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Preference aggregation theory without acyclicity: The core without majority dissatisfaction

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Abstract

Acyclicity of individual preferences is a minimal assumption in social choice theory. We replace that assumption by the direct assumption that preferences have maximal elements on a fixed agenda. We show that the core of a simple game is nonempty for all profiles of such preferences if and only if the number of alternatives in the agenda is less than the Nakamura number of the game. The same is true if we replace the core by the *core without majority dissatisfaction*, obtained by deleting from the agenda all the alternatives that are non-maximal for all players in a winning coalition. Unlike the core, the core without majority dissatisfaction depends only on the players' sets of maximal elements and is included in the union of such sets. A result for an extended framework gives another sense in which the core without majority dissatisfaction behaves better than the core.

Journal of Economic Literature Classifications: C71, D71, C02.

Keywords: Core, Nakamura number, kappa number, simple games, voting games, maximal elements, acyclic preferences, limit ordinals.

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URL: <http://econpapers.repec.org/RAS/pmi193.htm> (H.R. Mihara).

1 Introduction

1.1 Preference aggregation theory for acyclic individual preferences

Preference aggregation theory is concerned with aggregating individual preferences into a (collective) social preference, which is then maximized to yield a set of best alternatives. The theory investigates the extent to which social preferences inherit desirable properties from individual preferences. We typically restrict (strict) individual and social preferences to those asymmetric relations \succ on a set X of alternatives that are either (i) acyclic or (ii) transitive or (iii) negatively transitive.¹

Of the properties (i), (ii), and (iii) for asymmetric preferences, *negative transitivity* is the most demanding. Arrow's Theorem (1963) points out the difficulty of aggregating preferences for more than two alternatives while preserving asymmetry and negative transitivity.²

Acyclicity is the least demanding of the properties; it is necessary and sufficient for the existence of a maximal element on *every finite subset* of alternatives. The *Nakamura number* plays a critical role in the study of preference aggregation rules with *acyclic* social preferences.³ Consider a *simple game* (voting game) \mathcal{W} , a collection of "winning" coalitions in a set N of players. Combining the game with a set X of alternatives and a profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$ of individual preferences, one obtains a *simple game with preferences* $(\mathcal{W}, X, \mathbf{p})$, for which one can define the social preference $\succ_{\mathcal{W}}^{\mathbf{p}}$ (dominance relation) and the *core* $C(\mathcal{W}, X, \mathbf{p})$ (the set of maximal alternatives with respect to $\succ_{\mathcal{W}}^{\mathbf{p}}$). Nakamura's theorem (1979) gives a restriction on the number of alternatives that the set of players can deal with rationally (Theorem 1): the core of a simple game with preferences is always (i.e., for all profiles of acyclic preferences) nonempty if and only if the number of alternatives is finite and below a certain number, called the *Nakamura number* of the simple game. The theorem thus gives a condition (Corollary 1) for the social preferences to inherit acyclicity from individual preferences.

¹Define the weak preference \succeq by $x \succeq y \Leftrightarrow y \not\succeq x$. \succ is asymmetric iff \succeq is complete (reflexive and total). (i) \succ is *acyclic* if for any finite set $\{x_1, x_2, \dots, x_m\} \subseteq X$, whenever $x_1 \succ x_2, \dots, x_{m-1} \succ x_m$, we have $x_m \not\succeq x_1$. If \succ is acyclic, it is asymmetric and irreflexive. (ii) \succ is *transitive* if $x \succ y$ and $y \succ z$ imply $x \succ z$. When \succ is transitive, we say \succeq is *quasi-transitive*. (iii) \succ is *negatively transitive* if $x \not\succeq y$ and $y \not\succeq z$ imply $x \not\succeq z$. \succ is negatively transitive iff \succeq is transitive.

²The restriction to two alternatives disappears when there are infinitely many individuals (Fishburn, 1970), but such a resolution relies on highly nonconstructive mathematical objects (Mihara, 1997, 1999).

³Banks (1995), Truchon (1995), Andjiga and Mbih (2000), and Kumabe and Mihara (2008a) are recent contributions to the literature. Earlier papers on acyclic rules can be found in Truchon (1995) and Austen-Smith and Banks (1999). Kumabe and Mihara (2008b) comprehensively study the restrictions that various properties for a simple game impose on its Nakamura number.

To deal with an empty core (or cycles in social preferences), several authors have investigated solutions different from the core. We pick two examples for which there have been recent developments. First, Duggan (2007) proposes a procedure in which one deletes particular instances of preferences until the resulting subrelation is acyclic (alternatively, transitive or negatively transitive), and collects the maximal elements of all such (maximal acyclic) subrelations.⁴ Second, taking voters' foresight into consideration, Rubinstein (1980) proposes the *stability set*, a superset of the core.⁵ All these investigations focus on treating cyclic *social* preferences, assuming that *individual* preferences satisfy a rationality property at least as strong as acyclicity.

1.2 Preference aggregation theory without acyclicity

In this paper, we propose a *preference aggregation theory without acyclicity*. In contrast to the authors cited in the preceding paragraph, we do not attempt to remove cycles in (social and even individual) preferences.

We retain the usual framework (e.g., Arrow, 1963, Section II.2) in which a set X of underlying alternatives is distinguished from an *agenda* (opportunity set) $B \subseteq X$ with which a group N of players are confronted. In particular, fixing a simple game \mathcal{W} and a set X , we focus on the aggregation methods that assigns the core $C(\mathcal{W}, B, \mathbf{p})$ or the *core without majority dissatisfaction* $C^+(\mathcal{W}, B, \mathbf{p})$ (introduced later)⁶ to each $(\mathcal{W}, B, \mathbf{p})$ satisfying a certain assumption. The assumption, which is actually a condition concerning a pair (B, \mathbf{p}) , is that *every player's preference $\succ_i^{\mathbf{p}}$ has a maximal element in B : the maximal set $\max_B \succ_i^{\mathbf{p}}$ (the set of maximal elements of the preference) is nonempty for each i .*⁷ This is a rather direct assumption at the individual level, which is to be inherited to the requirement at the social level that there is a chosen alternative for the given pair.

The assumption is distinctive in that it involves an agenda B as well as a profile \mathbf{p} . For this reason, it does not fit the following *Standard Scenario*

⁴A related procedure is to collect the maximal elements of all maximal *chains* (subsets of alternatives on which the majority preference is a linear order), which yields the Banks set (Banks, 1985; Penn, 2006); the set consists of the sophisticated voting outcomes of some binary agenda.

⁵Le Breton and Salles (1990) provide a sufficient condition for the general nonemptiness of the stability set in terms of the Nakamura number. Using more complex characteristic numbers, Martin and Merlin (2006) propose a weaker sufficient condition and Andjiga and Moyouwou (2006) a necessary and sufficient condition for the general nonemptiness of the stability set.

⁶As the notation suggests, we define the *core (without majority dissatisfaction)* relative to an agenda B .

⁷Acyclicity of a preference is independent of the property of having a maximal element. There is a cyclic preference that has a maximal element on X . When X is infinite, there is an acyclic preference that has no maximal element on X . Similarly, the property of having a maximal element on B is independent of the property of having a maximal element on X .

in social choice theory (Arrow, 1963, page 12) very well: before knowing an agenda, the players discover and report their own preferences (on X) to the planner; having obtained a choice rule that assigns a nonempty subset to every agenda, the planner applies the rule to a particular agenda B . Since the planner does not generally know whether a pair (B, \mathbf{p}) satisfies the assumption until she faces the agenda B , what she obtains immediately after learning the profile \mathbf{p} of preferences is only a *partial* choice rule, which might assign an empty set to some agenda.⁸

An *Alternative Scenario* that the assumption fits well is the following: the planner first presents an agenda B to the players, who then discover and report their own preferences (for alternatives in B at least) to the planner; the planner then makes a choice. This is perhaps a closer description of actual collective decisions. While failing to produce a choice rule at an intermediate stage, the scenario has some advantages over the Standard Scenario.

First, the Alternative Scenario can deal with context-dependent choices based on multiple *rationales* (preferences belonging to the same individual) more easily, where the context is given by an agenda (Kalai et al., 2002; Ambrus and Rozen, 2008). The problem with the Standard Scenario is that a player is supposed to report a single preference for the whole set X , when she might actually have an irreducible set of multiple rationales.

Second, as Arrow (1963, page 110) writes, “ideally, one could observe all preferences among the available alternatives, but there would be no way to observe preferences among alternatives not feasible for society,” even if each player has a single preference. This argument justifies the Alternative Scenario either directly or via *Arrow’s IIA* (Independence of Irrelevant Alternatives), which requires that social choice from an agenda depend only on the individual preferences restricted to the agenda. Both the core and the core without majority dissatisfaction satisfy Arrow’s IIA.⁹

1.3 The core without majority dissatisfaction

The *core without majority dissatisfaction* (Definition 1) is obtained by deleting from an agenda all the alternatives that are non-maximal for all *individual* players in a large (winning) coalition. It is (Lemma 5) a strengthening or subset of the *core*, obtained by deleting from the agenda all the alternatives that are non-maximal for a large (winning) coalition of players *collectively*.

⁸This is not to say that *partial* rules are uninteresting as an object of study. Computability theory (e.g., Odifreddi, 1992), for example, is powerful precisely because it includes partial functions in its scope.

⁹To be precise, whenever $\succ_i^{\mathbf{p}} \cap (B \times B) = \succ_i^{\mathbf{p}'} \cap (B \times B)$ for all i , we have $C(\mathcal{W}, B, \mathbf{p}) = C(\mathcal{W}, B, \mathbf{p}')$ and $C^+(\mathcal{W}, B, \mathbf{p}) = C^+(\mathcal{W}, B, \mathbf{p}')$. The assertion for C^+ is a corollary of Lemma 4. The reader should not confuse Arrow’s IIA with the IIA (sometimes called *property α*) for choice rules, discussed by Kalai et al. (2002), for example.

Consequently, it only chooses Pareto efficient alternatives from an agenda, unlike other solutions such as the stability set (Rubinstein, 1980, page 153).

The core without majority dissatisfaction is a simple solution concept that treats the maximal sets $\max_B \succ_i^{\mathbf{p}}$ of the players in a better-behaved way than the core does. (It is reasonable to pay attention to such sets, since they are the very objects that we assume to be nonempty.) First, unlike the core, the core without majority dissatisfaction depends only on the players' maximal sets (Lemma 4).¹⁰ Second, each alternative in the core without majority dissatisfaction belongs to someone's maximal set (Lemma 6). The same is not true for the core (Examples 1 and 2).

1.4 Overview of the results

The main results of the paper are similar to Nakamura's theorem (1979), except that they consider *profiles for* an agenda B —profiles of (not necessarily acyclic) preferences that have maximal elements on the agenda.

The first main result, Theorem 2, is about the original Nakamura number for simple games \mathcal{W} defined on an algebra of coalitions. It asserts that the following statements are equivalent: (i) the number of alternatives in the agenda B is less than the Nakamura number $\nu(\mathcal{W})$; (ii) the core $C^+(\mathcal{W}, B, \mathbf{p})$ without majority dissatisfaction is nonempty for all profiles \mathbf{p} for the agenda; (iii) the core $C(\mathcal{W}, B, \mathbf{p})$ is nonempty for all profiles \mathbf{p} for the agenda. Regardless of which choice rule is used, the Nakamura number therefore measures the extent (the size of an agenda) to which simple games will assuredly choose some alternative from the agenda, whether individual preferences are assumed to be acyclic or to have maximal elements.

Theorem 2 is remarkable for the following reasons: First, it demonstrates that one can obtain a significant result without assuming acyclic preferences. Though neglected in the literature, a preference aggregation theory without acyclicity has potential. Second, the *general* nonemptiness of the core implies the *general* nonemptiness of the *strengthening* of the core. The core without majority dissatisfaction is as satisfactory as the core according to this criterion. Third, restricting preferences to those with maximal elements allows us to drop the awkward condition that the agenda is finite. Unlike Theorem 1, Theorem 2 gives a condition for the general nonemptiness of the core for *infinite*, as well as finite, agenda. Fourth, it fits the Alternative Scenario. It gives a condition for the planner to be assured of the existence of a chosen alternative as soon as she presents an agenda to the players (i.e., before she learns their preferences, supposing that they have maximal elements). Fifth, our framework is very general. Like Nakamura (1979), we impose no conditions such as monotonicity or properness on simple games.

¹⁰This property is sometimes called “tops-only”; it is investigated in an abstract social choice framework by Mihara (2000), for example.

Unlike Nakamura (1979), we consider arbitrary sets of players and *arbitrary algebras* of coalitions.¹¹

The *second main result*, Theorem 3, is about the *kappa number* (Definition 2), an extension of the Nakamura number to the even more general framework that distinguishes the collection \mathcal{B}' of the sets of players for which one can assign winning/losing status from the algebra \mathcal{B} of (identifiable) coalitions. The kappa number $\kappa(\mathcal{W}')$ is defined for a collection $\mathcal{W}' \subseteq \mathcal{B}'$ of winning sets. The result asserts that the following two statements are equivalent: (i) the number of alternatives in the agenda B is less than the kappa number $\kappa(\mathcal{W}')$; (ii) the core $C^+(\mathcal{W}', B, \mathbf{p})$ without majority dissatisfaction is nonempty for all profiles \mathbf{p} for the agenda. It also asserts that the above two statements imply, but are not implied by: (iii) the core $C(\mathcal{W}', B, \mathbf{p})$ is nonempty for all profiles \mathbf{p} for the agenda.

Theorem 3 gives another sense in which the core without majority dissatisfaction behaves better than the core. Computing the kappa number is not an easy task in general. So, in applying the theorem, Lemma 9 is useful; it estimates the kappa number from above and below in terms of the Nakamura numbers.

2 Preliminaries

2.1 Some facts about ordinal numbers

The notion of *ordinal numbers* (or *ordinals*) generalizes that of natural numbers. This section presents some facts about ordinals.¹² Consult a textbook for axiomatic set theory (e.g., Hrbacek and Jech (1984)) for more systematic treatment.

We start by introducing the first “few” ordinals. The natural numbers are constructed from \emptyset as follows: $0 = \emptyset$, $1 = 0 \cup \{0\} = \{0\} = \{\emptyset\}$, $2 = 1 \cup \{1\} = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, $3 = 2 \cup \{2\} = \{0, 1, 2\}$, $4 = 3 \cup \{3\} = \{0, 1, 2, 3\}$, etc. The first ordinal number that is not a natural number is the set $\omega = \{0, 1, 2, 3, \dots\}$ of natural numbers. We can continue the process to obtain $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \dots, \omega\}$, $\omega + 2 = (\omega + 1) + 1 = (\omega + 1) \cup \{\omega + 1\} = \{0, 1, 2, \dots, \omega, \omega + 1\}$, etc. We then have $\omega \cdot 2 = \omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$, $\omega \cdot 2 + 1, \dots, \omega \cdot 3, \dots, \omega \cdot \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \omega \cdot 3, \dots, \omega \cdot 4, \dots\}$.

For a given ordinal α , its *successor* $\alpha + 1 = \alpha \cup \{\alpha\}$ always exists and is

¹¹Most works in this literature consider finite sets of players. Nakamura (1979) considers arbitrary (possibly infinite) sets of players and the algebra of all subsets of players.

¹²The paper does not require much knowledge of the theory of ordinal numbers. Understanding a few notions (such as *limit ordinals* and *cardinal numbers*) and facts will suffice to understand details of the paper (mostly in footnotes). A deeper application to economic theory can be found in Lipman (1991), who uses this theory to find a fixed point of an “infinite regress” that a modeler faces.

an ordinal. For a given ordinal α , $\alpha - 1$ does not necessarily exist: if there is an ordinal β such that $\alpha = \beta + 1$, then α is a *successor ordinal*; otherwise, it is a *limit ordinal*. Every natural number except 0 is a successor ordinal. Both ω and $\omega + \omega$ are limit ordinals. But $\omega + 1$, $\omega + 2$, etc. are successor ordinals.

Define $\alpha < \beta$ if and only if $\alpha \in \beta$. $<$ has all the properties of a linear order. Every set A of ordinal numbers are *well-ordered* by $<$, that is, every nonempty subset of A has a $<$ -least element.

Each ordinal α has the property that

$$\alpha = \{\beta : \beta \text{ is an ordinal and } \beta < \alpha\}.$$

If α is a successor ordinal, say $\beta + 1$, then as a set, it has a greatest element, namely β . If α is a limit ordinal, then it does not have a greatest element, and $\alpha = \sup\{\beta : \beta < \alpha\}$.

A function whose domain is an ordinal α is called a (*transfinite*) *sequence* of length α .

Two sets Y and Y' are *equipotent* if there is a bijection (one-to-one and onto function) from Y to Y' . An ordinal number α is an *initial* ordinal if it is not equipotent to any $\beta < \alpha$. For example, ω is an initial ordinal, because it is not equipotent to any natural number. $\omega + 1$ not initial, because it is equipotent to ω . Similarly, none of countable ordinals $\omega + 2$, $\omega + 3$, $\omega + \omega$, $\omega \cdot \omega$, ω^ω , ... is initial.

The *cardinal number* of a set Y , denoted $\#Y$, is the unique initial ordinal equipotent to Y . In particular, if Y is countable, then $\#Y = \omega$. There are arbitrarily large cardinal numbers. Infinite cardinal numbers form a transfinite sequence of *alephs* \aleph_α , with α ranging over all ordinal numbers. We have $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$. Appendix A.1 gives a proof of the following:

Lemma 1 *An infinite cardinal number is a limit ordinal.*

Without defining the *ordinal sum* and the *cardinal sum* here, let us just mention the following useful lemma (proved in Appendix A.2):

Lemma 2 $\#(\alpha + \beta) = \#\alpha + \#\beta$, where the sum on the left side is the *ordinal sum* and the sum on the right is the *cardinal sum*.

2.2 Framework

Let N be an arbitrary nonempty set of players and $\mathcal{B} \subseteq 2^N$ an arbitrary Boolean algebra of subsets of N (so \mathcal{B} includes N and is closed under union, intersection, and complementation). The elements of \mathcal{B} are called **coalitions**. Intuitively, they are the observable or identifiable or describable subsets of players. A (\mathcal{B})-**simple game** \mathcal{W} is a subcollection of \mathcal{B} such that

$\emptyset \notin \mathcal{W}$ and $\mathcal{W} \neq \emptyset$. The elements of \mathcal{W} are said to be **winning**, and the other elements in \mathcal{B} are **losing**. A simple game \mathcal{W} is **weak** if the intersection $\cap \mathcal{W} = \cap_{S \in \mathcal{W}} S$ of the winning coalitions is nonempty.

Let X be a (finite or infinite) set of **alternatives**, with cardinal number $\#X \geq 2$. In this paper, a (*strict*) **preference** is an asymmetric relation \succ on X : if $x \succ y$ (“ x is preferred to y ”), then $y \not\succeq x$. A relation \succ is **total** if $x \neq y$ implies $x \succ y$ or $y \succ x$. An asymmetric relation is a **linear order** if it is transitive and total. A binary relation \succ on X is *acyclic* if for any finite set $\{x_1, x_2, \dots, x_m\} \subseteq X$, whenever $x_1 \succ x_2, \dots, x_{m-1} \succ x_m$, we have $x_m \not\succeq x_1$. Acyclic relations are preferences since they are asymmetric (and irreflexive). Let \mathcal{A} be the set of *acyclic* preferences on X .

A **profile** is a list $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$ of individual preferences $\succ_i^{\mathbf{p}}$. Intuitively, $x \succ_i^{\mathbf{p}} y$ means that player i prefers x to y at profile \mathbf{p} . A profile \mathbf{p} is (\mathcal{B})-**measurable** if $\{i : x \succ_i^{\mathbf{p}} y\} \in \mathcal{B}$ for all $x, y \in X$. Denote by $\mathcal{A}_{\mathcal{B}}^N$ the set of all measurable profiles of *acyclic* preferences.

An **agenda** (or “budget set” or “opportunity set”) B is a subset of X . Note that a preference, when restricted to the elements in B , defines an asymmetric relation on B . Let $B \subseteq X$ be an agenda. An alternative $x \in B$ is said to be a **maximal** element of B with respect to a binary relation \succ (written $x \in \max_B \succ$ though B is often dropped) if there does not exist an alternative $y \in B$ such that $y \succ x$. Let $\mathcal{M}(B)$ be the set of **preferences for B** , i.e., *asymmetric* relations \succ on X that has a *maximal element of B* .¹³ Let $\mathcal{M}(B)_{\mathcal{B}}^N$ be the set of **profiles for B** , i.e., measurable profiles $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N} \in \mathcal{M}(B)^N$ of preferences $\succ_i^{\mathbf{p}}$ for B .

A (\mathcal{B})-**simple game with (ordinal) preferences** is a list $(\mathcal{W}, B, \mathbf{p})$ of a \mathcal{B} -simple game $\mathcal{W} \subseteq \mathcal{B}$, a subset B of alternatives, and a profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$. Given the simple game with preferences, we define the (not necessarily asymmetric) **dominance** relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ on X by $x \succ_{\mathcal{W}}^{\mathbf{p}} y$ if and only if there is a winning coalition $S \in \mathcal{W}$ such that $x \succ_i^{\mathbf{p}} y$ for all $i \in S$.¹⁴ The **core** $C(\mathcal{W}, B, \mathbf{p})$ of the simple game with preferences is the set $\max_B \succ_{\mathcal{W}}^{\mathbf{p}}$ of undominated alternatives:

$$C(\mathcal{W}, B, \mathbf{p}) = \{x \in B : \nexists y \in B \text{ such that } y \succ_{\mathcal{W}}^{\mathbf{p}} x\}.$$

An alternative $x \in B$ is **Pareto in B** if there exists no $y \in B$ such that $\{i : y \succ_i^{\mathbf{p}} x\} = N$. It is easy to prove that any alternative in $C(\mathcal{W}, B, \mathbf{p})$ or in $\bigcup_i \max \succ_i^{\mathbf{p}}$ is Pareto in B .

A (**preference**) **aggregation rule** is a map $\succ : \mathbf{p} \mapsto \succ^{\mathbf{p}}$ from profiles \mathbf{p} of preferences to binary relations (*social preferences*) $\succ^{\mathbf{p}}$ on the set X of

¹³We define a preference for B on X , despite the fact that the set of maximal elements of B with respect to \succ depends only on the restricted relation $\succ \cap (B \times B)$.

¹⁴In this definition, $\{i : x \succ_i^{\mathbf{p}} y\}$ need not be winning since we do not assume \mathcal{W} is monotonic. Andjiga and Mbih (2000) study Nakamura’s theorem, adopting the notion of dominance that requires the above coalition to be winning.

alternatives. For example, the mapping $\succ_{\mathcal{W}}$ from profiles $\mathbf{p} \in \mathcal{A}_{\mathcal{B}}^N$ of acyclic preferences to dominance relations $\succ_{\mathcal{W}}^{\mathbf{p}}$ is an aggregation rule.

2.3 Nakamura's theorem for acyclic preferences

Nakamura (1979) gives a condition for a 2^N -simple game with preferences to have a nonempty core for any profile \mathbf{p} of *acyclic* preferences. To state Nakamura's theorem, we define the **Nakamura number** $\nu(\mathcal{W})$ of a \mathcal{B} -simple game \mathcal{W} to be the size of the smallest collection of winning coalitions having empty intersection¹⁵

$$\nu(\mathcal{W}) = \min\{\#\mathcal{W}' : \mathcal{W}' \subseteq \mathcal{W} \text{ and } \bigcap \mathcal{W}' = \emptyset\}$$

if $\bigcap \mathcal{W} := \bigcap_{S \in \mathcal{W}} S = \emptyset$ (i.e., if \mathcal{W} is *nonweak*); otherwise, set $\nu(\mathcal{W}) = +\infty$, which is understood to be greater than any cardinal number. By the assumption that $\emptyset \notin \mathcal{W}$ and $\mathcal{W} \neq \emptyset$, we have $\nu(\mathcal{W}) \geq 2$. It is easy to prove the following lemma:¹⁶

Lemma 3 *If \mathcal{W} is a nonweak \mathcal{B} -simple game, then $2 \leq \nu(\mathcal{W}) \leq \min\{\#S : S \in \mathcal{W}\} + 1$ and $\nu(\mathcal{W}) \leq \#N$.*

The following theorem extends Nakamura's result (Nakamura, 1979) for $\mathcal{B} = 2^N$:

Theorem 1 (Kumabe and Mihara (2008a)) *Let $\mathcal{W} \subseteq \mathcal{B}$ be a simple game and $B \subseteq X$ an agenda. Then the core $C(\mathcal{W}, B, \mathbf{p})$ is nonempty for all measurable profiles $\mathbf{p} \in \mathcal{A}_{\mathcal{B}}^N$ of acyclic preferences if and only if B is finite and $\#B < \nu(\mathcal{W})$.*

Since $\succ_{\mathcal{W}}^{\mathbf{p}}$ is acyclic if and only if the set $C(\mathcal{W}, B, \mathbf{p})$ of maximal elements of B with respect to $\succ_{\mathcal{W}}^{\mathbf{p}}$ is nonempty for every finite subset B of X , we have:

Corollary 1 *The dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ is acyclic for all $\mathbf{p} \in \mathcal{A}_{\mathcal{B}}^N$ if and only if $\#B < \nu(\mathcal{W})$ for all finite $B \subseteq X$.*

3 Two notions of the core

In this section, we first introduce the notion of the *core without majority dissatisfaction*, a strengthening of the core. We then compare the two notions of the core, focusing on how they treat the maximal elements of individual preferences.

We consider \mathcal{B} -simple games $(\mathcal{W}, B, \mathbf{p})$ with preferences, given for each profile \mathbf{p} (not necessarily in $\mathcal{M}(B)_{\mathcal{B}}^N$). An alternative $x \in B$ is *not* in the

¹⁵The minimum of the following set of cardinal numbers exists since every set of ordinal numbers has a $<$ -least element.

¹⁶This result can be found in Nakamura (1979, Lemma 2.1 and Corollary 2.2).

core $C(\mathcal{W}, B, \mathbf{p})$ if x is not maximal with respect to the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$: there are $y \in B$ and a winning coalition $S \in \mathcal{W}$ such that for all $i \in S$, $y_i = y$ and $y_i \succ_i^{\mathbf{p}} x$. (That is, $x \in B$ is not in $C(\mathcal{W}, B, \mathbf{p})$ if for some $y \in B$, the set $\{i : y \succ_i^{\mathbf{p}} x\}$ contains a winning coalition.) So, an alternative x (e.g., d in Examples 1 and 2 below) can be in the core even if there is a winning coalition contained in the set of players i that prefer some y_i to x , as long as y_i is different among the players. To exclude such an x from the core, we modify the definition:

Definition 1 An alternative $x \in B$ is in the **core** $C^+(\mathcal{W}, B, \mathbf{p})$ **without majority dissatisfaction** if there is no winning coalition $S \in \mathcal{W}$ such that for all $i \in S$, there exists some $y_i \in B$ satisfying $y_i \succ_i^{\mathbf{p}} x$. In other words, $x \in B$ is in $C^+(\mathcal{W}, B, \mathbf{p})$ if the set $\{i : x \notin \max_B \succ_i^{\mathbf{p}}\} = \{i : y \succ_i^{\mathbf{p}} x \text{ for some } y \in B\}$ of players for whom x is non-maximal (players “dissatisfied with x ”) contains no winning coalition.

Remark 1 The word “core” usually refers to the set of maximal alternatives with respect to some relation. We adopt the word since the core without majority dissatisfaction is indeed the set of maximal elements with respect to the following extended dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$. First, we extend i ’s preference $\succ_i^{\mathbf{p}} \subseteq X \times X$ to a relation $\succ_i^{\mathbf{p}} \subseteq 2^X \times X$ (where 2^X is the power set of X): $Y \succ_i^{\mathbf{p}} x$ if and only if there is $y \in Y$ such that $y \succ_i^{\mathbf{p}} x$.¹⁷ Next, we extend the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}} \subseteq X \times X$ to a relation $\succ_{\mathcal{W}}^{\mathbf{p}} \subseteq 2^X \times X$: $Y \succ_{\mathcal{W}}^{\mathbf{p}} x$ if and only if there is $S \in \mathcal{W}$ such that for all $i \in S$, $Y \succ_i^{\mathbf{p}} x$. Then, $x \in C^+(\mathcal{W}, B, \mathbf{p})$ if and only if there is no $Y \subseteq B$ such that $Y \succ_{\mathcal{W}}^{\mathbf{p}} x$ (if and only if $B \not\succeq_{\mathcal{W}}^{\mathbf{p}} x$).

The following two lemmas are immediate from the definition. The first one says that the core without majority dissatisfaction depends only on the set of maximal elements of each player.

Lemma 4 Let \mathbf{p}, \mathbf{p}' be two profiles satisfying $\max_B \succ_i^{\mathbf{p}} = \max_B \succ_i^{\mathbf{p}'}$ for all i . Then $C^+(\mathcal{W}, B, \mathbf{p}) = C^+(\mathcal{W}, B, \mathbf{p}')$.

Lemma 5 For each profile \mathbf{p} , we have $C^+(\mathcal{W}, B, \mathbf{p}) \subseteq C(\mathcal{W}, B, \mathbf{p})$. The inclusion \subseteq is strict for some profile.

Each of Examples 1 and 2 below¹⁸ shows that the inclusion is sometimes strict. Example 1 also shows that *an alternative can be in the core even if it is not maximal with respect to anyone’s preference*:

¹⁷We are following the consequentialist approach of extending preferences on X to ones on its power set—the set of opportunity sets. Unless one is concerned with preferences for flexibility (e.g., Kreps, 1979) or freedom of choice, this approach is standard.

¹⁸See Appendix A.4 for graph representations of the profiles in these examples.

Example 1 Let $N = \{1, 2, 3\}$ and let \mathcal{W} consist of the coalitions having a majority (i.e., having at least two players). Let $X = \{a, b, c, d, e\}$. Define a profile by $\succ_1 = \{(a, d), (e, b), (e, c)\}$, $\succ_2 = \{(b, d), (e, a), (e, c)\}$, and $\succ_3 = \{(c, d), (e, a), (e, b)\}$. Then the sets $\max \succ_i$ of maximal elements of X with respect to \succ_i are given by $\max \succ_1 = \{a, e\}$, $\max \succ_2 = \{b, e\}$, and $\max \succ_3 = \{c, e\}$. But the core C is $\{d, e\}$. So, the core is neither a subset nor a superset of $\bigcup_i \max \succ_i = \{a, b, c, e\}$. The core C^+ without majority dissatisfaction is $\{e\}$, which is a subset of $\bigcup_i \max \succ_i$.

Example 2 also shows that *the core, even if it is nonempty, does not necessarily intersect the union of the maximal elements of individual preferences*:

Example 2 We modify the simplest voting paradox (a cycle involving three alternatives and three players) by adding an alternative d . Let $N = \{1, 2, 3\}$ and let \mathcal{W} consist of the coalitions having a majority. Let $X = \{a, b, c, d\}$. Define a profile by $\succ_1 = \{(a, b), (b, c), (a, d)\}$, $\succ_2 = \{(b, c), (c, a), (b, d)\}$, and $\succ_3 = \{(c, a), (a, b), (c, d)\}$. Then the core C is $\{d\}$. So it does not intersect $\bigcup_i \max \succ_i = \{a, b, c\}$. The core C^+ without majority dissatisfaction is empty.

Unlike the core, the core without majority dissatisfaction is always included in the union of the maximal elements of individual preferences:

Lemma 6 *For each profile \mathbf{p} , we have $C^+(\mathcal{W}, B, \mathbf{p}) \subseteq C(\mathcal{W}, B, \mathbf{p}) \cap (\bigcup_i \max_B \succ_i^{\mathbf{p}})$.*¹⁹

Proof. By Lemma 5, it suffices to show that C^+ is a subset of $\bigcup_i \max_B \succ_i^{\mathbf{p}}$. Suppose $x \in B$ but $x \notin \bigcup_i \max_B \succ_i^{\mathbf{p}}$. Then, $x \notin \max_B \succ_i^{\mathbf{p}}$ for any $i \in N$. This implies that $\{i : x \notin \max_B \succ_i^{\mathbf{p}}\} = N \supseteq S$ for any winning coalition $S \in \mathcal{W}$. (Such an S exists since $\mathcal{W} \neq \emptyset$.) By the definition of C^+ , we have $x \notin C^+$. ■

4 Preferences with Maximal Alternatives

We consider in the rest of the paper profiles *for* a set B of alternatives, that is, measurable profiles consisting of *preferences that have a maximal element of B* .

4.1 The results for the core of a simple game

We now give a version of Nakamura's theorem for profiles *for* a set B of alternatives:

¹⁹Appendix A.3 shows that the inclusion is strict for some profile.

Theorem 2 Let $\mathcal{W} \subseteq \mathcal{B}$ be a simple game and $B \subseteq X$ an agenda. Let $\mathcal{M}(B)_B^N$ be the set of measurable profiles of preferences having a maximal element of B . Then the following statements are equivalent:²⁰

- (i) $\#B < \nu(\mathcal{W})$;
- (ii) the core $C^+(\mathcal{W}, B, \mathbf{p})$ without majority dissatisfaction is nonempty for all $\mathbf{p} \in \mathcal{M}(B)_B^N$;
- (iii) the core $C(\mathcal{W}, B, \mathbf{p})$ is nonempty for all $\mathbf{p} \in \mathcal{M}(B)_B^N$.

Proof. (i) \Rightarrow (ii). Suppose $C^+(\mathcal{W}, B, \mathbf{p}) = \emptyset$ for some profile \mathbf{p} for B . By Definition 1, for each $x \in B$, there is a winning coalition $S_x \in \mathcal{W}$ such that $S_x \subseteq \{i : x \notin \max \succ_i^{\mathbf{p}}\}$. We claim that $\bigcap_{x \in B} S_x = \emptyset$. (Otherwise, there is an i who is in S_x for all $x \in B$. It follows that $\succ_i^{\mathbf{p}}$ has no maximal element of B .) The claim shows that $\nu(\mathcal{W}) \leq \#B$.

(ii) \Rightarrow (iii). Immediate from Lemma 5.

(iii) \Rightarrow (i). Suppose $\#B \geq \nu(\mathcal{W})$. We construct a profile \mathbf{p} for B such that $C(\mathcal{W}, B, \mathbf{p}) = \emptyset$. Let $\nu = \nu(\mathcal{W}) \geq 2$.

Step 1, Case (a): ν is finite. We construct a profile \mathbf{p} such that the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ has a cycle consisting of ν alternatives (and the cycle contains an alternative x_0 greater than any alternative y not belonging to the cycle).

By the definition of the Nakamura number, there is a collection $\mathcal{W}' = \{L_0, \dots, L_{\nu-1}\}$ of winning coalitions such that $\bigcap \mathcal{W}' = \bigcap_{k=0}^{\nu-1} L_k = \emptyset$.

Choose a subset $B' = \{x_0, x_1, \dots, x_{\nu-1}\} \subseteq B$ and write $x_\nu = x_0$. Fix a cycle (noting that $(x, y) \in \succ$ means $x \succ y$)

$$\succ = \{(x_{k+1}, x_k) : k \in \{0, \dots, \nu - 1\}\}$$

and a relation $\succ' = \{(x_0, y) : y \notin B'\}$. Now, go to Step 2.

Step 1, Case (b): ν is infinite. We construct a profile \mathbf{p} such that the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ defines an increasing transfinite sequence of length ν (and the sequence contains an alternative x_0 greater than any alternative y not belonging to the sequence).

Recall that if ν is an ordinal, then $\nu = \{\alpha : \alpha \text{ is an ordinal and } \alpha < \nu\}$. By the definition of the Nakamura number, there is a collection²¹ $\mathcal{W}' = \{L_\alpha : \alpha \in \nu\}$ of winning coalitions L_α such that $\bigcap \mathcal{W}' = \bigcap_{\alpha < \nu} L_\alpha = \emptyset$.

Choose a subset $B' = \{x_\alpha : \alpha \in \nu\} \subseteq B$. Fix a relation

$$\succ = \{(x_{\alpha+1}, x_\alpha) : \alpha \in \nu\},$$

²⁰ The implication (iii) \Rightarrow (ii) is *not* the same as the following statement (Example 2): for each $\mathbf{p} \in \mathcal{M}(B)_B^N$, if $C(\mathcal{W}, B, \mathbf{p}) \neq \emptyset$, then $C^+(\mathcal{W}, B, \mathbf{p}) \neq \emptyset$.

²¹ Since \mathcal{W}' is a (well-orderable) set whose cardinal number is ν , there is a bijection that maps each $\alpha \in \nu$ into an element L_α of \mathcal{W}' . So we can write $\mathcal{W}' = \{L_\alpha : \alpha \in \nu\}$.

which defines an increasing transfinite sequence of alternatives,²² and another relation $\succ' = \{(x_0, y) : y \notin B'\}$.

Step 2. We define a profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$ by specifying, for each pair $(x, y) \in X^2$, the set $\{i : x \succ_i^{\mathbf{p}} y\}$ of players that prefer x to y . Note that $\succ \cap \succ' = \emptyset$. (In the following, the letter α denotes an ordinal number, including a natural number denoted k in Case (a) above.) If $(x, y) = (x_{\alpha+1}, x_\alpha) \in \succ$, let $\{i : x \succ_i^{\mathbf{p}} y\} = L_\alpha$. If $(x, y) \in \succ'$, let $\{i : x \succ_i^{\mathbf{p}} y\} = N$. If $(x, y) \notin \succ \cup \succ'$, let $\{i : x \succ_i^{\mathbf{p}} y\} = \emptyset$. The profile \mathbf{p} is measurable since $L_\alpha, N, \emptyset \in \mathcal{B}$.

The profile \mathbf{p} is for B (i.e., $\mathbf{p} \in \mathcal{M}(B)_{\mathcal{B}}^N$), since we can show that each i 's preference $\succ_i^{\mathbf{p}}$ has a maximal element of B . For example, if $i \in L_2 \cap L_3 \cap L_5$, but $i \notin L_k$ for $k \notin \{2, 3, 5\}$, then every alternative in B' except x_2, x_3, x_5 is a maximal element of $\succ_i^{\mathbf{p}}$. (More formally, the set of maximal elements of $\succ_i^{\mathbf{p}}$ is $\{x_\alpha \in B' : i \notin L_\alpha\}$, which is nonempty since $\bigcup L_\alpha^c = N$, where $L_\alpha^c = N \setminus L_\alpha$.)

The dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ is clearly $\succ \cup \succ'$, since L_α is winning, N contains a winning coalition, and \emptyset is losing. It follows that the core $C(\mathcal{W}, B, \mathbf{p})$ is empty. ■

The following example is an application of Theorem 2. It gives a condition for an *infinite* set of alternatives to have an element in the core, that is, a maximal element with respect to the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$.

Example 3 Let $N = [0, 1]$ be the unit interval on the set of real numbers and let \mathcal{B} consist of all subsets of N . Define a simple game \mathcal{W} by $S \in \mathcal{W}$ if and only if S^c is countable. Then, it is easy to show that $\nu(\mathcal{W})$ is uncountable. Let X be a countable set of alternatives (e.g., the set of rational numbers in $[0, 1]$). Suppose that for each i , her preference $\succ_i^{\mathbf{p}}$ has a maximal alternative (e.g., a utility function representing $\succ_i^{\mathbf{p}}$ has finitely many ‘‘peaks,’’ all corresponding to rational numbers). Then, Theorem 2 implies that $C^+(\mathcal{W}, X, \mathbf{p})$ and $C(\mathcal{W}, X, \mathbf{p})$ both contain alternatives.

The profiles constructed in the proof of (iii) \Rightarrow (i) of Theorem 2 consist of individual preferences that may have more than one maximal alternative. However, we can modify the proof in such a way that each individual preference has *exactly one* maximal alternative. That gives the following proposition:

Proposition 1 *Let $\mathcal{W} \subseteq \mathcal{B}$ be a simple game and $B \subseteq X$ an agenda. Then the three equivalent statements (i), (ii), and (iii) in Theorem 2 are equivalent to the following statements:*

(ii.a) *The core $C^+(\mathcal{W}, B, \mathbf{p})$ without majority dissatisfaction is nonempty for*

²²The sequence $(x_\alpha)_{\alpha \in \nu}$ does not end: if $\alpha \in \nu$, then $\alpha + 1 \in \nu$. This is because ν , being a Nakamura number, is a cardinal number and any infinite cardinal number is a limit ordinal by Lemma 1.

all measurable profiles \mathbf{p} of preferences having exactly one maximal element of B ,

(ii.b) The core $C^+(\mathcal{W}, B, \mathbf{p})$ without majority dissatisfaction is nonempty for all measurable profiles \mathbf{p} of linear orders having a maximal element of B ,

(iii.a) The core $C(\mathcal{W}, B, \mathbf{p})$ is nonempty for all measurable profiles \mathbf{p} of preferences having exactly one maximal element of B ,

(iii.b) The core $C(\mathcal{W}, B, \mathbf{p})$ is nonempty for all measurable profiles \mathbf{p} of linear orders having a maximal element of B .

Proof. (ii) \Rightarrow (ii.a) \Rightarrow (ii.b) and (iii) \Rightarrow (iii.a) \Rightarrow (iii.b) are obvious. (ii.b) \Rightarrow (iii.b) is immediate from Lemma 5.

(iii.b) \Rightarrow (i). Suppose $\#B \geq \nu(\mathcal{W})$. We construct a profile \mathbf{p} satisfying the condition such that $C(\mathcal{W}, B, \mathbf{p}) = \emptyset$. Let $\nu = \nu(\mathcal{W}) \geq 2$. In *Step 1* of the proof of Theorem 2, replace the relation \succ' by an arbitrary asymmetric subrelation $\succ' \subset X \times (X \setminus B')$ that is linear on $X \setminus B'$ and satisfies $x \succ' y$ whenever $x \in B'$ and $y \in X \setminus B'$. We replace *Step 2* of the proof by the following argument.

Case: ν is finite. Define $L_{-1} = N$ and for all $k \in \{0, \dots, \nu - 1\}$,

$$D_k = (L_{-1} \cap L_0 \cap \dots \cap L_{k-1}) \setminus L_k.$$

Then $\{D_0, \dots, D_{\nu-1}\}$ is a family of (possibly empty) pairwise disjoint coalitions in \mathcal{B} such that $L_k \subseteq D_k^c := N \setminus D_k$ for all k and $\bigcup_{k=0}^{\nu-1} D_k = N$ ($i \in N$ is in the first D_k such that $i \notin L_k$).

Define \mathbf{p} as follows: for each k , all players i in D_k have the same linearly ordered preference $\succ_i^{\mathbf{p}}$ with maximal element x_k , obtained by taking the transitive closure of $\succ \setminus \{(x_{k+1}, x_k)\} \cup \succ'$. Obviously, \mathbf{p} is measurable since for each $x, y \in X$, $\{i : x \succ_i^{\mathbf{p}} y\}$ is a finite union of D_k 's. Also, $\succ_{\mathcal{W}}^{\mathbf{p}}$ includes $\succ \cup \succ'$. (If $(x, y) = (x_{k+1}, x_k) \in \succ$, we have $\{i : x \succ_i^{\mathbf{p}} y\} = D_k^c \supseteq L_k \in \mathcal{W}'$. If $(x, y) \in \succ'$, then $\{i : x \succ_i^{\mathbf{p}} y\} = N$.) It follows that $C(\mathcal{W}, X, \mathbf{p}) = \emptyset$.

Case: ν is infinite. Note that $(B' \times B') \cap \succ' = \emptyset$. Define \mathbf{p} as follows: If $(x, y) \in \succ'$, then $\{i : x \succ_i^{\mathbf{p}} y\} = N$. If $(x, y) \notin (B' \times B') \cup \succ'$, then $\{i : x \succ_i^{\mathbf{p}} y\} = \emptyset$. If $(x, y) = (x_\alpha, x_\beta) \in B' \times B'$, $x_\alpha \succ_i^{\mathbf{p}} x_\beta$ if and only if either²³

- $i \in L_\alpha^c \cap L_\beta^c$ and $\alpha < \beta$; or
- $i \in L_\alpha^c \cap L_\beta$ and $\alpha \neq \beta$; or

²³For each i , $\{x_\alpha : i \in L_\alpha^c\} \neq \emptyset$ is i 's preferred class of alternatives in B' and $\{x_\alpha : i \in L_\alpha\}$ is her less-preferred class. Her preference $\succ_i^{\mathbf{p}}$ is linear on B' satisfying the following conditions: (a) if two alternatives belong to her preferred class, then i prefers the one with the smaller index; (b) i prefers each alternative in her preferred class to any in her less-preferred class; (c) if two alternatives belong to her less-preferred class, then i prefers the one with the greater index. For example, if $i \in L_2 \cap L_3 \cap L_5$, but $i \notin L_\alpha$ for $\alpha \notin \{2, 3, 5\}$, then i ranks the alternatives in the following order: $x_0, x_1, x_4, x_6, x_7, \dots, x_5, x_3, x_2$.

- $i \in L_\alpha \cap L_\beta$ and $\alpha > \beta$.

This is equivalent to saying that ²⁴

$$\{i : x \succ_i^{\mathbf{p}} y\} = \{i : x_\alpha \succ_i^{\mathbf{p}} x_\beta\} = \begin{cases} L_\alpha^c & \text{if } \alpha < \beta, \\ L_\beta & \text{if } \alpha > \beta, \\ \emptyset & \text{if } \alpha = \beta. \end{cases}$$

The profile \mathbf{p} is measurable since $N, \emptyset, L_\alpha^c, L_\beta \in \mathcal{B}$.

Clearly, each i 's preference $\succ_i^{\mathbf{p}}$ is a linear order on X that has a unique maximal element of B , namely the alternative in her preferred class $\{x_\alpha : i \in L_\alpha^c\} \neq \emptyset$ with the smallest index.

The dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ clearly includes $\succ \cup \succ'$, since $\{i : x_{\beta+1} \succ_i^{\mathbf{p}} x_\beta\} = L_\beta \in \mathcal{W}$ for each $\beta \in \nu$, for example. It follows that $C(\mathcal{W}, X, \mathbf{p}) = \emptyset$. ■

Remark 2 The argument for the finite ν case of the proof does not go through for the infinite case, since D_k does not necessarily belong to \mathcal{B} when k is infinite. The argument for the infinite case causes difficulty when applied to the finite case. For example, suppose $\nu = 4$ and $i \in L_0 \cap L_3$. Since $i \in L_3$ and $4 > 3$, we have $x_4 \succ_i^{\mathbf{p}} x_3$ by the definition of \mathbf{p} . Since $i \in L_0$ and $3 > 0$, we have $x_3 \succ_i^{\mathbf{p}} x_0$. Since $x_0 = x_4$, $\succ_i^{\mathbf{p}}$ is not asymmetric.

4.2 The results for the core of a collection of winning sets

4.2.1 Extended framework

We extend the framework by introducing a collection \mathcal{B}' , consisting of subsets of the set N of players. Recall that \mathcal{B} consists of the *coalitions*—intuitively, they are the observable or identifiable sets of players. In contrast, \mathcal{B}' consists of the sets *for which one can assign winning/losing status*—sometimes they are the sets whose size is well-defined (Example 4); other times they are the sets that are half-identifiable or listable (Example 5). We assume $\mathcal{B} \subseteq \mathcal{B}'$, which means that one can assign such a status for any coalition.

A **collection \mathcal{W}' of winning sets** is a subcollection of \mathcal{B}' satisfying $\emptyset \notin \mathcal{W}'$ and $\mathcal{W}' \neq \emptyset$. Given \mathcal{W}' , the most natural simple game one can define is the following: the **\mathcal{B} -simple game \mathcal{W} induced by \mathcal{W}'** consists of the winning coalitions (winning sets that are also coalitions), that is, $\mathcal{W} = \mathcal{W}' \cap \mathcal{B}$. We can define the *core* and the *core without majority dissatisfaction* as before, with \mathcal{W} replaced by \mathcal{W}' . Lemma 5 and $\mathcal{W} \subseteq \mathcal{W}'$ imply the following.

²⁴ $x_\alpha \succ_i^{\mathbf{p}} x_\beta$ iff either (1) $\alpha < \beta$ and $i \in (L_\alpha^c \cap L_\beta^c) \cup (L_\alpha^c \cap L_\beta) = L_\alpha^c \cap (L_\beta^c \cup L_\beta) = L_\alpha^c$ or (2) $\alpha > \beta$ and $i \in (L_\alpha^c \cap L_\beta) \cup (L_\alpha \cap L_\beta) = L_\beta$. Note that the first two cases can be restated: if $\alpha < \beta$, then $\{i : x_\beta \succ_i^{\mathbf{p}} x_\alpha\} = L_\alpha$ and $\{i : x_\alpha \succ_i^{\mathbf{p}} x_\beta\} = L_\alpha^c$.

Lemma 7 *Let $\mathcal{W}' \subseteq \mathcal{B}'$ and $\mathcal{W} = \mathcal{W}' \cap \mathcal{B}$. Then the following statements are true:*

- (i) $C^+(\mathcal{W}', B, \mathbf{p}) \subseteq C^+(\mathcal{W}, B, \mathbf{p}) \subseteq C(\mathcal{W}, B, \mathbf{p})$.
- (ii) $C^+(\mathcal{W}', B, \mathbf{p}) \subseteq C(\mathcal{W}', B, \mathbf{p}) \subseteq C(\mathcal{W}, B, \mathbf{p})$.

Since we had better be able to identify who prefers a given alternative to another, we leave the definition of *measurable* profiles and *profiles for an agenda* unaltered.

4.2.2 Justification for the extended framework

We have assumed $\mathcal{B}' = \mathcal{B}$ in the previous sections, but there are reasons for distinguishing the collection \mathcal{B}' from the Boolean algebra \mathcal{B} of coalitions. We illustrate that by two examples in this section.

The following example gives some, though limited, justification for the extended framework:

Example 4 Let $N = [0, 1]$ be the unit interval on the set of real numbers, \mathcal{B} the σ -algebra of Borel sets (i.e., \mathcal{B} is the smallest σ -algebra containing all open sets in $[0, 1]$), \mathcal{B}' the σ -algebra of Lebesgue measurable sets, and μ Lebesgue measure. Let $\mathcal{W}' \subseteq \mathcal{B}'$ be any collection of winning sets defined in terms of the measure alone. (For example, \mathcal{W}' consists of the sets $S \in \mathcal{B}'$ satisfying $\mu(S) > 1/2$. Alternatively, \mathcal{W}' consists of the sets $S \in \mathcal{B}'$ satisfying $\mu(S) \geq 2/3$.) We do have $\mathcal{B}' \supsetneq \mathcal{B}$, though this fact is not often exploited.²⁵ Observe that for each $S' \in \mathcal{B}'$ there is $S \in \mathcal{B}$ such that $S \subseteq S'$ and $\mu(S) = \mu(S')$.²⁶

The last observation in Example 4 explains why it fails to give a compelling justification for the extended framework, because of the following lemma. The lemma suggests that under a certain condition, it is not very meaningful to introduce a collection $\mathcal{W}' \subseteq \mathcal{B}'$, even when it makes sense to extend \mathcal{B} to \mathcal{B}' . The proof is straightforward.

Lemma 8 *Let $\mathcal{W}' \subseteq \mathcal{B}'$ and $\mathcal{W} = \mathcal{W}' \cap \mathcal{B}$. Suppose that for each winning set $S' \in \mathcal{W}'$, there exists a winning coalition $S \in \mathcal{W}$ satisfying $S \subseteq S'$. Then $C(\mathcal{W}, B, \mathbf{p}) = C(\mathcal{W}', B, \mathbf{p})$ and $C^+(\mathcal{W}, B, \mathbf{p}) = C^+(\mathcal{W}', B, \mathbf{p})$.*

Now we give a more compelling justification for the framework in which \mathcal{B}' and \mathcal{W}' are introduced:

²⁵For example, Dasgupta and Maskin (2008) adopt this framework to formalize the concept that an axiom is satisfied for almost all profiles. However, their focus is measurable profiles, which means they are mainly interested in certain coalitions in \mathcal{B} . Banks et al. (2006), when restricted to our framework, focus on simple games $\mathcal{W} \subseteq \mathcal{B}$, rather than on $\mathcal{W}' \subseteq \mathcal{B}'$.

²⁶Let $S' \in \mathcal{B}'$. Let $E' = S'^c \in \mathcal{B}'$. Apply the following proposition (Royden, 1988, page 293) and let $S = E^c$: If $E' \subseteq [0, 1]$ is any set, then there is a Borel set $E \in \mathcal{B}$ such that $E' \subseteq E$ and $\mu^*(E') = \mu(E)$, where μ^* is Lebesgue outer measure.

Example 5 Let $N = \{0, 1, 2, \dots\}$ be the set of natural numbers. A natural and faithful way to model *identifiable* or *half-identifiable* sets of players is to let \mathcal{B} be the algebra of recursive sets (coalitions) and \mathcal{B}' the lattice of r.e. sets.²⁷

The *first reason* for introducing $\mathcal{B}' \supseteq \mathcal{B}$ in our framework is that we cannot let \mathcal{B} be the lattice of r.e. sets, since the lattice is not a Boolean algebra.

The *second reason* is that the natural system $(W_e)_{e \in N}$ for indexing r.e. sets is easier to handle than the system $(\varphi_e)_{e \in N}$ that can be used for indexing recursive sets (where φ_e is the e th partial recursive function and W_e its domain). For example, while φ_e corresponds to (i.e., is the characteristic function of) no recursive set for some $e \in N$ (and there is no algorithm to decide whether a given e corresponds to some recursive set), W_e corresponds to (i.e., is) an r.e. set for any $e \in N$. One can therefore write any class of r.e. sets as $\{W_e : e \in I\}$ for some (not necessarily r.e.) set I of indices, without worrying that some W_e might correspond to no r.e. set. Odifreddi (1992, page 226) gives more reasons for preferring $(W_e)_{e \in N}$ to $(\varphi_e)_{e \in N}$.

We now give a *reason for introducing* $\mathcal{W}' \subseteq \mathcal{B}'$ into our framework. We exhibit \mathcal{W}' , $\mathcal{W} = \mathcal{W}' \cap \mathcal{B}$, and \mathbf{p} for which $C^+(\mathcal{W}, X, \mathbf{p}) \neq C^+(\mathcal{W}', X, \mathbf{p})$.

A set is *cofinite* if it is the complement of a finite set; otherwise, it is *coinfinite*. A coinfinite r.e. set T is **maximal coinfinite**²⁸ if it has only trivial supersets: if S is a coinfinite r.e. set satisfying $S \supseteq T$, then $S \setminus T$ is finite.

Let \mathcal{W}'_0 be the collection of all maximal coinfinite sets. Since maximal coinfinite sets are nonrecursive, we have $\mathcal{W}_0 = \mathcal{W}'_0 \cap \mathcal{B} = \emptyset$, which is not a simple game according to our definition. We can nevertheless conclude $\mathcal{W}_0 \neq \mathcal{W}'_0$ and define the core, obtaining $C(\mathcal{W}_0, X, \mathbf{p}) = C^+(\mathcal{W}_0, X, \mathbf{p}) = X$ for any profile \mathbf{p} . Let $X = \{a, b, c\}$ and define a profile \mathbf{p} by $\succ_i^{\mathbf{p}} = \{(a, b)\}$ for all $i \in N$. Then, $C(\mathcal{W}'_0, X, \mathbf{p}) = C^+(\mathcal{W}'_0, X, \mathbf{p}) = \{a, c\} \neq X$.

Let $\mathcal{W}'_1 = \{S' \in \mathcal{B}' : S' \supseteq T \text{ for some maximal coinfinite } T\}$. Then we can easily show that $\mathcal{W}_1 = \mathcal{W}'_1 \cap \mathcal{B}$ consists of all cofinite sets; therefore, $\mathcal{W}_1 \neq \mathcal{W}'_1$. Let $X = \{x_{-1}, x_0, x_1, x_2, \dots\}$ be a countable set. Let $T \in \mathcal{B}'$ be a maximal coinfinite set. Define a profile \mathbf{p} as follows: $\succ_i^{\mathbf{p}} = \{(x_i, x_{-1})\}$ if $i \in T$; $\succ_i^{\mathbf{p}} = \emptyset$ otherwise. Then, $\{i : x \succ_i^{\mathbf{p}} y\}$ is $\{i\}$ if $x = x_i$ for $i \in T$

²⁷According to *Church's thesis* (Odifreddi, 1992), a set of players (natural numbers) is *recursive* if there is an algorithm that, given any player, will decide whether she is in the set. A set of players is *r.e. (recursively enumerable)* if there is an algorithm that lists, in some order, the members of the set; the condition in general does not mean that there is an algorithm to decide whether a given player belongs to it. A set $A \subseteq N$ is recursive if and only if A and A^c are both r.e. Odifreddi (1992) gives detailed discussion of recursion theory (computability theory). The papers by Mihara (1997, 1999) contain short reviews of recursion theory.

²⁸What we call *maximal coinfinite sets* are known as *maximal sets* in recursion theory (Odifreddi, 1992, page 288). Friedberg (1958) constructively proves the existence of such sets.

and $y = x_{-1}$; it is \emptyset otherwise. So \mathbf{p} is measurable and each player has a maximal alternative. We have $C(\mathcal{W}_1, X, \mathbf{p}) = C(\mathcal{W}'_1, X, \mathbf{p}) = X$. Also, $\{i : x \notin \max \succ_i^{\mathbf{p}}\}$ is T if $x = x_{-1}$; it is \emptyset otherwise. So, $C^+(\mathcal{W}_1, X, \mathbf{p}) = X$ but $C^+(\mathcal{W}'_1, X, \mathbf{p}) = X \setminus \{x_{-1}\}$.

4.2.3 The results for the extended framework

Before stating the main result for the extended framework, we need to extend the notion of the Nakamura number.

Let $\mathcal{B}' \supseteq \mathcal{B}$ be a collection that includes \mathcal{B} , the Boolean algebra of coalitions. Let $\mathcal{W}' \subseteq \mathcal{B}'$ be a collection of winning sets. A nonempty collection $\mathcal{Z} \subseteq \mathcal{B}$ is a (\mathcal{B}) -cover of $S' \in \mathcal{B}'$ if $\bigcup \mathcal{Z} := \bigcup_{S \in \mathcal{Z}} S \supseteq S'$. If \mathcal{Z} is a finite cover, then $\bigcup \mathcal{Z} \in \mathcal{B}$, since \mathcal{B} is a Boolean algebra. So, we can assume without loss of generality that $\#\mathcal{Z}$ is either 1 or infinite. Let $M(\mathcal{W}')$ be the collection of pairs $(\mathcal{Y}, \mathcal{Z})$ such that

- (a) $\mathcal{Y} \subseteq \mathcal{W}'$ and
- (b) $\mathcal{Z}: \mathcal{Y} \rightarrow \mathcal{B}$ is a correspondence that maps each winning set $W \in \mathcal{Y}$ to a \mathcal{B} -cover $\mathcal{Z}(W)$ of W and that satisfies $\bigcap_{W \in \mathcal{Y}} \bigcup \mathcal{Z}(W) = \emptyset$.

Definition 2 The **kappa number** $\kappa(\mathcal{W}')$ of a collection \mathcal{W}' of winning sets relative to \mathcal{B} is $+\infty$ (greater than any cardinal number) if $M(\mathcal{W}') = \emptyset$; otherwise, it is the cardinal number given by²⁹

$$\kappa(\mathcal{W}') = \min_{(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')} \max\{\#\mathcal{Y}, \sup\{\#\mathcal{Z}(W) : W \in \mathcal{Y}\}\}.$$

There is another, obvious extension ν' (defined for collections of winning sets) of the Nakamura number ν (defined for simple games). Let $\nu'(\mathcal{W}')$ be defined exactly as before, with \mathcal{W} replaced by \mathcal{W}' .

Lemma 9 $2 \leq \nu'(\mathcal{W}') \leq \kappa(\mathcal{W}') \leq \nu(\mathcal{W})$ for $\mathcal{W} = \mathcal{W}' \cap \mathcal{B}$.

Proof. $2 \leq \nu'(\mathcal{W}')$ is a consequence of the assumptions $\emptyset \notin \mathcal{W}'$ and $\mathcal{W}' \neq \emptyset$.

We show $\nu'(\mathcal{W}') \leq \kappa(\mathcal{W}')$ next. If $\kappa := \kappa(\mathcal{W}') = +\infty$, the result is obvious. So, suppose otherwise. By Definition 2, there exists $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$ satisfying (a) and (b) in the definition of $M(\mathcal{W}')$ such that

$$\kappa = \max\{\#\mathcal{Y}, \sup\{\#\mathcal{Z}(W) : W \in \mathcal{Y}\}\} \geq \#\mathcal{Y}.$$

²⁹The right-hand side is well-defined, since every *set* of ordinals has a supremum (Hrbacek and Jech, 1984, page 141) as well as the least element. (The same is not true for *classes* that are not sets, like the class of *all* cardinal numbers.) To see the supremum is a cardinal number, let $\sup_{\gamma} \alpha_{\gamma} = \alpha$, where each α_{γ} is a cardinal number. It suffices to show that α is an initial ordinal. Suppose $\beta < \alpha$. Then, by the definition of the supremum, $\beta < \alpha_{\gamma}$ for some γ . Since α_{γ} is a cardinal, $\#\beta < \alpha_{\gamma}$. But $\alpha_{\gamma} \leq \alpha$ implies that α is not equipotent to β .

Since $W \subseteq \bigcup \mathcal{Z}(W)$ for all $W \in \mathcal{Y} \subseteq \mathcal{W}'$, we have $\bigcap_{W \in \mathcal{Y}} W \subseteq \bigcap_{W \in \mathcal{Y}} \bigcup \mathcal{Z}(W) = \emptyset$. It follows from the definition of $\nu'(\mathcal{W}')$ that $\nu'(\mathcal{W}') \leq \#\mathcal{Y} \leq \kappa$.

We show $\kappa(\mathcal{W}') \leq \nu(\mathcal{W})$ finally. Suppose $\nu := \nu(\mathcal{W}) \neq +\infty$ as above. Then, by the definition of ν , there is a collection $\mathcal{Y} \subseteq \mathcal{W}$ such that $\#\mathcal{Y} = \nu \geq 2$ and $\bigcap_{W \in \mathcal{Y}} W = \emptyset$. For each $W \in \mathcal{Y}$, let $\mathcal{Z}(W) = \{W\} \subseteq \mathcal{B}$. We claim that $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$. ((a) $\mathcal{Y} \subseteq \mathcal{W} \subseteq \mathcal{W}'$. (b) $\mathcal{Z}(W)$ is a cover of W since $\bigcup \mathcal{Z}(W) = W$. Also, $\bigcap_{W \in \mathcal{Y}} \bigcup \mathcal{Z}(W) = \bigcap_{W \in \mathcal{Y}} W = \emptyset$.) It follows from Definition 2 that $\kappa(\mathcal{W}') \leq \max\{\#\mathcal{Y}, \sup\{\#\mathcal{Z}(W) : W \in \mathcal{Y}\}\} = \#\mathcal{Y} = \nu$. ■

The inequalities in Lemma 9 are sometimes strict. In the following example, we have $\nu'(\mathcal{W}') < \kappa(\mathcal{W}') < \nu(\mathcal{W})$.

Example 6 Consider $N = \aleph_\omega := \bigcup_{i \in \omega} \aleph_i$. (\aleph_ω is an example of a *singular* cardinal.) Let \mathcal{B}' be the collection of all subsets of N . Let $\mathcal{W}' = \{S \subseteq N : \bigcup S = \aleph_\omega\}$ (it consists of the sets *cofinal* in \aleph_ω). For example, both $W = \{\aleph_0, \aleph_1, \dots\}$ and $W' = \{\aleph_0 + 1, \aleph_1 + 1, \dots\}$ belong to \mathcal{W}' . We have $\nu'(\mathcal{W}') = 2$, since $W \cap W' = \emptyset$. Let \mathcal{B} be the collection of all finite or cofinite sets $S \subseteq N$. Let \mathcal{W} be the collection of all cofinite sets $S \subseteq N$.

We prove that $\mathcal{W} = \mathcal{W}' \cap \mathcal{B} \subsetneq \mathcal{W}'$. (\subseteq): Suppose $S \in \mathcal{W}$. Then, $S \in \mathcal{B}$. Since S^c is finite, we have $\bigcup S = \aleph_\omega$. (\supseteq): Suppose $S \in \mathcal{W}' \cap \mathcal{B}$. Since $S \in \mathcal{B}$, either S is finite or cofinite. To show it is cofinite, suppose it is finite. Then, since S consists of ordinals less than \aleph_ω , $\bigcup S$ is the $<$ -maximum of such ordinals. So it cannot be equal to \aleph_ω .

We prove that $\nu(\mathcal{W}) = \aleph_\omega$. It suffices to show that $\bigcap_{\alpha < \nu} S_\alpha = \emptyset$ for a collection $\{S_\alpha \in \mathcal{W} : \alpha < \nu\}$ implies $\nu \geq \aleph_\omega$. Taking the complement, $\aleph_\omega = N = \#(\bigcup_{\alpha < \nu} S_\alpha^c) \leq \nu \cdot \sup\{\#S_\alpha^c : \alpha < \nu\}$ (Hrbacek and Jech, 1984, Theorem 1.3, page 188). Since the supremum is at most ω , we have $\nu \geq \aleph_\omega$.

Finally, we prove that $\kappa(\mathcal{W}') = \omega$.

We first show that $\kappa(\mathcal{W}') \leq \omega$. Let $\mathcal{Y} = \{W, W'\}$, where $W, W' \in \mathcal{W}'$ are the winning sets given above. Let $\mathcal{Z}(W) = \{\{\aleph_i\} : i \in \omega\} \subset \mathcal{B}$ and $\mathcal{Z}(W') = \{\{\aleph_i + 1\} : i \in \omega\} \subset \mathcal{B}$. It is straightforward to show that $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$. Since $\#\mathcal{Y} = 2$ and $\sup\{\#\mathcal{Z}(W), \#\mathcal{Z}(W')\} = \omega$, we have $\kappa(\mathcal{W}') \leq \omega$.

We next show that $\kappa(\mathcal{W}')$ is not finite. Suppose it is finite. Then, there exists $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$ such that \mathcal{Y} is finite and $\mathcal{Z}(W) \subset \mathcal{B}$ is finite for each $W \in \mathcal{Y}$. Since \mathcal{B} is a Boolean algebra, this implies that $\bigcup \mathcal{Z}(W) \in \mathcal{B}$, which in turn implies that $\bigcup \mathcal{Z}(W)$ is either finite or cofinite. Suppose $\bigcup \mathcal{Z}(W)$ is finite for some $W \in \mathcal{Y} \subseteq \mathcal{W}'$. Then $\bigcup \mathcal{Z}(W) \supseteq W$ implies that $W \in \mathcal{W}'$ is finite. It follows that $W \in \mathcal{W}' \cap \mathcal{B} = \mathcal{W}$, a contradiction. Hence $\bigcup \mathcal{Z}(W)$ is cofinite for all $W \in \mathcal{Y}$. Being a finite intersection of cofinite sets, $\bigcap_{W \in \mathcal{Y}} \bigcup \mathcal{Z}(W)$ is nonempty, contrary to assumption.

Lemma 10 κ is an extension of ν . That is, if $\mathcal{W}' \subseteq \mathcal{B} \subseteq \mathcal{B}'$, then $\kappa(\mathcal{W}') = \nu(\mathcal{W}')$.

Proof. Suppose $\mathcal{W}' \subseteq \mathcal{B}$. Then, $\mathcal{W} = \mathcal{W}' \cap \mathcal{B} = \mathcal{W}'$. Lemma 9 implies $\nu'(\mathcal{W}') \leq \kappa(\mathcal{W}') \leq \nu(\mathcal{W}')$. Since ν' extends ν and \mathcal{W}' is a simple game in this case, we have $\nu'(\mathcal{W}') = \nu(\mathcal{W}')$, from which the conclusion follows. ■

We now give the main result for the extended framework.

Theorem 3 *Let $\mathcal{W}' \subseteq \mathcal{B}'$ be a collection of winning sets and $B \subseteq X$ an agenda. Let $\mathcal{M}(B)_{\mathcal{B}}^N$ be the set of measurable profiles of preferences having a maximal element of B . Then the following two statements are equivalent:*

- (i) $\#B < \kappa(\mathcal{W}')$;
- (ii) *the core $C^+(\mathcal{W}', B, \mathbf{p})$ without majority dissatisfaction is nonempty for all $\mathbf{p} \in \mathcal{M}(B)_{\mathcal{B}}^N$.*

Moreover, these equivalent statements imply, but are not implied by

- (iii) *the core $C(\mathcal{W}', B, \mathbf{p})$ is nonempty for all $\mathbf{p} \in \mathcal{M}(B)_{\mathcal{B}}^N$.*

*Proof.*³⁰ (i) \Rightarrow (ii). Suppose $C^+(\mathcal{W}', B, \mathbf{p}) = \emptyset$ for some profile $\mathbf{p} \in \mathcal{M}(B)_{\mathcal{B}}^N$. We show that $\#B \geq \kappa(\mathcal{W}')$.

By the definition (Definition 1) of C^+ , for each $x \in B$, there is a winning set $W_x \in \mathcal{W}'$ such that $W_x \subseteq \{i : x \notin \max_B \succ_i^{\mathbf{p}}\}$. We claim that $\bigcap_{x \in B} \{i : x \notin \max_B \succ_i^{\mathbf{p}}\} = \emptyset$. (Otherwise, there is a player i whose preference has no maximal element of B .)

Let $\mathcal{Y} = \{W_x : x \in B\} \subseteq \mathcal{W}'$. We have $\#\mathcal{Y} \leq \#B$. For each $W_x \in \mathcal{Y}$, let

$$\mathcal{Z}(W_x) = \{\{i : y \succ_i^{\mathbf{p}} x\} : y \in B\}.$$

We have $\#\mathcal{Z}(W_x) \leq \#B$.

We claim that $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$. (*Details.* We verify (b) of the definition of $M(\mathcal{W}')$. First, $\mathcal{Z}(W_x) \subseteq \mathcal{B}$ since \mathbf{p} measurable implies $\{i : y \succ_i^{\mathbf{p}} x\} \in \mathcal{B}$. Second, $\mathcal{Z}(W_x)$ is a cover of W_x since $\bigcup \mathcal{Z}(W_x) = \bigcup_{y \in B} \{i : y \succ_i^{\mathbf{p}} x\} = \{i : x \notin \max_B \succ_i^{\mathbf{p}}\} \supseteq W_x$. Third, $\bigcap_{W_x \in \mathcal{Y}} \bigcup \mathcal{Z}(W_x) = \bigcap_{x \in B} \{i : x \notin \max_B \succ_i^{\mathbf{p}}\} = \emptyset$ by the claim above.)

By the definition (Definition 2) of κ , we get

$$\kappa(\mathcal{W}') \leq \max\{\#\mathcal{Y}, \sup\{\#\mathcal{Z}(W_x) : x \in B\}\} \leq \#B.$$

(ii) \Rightarrow (i). Suppose $\#B \geq \kappa := \kappa(\mathcal{W}')$. We construct a profile $\mathbf{p} \in \mathcal{M}(B)_{\mathcal{B}}^N$ such that $C^+(\mathcal{W}', B, \mathbf{p}) = \emptyset$. By the definition (Definition 2) of $\kappa(\cdot)$, there exists $(\mathcal{Y}, \mathcal{Z}) \in M(\mathcal{W}')$ such that³¹

- (a) $\mathcal{Y} = \{L_\alpha : \alpha < \kappa'\} \subseteq \mathcal{W}'$, where $\kappa' := \#\mathcal{Y}$;
- (b) \mathcal{Z} maps each $L_\alpha \in \mathcal{Y}$ to $\mathcal{Z}(L_\alpha) = \{L_\alpha^\beta : \beta < \beta(\alpha)\} \subseteq \mathcal{B}$ (where $\beta(\alpha) := \#\mathcal{Z}(L_\alpha)$) satisfying $L'_\alpha := \bigcup \mathcal{Z}(L_\alpha) = \bigcup_{\beta < \beta(\alpha)} L_\alpha^\beta \supseteq L_\alpha \in \mathcal{W}'$ and $\bigcap_{L_\alpha \in \mathcal{Y}} \bigcup \mathcal{Z}(L_\alpha) = \bigcap_{\alpha < \kappa'} L'_\alpha = \emptyset$;
- (c) $\kappa = \max\{\kappa', \sup\{\beta(\alpha) : \alpha < \kappa'\}\} \leq \#B$.

³⁰Lemmas 7 and 9 are not useful for proving Theorem 3 from Theorem 2, and vice versa.

³¹We can write \mathcal{Y} and $\mathcal{Z}(L_\alpha)$ in the form below (footnote 21).

Write $B = \{x_\alpha : \alpha < \#B\}$ and let $B' = \{x_\alpha : \alpha < \kappa'\}$.

Case: κ is finite. We construct a profile \mathbf{p} such that the dominance relation $\succ_{\mathcal{W}}^{\mathbf{p}}$ has a cycle consisting of κ alternatives. Since $\beta(\alpha) \leq \kappa$ is finite for all $\alpha < \kappa'$ in this case, $L'_\alpha = \bigcup_\beta L_\alpha^\beta$ is a finite union of elements of the Boolean algebra \mathcal{B} . So we can assume $L'_\alpha \in \mathcal{B}$ and $\beta(\alpha) = 1$ without loss of generality. By (c), we have $\kappa' = \kappa$. Write $x_\kappa = x_0$ and fix a relation $\succ' = \{(x_0, y) : y \notin B'\}$. Let

$$\succ_i^{\mathbf{p}} = \{(x_{\alpha+1}, x_\alpha) : L'_\alpha \ni i\} \cup \succ'$$

for all $i \in N$. The profile \mathbf{p} is the same as that in the proof of Theorem 2, except that L_α is replaced by $L'_\alpha \in \mathcal{B}$. The rest of the proof runs as before.

Case: κ is infinite. We construct a profile \mathbf{p} such that for each alternative $x_\alpha \in B$, there is a winning set of players i who prefer another alternative $x_{\alpha+\beta_i} \in B$ to x_α .³² Since preferences involving alternatives outside B are irrelevant, we construct the profile in such a way that it satisfies $\succ_i^{\mathbf{p}} \subseteq B \times B$ for all i .

We define \mathbf{p} by specifying $\{i : x \succ_i^{\mathbf{p}} y\}$ for each pair $(x, y) = (x_{\alpha'}, x_\alpha) \in B \times B$ satisfying $\alpha' > \alpha$. (If (x, y) does not satisfy the conditions concerning the indices, then $\{i : x \succ_i^{\mathbf{p}} y\} = \emptyset$.) Note that each such pair can be uniquely written as $(x, y) = (x_{\alpha+\beta}, x_\alpha)$ for some $\beta > 0$.³³ Let $L_\alpha^1 = \emptyset$ if $\beta(\alpha) = 1$; otherwise, we can assume $\beta(\alpha)$ is infinite. Define \mathbf{p} by the following, which immediately establishes that \mathbf{p} is measurable:³⁴

$$\{i : x_{\alpha+\beta} \succ_i^{\mathbf{p}} x_\alpha\} = \begin{cases} L_\alpha^0 \cup L_\alpha^1 \in \mathcal{B} & \text{if } \alpha < \kappa' \text{ and } \beta = 1, \\ L_\alpha^\beta \in \mathcal{B} & \text{if } \alpha < \kappa' \text{ and } 1 < \beta < \beta(\alpha), \\ N \in \mathcal{B} & \text{if } \kappa' \leq \alpha < \#B \text{ and } \beta = 1, \\ \emptyset \in \mathcal{B} & \text{otherwise.} \end{cases}$$

For each $x_\alpha \in B$, we have $\{i : x_\alpha \notin \max_B \succ_i^{\mathbf{p}}\} = \bigcup_\beta \{i : x_{\alpha+\beta} \succ_i^{\mathbf{p}} x_\alpha\}$, which equals $\bigcup_{\beta < \beta(\alpha)} L_\alpha^\beta = L'_\alpha$ (if $\alpha < \kappa'$; the equality holds whether

³²Though not required in our setting, we construct the profile so that individual preferences will be acyclic. This is achieved by the following: no player prefers an alternative $x_{\alpha'} \in B$ with smaller index to x_α (if $\alpha' \leq \alpha$, then $x_{\alpha'} \not\succ_i^{\mathbf{p}} x_\alpha$).

³³ β is the unique ordinal isomorphic to $\{\gamma : \alpha \leq \gamma < \alpha'\}$.

³⁴See Appendix A.4 for a graph representation of the profile. For the subsequent argument, we need to verify that $x_{\alpha+\beta}$ and x_α corresponding to the first two cases do exist in B , that is, $\alpha + \beta < \#B$ (and $\alpha < \#B$, which is obvious). For the first and the third cases, since $\alpha < \#B$, we have $\alpha + \beta = \alpha + 1 < \#B$. This is because $\#B$, being an infinite cardinal number, is a limit ordinal by Lemma 1. For the second case, since $\alpha < \kappa' \leq \#B$, we have $\#\alpha < \#B$. This is because $\#B$, being a cardinal number, is not equipotent to $\alpha < \#B$. Similarly, since $\beta < \beta(\alpha) \leq \#B$, we have $\#\beta < \#B$. If α and β are finite, the conclusion is obvious since $\#B$ is infinite in this case. Suppose otherwise. Then, by Lemma 2, we have $\#(\alpha + \beta) = \#\alpha + \#\beta = \max\{\#\alpha, \#\beta\} < \#B = \#(\#B)$. This implies that there is no one-to-one function from $\#B$ to $\alpha + \beta$. Therefore, $\alpha + \beta < \#B$.

$\beta(\alpha)$ is 1 or infinite) or N (otherwise). In either case, the set contains a winning set $L_\alpha \in \mathcal{W}'$. Therefore, $x_\alpha \notin C^+(\mathcal{W}', B, \mathbf{p})$. This establishes that $C^+(\mathcal{W}', B, \mathbf{p}) = \emptyset$.

To establish that $\mathbf{p} \in \mathcal{M}(B)_B^N$, we need to show that each i 's preference $\succ_i^{\mathbf{p}}$ has a maximal element of B . Since $\bigcap_{\alpha < \kappa'} L'_\alpha = \emptyset$, we have $\bigcup_{\alpha < \kappa'} (L'_\alpha)^c = N$. So, for each $i \in N$, there is an $\alpha < \kappa'$ such that $i \notin L'_\alpha$. Since $i \notin L_\alpha^\beta$ for any $\beta < \beta(\alpha)$, we have $x_\alpha \in \max_B \succ_i^{\mathbf{p}}$ by the definition of \mathbf{p} .

(ii) \Rightarrow (iii). Immediate from Lemma 7.

(iii) \nRightarrow (i). Consider Example 6. We first prove that $C(\mathcal{W}', B, \mathbf{p}) = C(\mathcal{W}, B, \mathbf{p})$ for all $\mathbf{p} \in \mathcal{M}(B)_B^N$ and all $B \subseteq X$. By Lemma 7, it suffices to show that $C(\mathcal{W}, B, \mathbf{p}) \subseteq C(\mathcal{W}', B, \mathbf{p})$. Suppose $x \in B$ is not in $C(\mathcal{W}', B, \mathbf{p})$. Then, there is $y \in B$ and $S \in \mathcal{W}'$ such that $S \subseteq \{i : y \succ_i^{\mathbf{p}} x\}$. Since \mathbf{p} is measurable, $\{i : y \succ_i^{\mathbf{p}} x\} \in \mathcal{B}$ is either finite or cofinite. If it is finite, then S is finite, implying $S \in \mathcal{W}' \cap \mathcal{B} = \mathcal{W}$, a contradiction. It follows that $\{i : y \succ_i^{\mathbf{p}} x\}$ is cofinite, hence it belongs to \mathcal{W} . So $x \notin C(\mathcal{W}, B, \mathbf{p})$.

Choose an agenda B satisfying $\#B = \omega$. (i) is violated since $\#B = \omega \not\leq \omega = \kappa(\mathcal{W}')$. On the other hand, since $\#B = \omega < \aleph_\omega = \nu(\mathcal{W})$, Theorem 2 implies that $C(\mathcal{W}, B, \mathbf{p}) \neq \emptyset$ for all $\mathbf{p} \in \mathcal{M}(B)_B^N$. Then (iii) is satisfied since $C(\mathcal{W}', B, \mathbf{p}) = C(\mathcal{W}, B, \mathbf{p})$. ■

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A Not to be Published

A.1 Proof of Lemma 1

We show that an infinite cardinal number is a limit ordinal.

Suppose α is an infinite cardinal number that is not a limit ordinal. Then, being a successor ordinal, $\alpha = \beta + 1 = \beta \cup \{\beta\}$ for some ordinal β . Clearly, $\beta < \alpha$. Since α is initial, there is no bijection between $\alpha = \beta \cup \{\beta\}$ and β . But we can construct such a bijection f as follows: $f(\gamma) = 0$ if $\gamma = \beta$; $f(\gamma) = \gamma + 1$ if $\gamma \in \omega$; $f(\gamma) = \gamma$ if $\gamma \in \beta$ but $\gamma \notin \omega$.

A.2 Proof of Lemma 2

We show that $\#(\alpha + \beta) = \#\alpha + \#\beta$, where the sum on the left side is the ordinal sum and the sum on the right is the cardinal sum.

Pick disjoint well-ordered sets $(A, <_A)$ and $(B, <_B)$ isomorphic to ordinals α and β , respectively. Then, we have $\#(\alpha + \beta) = \#(A \cup B)$ (Hrbacek and Jech, 1984, Theorem 5.5, page 152). This is equal to $\#\alpha + \#\beta$ by the definition of the cardinal sum.

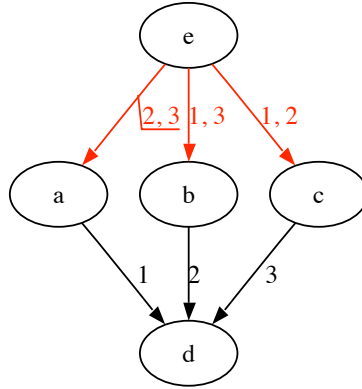
A.3 The inclusion of Lemma 6 is strict for some profile

We show that there is a profile \mathbf{p} such that there is an alternative not in $C^+(\mathcal{W}, B, \mathbf{p})$ but in $C(\mathcal{W}, B, \mathbf{p}) \cap (\bigcup_i \max_B \succ_i^{\mathbf{p}})$.

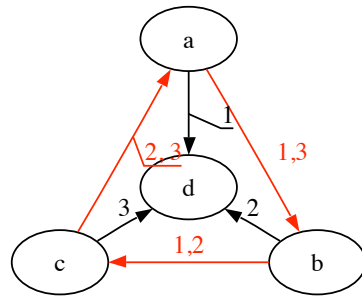
We “replicate” sufficiently many times an example for which C^+ is a proper subset of C , and then add an “insignificant” player whose maximal alternative contains an alternative belonging to the difference $C \setminus C^+$. We build on Example 1 here. Let $N = \{1, 1', 2, 2', 3, 3', 4\}$ and let \mathcal{W} consist of the coalitions having a majority. Let $X = \{a, b, c, d, e\}$. Define a profile by $\succ_1 = \succ_{1'} = \{(a, d), (e, b), (e, c)\}$, $\succ_2 = \succ_{2'} = \{(b, d), (e, a), (e, c)\}$, $\succ_3 = \succ_{3'} = \{(c, d), (e, a), (e, b)\}$, and $\succ_4 = \emptyset$ (or any preference such that d is a maximal). Then, as before, the core C is $\{d, e\}$ and the core C^+ without majority dissatisfaction is $\{e\}$. So C^+ is a proper subset of $C \cap (\bigcup_i \max \succ_i) = \{d, e\}$.

A.4 Figures

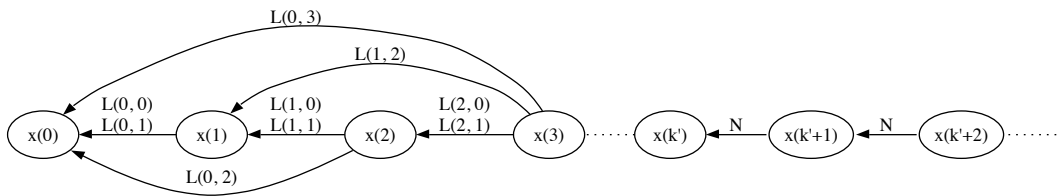
A.4.1 Example 1



A.4.2 Example 2



A.4.3 Profile for the proof (ii) \Rightarrow (i) of Theorem 3, when κ is infinite



For each $\alpha, \beta < \#B$, $x(\alpha)$ denotes x_α , $L(\alpha, \beta)$ denotes L_α^β , and k' denotes κ' .