The valuation of barrier options prices : A methods review

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THE VALUATION OF BARRIER OPTIONS PRICES: A METHODS REVIEW

INTRODUCTION

In the Over-Counter-Market [OTM], barrier options become extremely popular. They are the path dependent options. The existence of the option is depending upon whether the underlying asset price has touched the barrier which is the critical value during the lifetime of the option.

In recent years, barrier options are used as a useful hedging risk management instrument by the trader. Also, barrier options have another advantage. In the trading of barrier options, the holder or buyer finds that they provide more flexibility in tailoring the returns of the portfolio. They are less expensive than standard options.

Generally, different types of barrier options are known such as a Knock- in feature.

This means that the option is activated if only the underlying asset price first hits the value called barrier. However, a Knock- out feature is met in the contrary case.

The option is cancelled if the underlying asset price hits the barrier. Hence, there are eight different types of barrier options which are as follows:

- Down – and - Out call option;
- Down – and – Out put option;
- Up – and – Out call option;
- Up – and – Out put option;
- Down – and – In call option;
- Down – and – In put option;

- Up – and – In call option;
- Up – and – In put option.

The valuation of barrier option is the main part of our research. Let's write that Merton was first to value in 1973 the barrier option. He used the BS Partial Differential Equation Method (PDEM). This is the first section (I). The remaining of the article is organized in three sections. The next one is on the Tree Methods (TM) (II) while the third section examines the simulation methods (III). Finally the last and fourth section analyses the recent methods like the Fast Fourier Transform methods (FFTM) and Laplace Transform Methods (LTM), the Adaptive Mesh Methods (AMM) (IV), BSDE methods and it concludes (Figlewski & Gao, 1999). The next part is about the empirical performance of barrier options evaluation models in which we are going to use the numerical applications. We are discussing now about the Partial Differential Equation Methods.

I. – PARTIAL DIFFERENTIAL EQUATION METHODS

Barrier options are exotic options. Their evaluation is made with the BS model. Before this model, let's talk about the oldest model known under the name of Gaussian model from its founder a French mathematician Louis Bachelier. He received his Ph.D on March 29, 1900 under the leadership of Pr. Henri Poincaré from Sorbonne University. He was first who used advanced mathematics to price options. Through his thesis, titled "*Theory of speculation*", he used a new concept known as a Brownian motion. The concept from physics is also called a Wiener process. It is giving the foundation of BS and many others know financial models.

Fisher Black born in 1938 and died in 1995 due to the throat cancer received his Ph.D in applied mathematics from Harvard University in 1964. Meanwhile time, Myron Scholes who was born in 1941 received his PhD in 1969 from University of

Chicago. In 1968, Myron Scholes joined the Massachusetts Institute of Technology (MIT) at Sloan School of Management. He took occasion to meet Black. Robert Merton was born in 1944. And he got his Ph.D in 1970 under the mentorship of Paul Samuelson. These three scientific played a key role in the valuation formula for options.

The BS model is published in 1973 by Fisher Black and Myron Scholes. The publishing journal was named Political economy accepted the article "*the pricing of options and Corporate liabilities"* after Eugene Fama and Merton Miller offered a second look on the article. It was in the May-June 1973 issue.

Black and Scholes (1973) built the derivative pricing theory on the Geometric Brownian Motion (GBM). The price of an asset follows a GBM if it satisfies the stochastic differential equation (SDE) below:

$$\mathrm{dS} = \mu S \mathrm{dt} + \sigma S \mathrm{dW} \tag{1}$$

where :

W_t : Weiner process or Brownian motion;
μ : Drift;
σ : Volatility.

Considering the model, it was to price the European option from which the BS partial Differential Equation is given below:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mathrm{r}\, \mathrm{S}\frac{\partial f}{\partial S} - \frac{\partial f}{\partial t} - \mathrm{rf} = 0 \tag{2}$$

The solution of the equation (2) is giving the theoretical price of the European call option as follows:

$$\mathrm{C(S, t)} = \mathrm{SN}\,(d_1) - \mathrm{K}e^{-T}\mathrm{N}\,(d_2) \tag{3}$$

where:

C : Call option;
S : Stock price;
K : Strike price;
r : Risk free rate;
σ : Volatility of the stock;
T : Time to maturity;
N (.): Cumulative distribution function.

From the call option, we are able to deduct its opposite, the put option.

$$P(S,t) = - S\,N(d_1) + Ke^{-T}N(d_2) \quad (4)$$

where:

P(S,t) : Put option

Let's explain d_1 *and* d_2 :

$$d_1 = [\ln(S/K) + (r + \sigma^2/2)T]\sigma\sqrt{T} \quad (5)$$

$$d_2 = [\ln(S/K) + (r - \sigma^2/2)T]\sigma\sqrt{T} \quad (6)$$

In fact, considering their impact on over counter market, some authors have written articles on PDE method to price barrier options Table 1.1.

Table 1.1.: Articles on PDE methods to price barrier options

Years	Authors	Country	Subject	Observation
1969	Snyder	USA	Description	Down and Out options
1973	Merton R.	USA	Closed form solution	Call down and out
1978	Brennan	USA	valuation	Real option

1983	Johnson		Valuation	Put option
1983	Ewnine J.	USA	Analytical expression	Option
1983	Bergman		Pricing	Put option asset
1991	Rubinstein M & Reiner	USA	Pricing	Option asset

The PDE method has not been the only one of methods to price barrier options. We are going to study here a specific barrier options valuation method called "tree methods".

II.- TREE METHODS

Barrier options are different from standard options. Tree methods are used to value them. In this category of valuation methods, we have two groups of models. They are grouped in the binomial option-pricing model and the trinomial option-pricing model. What kind of definition we could give to the binomial tree model?

2.1.- Binomial tree model

The binomial tree model was developed in 1979 by three researchers who are John, C. Cox, Stephen A. Ross and Mark Rubinstein in their paper titled:"*Option pricing: A simplified approach"*. Valuing the option, the model has an iterative approach. There are two possible outcomes with for each interaction a move up or a move down. The probability of an up move is p, so that the probability of a down move is $1-$ p. The stock price process is illustrated as:

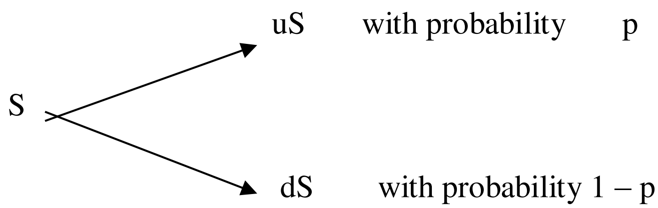


Figure 1 : Binomial stock tree

And the option price process is represented as::

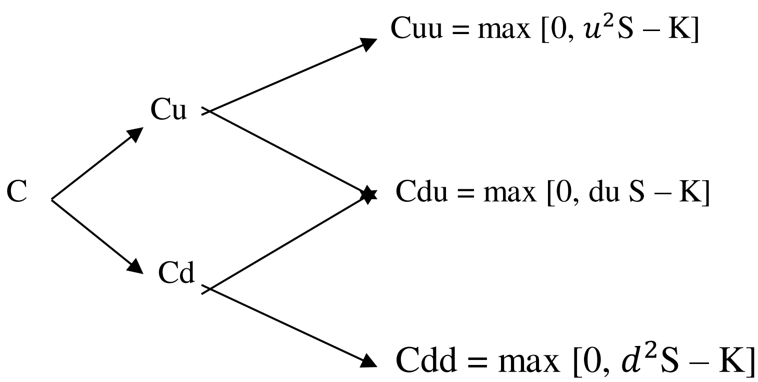


Figure 2: Binomial price tree

Due to the limitations of the BTM, a researcher known as a professor at Waterloo University in Canada and named Phelim P. Boyle proposed (1941) in 1986 a trinomial tree model (TTM).

2.2.- Trinomial Tree Model

In this model, the underlying stock price has three possible steps: an up, down and stable or middle with:

$$u = e^{\sigma\sqrt{2\Delta}t} \quad (7)$$

$$d = e^{-\sigma\sqrt{2\Delta}t} = \frac{1}{u} \quad (8)$$

$$m = 1 \quad (9)$$

The transition probabilities corresponding at each node are follows:

$$p_u = (\frac{e^{(r-q)\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}}} - \frac{e^{-\sigma\sqrt{\Delta t/2}}}{e^{-\sigma\sqrt{\Delta t/2}}})^2 \quad (10)$$

$$p_d = (\frac{e^{\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}}} - \frac{e^{(r-q)\Delta t/2}}{e^{-\sigma\sqrt{\Delta t/2}}})^2 \quad (11)$$

$$p_m = 1 - (p_u + p_d) \text{ or } 1 - p_u - p_d \quad (12)$$

where:

u : Up, d = down, m = middle:

Δt : Length of time per step;

r : Risk-free interest rate;

σ : Volatility of the underlying;

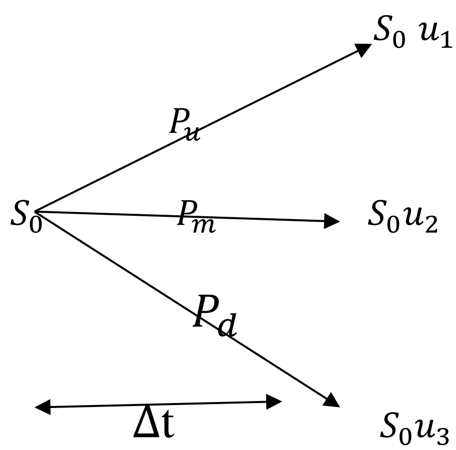
q : Dividend yield;

P : Probability.

p_u, p_d and p_m have to be in the interval [0, 1]. To satisfy at the above condition, the Δt must be:

$$\Delta t < 2\frac{\sigma^2}{(r-q)^2} \quad (13)$$

So, the tree of prices can be calculated at each node. Here are built examples of TTM.



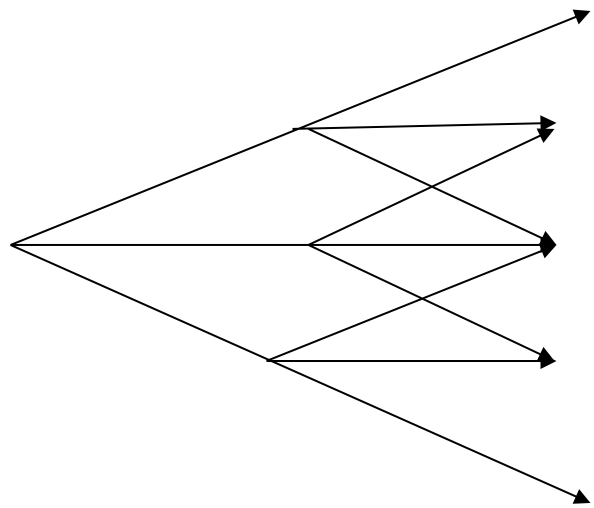


Figure 3: Trinomial stock tree. Figure 4: Trinomial stock price tree

In this context, the Boyle's model is different from the model developed in 1991 by Kamrad-Ritchken and the other one proposed by Hull-White.

We have selected authors having written articles on the tree model in the table 2.2.

Table 2: 2: Articles on Tree model to valuate barrier options

Years	Authors	Country	Subject	Observations
1994	Boyle and Lau	North Ireland	Pricing	Binomial lattice
1995	Ritchken	USA	Pricing	Trinomial lattice
1996	Cheuk and Vost	USA	Valuation	Trinomial lattice
1997	Gao	USA	Analytic High Order Trinomial	Double barrier

1999	Boyle and Tiam	North Ireland	Constant Elasticity variance	Trinomial lattice
1999	Figlewski and Gao	USA	Valuation	Trinomial lattice

For solving option valuation problems, the theory of theory in finance has been extended to the Monte-Carlo simulations methods.

III.- MONTE-CARLO SIMULATIONS METHODS

The first application of Monte Carlo simulations methods to price option was made by Phelim Boyle in 1977. This application was made on European option. In 1996, Broadie M. and Glasserman P. used them to price Asian options. Finally, in 2001, Longstaf F. A. and Schwartz E. S. developed the practical use of these methods to price American options. However, MCM was first introduced in finance by David B. Hertz (1964). As the methodology of the technique of Monte-Carlo methods, the value of the option is the result of the three steps:

1) *Calculate potential future prices of the underlying asset;*
2) *Calculate the payoff of the option for each of the potential underlying price paths;*
3) *Discount the payoffs back to today and average then to determine the expected price.*

The stock price evaluation equation following is the standard model to price any equity. It is given by the wiener process.

$$Sk_{\Delta t} = S_0 \exp(\textstyle\sum_{i=1}^{k} \left[\!\left[\left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \left(\sigma\sqrt{\Delta t}\right)\varepsilon_i \right]\!\right]) \qquad (14)$$

where:

S_0 : Stock price today;

$S_{\Delta t}$: Stock price at a small time into the future;

Δt : Small increment of time;

μ : Expected return;

σ: : Expected volatility;

ε : (Randon) number sampled from a standard normal distribution.

The pricing of the option gives for call and put option the payoffs:

$$\text{Payoff of call option} = \text{Max}\,(S - K, 0) \quad (15)$$

$$\text{Payoff of put option} = \text{Max}\,(K - S, 0) \quad (16)$$

where :

K : Strike price;

S : Average value of the asset price

Then,

$$G = E(g(x)) = \int_a^b g(x) f(x)\,dx = \frac{1}{N}\sum_{i=1}^{N} g(x_i) \quad (17)$$

with :

G(.) : Generator function;

E : Mathematical expectation.

In our reading of the literature on the Monte Carlo simulations methods, some articles and their authors are reported in the table 3.3.

Table 3.3.: Articles on Monte Carlo simulations Methods

Years	Authors	Country	Subject	Observations
1977	Boyle P.	North Ireland	Valuation	Efficiency of the method
1977	Vasicek		Valuation	Interest Rate Model Parameters
1992	Chan Karolyi, Longstaff and Sanders		CKLS model	Factor interest rate model
1996	Andersen and Brothertom-Rotcliffe		Brownian bridge	Simple constant barriers
1997	Beaglehole, Doyovig and Zhou		Valuation	Simple constant barriers
1999	Baldi, Caramellino and Iovino		Sharp large deviation techniques	General diffusion process

Different researchers tried to find others methods to valuate barrier options that the literature in derivatives calls recent methods.

IV.- RECENT METHODS

They are composed of two (02) models which are the Fast Fourier Transform Method (FFT) and the Adaptive Mesh Model (AMM). The foundation of the method is based on the mathematical tool where there are integrals on the Fourier Transform.

4. 1. Fast Fourier Transform (FFT) Methods

More researchers have used Fast Fourier Transform in their works (Cooley and Turkey, 1995; Gauss, 1805; Beagland, 1969; Strang, 1993). This method takes its origin from Fourier series having for their inventor a French mathematician Jean-Baptiste Joseph Fourier (1768-1830).

The Fourier series are including in their form a constant. They are written as follows:

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_0 \sin\left(\frac{2\pi nt}{T}\right) =$$

$$= \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\tau mt}{T}\right) + \sum_{n=1}^{\infty} b_0 \sin\left(\frac{2\pi nt}{T}\right) \tag{18}$$

where :

a, m, b and n : coefficients of the Fourier series.

Their role consists to determine the weights for each of the sinusoids.

As the generalization of the complex Fourier series, the Fourier Transform can be defined as the function of $F_{(\omega)}$ from the function $f_{(x)}$. The function is represented by:

$$F_{(\omega)} = \int_{-\infty}^{\infty} f_{(x)}\, e^{-i\omega x} d_{(x)} \tag{19}$$

Then, the inverse of Fourier Transform is:

$$f_{(-x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{(\omega)}\, e^{i\omega x} d_{\omega} \tag{20}$$

where :

$i = \sqrt{-1}$;

$\cos\theta = \cos\theta + \sin\theta.$

Fourier Transform methods are widely used to valuate options as they have been applied also to solve a great number of problems in mathematics and physical sciences by different researchers. In options prices, much in the recent literature consider the preliminaries works of Bakshi and Chen, 1997, Scott, 1997, Bates, 1996, Heston, 1993, Chen and Scott, 1992, Walker, 1996, Bakshi and Madan, 1999. Additionally, it is the case of Carr P. and Madan D. (1999) who have been first to illustrate the technique. After their seminal article "*Option valuation using the fast Fourier transform*", many others articles are published by authors such as Duffie and al., 2000; Hubalek and al., 2006; Borowkov and Novikov, 2012; Kwok Y.Y. and al., 2012.

In fact, Selby, 1983 and Buser, 1986 have introduced first in finance to value options the Laplace Transform methods.

4. 2. Laplace Transform (LT) Methods

For their origin, they are named after the invention of an integral transform by Pierre-Simon Laplace (1749-1827) who was a French mathematician and astronomer. The integral transforms a function of a real variable to a function of a complex variable. The formula of the Laplace Transform of f(t) can be defined as :

$$\mathrm{F(s)} = \int_0^{\infty} f(t)\, e^{-st} \mathrm{dt} \tag{21}$$

where:

s: $\sigma + j\omega$:

j: $\sqrt{-}\ 1$.

The inverse of Laplace Transform is noted: $£^{-1}$. So, the inverse transform of F(s) is the equivalent of:

$$\text{F(s)} = £^{-1}[F(s)] = £^{-1}[£f(t)] = \text{f(t)} \tag{22}$$

In terms of the end on the Laplace Transform methods (LTM), let us write that they have been used in pricing of the exotic options like barrier options and Asian options. Geman and Yor, 1993, 1996; Pelsser, 2000; Sbuelz, 1999, 2005; Davydov and Linetsky, 2001a, 2001b; Leblanc and Scaillet, 1998 and Cathcart, 1998; Geman and Eydeland, 1995; Fu, Madan and Wang, 1998; Lipton, 1999; Fusai, 2000; Akahori, 1995; Ballotta, 2001; Ballotta and Kyprianou, 2001; Hesney et al., 1995; Chesney et al., 1997; Dassios, 1995; Hugonnier, 1999; Linetsky, 1999; Fusai and Tagliani, 2001; Fusai, 2004.

Then, it is useful to illustrate that LTM can be reconverted simply from PDE into an Ordinary Diffential Equation (ODE). This kind of equations is easier solvable. However, for many problems with the valuation of exotic options, people apply for the approach of the adaptive Mesh Methods.

4.3. Adaptive Mesh Methods

Figlewski and Gao, 1999 developed the AMM for exotic options. The principals of these methods are relatively simples. Let's consider the last paper of Dong-Hyun Ahn, Sephen Figlewski and Bin Gao (1999), the AMM is presented in step by sep. Each step represents an equation.The base lattice is a trinomial set up to approximate the risk neutralized price process for the underlying asset. The asset price S follows the standard diffusion process.

$d\ln S = \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma dw$ (23)

where :

S: Stock price;

r : Riskless interest rate;

q : Rate of dividend yield;

σ: Volatility;

w : Brownian motion.

It is convenient to define X.ln S and the drift $(r - q - \frac{\sigma^2}{2})$, thus we have the equation following:

$dX = \mu dt + \sigma dw$ (24)

The process is discretized approximated by a trinomial process:

$X_{t+k} - X_t = ak + \sigma h$ with the probability $p_u = \frac{k}{2h^2}$: ak with the probability $p_m = 1 - \frac{k}{h^2}$: ak – σh with the probability $p_d = \frac{k}{2h^2}$. (25)

where :

k : Discrete time sep;

h : Price step:

p_u: Probability for up;

p_m: Probability for middle:

p_d: Probability for down.

The value of the option for a given asset, price and time for each node becomes:

$V(X,t) = exp(-rk)[p_u V(X + rk\sigma h, t+k) + p_m(X + rk, t + k) + p_d V(X + rk \sigma h, t + k)]$ (26)

where :

X : Stock price

r : $0\sqrt{k}$;

h : $\sqrt{3k}$;

$p_u = p_d = \frac{1}{6}$;

$p_m = \frac{2}{3}$.

Finally, the above model is the High Order Trinomial (HOT) model proposed by Gao B. in 1997.

Our proposal on the barrier option valuation is the use of the Backward Stochastic Differential Equations. We call the model the BSDE methods.

4.4.- BSDE Methods

The model to value barrier options from BSDE methods depend on the development of the backward stochastic differential equations. The pioneers in this domain are El Karoui N. (1997), Peng S. (1990), Pardoux M. (1990), Quenez M. C. (1990) and Bismut (1973). Recently the BSDEs have their applications in finance problems such as the valuation of options. As the introduction, we will start to define what the BSDE are.

$$-\,\mathrm{d}Y_t = \mathrm{f}(\mathrm{t},, Y_t, Z_t)\mathrm{dt} - Z_t \mathrm{d}w_t \qquad Y_t = \varepsilon \qquad (27)$$

In other hand, the formula can be written as:

$$\mathrm{d}Y_t = -\,\mathrm{f}(\mathrm{t}, Y_t, Z_t)\,\mathrm{dt} + Z_t \mathrm{d}W_t \qquad (28)$$

$$Y_t = \mathrm{g}(S_t)_{t\,\in[0,T]}) \qquad (29)$$

$$Y_S = \mathrm{E}\,(Y_u + \int_S^u f(Y_t)\,\mathrm{dt}|F_t) \qquad (30)$$

where :

f : Generator;
ε : Terminal condition.

CONCLUSIONS AND DISCUSSIONS

The methods of the evaluation that we have discussed in this research can be applied to value all versions of the barrier options: PDE methods, Tree methods, Monte Carlo Methods and recent methods in which we have proposed a new model having a basis the BSDE. Although different, all the methods have to give the same results.

FURTHER WORK

For future work, we will discuss about the empirical performance of each barrier options evaluation model in which we are will use the numerical applications.

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