(1-s)(1-\theta)[4+(1+\theta)s-(1-\theta)s^2]a}{[8(1+\theta) - \{4+(1-\theta)^2\}s^2 + (1-\theta)^2 s^4]} > 0 \tag{38}$$

Combining condition (37) and (38) we have $w_1^{BD}(\theta) > a/2 > w_0^{BD}(\theta)$. Hence proved.

Proof of Lemma 1 (iii): Finally, the ratio $\omega(\theta) = (w_1^{BD}(\theta) - w_1^{BI})/(w_0^{BI} - w_0^{BD}(\theta)) = [4 + (1+\theta)s - (1-\theta)s^2]/(2+s)[1+\theta - s(1-\theta)]$. Hence, differentiating $\omega(\theta)$ with respect to θ we get $\partial\omega/\partial\theta = -4(1+s)/(2+s)[1+\theta-s(1-\theta)]^2 < 0$. Hence, the result.

Proof of Proposition 1:

Proof of Proposition 1 (i): Substituting $w_0 = w_0^{CD}$ and $w_1 = w_1^{CD}$ in the condition (10) and (11) we will respectively have the resulting quantities $\hat{q}_0^{CD}(\theta) = q_0^{CD}(w_0^{CD}, w_1^{CD}, \theta) = (2-s)a/2\left[2(1+\theta) - s^2\right]$ and $\hat{q}_1^{CD}(\theta) = q_1^{CD}(w_0^{CD}, w_1^{CD}, \theta) = (1+\theta-s)a/2\left[2(1+\theta)-s^2\right]$. Substituting $q_0 = \hat{q}_0^{CD}(\theta)$, $q_1 = \hat{q}_1^{CD}(\theta)$, $w_0 = w_0^{CD}$ and $w_1 = w_1^{CD}$ in (8) we have the resulting welfare

$$\hat{W}^{CD}(\theta) = \left[3\theta^2 + 2(7-5s)\theta + (7-6s-2s^2+2s^3)\right] a^2/8\left[2(1+\theta)-s^2\right]^2.$$

Therefore if θ^{CD} be the Stage-I optimum privatization then θ^{CD} satisfy the first order condition $\partial\hat{W}^{CD}\left(\theta = \theta^{CD}\right)/\partial\theta = (2-s)\left[s(1-s) - (4-3s)\theta^{CD}\right]/4\left[2(1+\theta^{CD})-s^2\right]^3 = 0$. Solving for θ^{CD} we get $\theta^{CD} = \theta^{CI} = s(1-s)/(4-3s)$.

Proof of Proposition 1 (ii) Substituting $w_0 = w_0^{BD}(\theta)$ and $w_1 = w_1^{BD}(\theta)$ in the condition (12) and (13) we will respectively have the resulting quantities $\hat{q}_0^{BD}(\theta) = q_0^{BD}\left(w_0^{BD}(\theta), w_1^{BD}(\theta), \theta\right) = \left[(4+s-s^2) + s(1+s)\theta\right] a/(1+s)\left[(8-5s^2+s^4) + 2(4+s^2-s^4)\theta - s^2(1-s^2)\theta^2\right]$ and $\hat{q}_1^{BD}(\theta) = q_1^{BD}\left(w_0^{BD}(\theta), w_1^{BD}(\theta), \theta\right) = (2+s)\left[(1-s)+(1+s)\theta\right]a/(1+s)\left[(8-5s^2+s^4)+2(4+s^2-s^4)\theta - s^2(1-s^2)\theta^2\right]$. Substituting $q_0^{BD} = \hat{q}_0^{BD}(\theta)$, $q_1 = \hat{q}_1^{BD}(\theta)$, $w_0 = w_0^{BD}(\theta)$ and $w_1 = w_1^{BD}(\theta)$ in condition (8) we get the resulting welfare is $\hat{W}^{BD}(\theta) = W(\hat{q}_0^{BD}(\theta), \hat{q}_1^{BD}(\theta), w_0^{BD}(\theta), w_1^{BD}(\theta))$. In the Stage-I the social planner will maximize $\hat{W}^{BD}(\theta)$ by choosing $\theta \in [0,1]$. Differentiating $\hat{W}^{BD}(\theta)$ with respect to θ we have

$$\frac{\partial\hat{W}^{BD}}{\partial\theta} = \frac{(1-s)\mathcal{P}_1(\theta,s)a^2}{(1+s)\left[(8-5s^2+s^4)+2(4+s^2-s^4)\theta-s^2(1-s^2)\theta^2\right]^3}. \tag{39}$$

where for any $\theta \in [0,1]$ we have $\mathcal{P}_1(\theta,s) = C_4(s)\theta^4 + C_3(s)\theta^3 + C_2(s)\theta^2 + C_1(s)\theta + C_0(s)$ in which we have $C_0(s) = s(1+s)(2+s)(16+2s-19s^2+10s^3+2s^4-4s^5+s^6)$; $C_1(s) = -128 + 32s + 44s^2 - 36s^3 + 30s^4 - 6s^5 - 2s^6 + 22s^7 - 4s^9$; $C_2(s) = 6s^2(1-s^2)(10+4s+2s^3-s^4-s^5)$; $C_3(s) = 2s^2(1+s)(1-s^2)(2+4s-s^2+2s^3+s^4+s^5)$; and $C_4(s) = -s^4(1-s^2)(1+s)$. Given for all $\theta \in [0,1]$ and $s \in (0,1)$, $(8-5s^2+s^4)+2(4+s^2-s^4)\theta - s^2(1-s^2)\theta^2 > 0$, therefore, form condition (39) we have $sign(\partial\hat{W}^{BD}/\partial\theta) = sign(\mathcal{P}_1(\theta,s))$. Therefore at the optimality, we must have $\mathcal{P}_1(\theta^{BD},s) = 0$. Now, for all $s \in (0,1)$, $\mathcal{P}_1(\theta = 0, s) = C_{10}(s) = s(1+s)(2+s)(16+2s-19s^2+10s^3+2s^4-4s^5+s^6) > 0$ implies at $\theta = 0$, $\partial\hat{W}^{BD}/\partial\theta > 0$. Therefore, full nationalization is not optimal. Further, for all $s \in (0,1)$, $\mathcal{P}_1(\theta = 1, s) = \sum_{i=0}^{4} C_{1i}(s) = -16(1-s)^2(2+s)^2 < 0$ implies at $\theta = 1$, $\partial\hat{W}^{BD}/\partial\theta < 0$. Therefore, full privatization is not optimal. Hence, we have the partial privatization. Therefore, given any $s \in (0,1)$, if θ^{BD} be any optimal privatization ratio then

$\mathcal{P}_1(\theta^{BD}, s) = 0$. Further, we have $\partial\mathcal{P}_1(\theta, s)/\partial\theta = 4C_{14}(s)\theta^3 + 3C_{13}(s)\theta^2 + 2C_{12}(s)\theta + C_{11}(s)$. Since, for all $s \in (0,1)$ we have $C_{12}(s) = 6s^2(1-s^2)(10+4s+2s^3-s^4-s^5) > 0$ and $C_{13}(s) = 2s^2(1+s)(1-s^2)(2+4s-s^2+2s^3+s^4+s^5) > 0$ therefore, $\mathcal{P}_2(\theta, s) = 3C_{13}(s)\theta^2 + 2C_{12}(s)\theta + C_{11}(s)$ is strictly increasing in θ. Further, for all $s \in (0,1)$, $\mathcal{P}_2(\theta = 1, s) = -4s^9 - 12s^8 - 8s^7 - 8s^6 - 60s^5 - 84s^4 + 48s^3 + 176s^2 + 32s - 128 < 0$, therefore, for all $s \in (0,1)$ and $\theta \in [0,1]$ we have $\mathcal{P}_2(\theta, s) < 0$. Hence, given $C_{14}(s) = -s^4(1-s^2)(1+s) < 0$ for all $s \in (0,1)$ $(\partial\mathcal{P}_1/\partial\theta) = 4C_{14}(s)\theta^4 + \mathcal{P}_2(\theta, s) < 0$ implies $\mathcal{P}_1(\theta, s)$ is monotonically decreasing in $\theta \in (0,1)$. Therefore, there exist an unique $\theta^{BD} \in (0,1)$ that maximizes $\hat{W}^{BD}(\theta)$ such that $\mathcal{P}_1(\theta^{BD}, s) = 0$.

Proof of Proposition 1 (iii): One can show that $\mathcal{P}_1\left(\theta^{CD}, s\right) = s^2\mathcal{P}_3(s)/(4-3s)^4$ where $\mathcal{P}_3(s) = (1-s)^3\mathcal{P}_4(s) + \mathcal{P}_5(s)$ in which

$$\mathcal{P}_4(s) = \left[\begin{array}{c} 6402 + 6914(1-s) + 2523(1-s)^2 - 1398(1-s)^3 + 711(1-s)^4 - 1596(1-s)^5 \\ +438(1-s)^6 - 204(1-s)^7 + 72(1-s)^8 + 10(1-s)^9 - (1-s)^{10} + 2(1-s)^{11} - (1-s)^{12} \end{array}\right] > 0$$

and $\mathcal{P}_5(s) = 2880s^2 - 6368s + 3536 \geq (716/45)$. Therefore, $\mathcal{P}_3(s) > 0$ implies $\mathcal{P}_1\left(\theta^{CD}, s\right) > 0$. Given $\mathcal{P}_1(\theta, s)$ is decreasing in θ hence $\theta^{BD} > \theta^{CD}$.

Proof of the Proposition 2:

Proof of Proposition 2 (i): We will derive the Cournot Bertrand ranking corresponding to equilibrium output, price and profit of the Firm 0 one after another sequentially.

Cournot Bertrand ranking corresponding to equilibrium output of Firm 0: The equilibrium quantity that publicly regulated firm produces under Cournot competition is $\tilde{q}_0^{CD} = (4-3s)a/2(4-3s^2)$ and the resulting quantity of output that public firm produces under Bertrand competition evaluated at Stage-II optimum input prices is $\hat{q}_0^{BD}(\theta) = [4+(1+\theta)s-(1-\theta)s^2]a/(1+s)[8+8\theta-\{4+(1-\theta)^2\}s^2+(1-\theta)^2s^4]$. Consider the difference $\hat{q}_0^{BD}(\theta) - \tilde{q}_0^{CD} = \mathcal{P}_6(\theta, s)a/2(1+s)(4-3s^2)[8+8\theta-\{4+(1-\theta)^2\}s^2+(1-\theta)^2s^4]$ where $\mathcal{P}_6(\theta, s) = C_{26}(s)\theta^2 + C_{16}(s)\theta + C_{06}(s)$ where $C_{26}(s) = s^2(1-s)(4-3s)(1+s)^2 > 0$, $C_{16}(s) = -2(1+s)(16-16s+4s^2-4s^4+3s^5) < 0$ and $C_{06}(s) = s^2(1-s)(12+11s-2s^2-3s^3) > 0$. Therefore, $\hat{q}_0^{BD}(\theta) - \tilde{q}_0^{CD} \gtreqless 0$ if and only if $\mathcal{P}_6(\theta, s) = C_{26}(s)(\theta - \theta_1^{q_0}(s))(\theta - \theta_2^{q_0}(s)) \gtreqless 0$ where $\theta_1^{q_0}(s) = \frac{-C_{16}(s) - \sqrt{(C_{16}(s))^2 - 4C_{26}(s)C_{06}(s)}}{2C_{26}(s)}$ and $\theta_2^{q_0}(s) = \frac{-C_{16}(s) + \sqrt{(C_{16}(s))^2 - 4C_{26}(s)C_{06}(s)}}{2C_{26}(s)}$. We have the following about $\theta_1^{q_0}(s)$ and $\theta_2^{q_0}(s)$.

(i) One can show that $(C_{16}(s))^2 - 4C_{26}(s)C_{06}(s) > 0$ therefore both $\theta_1^{q_0}(s)$ and $\theta_2^{q_0}(s)$ are real.

(ii) Given $\theta_1^{q_0}(s)\theta_2^{q_0}(s) = C_{06}(s)/C_{26}(s) > 0$ and $\theta_1(s) + \theta_2(s) = -C_{16}(s)/C_{26}(s) > 0$ we have $\theta_1^{q_0}(s), \theta_2^{q_0}(s) > 0$

(iii) Given $C_{16}(s) < 0$ we have $\theta_2^{q_0}(s) > \theta_1^{q_0}(s) > 0$.

(iv) Given $C_{26}(s) - C_{16}(s) + C_{06}(s) > 0$ therefore, $\theta_1^{q_0}(s) < 1$. Further, $C_{26}(s) + C_{16}(s) + C_{06}(s) < 0$ implies $\theta_2^{q_0}(s) > 1$.

Hence, we have $\theta_2^{q_0}(s) > 1 > \theta_1^{q_0}(s) > 0$. Therefore, (a) for all $(\theta, s) \in [0,1] \times (0,1)$ if $\theta_1^{q_0}(s) < \theta \leq 1$ then we have $\mathcal{P}_6(\theta, s) < 0$. Further, given any $s \in (0,1)$ there exist an unique $\theta^{BD} \in (0,1)$ such that $\mathcal{P}_1(\theta^{BD}, s) = 0$. Finally, one can show that $\mathcal{P}_1(\theta_1(s), s) > 0$ for all $s \in (0,1)$. Therefore, using $(\partial \mathcal{P}_1/\partial\theta) < 0$ we have $1 > \theta^{BD} > \theta_1^{q_0}(s)$ for all $s \in (0,1)$. Hence, using (a) we have $\mathcal{P}_6(\theta^{BD}, s) < 0$ implies $\tilde{q}_0^{BD} = \hat{q}_0^{BD}(\theta^{BD}) < \hat{q}_0^{CD}$. Hence the ranking.

Cournot Bertrand ranking corresponding to equilibrium price of Firm 0: The equilibrium price that publicly regulated firm will charged under Cournot competition is $\tilde{p}_1^{CD} = (8 - 2s - 9s^2 + 4s^3)a/2(8 - 4s - 6s^2 + 3s^3)$ and the resulting price that publicly regulated firm will charged under Bertrand competition evaluated at Stage-II optimum input prices is $\hat{p}_1^{BD}(\theta) = [(1+s)(4 - 3s - s^2 + s^3) + (8 - 3s + s^2 - 2s^4)\theta - s^2(1 - s^2)\theta^2]a/[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$. Consider the difference $\hat{p}_1^{BD}(\theta) - \tilde{p}_1^{CD} = -(1-s)\mathcal{P}_7(\theta, s)a/2(4 - 3s^2)[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$ where $\mathcal{P}_7(\theta, s) = C_{27}(s)\theta^2 + C_{17}(s)\theta + C_{07}(s)$ in which $C_{27}(s) = s^2(1+s)(4 - s - 2s^2) >$, $C_{17}(s) = -2(2+s)(8 - 4s - 2s^2 + s^3 - 2s^4) <$ and $C_{07}(s) = s^2(4 + 5s - 3s^2 - 2s^3) > 0$. Therefore, $\hat{p}_1^{BD}(\theta) - \tilde{p}_1^{CD} \gtreqless 0$ if and only if $\mathcal{P}_7(\theta, s) = C_{27}(s)(\theta - \theta_1^{p_0}(s))(\theta - \theta_2^{p_0}(s)) \lesseqgtr 0$ where $\theta_1^{p_0}(s) = \frac{-C_{17}(s) - \sqrt{(C_{17}(s))^2 - 4C_{27}(s)C_{07}(s)}}{2C_{27}(s)}$ and $\theta_2^{p_0}(s) = \frac{-C_{17}(s) + \sqrt{(C_{17}(s))^2 - 4C_{27}(s)C_{07}(s)}}{2C_{27}(s)}$. We have the following observations about $\theta_1^{p_0}(s)$ and $\theta_2^{p_0}(s)$.

(i) One can show that $(C_{17}(s))^2 - 4C_{27}(s)C_{07}(s) > 0$ therefore both $\theta_1^{p_0}(s)$ and $\theta_2^{p_0}(s)$ are real.

(ii) Given $\theta_1^{p_0}(s)\theta_2^{p_0}(s) = C_{07}(s)/C_{27}(s) > 0$ and $\theta_1^{p_0}(s) + \theta_2^{p_0}(s) = -C_{17}(s)/C_{27}(s) > 0$ we have $\theta_1^{p_0}(s), \theta_2^{p_0}(s) > 0$.

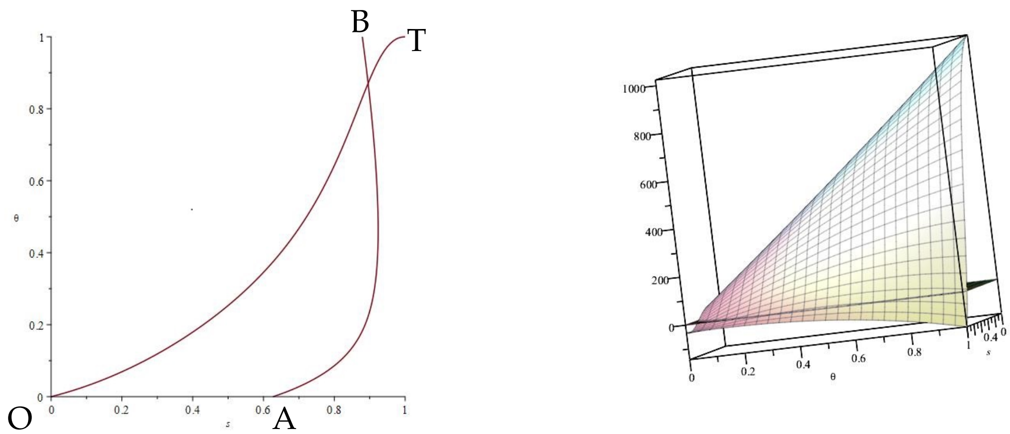
(iii) Given $C_{17}(s) < 0$ we have $\theta_2^{p_0}(s) > \theta_1^{p_0}(s) > 0$.

(iv) Given $C_{27}(s) - C_{17}(s) + C_{07}(s) > 0$ therefore, $\theta_1^{p_0}(s) < 1$. Further, $C_{27}(s) + C_{17}(s) + C_{07}(s) < 0$ implies $\theta_2^{p_0}(s) > 1$.

Hence, we have $\theta_2^{p_0}(s) > 1 > \theta_1^{p_0}(s) > 0$. Therefore, (a_2) for all $(\theta, s) \in [0,1] \times (0,1)$ if $\theta_1^{p_0}(s) < \theta \leq 1$ then we have $\mathcal{P}_7(\theta, s) < 0$. Further, given any $s \in (0,1)$ there exist an unique $\theta^{BD} \in (0,1)$ such that $\mathcal{P}_3(\theta^{BD}, s) = 0$. Finally, one can show that $\mathcal{P}_1(\theta_1^{p_0}(s), s) > 0$ for all $s \in (0,1)$. Therefore, using $(\partial \mathcal{P}_1/\partial\theta) < 0$ we have $\theta^{BD} > \theta_1^{p_0}(s)$ for all $s \in (0,1)$. Hence, using (a_2) we have $\mathcal{P}_7(\theta^{BD}, s) < 0$ implies $\tilde{p}_0^{BD} = \hat{p}_0^{BD}(\theta^{BD}) > \hat{p}_0^{CD}$. Hence the ranking.

Ranking of Profit of Firm 0: The equilibrium profit that public firm will earn under Cournot competition is $\tilde{\pi}_0^{CD} = s(1-s)(4-3s)a^2/4(4-3s^2)^2$ and the profit that public firm will earn at Stage-II optimum input prices under Bertrand competition is $\hat{\pi}_0^{BD}(\theta) = (1-s)[4 + (1+\theta)s - (1-\theta)s^2][4\theta + (2 + \theta - \theta^2)s - (1 - 3\theta + 2\theta^2)s^2 - (1-\theta)^2 s^3]a/(1+s)(4-3s^2)^2[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]^2$. Consider the difference $\hat{\pi}_0^{BD}(\theta) - \tilde{\pi}_0^{CD} = (1-s)\mathcal{P}_8(\theta, s)a^2/4(1+s)(4 - 3s^2)^2[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]^2$ where $\mathcal{P}_8(\theta, s) = C_{48}(s)\theta^4 + C_{38}(s)\theta^3 + C_{28}(s)\theta^2 + C_{18}(s)\theta + C_{08}(s)$ in which $C_{48}(s) = -s^5(1+s)(4-3s)(1-s^2)^2$, $C_{38}(s) = -4s^2(1+s)^2(16 + 4s^2 -$

$40s^3 + 16s^4 + 10s^5 - 7s^6 + 3s^7)$, $C_{28}(s) = 2s(1+s)(9s^9 - 12s^8 + 24s^7 + 94s^6 - 219s^5 - 140s^4 + 312s^3 + 64s^2 - 32s - 128)$, $C_{18}(s) = -12s^{11} + 4s^{1}0 - 20s^9 - 132s^8 + 480s^7 + 720s^6 - 960s^5 - 624s^4 + 512s^3 - 960s^2 + 1024$ and $C_{08} = s(3s^{10} - s^9 + 2s^8 + 10s^7 - 185s^6 - 113s^5 + 620s^4 + 272s^3 - 704s^2 - 192s + 256)$. In Figure 4a we have the implicit plot of the equations $\mathcal{P}_3(\theta, s) = 0$ (the curve OT) and $\mathcal{P}_8(\theta, s) = 0$ (the curve AB). From the 3D plot of $\mathcal{P}_8(\theta, s)$ to the left (right) of the curve AB we have $\mathcal{P}_8(\theta, s) > 0 (< 0)$. Hence using the curve OT one can conclude that for low value of s we have $\tilde{\pi}_0^{BD} = \hat{\pi}_0^{BD}(\theta^{BD}) > \tilde{\pi}_0^{CD}$ and for high value of s we have $\tilde{\pi}_0^{BD} = \hat{\pi}_0^{BD}(\theta^{BD}) < \tilde{\pi}_0^{CD}$.



(a) Implicit plot of $\mathcal{P}_1(\theta, s) = 0$ and $\mathcal{P}_8(\theta, s) = 0$

(b) 3D plot of $\mathcal{P}_8(\theta, s)$

Figure 4

Proof of Proposition 2 (ii): Similar to the Proposition 2 (i) here also we will derive the Cournot Bertrand ranking corresponding to equilibrium output, price and profit of the Firm 1 one after another sequentially.

Cournot Bertrand ranking corresponding to equilibrium output of Firm 1: The equilibrium quantity of Firm 1 under Cournot competition is $\tilde{q}_1^{CD} = (2 - 3s + s^2)a/(8 - 4s - 6s^2 + 3s^3)$ and the resulting quantity of output that Firm 1 produces under Bertrand competition evaluated at Stage-II optimum input price is $\hat{q}_1^{BD}(\theta) = (2+s)[(1+\theta - s(1-\theta))]a/(1+s)[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$. Consider the difference $\hat{q}_1^{BD}(\theta) - \tilde{q}_1^{CD} = s\mathcal{P}_9(\theta, s)a/2(1+s)(4-3s^2)[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$ where $\mathcal{P}_9(\theta, s) = C_{29}(s)\theta^2 + C_{19}(s)\theta + C_{09}(s)$ where $C_{29}(s) = s(1-s^2)^2 > 0$, $C_{19}(s) = (1+s)(12 - 8s - s^2 + 2s^3 - 2s^4) > 0$ and $C_{09}(s) = -(1-s)(4 + s - 2s^2 + s^3 + s^4) < 0$. Therefore, $\hat{q}_1^{BD}(\theta) - \tilde{q}_1^{CD} \gtreqless 0$ if and only if $\mathcal{P}_9(\theta, s) = C_{29}(s)(\theta - \theta_1^{q_1}(s))(\theta - \theta_2^{q_1}(s)) \gtreqless 0$ where $\theta_1^{q_1}(s) = \frac{-C_{19}(s) - \sqrt{(C_{19}(s))^2 - 4C_{29}(s)C_{09}(s)}}{2C_{29}(s)}$ and $\theta_2^{q_1}(s) = \frac{-C_{19}(s) + \sqrt{(C_{19}(s))^2 - 4C_{29}(s)C_{09}(s)}}{2C_{29}(s)}$. We have the following about $\theta_1^{q_1}(s)$ and $\theta_2^{q_1}(s)$.

(i) One can show that $(C_{19}(s))^2 - 4C_{29}(s)C_{09}(s) > 0$ therefore both $\theta_1^{q_1}(s)$ and $\theta_2^{q_1}(s)$ are real.

(ii) Given $\theta_1^{q_1}(s)\theta_2^{q_1}(s) = C_{09}(s)/C_{29}(s) < 0$ therefore we have either $\theta_1^{q_1}(s) < 0$ and $\theta_2^{q_1}(s) > 0$ or $\theta_1^{q_1}(s) > 0$ and $\theta_2^{q_1}(s) < 0$.

(iii) Given $C_{16}(s) > 0$ and $C_{29}(s) > 0$ we have $\theta_1^{q_1}(s) < 0$. Therefore, using (ii) we have $\theta_2^{q_1}(s) > 0$.

(iv) Given $C_{26}(s) + C_{16}(s) + C_{06}(s) = 8 + 8s - 6s^2 - 4s^3 > 0$ therefore, $\theta_2^{q_1}(s) < 1$.

Hence, we have $1 > \theta_2^{q_0}(s) > 0 > \theta_1^{q_0}(s)$. Therefore, given $C_{29}(s) > 0$ and $\theta_1^{q_1}(s) < 0$ we can conclude that $\mathcal{P}_9(\theta, s) \gtreqless 0$ if and only if $\theta \gtreqless \theta_2^{q_1}(s)$. One can show that $\mathcal{P}_1(\theta_2^{q_1}(s), s) \gtreqless 0$ if and only if $s \gtreqless \hat{s}_3 \in (0.4255, 0.4256)$ (six decimal estimation of $\hat{s}_3$ is 0.425557.). Therefore, for all $s < \hat{s}_3$ we have $\mathcal{P}_1(\theta_2^{q_1}(s), s) < 0$ implies $\theta_2^{q_1}(s) > \theta^{BD}$ leads to $\mathcal{P}_9(\theta^{BD}, s) < 0$ implies $\tilde{q}_1^{BD} = \hat{q}_1^{BD}(\theta^{BD}) < \tilde{q}_1^{CD}$.

Cournot Bertrand ranking corresponding to equilibrium price of Firm 1: The equilibrium price that private firm will charged under Cournot competition is $\tilde{p}_1^{CD} = (12 - 10s - 4s^2 + 3s^3)a/2(8 - 4s - 6s^2 + 3s^3)$ and the resulting price that private firm will charged under Bertrand competition evaluated at Stage-II optimum input prices is $\hat{p}_1^{BD}(\theta) = [(6-s)(1+\theta) - (4 - \theta + \theta^2)s^2 + (1-\theta)^2 s^4]a/[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$. Consider the difference $\hat{p}_1^{BD}(\theta) - \tilde{p}_1^{CD} = -s(1-s)\mathcal{P}_{10}(\theta, s)a/2(4 - 3s^2)[8 + 8\theta - \{4 + (1-\theta)^2\}s^2 + (1-\theta)^2 s^4]$ where $\mathcal{P}_{10}(\theta, s) = C_{2,10}(s)\theta^2 + C_{1,10}(s)\theta + C_{0,10}(s)$ in which $C_{2,10}(s) = s(1+s)(2 + 2s - 3s^2)$, $C_{1,10}(s) = -2(4 - 4s + s^2 - s^3 - 3s^4)$ and $C_{0,10}(s) = -(8 - 6s - 10s^2 + s^3 + 3s^4)$. Therefore, $\hat{p}_1^{BD}(\theta) - \tilde{p}_1^{CD} \gtreqless 0$ if and only if $\mathcal{P}_{10}(\theta, s) \lesseqgtr 0$. Let us first note the following.

(i) Given any $s \in (0, 1)$, $(\partial \mathcal{P}_{10}(\theta, s)/\partial s) = -12(1-\theta)^2 s^3 - 3(1-\theta)^2 s^2 + 8(1-\theta)^2 s + 12(1+\theta)s + 2(1+\theta)(3+\theta) > 0$ therefore for all $s \in (0, 1)$ we have $\mathcal{P}_{10}(\theta, s)$ is increasing in s.

(ii) Further, $D_{10} = (C_{1,10}(s))^2 - 4C_{2,10}(s)C_{0,10}(s) = 4(36s^6 + 60s^5 - 71s^4 - 68s^3 + 44s^2 - 16s + 16)$. One can show that there exist $\hat{s}_4 \in (0.707, 0.708)$ and $\hat{s}_5 \in (0.9893, 0.9894)$ such that we have the following (a) $D_{10} > 0$ for all $s \in (0, \hat{s}_4) \cup (\hat{s}_5, 1)$, and (b) $D_{10} < 0$ for all $s \in (\hat{s}_4, \hat{s}_5)$.

Hence given $C_{2,10}(s) > 0$ for all $s \in (0, 1)$ and (ii) (b) one can conclude that for all $s \in (\hat{s}_4, \hat{s}_5)$ we have $\mathcal{P}_{10}(\theta, s) > 0$. Finally using (i) we can conclude that $\mathcal{P}_{10}(\theta, s) > 0$ for all $s \in (\hat{s}_4, 1)$. Observe that we can express $\mathcal{P}_{10}(\theta, s) = C_{2,10}(s)\left(\theta - \theta_1^{p_1}(s)\right)\left(\theta - \theta_2^{p_1}(s)\right)$ where $\theta_1^{p_1}(s) = \frac{-C_{1,10}(s) - \sqrt{D_{10}}}{2C_{2,10}(s)}$ and $\theta_2^{p_1}(s) = \frac{-C_{1,10}(s) + \sqrt{D_{10}}}{2C_{2,10}(s)}$. We have the following about $\theta_1^{p_1}(s)$ and $\theta_2^{p_1}(s)$ for all $s \in (0, \hat{s}_4)$.

(i) Given $D_{10} > 0$ therefore both $\theta_1^{p_1}(s)$ and $\theta_2^{p_1}(s)$ are real.

(ii) For all $s \in (0, \hat{s}_4)$ we have one can show that $C_{1,10}(s) < 0$, therefore $\theta_1^{p_0}(s) < \theta_2^{p_0}(s)$.

Hence, we have $\mathcal{P}_{10}(\theta,s) < 0$ if and only if $\theta \in (\theta_1^{p_1}(s), \theta_2^{p_1}(s))$. One can show that $\mathcal{P}_1(\theta_1^{p_1}(s),s) > 0$ for all $s \in (0,\hat{s}_4)$. Therefore, given $\partial\mathcal{P}_1(\theta,s)/\partial\theta = 0$ and $\mathcal{P}_1(\theta^{BD},s) = 0$ we have $\theta_1^{p_1}(s) < \theta^{BD}$. Further, one can show that there exist a $\hat{s}_6 \in (0,\hat{s}_4)$ such that we have $\mathcal{P}_1(\theta_2^{p_1}(s),s) \gtreqless 0$ if and only if $s \gtreqless \hat{s}_6$. Therefore, $\theta^{BD} \lesseqgtr \theta_2^{p_1}(s)$ such that $s \lesseqgtr \hat{s}_6$. Hence, we have $\mathcal{P}_{10}(\theta^{BD},s) \lesseqgtr 0$ if and only if $s \lesseqgtr \hat{s}_6$. Therefore, $\tilde{p}_1^{BD} = \hat{p}_1^{BD}(\theta^{BD}) \gtreqless \tilde{p}_1^{CD}$ if and only if $s \lesseqgtr \hat{s}_6$. Hence the ranking.

Cournot Bertrand ranking corresponding to equilibrium profit of Firm 1: The equilibrium profit that private firm will earn under Cournot competition is $\tilde{\pi}_1^{CD} = (1-s)^2a^2/(4-3s^2)^2$ and the profit that private firm will earn at Stage-II optimum input prices under Bertrand competition is $\hat{\pi}_1^{BD}(\theta) = (1-s)(2+s)^2[1+\theta-s(1-\theta)]^2a^2/(1+s)[8(1+\theta)-(4+(1-\theta)^2)s^2+(1-\theta)^2s^4]^2$. Consider the difference $\sqrt{\hat{\pi}_1^{BD}(\theta)} - \sqrt{\tilde{\pi}_0^{CD}} = \mathcal{P}_{11}(\theta,s)a/\sqrt{1+s}(4-3s^2)[8(1+\theta)-(4+(1-\theta)^2)s^2+(1-\theta)^2s^4]$ where $\mathcal{P}_{11}(\theta,s) = C_{2,11}(s)\theta^2 + C_{1,11}(s)\theta + C_{0,11}(s)$ in which $C_{2,11}(s) = s^2(1-s)^2(1+s)^{\frac{3}{2}} > 0$, $C_{1,11}(s) = (8+12s-2s^2-9s^3-3s^4)\sqrt{1-s} - 2(1-s)(4+s^2-s^4)\sqrt{1+s} > 0$ and $C_{0,11}(s) = -(1-s)[(8-5s^2+s^4)\sqrt{1+s} - (8+4s-6s^2-3s^3)\sqrt{1-s}] < 0$. Therefore, $D_{11}(s) = \{C_{1,11}\}^2 - 4C_{2,11}C_{0,11} > 0$ implies $\theta_1^{\pi_1} = (-C_{1,11(s)} + \sqrt{D_{11}(s)})/2C_{2,11}(s)$ and $\theta_2^{\pi_1} = (-C_{1,11}(s) - \sqrt{D_{11}(s)})/2C_{2,11}(s)$ both are real. Further, $C_{1,11}(s) > 0$ implies $\theta_2^{\pi} < 0$. Moreover, $C_{0,11}(s) < 0$ implies $D_{11}(s) > \{C_{1,11}(s)\}^2$ implies $\theta_1^{\pi_1} > 0$. Finally, $C_{0,11}(s) + C_{1,11}(s) + C_{2,11}(s) > 0$ implies $\theta_1^{\pi_1} < 1$. Therefore, we have $1 > \theta_1^{\pi_1} > 0 > \theta_2^{\pi_1}$. Given, $\mathcal{P}_{11}(\theta,s) = C_{2,11}(s)(\theta-\theta_1^{\pi_1})(\theta-\theta_2^{\pi_1})$, $C_{2,11}(s) > 0$ and $1 > \theta_1^{\pi_1} > 0 > \theta_2^{\pi_1}$, one can conclude that $\mathcal{P}_{11}(\theta,s) \gtreqless 0$ if and only if $\theta \gtreqless \theta_1^{\pi_1}$. One can show that $\mathcal{P}_1(\theta_1^{\pi_1},s) \gtreqless 0$ if and only if $s \gtreqless \hat{s}_7$ with $\hat{s}_7 \in (0.62, 0.63)$. Therefore, for all $s < \hat{s}_7$ we have $\mathcal{P}_1(\theta_1^{\pi_1},s) < 0$ implies $\theta_1^{\pi_1} > \theta^{BD}$ implies $\mathcal{P}_{11}(\theta^{BD},s) < 0$ therefore $\tilde{\pi}_1^{BD} = \hat{\pi}_1^{BD}(\theta^{BD}) < \tilde{\pi}_1^{CD}$. Hence, the ranking.

Proof of Proposition 2 (iii): The equilibrium societies welfare under Cournot competition is $\tilde{W}^{CD} = (7-6s)a^2/8(4-3s^2)$ and the society's welfare at Stage-II optimum input prices under Bertrand competition is $\hat{W}^{BD}(\theta) = \mathcal{P}_{12}(\theta,s)a^2/(1+s)[8+8\theta-\{4+(1-\theta)^2\}s^2+(1-\theta)^2s^4]^2$ where $\mathcal{P}_{12}(\theta,s) = C_{3,12}(s)\theta^3 + C_{2,12}(s)\theta^2 + C_{1,12}(s)\theta + C_{0,12}$ in which $C_{3,12}(s) = -s^2(1-s)(1+s)^3$; $C_{2,12}(s) = (1+s)(6+6s-4s^2+4s^3-s^4-3s^5)$; $C_{1,12}(s) = (28+4s-5s^2-6s^3-10s^4+2s^5+3s^6)$ and $C_{0,12}(s) = (14-16s^2+7s^4-s^6)$. Consider the difference $\hat{W}^{BD}(\theta) - \tilde{W}^{CD} = -(1-s)\mathcal{P}_{13}(\theta,s)a^2/8(1+s)(4-3s^2)^2[8+8\theta-\{4+(1-\theta)^2\}s^2+(1-\theta)^2s^4]^2$ where $\mathcal{P}_{13}(\theta,s) = C_{4,13}(s)\theta^4 + C_{3,13}(s)\theta^3 + C_{2,13}(s)\theta^2 + C_{1,13}(s)\theta + C_{0,13}(s)$ in which $C_{4,13}(s) = s^4(1-s)(7-6s)(1+s)^3$, $C_{3,13}(s) = -4s^2(1-s)(1+s)^2(20-12s+s^2+s^3-6s^4)$, $C_{2,13}(s) = 2(1+s)(128-160s+32s^2-8s^3-47s^4+55s^5-27s^6-3s^7+18s^8)$, $C_{1,13}(s) = -4s^2(68+8s-51s^2+10s^3-2s^4-20s^5+5s^6+6s^7)$ and $C_{0,13} = s(1-s)(64+32s-80s^2-33s^3+55s^4+27s^5-11s^6-6s^7)$. In Figure 5a we have the implicit plot of the equations $\mathcal{P}_1(\theta,s) = 0$ (the curve OT) and $\mathcal{P}_{13}(\theta,s) = 0$ (the curve AB). Moreover, in the Figure 5b we have the 3D plot of the polynomial $\mathcal{P}_{13}(\theta,s)$ for all $(\theta,s) \in [0,1] \times (0,1)$. From the 3D plot of $\mathcal{P}_{13}(\theta,s)$ to the left (right) of the curve AB we have $\mathcal{P}_{13}(\theta,s) > 0 (< 0)$. Hence using the curve OT one can conclude that for low value of s we have

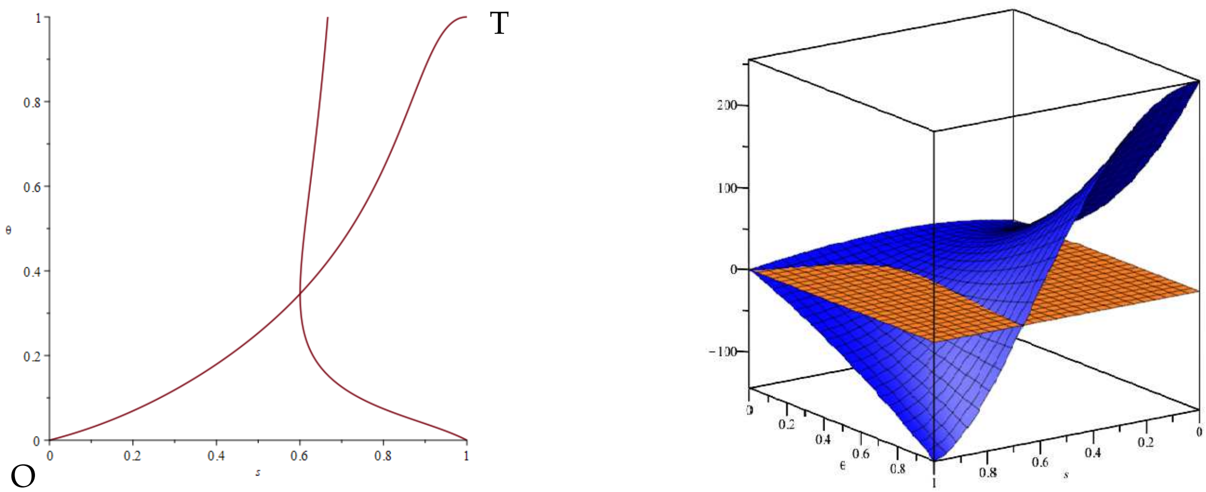
$\tilde{W}^{BD} = \hat{W}^{BD}(\theta^{BD}) < \tilde{w}_0^{CD}$ and for high value of s we have the opposite.

m \ s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.03	0.07	0.12	0.18	0.25	0.34	0.46	0.64	0.88
$\frac{(1-s)}{10}$	0.033	0.077	0.133	0.2	0.29	0.41	0.58	0.91	1
$\frac{(1-s)}{9}$	0.0334	0.078	0.135	0.21	0.30	0.42	0.60	0.98	1
$\frac{(1-s)}{8}$	0.034	0.079	0.137	0.21^{+}	0.303	0.43	0.62	1	1
$\frac{(1-s)}{7}$	0.0347	0.081	0.14	0.216	0.31	0.44	0.66	1	1
$\frac{(1-s)}{6}$	0.0357	0.083	0.145	0.224	0.33	0.47	0.72	1	1
$\frac{(1-s)}{5}$	0.0372	0.087	0.15	0.236	0.35	0.51	0.86	1	1
$\frac{(1-s)}{4}$	0.0398	0.093	0.16	0.257	0.38	0.59	1	1	1
$\frac{(1-s)}{3}$	0.0449	0.106	0.19	0.304	0.47	1	1	1	1
$\frac{(1-s)}{2}$	0.0609	0.149	0.28	0.51	1	1	1	1	1

Table 1: Simulation Results for Bertrand Competition

m \ s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.024	0.047	0.067	0.085	0.1	0.109	0.11	0.100	0.069
$\frac{(1-s)}{10}$	0.0272	0.0529	0.076	0.097	0.114	0.125	0.126	0.114	0.0778
$\frac{(1-s)}{9}$	0.0275	0.0537	0.0778	0.0989	0.116	0.1265	0.1281	0.1152	0.0788
$\frac{(1-s)}{8}$	0.028	0.0546	0.0792	0.1008	0.1180	0.129	0.1305	0.1173	0.0801
$\frac{(1-s)}{7}$	0.0286	0.0559	0.0811	0.1032	0.1209	0.1321	0.1336	0.12	0.0817
$\frac{(1-s)}{6}$	0.02951	0.0576	0.0837	0.1066	0.125	0.1365	0.1379	0.1236	0.0839
$\frac{(1-s)}{5}$	0.0308	0.0602	0.0876	0.1117	0.1309	0.1402	0.1442	0.1288	0.0870
$\frac{(1-s)}{4}$	0.0329	0.0646	0.094	0.1200	0.1406	0.1533	0.1542	0.1371	0.0918
$\frac{(1-s)}{3}$	0.0372	0.0731	0.1066	0.136	0.159	0.1728	0.1726	0.152	0.1002
$\frac{(1-s)}{2}$	0.0497	0.0977	0.1420	0.18	0.2083	0.2228	0.2180	0.1867	0.1186

Table 2: Simulation Results for Cournot Competition



(a) Implicit plot of $\mathcal{P}_1(\theta, s) = 0$ and $\mathcal{P}_{11}(\theta, s) = 0$

(b) 3D plot of $\mathcal{P}11(\theta, s)$

Figure 5

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